Short HW1 - Preparing for the course

Useful python libraries, Probability, and Linear algebera

Instructions

General

- **First, don't panic!**
  - This assignment seems longer than it actually is.
  - In the first part, you are mostly required to run *existing* code and complete short python commands here and there.
  - In the two other parts you need to answer overall 4 analytic questions.
  - **Note**: The other 3 *short* assignments will be shorter and will *not* require programming.
- **Individually or in pairs?** Individually only.
- **Where to ask?** In the [Piazza forum](#).
- **How to submit?** In the webcourse.
- **What to submit?** A pdf file with the completed jupyter notebook (including the code, plots and other outputs) and the answers to the probability/algebra questions (Hebrew or English are both fine).
  Or two separate pdf files in a zip file. All submitted files should contain **your ID number** in their names.
- **When to submit?** Sunday 09.06.2024 at 23:59.
- **Important!** Note that any deviation from the aforementioned guidelines will result in points deduction.

Specific

- First part: get familiar with popular python libraries useful for machine learning and data science. We will use these libraries heavily throughout the major programming assignments.
  - You should read the instructions and run the code blocks sequentially.
    In 10 places you are reqired to complete missing python commands or answer short questions (look for the **TODO** comments, or notations like **(T3)** etc.). Try to understand the flow of this document and the code you run.
  - Start by loading the provided jupyter notebook file (*Short_HW1.ipynb*) to [Google Colab](#), which is a very convenient online tool for running python scripts combined with text, visual plots, and more.
  - Alternatively, you can [install jupyter](#) locally on your computer and run the provided notebook there.
- Second and third parts: questions on probability and linear algebra to refresh your memory and prepare for the rest of this course.
  The questions are mostly analytic but also require completing and running simple code blocks in the jupyter notebook.
  - Forgot your linear algebra? Try watching [Essence of LA](#) or reading [The Matrix Cookbook](#).
  - Forgot your probability? Try reading [Probability Theory Review for Machine Learning](#).

Important: How to submit the notebook's output?

You should only submit PDF file(s). In the print dialog of your browser, you can choose to `Save as PDF`. However, notice that some of the outputs may be cropped (become invisible), which can harm your grade.

To prevent this from happening, tune the "scale" of the printed file, to fit in the *entire* output. For instance, in Chrome you should lower the value in `More settings->Scale->Custom` to contain the entire output (50%~ often work well).

Good luck!

What is pandas?

Python library for Data manipulation and Analysis

- Provide expressive data structures designed to make working with “relational” or “labeled” data both easy and intuitive.
- Aims to be the fundamental high-level building block for doing practical, real world data analysis in Python.
- Built on top of NumPy and is intended to integrate well within a scientific computing.
- Inspired by R and Excel.

Pandas is well suited for many different kinds of data:

- **Tabular data** with heterogeneously-typed columns, as in an SQL table or Excel spreadsheet
- Ordered and unordered (not necessarily fixed-frequency) **time series data**.
- **Arbitrary matrix data** (homogeneously typed or heterogeneous) with row and column labels
- Any other form of observational / statistical data sets (can be unlabeled)

Two primary data structures

- **Series** (1-dimensional) – Similar to a column in Excel’s spreadsheet
- **Data Frame** (2-dimensional) – Similar to R’s data frame

A few of the things that Pandas does well

- Easy handling of **missing data** (represented as NaN)
- Automatic and explicit **data alignment**
- Read and Analyze **CSV** , Excel Sheets Easily
- Operations
- Filtering, Group By, Merging, Slicing and Dicing, Pivoting and Reshaping
- Plotting graphs

Pandas is very useful for interactive data exploration at the data preparation stage of a project

The offical guide to Pandas can be found [here](#)

Pandas Objects

```
import pandas as pd
import numpy as np
```

Series is like a column in a spreadsheet.

```
s = pd.Series([1,3.2,np.nan,'string'])
s
```

```
0         1
1       3.2
2       NaN
3    string
dtype: object
```

DataFrame is like a spreadsheet – a dictionary of Series objects

```
data = [['ABC', -3.5, 0.01], ['ABC', -2.3, 0.12], ['DEF', 1.8, 0.03],
['DEF', 3.7, 0.01], ['GHI', 0.04, 0.43], ['GHI', -0.1, 0.67]]

df = pd.DataFrame(data, columns=['gene', 'log2FC', 'pval'])

df
```

	gene	log2FC	pval
0	ABC	-3.50	0.01
1	ABC	-2.30	0.12
2	DEF	1.80	0.03
3	DEF	3.70	0.01
4	GHI	0.04	0.43
5	GHI	-0.10	0.67

Next steps: Generate code with `df` View recommended plots

Input and Output

How do you get data into and out of Pandas as spreadsheets?

- Pandas can work with XLS or XLSX files.
- Can also work with CSV (comma separated values) file
- CSV stores plain text in a tabular form
- CSV files may have a header
- You can use a variety of different field delimiters (rather than a 'comma'). Check which delimiter your file is using before import!

Import to Pandas

> `df = pd.read_csv('data.csv', sep='\t', header=0)`

For Excel files, it's the same thing but with read_excel

Export to text file

> `df.to_csv('data.csv', sep='\t', header=True, index=False)`

The values of header and index depend on if you want to print the column and/or row names

Case Study – Analyzing Titanic Passengers Data

```
import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np
import pandas as pd
import os


#set your working_dir
working_dir = os.path.join(os.getcwd(), 'titanic')

url_base = 'https://github.com/Currie32/Titanic-Kaggle-Competition/raw/master/{}.csv'
train_url = url_base.format('train')
test_url = url_base.format('test')

# For .read_csv, always use header=0 when you know row 0 is the header row
train = pd.read_csv(train_url, header=0)
test = pd.read_csv(test_url, header=0)
# You can also load a csv file from a local file rather than a URL
```

(T1) Use [`pandas.DataFrame.head`] to display the top 6 rows of the `train` table

```
# DONE: print the top 6 rows of the table

train.head(6)
```

	PassengerId	Survived	Pclass	Name	Sex	Age	SibSp	Parch	Ticket	Fare	Cabin	Embarked
0	1	0	3	Braund, Mr. Owen Harris	male	22.0	1	0	A/5 21171	7.2500	NaN	S
1	2	1	1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1	0	PC 17599	71.2833	C85	C
2	3	1	3	Heikkinen, Miss. Laina	female	26.0	0	0	STON/O2. 3101282	7.9250	NaN	S
3	4	1	1	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1	0	113803	53.1000	C123	S
4	5	0	3	Allen, Mr. William Henry	male	35.0	0	0	373450	8.0500	NaN	S
5	6	0	3	Moran, Mr. James	male	NaN	0	0	330877	8.4583	NaN	Q

Next steps: Generate code with `train` View recommended plots

VARIABLE DESCRIPTIONS:

Survived - 0 = No; 1 = Yes
Age - Passenger's age
Pclass - Passenger Class (1 = 1st; 2 = 2nd; 3 = 3rd)
SibSp - Number of Siblings/Spouses Aboard
Parch - Number of Parents/Children Aboard
Ticket - Ticket Number
Fare - Passenger Fare
Cabin - Cabin ID
Embarked - Port of Embarkation (C = Cherbourg; Q = Queenstown; S = Southampton)

```
train.columns
```

```
Index(['PassengerId', 'Survived', 'Pclass', 'Name', 'Sex', 'Age', 'SibSp',
       'Parch', 'Ticket', 'Fare', 'Cabin', 'Embarked'],
      dtype='object')
```

Understanding the data (Summarizations)

```
train.info()
```

```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 891 entries, 0 to 890
Data columns (total 12 columns):
 #   Column       Non-Null Count  Dtype
```

```
---  ------       --------------  -----
 0   PassengerId  891 non-null    int64
 1   Survived     891 non-null    int64
 2   Pclass       891 non-null    int64
 3   Name         891 non-null    object
 4   Sex          891 non-null    object
 5   Age          714 non-null    float64
 6   SibSp        891 non-null    int64
 7   Parch        891 non-null    int64
 8   Ticket       891 non-null    object
 9   Fare         891 non-null    float64
 10  Cabin        204 non-null    object
 11  Embarked     889 non-null    object
dtypes: float64(2), int64(5), object(5)
memory usage: 83.7+ KB
```

```
train.shape
```

```
(891, 12)
```

```
# Count values of 'Survived'
train.Survived.value_counts()
```

```
Survived
0    549
1    342
Name: count, dtype: int64
```

```
# Calculate the mean fare price
train.Fare.mean()
```

```
32.204207968574636
```

```
# General statistics of the dataframe
train.describe()
```

	PassengerId	Survived	Pclass	Age	SibSp	Parch	Fare
count	891.000000	891.000000	891.000000	714.000000	891.000000	891.000000	891.000000
mean	446.000000	0.383838	2.308642	29.699118	0.523008	0.381594	32.204208
std	257.353842	0.486592	0.836071	14.526497	1.102743	0.806057	49.693429
min	1.000000	0.000000	1.000000	0.420000	0.000000	0.000000	0.000000
25%	223.500000	0.000000	2.000000	20.125000	0.000000	0.000000	7.910400
50%	446.000000	0.000000	3.000000	28.000000	0.000000	0.000000	14.454200
75%	668.500000	1.000000	3.000000	38.000000	1.000000	0.000000	31.000000
max	891.000000	1.000000	3.000000	80.000000	8.000000	6.000000	512.329200

Selection examples

Selecting columns

```
# Selection is very similar to standard Python selection
df1 = train[["Name", "Sex", "Age", "Survived"]]
df1.head()
```

	Name	Sex	Age	Survived
0	Braund, Mr. Owen Harris	male	22.0	0
1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1
2	Heikkinen, Miss. Laina	female	26.0	1
3	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1
4	Allen, Mr. William Henry	male	35.0	0

Next steps: Generate code with df1 | View recommended plots

Selecting rows

```
df1[10:15]
```

	Name	Sex	Age	Survived
10	Sandstrom, Miss. Marguerite Rut	female	4.0	1
11	Bonnell, Miss. Elizabeth	female	58.0	1
12	Saundercock, Mr. William Henry	male	20.0	0
13	Andersson, Mr. Anders Johan	male	39.0	0
14	Vestrom, Miss. Hulda Amanda Adolfina	female	14.0	0

Filtering Examples

Filtering with one condition

```
# Filtering allows you to create masks given some conditions
df1.Sex == 'female'
```

```
0      False
1       True
2       True
3       True
4      False
       ...
886    False
887     True
888     True
889    False
890    False
Name: Sex, Length: 891, dtype: bool
```

```
onlyFemale = df1[df1.Sex == 'female']
onlyFemale.head()
```

	Name	Sex	Age	Survived
1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1
2	Heikkinen, Miss. Laina	female	26.0	1
3	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1
8	Johnson, Mrs. Oscar W (Elisabeth Vilhelmina Berg)	female	27.0	1
9	Nasser, Mrs. Nicholas (Adele Achem)	female	14.0	1

Next steps: Generate code with `onlyFemale` View recommended plots

Filtering with multiple conditions

(T2) Alter the following command so `adultFemales` will contain only females whose age is 18 and above.
You need to filter using a **single** mask with multiple conditions (google it!), i.e., without creating any temporary dataframes.

Additionally, update the `survivalRate` variable to show the correct rate.

```
# DONE: update the mask
adultFemales = df1[(df1.Sex == 'female') & (df1.Age >= 18)]

# DONE: Update the survival rate
# survived / total
survivalRate = len(adultFemales[(adultFemales.Survived == 1)]) / len(adultFemales)
print("The survival rate of adult females was: {:.2f}%".format(survivalRate * 100))
```

```
The survival rate of adult females was: 77.18%
```

Aggregating

Pandas allows you to aggregate and display different views of your data.

```
df2 = train.groupby(['Pclass', 'Sex']).Fare.agg(np.mean)
df2
```

```
Pclass  Sex   
1       female    106.125798
        male       67.226127
2       female     21.970121
        male       19.741782
3       female     16.118810
        male       12.661633
Name: Fare, dtype: float64
```

```
pd.pivot_table(train, index=['Pclass'], values=['Survived'], aggfunc='count')
```

Pclass	Survived
1	216
2	184
3	491

The following table shows the survival rates for each combination of passenger class and sex.
(T3) Add a column showing the mean **age** for such a combination.

```
# DONE: Also show the mean age per group
pd.pivot_table(train, index=['Pclass', 'Sex'], values=['Survived', 'Age'], aggfunc='mean')
```

Pclass	Sex	Age	Survived
1	female	34.611765	0.968085
	male	41.281386	0.368852
2	female	28.722973	0.921053
	male	30.740707	0.157407
3	female	21.750000	0.500000
	male	26.507589	0.135447

(T4) Use [this]() question on stackoverflow, to find the mean survival rate for ages 0-10, 10-20, etc.).

Hint: the first row should roughly look like this:

```
                Age    Survived
Age
(0, 10]    4.268281    0.593750
```

```
# DONE: find the mean survival rate per age group
ageGroups = np.arange(0, 81, 10)

survivedMean = train.groupby(pd.cut(train['Age'], ageGroups)).Survived.mean()
ageMean = train.groupby(pd.cut(train['Age'], ageGroups)).Age.mean()

survivalPerAgeGroup = pd.concat([ageMean, survivedMean], axis = 1)

survivalPerAgeGroup
```

Age	Age	Survived
(0, 10]	4.268281	0.593750
(10, 20]	17.317391	0.382609
(20, 30]	25.423913	0.365217
(30, 40]	35.051613	0.445161
(40, 50]	45.372093	0.383721
(50, 60]	54.892857	0.404762
(60, 70]	63.882353	0.235294
(70, 80]	73.300000	0.200000

```
type(train.groupby(pd.cut(train.Age, ageGroups)).Survived.mean())
```

```
pandas.core.series.Series
def __init__(data=None, index=None, dtype: Dtype | None=None, name=None, copy: bool |
None=None, fastpath: bool=False) -> None

/usr/local/lib/python3.10/dist-packages/pandas/core/series.py
One-dimensional ndarray with axis labels (including time series).

Labels need not be unique but must be a hashable type. The object
supports both integer- and label-based indexing and provides a host of
methods for performing operations involving the index. Statistical
```

Filling missing data (data imputation)

Note that some passenger do not have age data.

```
print("{} out of {} passengers do not have a recorded age".format(df1[df1.Age.isna()].shape[0], df1.shape[0]))
```

```
177 out of 891 passengers do not have a recorded age
```

```
df1[df1.Age.isna()].head()
```

	Name	Sex	Age	Survived
5	Moran, Mr. James	male	NaN	0
17	Williams, Mr. Charles Eugene	male	NaN	1
19	Masselmani, Mrs. Fatima	female	NaN	1
26	Emir, Mr. Farred Chehab	male	NaN	0
28	O'Dwyer, Miss. Ellen "Nellie"	female	NaN	1

Let's see the statistics of the column **before** the imputation.

```
df1.Age.describe()
```

```
count    714.000000
mean      29.699118
std       14.526497
min        0.420000
25%       20.125000
50%       28.000000
75%       38.000000
max       80.000000
Name: Age, dtype: float64
```

Read about `pandas.Series.fillna`.
(T5) Replace the missing ages `df1` with the general age *median*, and insert the result into variable `filledDf` (the original `df1` should be left unchanged).

```
# TODO : Fill the missing values
filledDf = df1
values = {"Age": df1.Age.median()}
filledDf = filledDf.fillna(value = values)
```

```
print("{} out of {} passengers do not have a recorded age".format(filledDf[filledDf.Age.isna()].shape[0], filledDf.shape[0]))
```

```
0 out of 891 passengers do not have a recorded age
```

Let's see the statistics of the column **after** the imputation.

```
filledDf.Age.describe()
```

```
count    891.000000
mean      29.361582
std       13.019697
min        0.420000
25%       22.000000
50%       28.000000
75%       35.000000
max       80.000000
Name: Age, dtype: float64
```

(T6) Answer below: which statistics changed, and which did not? Why? (explain briefly, no need to be very formal.)

Answer:

Count : 714 -> 891.

Count is counting number of non null entries in Age column. Since we filled in all NA entries, we expected after change the count to be as the number of passangers which is 891.

Mean : 29.699118 -> 29.361582

Just a tiny slight changed due to filling in the median in NA entries.

STD : 14.526497 -> 13.019697

Represent how far the age from the mean (std = square root of the variance). Since we added more values close to the mean, overall move values are closer to the mean and directly the std gets smaller.

Min/Max : No change

We filled NA with the median value which is higher/lower than the min/max value thus the min/max remain unchanged.

25% lower quarter : 20.125000 -> 22.000000

We filled NA with the meadian value which is higher then the before low quarter value. Thus after change we have more values above the lower quarter to it's value has to change inorder to have exactly 25% of the entires below it's value.

50% median : No change

Even after filling the NA with the median value there are still 50% of the values above/equal and 50% of the values below/equal the median value

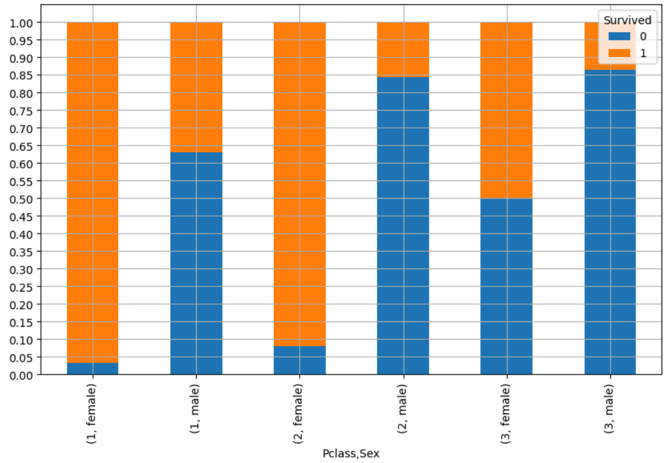
75% upper quarter : 38.000000 -> 35.000000

Filling NA with the median value which is smaller then the upper quarter value before change, will give more mass to values below the upper quarter which will lead to it's decrease.

Plotting

Basic plotting in pandas is pretty straightforward

```
new_plot = pd.crosstab([train.Pclass, train.Sex], train.Survived, normalize="index")
new_plot.plot(kind='bar', stacked=True, grid=False, figsize=(10,6))
plt.yticks(np.linspace(0,1,21))
plt.grid()
```



(T7) Answer below: which group (class × sex) had the best survival rate? Which had the worst?

Answer:

- Best survival rate: female from class 1
- Worst survival rate: male from class 3

What is Matplotlib

A 2D plotting library which produces publication quality figures.

- Can be used in python scripts, the python and IPython shell, web application servers, and more …
- Can be used to generate plots, histograms, power spectra, bar charts, errorcharts, scatterplots, etc.
- For simple plotting, pyplot provides a MATLAB-like interface
- For power users, a full control via OO interface or via a set of functions

There are several Matplotlib add-on toolkits

- Projection and mapping toolkits [basemap](#) and [cartopy](#).
- Interactive plots in web browsers using [Bokeh](#).
- Higher level interface with updated visualizations [Seaborn](#).

Matplotlib is available at [www.matplotlib.org](#)

```
import matplotlib.pyplot as plt
import numpy as np
```

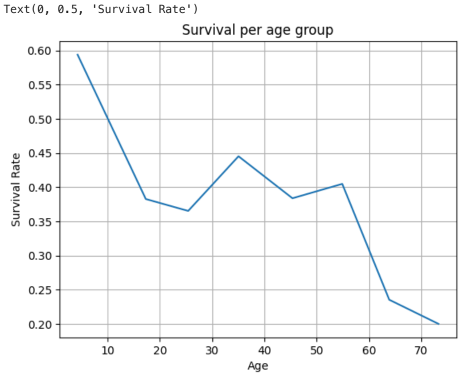
Line Plots

The following code plots the survival rate per age group (computed above, before the imputation).

(T8) Use the [matplotlib documentation](#) to add a grid and suitable axis labels to the following plot.

```
plt.plot(survivalPerAgeGroup.Age, survivalPerAgeGroup.Survived)
_ = plt.title("Survival per age group")
# DONE : Update the plot as required.
plt.grid()
plt.xlabel("Age")
plt.ylabel("Survival Rate")
```

```
Text(0, 0.5, 'Survival Rate')
```




```
survivalPerAgeGroup
```

Age	Age	Survived
(0, 10]	4.268281	0.593750
(10, 20]	17.317391	0.382609
(20, 30]	25.423913	0.365217
(30, 40]	35.051613	0.445161
(40, 50]	45.372093	0.383721
(50, 60]	54.892857	0.404762
(60, 70]	63.882353	0.235294
(70, 80]	73.300000	0.200000

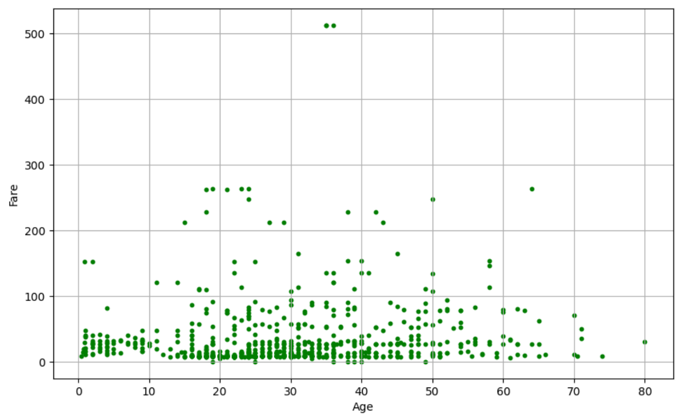
Scatter plots

(T9) Alter the matplotlib.pyplot.scatter command, so that the scattered dots will be `green`, and their size will be `10`.

Also, add a grid and suitable axis labels.

```
# TODO : Update the plot as required.
plt.figure(figsize=(10,6))
plt.scatter(train.Age, train.Fare, color='g', s=10)
plt.grid()
plt.xlabel("Age")
plt.ylabel("Fare")
```

```
Text(0, 0.5, 'Fare')
```



(T10) Answer below: approximately how old are the two highest paying passengers?

Answer: The two highest paying passengers are approx. 35 and 36/37 years old.

Probability refresher

Q1 - Variance of empirical mean

Let $X_1, \dots, X_m$ be i.i.d random variables with mean $\mathbb{E}[X_i] = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2$.
We would like to "guess", or more formally, estimate (לְשַׁעֲרֵךְ), the mean μ from the observations $x_1, \dots, x_m$.
We use the empirical mean $\overline{X} = \frac{1}{m}\sum_i X_i$ as an estimator for the unknown mean μ. Notice that $\overline{X}$ is itself a random variable.
Note: The instantiation of $\overline{X}$ is usually denoted by $\hat{\mu} = \frac{1}{m}\sum_i x_i$, but this is currently out of scope.

1. Express analytically the expectation of $\overline{X}$.
   Answer: $\mathbb{E}\left[\overline{X}\right] = \mathbb{E}\left[\frac{1}{m}\sum_i X_i\right] = \frac{1}{m}E[\sum_{i=1}^m X_i] = \frac{1}{m}\sum_{i=1}^m E[X_i] = \frac{1}{m}\sum_{i=1}^m \mu = \frac{1}{m}m\mu = \mu.$
   Using the linearity of the expectation in the first equality, independence of the X_i in the 2nd equality and the identical distribution in the 3rd equality.
2. Express analytically the variance of $\overline{X}$.
   Answer: $\mathrm{Var}\left[\overline{X}\right] = Var[\frac{1}{m}\sum_{i=1}^m X_i] = \frac{1}{m^2}Var[\sum_{i=1}^m X_i] = \frac{1}{m^2}\sum_{i=1}^m Var[X_i] = \frac{1}{m^2}\sum_{i=1}^m \sigma^2 = \frac{1}{m^2}m\sigma^2 = \frac{\sigma^2}{m}.$
   Using constant expenditure in variance in the 2nd equality, independence of the X_i in the 3rd equality and the identical distribution in the 3rd equality.

You will now verify the expression you wrote for the variance.
We assume $\forall i : X_i \sim \mathrm{N}(0, 1)$.
We compute the empirical mean's variances for sample sizes $m = 1, \dots, 30$.
For each sample size m, we sample m normal variables and compute their empirical mean. We repeat this step 50 times, and compute the variance of the empirical means (for each m).

3 . Complete the code blocks below according to the instructions and verify that your analytic function of the empirical mean's variance against as a function of m suits the empirical findings.

```
all_sample_sizes = range(1, 31)
repeats_per_size = 50

allVariances = []

for m in all_sample_sizes:
    empiricalMeans = []

    for _ in range(repeats_per_size):
        # Random m examples and compute their empirical mean
        X = np.random.randn(m)
        empiricalMeans.append(np.mean(X))

    # DONE: Using numpy, compute the variance of the empirical means that are in
    # the `empiricalMeans` list (you can google the numpy function for variance)
    variance = np.var(empiricalMeans)

    allVariances.append(variance)
```

Complete the following computation of the anayltic variance (according to the your answers above). You can try to use simple arithmetic operations between an `np.array` and a scalar, and see what happens! (for instance, `2 * np.array(all_sample_sizes)` .)

```
# DONE: compute the analytic variance
# (the current command wrongfully sets the variance of an empirical mean
#  of a sample with m variables simply as 2*m)
# 1 = var(X~N(0,1))
analyticVariance = 1 / np.array(all_sample_sizes).astype(float)
```

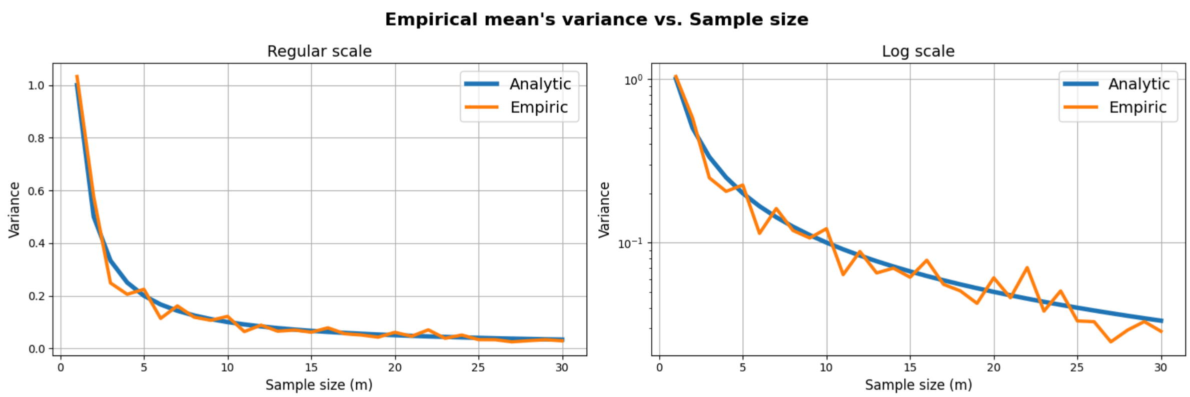
The following code plots the results from the above code. **Do not** edit it, only run it and make sure that the figures make sense.

```
fig, axes = plt.subplots(1,2, figsize=(15,5))
axes[0].plot(all_sample_sizes, analyticVariance, label="Analytic", linewidth=4)
axes[0].plot(all_sample_sizes, allVariances, label="Empiric", linewidth=3)
axes[0].grid()
axes[0].legend(fontsize=14)
axes[0].set_title("Regular scale", fontsize=14)
axes[0].set_xlabel("Sample size (m)", fontsize=12)
axes[0].set_ylabel("Variance", fontsize=12)

axes[1].semilogy(all_sample_sizes, analyticVariance, label="Analytic", linewidth=4)
axes[1].semilogy(all_sample_sizes, allVariances, label="Empiric", linewidth=3)
axes[1].grid()
axes[1].legend(fontsize=14)
axes[1].set_title("Log scale", fontsize=14)
axes[1].set_xlabel("Sample size (m)", fontsize=12)
axes[1].set_ylabel("Variance", fontsize=12)

_ = plt.suptitle("Empirical mean's variance vs. Sample size",
             fontsize=16, fontweight="bold")

plt.tight_layout()
```



Reminder - Hoeffding's Inequality

Let $\theta_1, \ldots, \theta_m$ be i.i.d random variables with mean $\mathbb{E}[\theta_i] = \mu$.

Additionally, assume all variables are bound in $[a, b]$ such that $\Pr[a \le \theta_i \le b] = 1$.

Then, for any $\epsilon > 0$, the empirical mean $\overline{\theta(m)} = \frac{1}{m}\sum_i \theta_i$ holds:

$$\Pr\left[\left|\overline{\theta(m)} - \mu\right| > \epsilon\right] \le 2\exp\left\{-\frac{2m\epsilon^2}{(b-a)^2}\right\}.$$

Q2 - Identical coins and the Hoeffding bound

We toss $m \in \mathbb{N}$ identical coins, each coin 40 times.
All coins have the same *unknown* probability of showing "heads", denoted by $p \in (0, 1)$.
Let θ_i be the (observed) number of times the i-th coin showed "heads".

1. What is the distribution of each θ_i?
   Answer: $\theta_i \sim \mathrm{Bin}(40{,}p)$.
2. What is the mean $\mu = \mathbb{E}[\theta_i]$?
   Answer: $\mathbb{E}[\theta_i] = 40\mathrm{p}$. (we know that if X~Bin(n,p) then E[X] = np)
3. We would like to use the empirical mean defined above as an estimator $\overline{\theta(m)}$ for μ.
   Use Hoeffding's inequality to compute the *smallest* error ϵ that can guaranteed given a sample size $m = 20$ with confidence 0.95 (notice that we wish to estimate μ, not p).
   That is, find the smallest ϵ that holds $\Pr\left[\left|\overline{\theta(20)} - \mu\right| > \epsilon\right] \le 0.05.$
   Answer:
   Given:

- [a,b] = [0,40] = #tosses for each m => (b-a) = 40
- m = 20

Thus using Hoeffding's Inequality:

$$0.05 = 2 \cdot e^{\frac{-2\cdot 20\cdot \epsilon^2}{40^2}}$$
$$\frac{1}{20} = \frac{2}{e^{\frac{2\cdot 20\cdot \epsilon^2}{40^2}}}$$
$$40 = e^{\frac{\epsilon^2}{40}}$$
$$\ln(40) = \frac{\epsilon^2}{40}$$
$$\epsilon = \sqrt{40\ln(40)} \approx 12.14$$

4 . The following code simulates tossing $m = 10^4$ coins, each 50 times. For each coin, we use the empirical mean as the estimator and save it in the `all_estimators` array. The (unknown) probability of each coin is 0.75.
Complete the missing part so that for each coin, an array of 50 binary `observations` will be randomized according to the probability p.

```
m = 10**4
tosses = 50
p = 0.75
all_estimators = []

# Repeat for n coins
for coin in range(m):
    # DONE: Use Google to find a suitable numpy.random function that creates
    # a binary array of size (tosses,), where each element is 1
    # with probability p, and 0 with probability (1-p).
    observations = np.random.binomial(1, p, tosses)

    # Compute and save the empirical mean
    estimator = np.mean(observations)
    all_estimators.append(estimator)
```

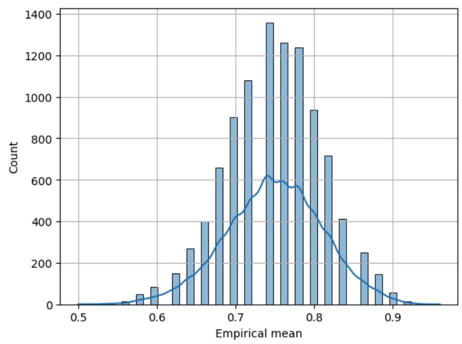
5 . The following code plots the histogram of the estimators (empirical means). Run it. What type of distribution is obtained (no need to specify the exact paramters of the distribution)? Explain **briefly** what theorem from probability explains this behavior (and why).

Answer: Consider the following:
$X_i \sim Ber(50, 0.75), X_i$ are IID.
Denote $S_m = \frac{1}{m}\sum_{i=1}^{m} X_i$.

Looks like the plot represents distrebution of $S_m \sim \mathrm{N}(\mu, \frac{\sigma^2}{m})$.
This is given due to the Low of Large Numbers which holds since X_i are IID and number of experiments($m = 10^4$) is large enough. LLN under our conditions claims, when m is large enough the empirical mean S_m converges to to the true mean μ. Moreover Var(S_m)= $\frac{\sigma^2}{m}$.

```
import seaborn as sns
sns.histplot(all_estimators, bins=tosses, kde=True)
plt.grid()
plt.xlabel("Empirical mean")
plt.ylabel("Count")
```

```
Text(0, 0.5, 'Count')
```



Linear Algebra and Multivariable Calculus refresher

Reminder - Positive semi-definite matrices

A symmetric real matrix $A \in \mathbb{R}^{n \times n}$ is called positive semi-definite (PSD) iff:

$$\forall x \in \mathbb{R}^n \setminus \{0_n\} : x^\top A x \geq 0.$$

If the matrix holds the above inequality *strictly*, the matrix is called positive definite (PD).

Q3 - PSD matrices

1. Let $A \succeq \mathbf{0}_{n \times n}$ be a symmetric PSD matrix in $\mathbb{R}^{n \times n}$.
   Recall that all eigenvalues of real symmetric matrices are real.
   Prove that all the eigenvalues of A are non-negative.
   Answer:
   Let A$\in \mathbb{R}^{n \times n}$ be a symmetric PSD martix, $\lambda \in \mathbb{R}$ an eigenvalue and X$\in \mathbb{R}^n$ the corresponding eigenvector.
   Since A is symmetric it is known that $A^T = A$ and also since A is PSD $\forall x \in \mathbb{R}_n : x^T A x \geq 0$. Thus:
   $0 \leq x^T A x = x^T \lambda x =< x, \lambda x > \overset{\lambda \in \mathbb{R}}{=} \lambda \overset{\leq 0}{< x, x >}$
   It follows that also $\lambda \geq 0$.

---

2. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric PSD matrix and $B \in \mathbb{R}^{n \times n}$ a square matrix.
   What can be said about the symmetric matrix $(B^\top A B)$? Specifically, is it necessarily PSD? is it necessarily PD? Explain.
   Answer:
   Let A$\in \mathbb{R}^{n \times n}$ be a symmetric PSD martix, B $\in \mathbb{R}^{n \times n}$ a square matrix and vector $x \in \mathbb{R}^n$.
   First, notice that $B^T A B$ is symetric because $(B^T A B)^T = B^T A^T B = B^T A B$.
   Moreover, since A is PSD:
   $x^T (B^T A B) x = (Bx)^T A (Bx) \geq 0$
   Hence, $B^T A B$ is PSD as well. Altought it is not necesery PD!
   If A is PD then $x^T A x > 0$ for any non zero vector x.
   So $x^T (B^T A B) x = (Bx)^T A (Bx) > 0$ iff $Bx \neq 0$ iff $kerB = \vec{0}$ iff B is invertable.
   If A is not PD, then there is some $0 \neq y \in \mathbb{R}^n$ s.t. $y^T A y = 0$. So if $Bx = y$ then $(Bx)^T A (Bx) = 0$ thus not PD.

Q4 - Gradients

Define $f : \mathbb{R}^d \to \mathbb{R}$, where $f(w) = w^\top x + b$, for some given vector $x \in \mathbb{R}^d$ and a scalar $b \in \mathbb{R}$.

Recall: the gradient vector is defined as $\nabla_w f = \left[\frac{\partial f}{\partial w_1}, \ldots, \frac{\partial f}{\partial w_d}\right]^\top \in \mathbb{R}^d$.

1. Prove that $\nabla_w f = x$.

Recall/read the definition of the Hessian matrix $\nabla_w^2 f \in \mathbb{R}^{d \times d}$.

2. Find the Hessian matrix $\nabla_w^2 f$ of the function f defined in this question.
3. Is the matrix you found positive semi-definite? Explain.

Now, define $g : \mathbb{R}^d \to \mathbb{R}$, where $\lambda > 0$ and $g(w) = \frac{1}{2}\lambda\|w\|^2$.

4. Find the gradient vector $\nabla_w g$.
5. Find the Hessian matrix $\nabla_w^2 g$.
6. Is the matrix you found positive semi-definite? is it positive definite? Explain.

Finally, define $h : \mathbb{R}^2 \to \mathbb{R}$, where $h(w_1, w_2) = 12w_1^3 - 36w_1w_2 - 2w_2^3 + 9w_2^2 - 72w_1 + 60w_2 + 5$.

7. Find all the critical points of the function h. That is, find all $\underline{w^* \in \mathbb{R}}$ s.t. $\underline{\nabla h_w(w^*) = 0.}$
8. Which of the critical points are maxima, minima, or saddle points? You may use the [second partial derivative test](), but state how h meets it's conditions.
9. Does h has a global maximum? global minimum? Prove your answer.

Answers:

1.

$$\begin{aligned}
&\text{Given } f(w) = w^T x + b : \\
&\text{for each i : } \frac{\partial f}{\partial w_i} = \frac{\partial (w_i x_i + b)}{\partial w_i} = x_i \\
&\text{It grants } \nabla_w f = \left[\frac{\partial f}{\partial w_1}, \dots, \frac{\partial f}{\partial w_d}\right]^\top = [x_1, \dots, x_d]^\top = x
\end{aligned}$$

2.

The Hessian matrix of f is matrix of all 2nd derivatives.
Since $f(w) = w^T x + b$ is linear, the 2nd derivetive would be 0 for each $1 \le i, j \le d$.
Hence, $\nabla^2_w f = 0 \in \mathrm{R}^{d \times d}$.

3.

Yes, the Hessian matrix of f is PSD since $\forall x \in \mathrm{R}^d, x^T 0 x = 0 \ge 0$.

4.

$$\begin{aligned}
&\text{Given } g(w) = \frac{1}{2}\lambda ||w||^2 = \frac{1}{2}\lambda w^T w \\
&\text{for each i : } \frac{\partial g}{\partial w_i} = \frac{\partial (\frac{1}{2}\lambda w_i^T w_i)}{\partial w_i} = \frac{\partial (\frac{1}{2}\lambda w_i^2)}{\partial w_i} = \frac{2}{2}\lambda w_i = \lambda w_i \\
&\text{Thus } \nabla_w g = \left[\frac{\partial g}{\partial \lambda w_1}, \dots, \frac{\partial g}{\partial w_d}\right]^\top = [\lambda w_1, \dots, \lambda w_d]^\top = \lambda w
\end{aligned}$$

5.

$\nabla^2_w g$ is the derivative of the linear function λw
Thus $\nabla^2_w g = \lambda I^{\,d \times d}$

6.

Given $x \in \mathbb{R}^d, x^T \lambda I x = x^T \lambda x = \lambda < x, x >$
$< x, x > \ge 0$ for every $x \in \mathbb{R}^d$ and $< x, x >> 0$ iff $x = 0$.
So if $\lambda > 0, \lambda I$ is PD.
If $\lambda = 0, \lambda I$ is PSD.
And if $\lambda < 0, \lambda I$ is not PSD neigher PD.

7.

$$\frac{\partial h}{\partial w_1} = 36w_1^2 - 36w_2 - 72$$
$$\frac{\partial h}{\partial w_2} = -36w_1 - 6w_2^2 + 18w_2 + 60$$

To find critical points : $\nabla_w h = 0$

$$\begin{cases} w_1^2 - w_2 - 2 = 0 \\ -6w_1 - w_2^2 + 3w_2 + 10 = 0 \end{cases}$$

From the first equastion one have $w_2 = w_1^2 - 2$
Substitute into the second equation:

$$\begin{aligned}
-6w_1 - (w_1^2 - 2)^2 + 3(w_1^2 - 2) + 10 &= 0 \\
-6w_1 - w_1^4 + 4w_1^2 - 4 + 3w_1^2 - 6 + 10 &= 0 \\
-w_1^4 + 7w_1^2 - 6w_1 &= 0 \\
w_1(w_1 - 1)(w_1 - 2)(w_1 + 3) &= 0
\end{aligned}$$

$$\Longrightarrow \begin{cases} w_1 = 0, w_2 = -2 \\ w_1 = 1, w_2 = -1 \\ w_1 = 2, w_2 = 2 \\ w_1 = -3, w_2 = 7 \end{cases}$$

In conclusion, the critical points are : (0,-2), (1,-1), (2,2), (-3,7).

8.

$$H(w_1, w_2) = \begin{bmatrix} \frac{\partial^2 h}{\partial w_1^2} & \frac{\partial^2 h}{\partial w_1 w_2} \\ \frac{\partial^2 h}{\partial w_2 w_1} & \frac{\partial^2 h}{\partial w_2^2} \end{bmatrix}$$

Since $h(w_1, w_2)$ is continous as composition of continous functions and 1st, 2nd derivatives exists and are continuous, one have:

$$H(w_1, w_2) = \begin{bmatrix} 72w_1 & -36 \\ -36 & -12w_2 + 18 \end{bmatrix}$$

$det(H(0, -2)) = -1296 < 0$ thus (0,-2) is a saddle point.
$det(H(2, 2)) = -2160 < 0$ thus (2,2) is a saddle point.
$det(H(1, -1)) = 864 > 0$ and $\frac{\partial^2 h}{\partial w_1^2} > 0$ thus (1,-1) is a local minimum point.
$det(H(-3, 7)) = 12960 > 0$ and $\frac{\partial^2 h}{\partial w_1^2} < 0$ thus (-3,7) is a local maximum point.

9.

The function h does not have global minimum of maximum.
This is due to it's behavior when each paramater goes to $\pm\infty$.
If $w_1 \to \infty$ and w_2 is fixed, then $h(w_1, w_2) \sim 12w_1^3 \to \infty$.
Similar if $w_1 \to -\infty$ and w_2 is fixed, then $h(w_1, w_2) \sim 12w_1^3 \to -\infty$.
If $w_2 \to \infty$ and w_1 is fixed, then $h(w_1, w_2) \sim -2w_2^3 \to -\infty$.
Similar if $w_2 \to -\infty$ and w_1 is fixed, then $h(w_1, w_2) \sim -2w_2^3 \to \infty$.
Either way, the function $h(w_1, w_2)$ is not bounded hence does not get global minimum of maximum.