ECE 742 Final Project

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1 Theory

1.1 PML

Perfectly Matched Layer (PML) boundary conditions are absorbing boundary conditions. PML BCs decay the wave within a boundary layer at the edge of the simulation. The edge of the simulation BC can be implemented as PEC. Well-implemented PML BCs completely decay the wave from the time it enters the boundary layer to the time after it reflects and attempts to leave.

1.2 Finite-Difference Derivation

Let's start with equation:

$$\nabla \times \vec{H} = j\omega\epsilon\bar{\bar{s}}\vec{E}$$

Evaluate the cross product and write in matrix form:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \mathrm{j}\omega\epsilon \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Where the values in the second rank tensor can be described by:

$$s_x = \kappa_x + \frac{\sigma_x}{j\omega\epsilon_0}$$

$$s_y = \kappa_y + \frac{\sigma_y}{j\omega\epsilon_0}$$

$$s_z = \kappa_z + \frac{\sigma_z}{j\omega\epsilon_0}$$

To make the calculation computationally more managable, we can define the following relations:

$$D_x = \epsilon\frac{s_z}{s_x}E_x$$

$$D_y = \epsilon \frac{s_x}{s_y} E_y$$

$$D_z = \epsilon \frac{s_y}{s_z} E_z$$

such that now:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \mathrm{j}\omega \begin{bmatrix} s_y & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_x \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Using the defined values of s and that $\frac{\partial}{\partial t} = j\omega$:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} \kappa_y & 0 & 0 \\ 0 & \kappa_z & 0 \\ 0 & 0 & \kappa_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} + \frac{1}{\epsilon_0} \begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

Discretize to find D

Discretized at point i,j,k

$$\begin{bmatrix} \frac{H_z^{n+1/2}(i,j+1,k) - H_z^{n+1/2}(i,j,k)}{\Delta_y} - \frac{H_y^{n+1/2}(i,j,k+1) - H_y^{n+1/2}(i,j,k)}{\Delta_z} \\ \frac{H_x^{n+1/2}(i,j,k+1) - H_x^{n+1/2}(i,j,k)}{\Delta_z} - \frac{H_z^{n+1/2}(i+1,j,k) - H_z^{n+1/2}(i,j,k)}{\Delta_x} \\ \frac{H_y^{n+1/2}(i+1,j,k) - H_y^{n+1/2}(i,j,k)}{\Delta_x} - \frac{H_x^{n+1/2}(i,j+1,k) - H_x^{n+1/2}(i,j,k)}{\Delta_y} \end{bmatrix} = \begin{bmatrix} \kappa_y & 0 & 0 \\ 0 & \kappa_z & 0 \\ 0 & 0 & \kappa_x \end{bmatrix} \begin{bmatrix} D_x^{n+1} - D_x^n \\ D_y^{n+1} - D_y^n \\ D_z^{n+1} - D_z^n \end{bmatrix}$$
$$+ \frac{1}{\epsilon_0} \begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{bmatrix} \begin{bmatrix} D_x^n \\ D_y^n \\ D_z^n \end{bmatrix}$$

Update Components for D

Update Components for E

Update Components for B

Update Components for H

1.3 Graded Conductivity

Reflection Factor

$$R(\theta) = \exp^{-2\eta cos(\theta) \int_0^d \sigma_x(x) dx}$$

Where σ_x is the graded conductivity of the PML material.

θ is the angle of incidence of the wave. So steeper angles of θ will result in higher values of reflection error.

We want to minimize relfection R but also make sure the wave decays completely in the PML boundary layer.

We are going to compare the error for different types of grading profiles. And/or we can use different values in the grading profile

1.3.1 Polynomial grading

Where the graded conductivity is:

$$\sigma_x = (\frac{x}{d})^m \sigma_{x,max}$$

And the graded value for κ_x is:

$$\kappa_x = 1 + (\kappa_{x,max} - 1)(\frac{x}{d})^m$$

Reflection factor simplifies to

1.3.2 Geometric grading

Where the graded conductivity is:

$$\sigma_x = (g^{\frac{1}{\Delta}})^x \sigma_{x,0}$$

$\sigma_{x,0}$ is the conductivity at the surface of the PML.
g is a scaling factor. Nearly optimal: $2 \leq g \leq 3$
Δ is spacing of FDTD lattice.

And the graded value for κ_x is:

$$\kappa_x = [(\kappa_{max})^{\frac{1}{d}} g^{\frac{1}{\Delta}}]^x$$

2 Code

3 Error Analysis

Insert Error Analysis Here

PMLs are exact for continuous functions, but error is introduced for discrete functions. Having a large step discontinuity can

3.0.1 Error of Polynomial Grading

3.0.2 Error of Geometric Grading

4 Fix me: Bibliography

Susan's book - third edition