ECE 742 Final Project

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1 Theory

1.1 PML

Perfectly Matched Layer (PML) boundary conditions are absorbing boundary conditions. PML BCs decay the wave within a boundary layer at the edge of the simulation. The edge of the simulation BC can be implemented as PEC. Well-implemented PML BCs completely decay the wave from the time it enters the boundary layer to the time after it reflects and attempts to leave.

1.2 Finite-Difference Derivation

Let's start with equation:

$$\nabla \times \vec{H} = j\omega\epsilon\bar{\bar{s}}\vec{E} \tag{1}$$

Evaluate the cross product and write in matrix form:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \mathrm{j}\omega\epsilon \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{2}$$

Where the values in the second rank tensor can be described by:

$$s_x = \kappa_x + \frac{\sigma_x}{j\omega\epsilon_0} \tag{3}$$

$$s_y = \kappa_y + \frac{\sigma_y}{j\omega\epsilon_0} \tag{4}$$

$$s_z = \kappa_z + \frac{\sigma_z}{j\omega\epsilon_0} \tag{5}$$

To make the calculation computationally more managable, we can define the following relations:

$$D_x = \epsilon\frac{s_z}{s_x}E_x \tag{6}$$

$$D_y = \epsilon \frac{s_x}{s_y} E_y \tag{7}$$

$$D_z = \epsilon \frac{s_y}{s_z} E_z \tag{8}$$

such that now:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \mathrm{j}\omega \begin{bmatrix} s_y & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \tag{9}$$

Using the defined values of s and that $\frac{\partial}{\partial t} = j\omega$:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} \kappa_y & 0 & 0 \\ 0 & \kappa_z & 0 \\ 0 & 0 & \kappa_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} + \frac{1}{\epsilon_0} \begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} \tag{10}$$

This has too much info. Since we are doing 2D, $\kappa_z = 1$, $\sigma_z = 0$ in this simulation. As well as $D_x = D_y = 0$ and $H_z = 0$. Equation reduces to:

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} \kappa_y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \kappa_x \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ D_z \end{bmatrix} + \frac{1}{\epsilon_0} \begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_x \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ D_z \end{bmatrix} \tag{11}$$

This leaves us with one equation to find an update quation for D_z:

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \kappa_x \frac{\partial}{\partial t} D_z + \frac{\sigma_x}{\epsilon_0} D_z \tag{12}$$

Using semi-implicit (???)

$$D^{n+1/2} = \frac{D^{n+1} + D^n}{2} \tag{13}$$

If we discretize around point i,j,k at timestep n
This equation discretizes to:

$$\begin{aligned} &\frac{1}{\Delta_x}(H_y^{n+1/2}(i+1/2,j) - H_y^{n+1/2}(i-1/2,j)) \\ &\quad - \frac{1}{\Delta_y}(H_x^{n+1/2}(i,j+1/2) - H_x^{n+1/2}(i,j-1/2)) = \\ &\frac{\kappa_x}{\Delta t}(D_z^{n+1}(i,j) - D_z^n(i,j)) + \frac{\sigma_x}{2\epsilon_0}(D_z^{n+1}(i,j) + D_z^n(i,j)) \end{aligned} \tag{14}$$

We can solve this to find update equation for D:

$$
\begin{aligned}
&\frac{1}{\Delta_x}(H_y^{n+1/2}(i+1/2,j) - H_y^{n+1/2}(i-1/2,j)) \\
&- \frac{1}{\Delta_y}(H_x^{n+1/2}(i,j+1/2) - H_x^{n+1/2}(i,j-1/2)) = \\
&(\frac{\sigma_x}{2\epsilon_0} + \frac{\kappa_x}{\Delta t})D_z^{n+1}(i,j) + (\frac{\sigma_x}{2\epsilon_0} - \frac{\kappa_x}{\Delta t})D_z^n(i,j)
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\frac{1}{\Delta_x}(H_y^{n+1/2}(i+1/2,j) - H_y^{n+1/2}(i-1/2,j)) \\
&- \frac{1}{\Delta_y}(H_x^{n+1/2}(i,j+1/2) - H_x^{n+1/2}(i,j-1/2)) = \\
&(\frac{\sigma_x \Delta t + 2\epsilon_0\kappa_x}{2\epsilon_0\Delta t})D_z^{n+1}(i,j) + (\frac{\sigma_x \Delta t - 2\epsilon_0\kappa_x}{2\epsilon_0\Delta t})D_z^n(i,j)
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
D_z^{n+1}(i,j) = &-\frac{\sigma_x - 2\epsilon_0\kappa_x}{\sigma_x\Delta t + 2\epsilon_0\kappa_x} D_z^n(i,j) \\
&+ \frac{2\epsilon_0\Delta t}{\Delta_x(\sigma_x\Delta t + 2\epsilon_0\kappa_x)}(H_y^{n+1/2}(i+1/2,j) - H_y^{n+1/2}(i-1/2,j)) \\
&- \frac{2\epsilon_0\Delta t}{\Delta_y(\sigma_x\Delta t + 2\epsilon_0\kappa_x)}(H_x^{n+1/2}(i,j+1/2) - H_x^{n+1/2}(i,j-1/2))
\end{aligned} \tag{17}
$$

Update Components for E

Start with rewriting 8:

$$s_z D_z = \epsilon s_y E_z \tag{18}$$

Using **??** and **??**:

$$(\kappa_z + \frac{\sigma_z}{j\omega\epsilon_0})D_z = \epsilon(\kappa_y + \frac{\sigma_y}{j\omega\epsilon_0})E_z \tag{19}$$

Take the partial time derivative and use $\frac{\partial}{\partial t} = j\omega$

$$\frac{\partial}{\partial t}(\kappa_z D_z) + \frac{\sigma_z}{\epsilon_0}D_z = \epsilon(\frac{\partial}{\partial t}(\kappa_y E_z) + \frac{\sigma_y}{\epsilon_0}E_z) \tag{20}$$

$$\frac{\partial}{\partial t}(\kappa_z D_z) + \frac{\sigma_z}{\epsilon_0}D_z = \epsilon(\frac{\partial}{\partial t}(\kappa_y E_z) + \frac{\sigma_y}{\epsilon_0}E_z) \tag{21}$$

Update Components for B
Update Components for H

1.3 Graded Conductivity

Reflection Factor

$$R(\theta) = \exp^{-2\eta cos(\theta) \int_0^d \sigma_x(x)dx}$$

Where σ_x is the graded conductivity of the PML material.
θ is the angle of incidence of the wave. So steeper angles of θ will result in higher values of reflection error.
We want to minimize relfection R but also make sure the wave decays completely in the PML boundary layer.

We are going to compare the error for different types of grading profiles. And/or we can use different values in the grading profile

1.3.1 Polynomial grading

Where the graded conductivity is:

$$\sigma_x = (\frac{x}{d})^m \sigma_{x,max}$$

And the graded value for κ_x is:

$$\kappa_x = 1 + (\kappa_{x,max} - 1)(\frac{x}{d})^m$$

Reflection factor simplifies to

1.3.2 Geometric grading

Where the graded conductivity is:

$$\sigma_x = (g^{\frac{1}{\Delta}})^x \sigma_{x,0}$$

$\sigma_{x,0}$ is the conductivity at the surface of the PML.
g is a scaling factor. Nearly optimal: $2 \leq g \leq 3$
Δ is spacing of FDTD lattice.

And the graded value for κ_x is:

$$\kappa_x = [(\kappa_{max})^{\frac{1}{d}} g^{\frac{1}{\Delta}}]^x$$

2 Code

3 Error Analysis

Insert Error Analysis Here

PMLs are exact for continuous functions, but error is introduced for discrete functions. Having a large step discontinuity can

3.0.1 Error of Polynomial Grading

3.0.2 Error of Geometric Grading

4 Fix me: Bibliography

Susan's book - third edition