2D-FDTD- UPML Simulation of Wave Propagation on Dispersive Media

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Abstract: **As one of the major computational electromagnetic tools, the finite-difference time-domain (FDTD) method finds widespread use as a solver for a variety of electromagnetic problems. In this paper we are interested in the implementation of a two-dimensional time-domain numerical scheme for simulation of wave propagation, on dispersive and inhomogeneous media with conductive loss which are based on Debye model and incorporated into the FDTD scheme by using the auxiliary differential equation (ADE) technique. The uniaxial perfectly matched layer (UPML) is used as an absorbing boundary condition to simulate an open space.**

Key words: ***FDTD, UPML, wave, propagation, dispersive.***

I. INTRODUCTION

Since the finite difference time domain (FDTD) algorithm was developed by Yee [1] in the middle 1960s, the FDTD method has proven to be an efficient technique for numerous applications in electromagnetic[2][3]. One of the greatest challenges of the FDTD technique has been the development of accurate solution of ectromagnetic wave interaction problems in unbounded regions. One of the most efficient methods to simulate the infinite space is the perfectly matched layer (PML). They were first developed by Berenger for the FDTD method [5] and are based on a field splitting in a Cartesian coordinate system. Based on the Berger's PML, S. Gedney introduced a uniaxial anisotropic dispersive media (UPML) [6] [7] which allowed a direct and easy implementation in the Maxwell's equations. In this paper, we present a 2D FDTD simulation of wave propagation on dispersive and lossy media with a lossy uniaxial perfectly matched layer-boundary condition. Different techniques were developed to incorporate dispersion effects into existing FDTD schemes. In this work, the dispersion is modelled by the Debye model and incorporated by an auxiliary differential equation (ADE)[2][3][4].

II. FORMULATION OF FDTD

The FDTD is a popular computational EM modelling technique. It is considered to be easy to understand and to implement in software [2]. Since it is a time domain method, solutions can cover a wide frequency range with a single simulation run. All macroscopic electromagnetic phenomena occurring in practice can be mathematically described with the complete set of Maxwell's equations. In Yee's scheme [1]; the computational domain is discretized using a rectangular grid. The electric fields are located along the edges of the electric elements, while the magnetic fields are sampled at the centers of the electric element surfaces and are oriented normal to these surfaces (fig.1) [1][2][4]. In addition, in the time domain, the electric fields are sampled at times $n\Delta t$, and are assumed to be uniform in the time period of $(n-1/2)\Delta t$ to $(n+1/2)\Delta t$. Similarly, the magnetic fields are sampled at $(n+1/2\Delta t)$, and are assumed to be uniform in the time period of $n\Delta t$ to $(n+1)\Delta t$ (fig. 2).

A space point in a uniform rectangular lattice is denoted as (i, j, k) = (iΔx, jΔy, kΔz). If any scalar function of space and time evaluated at a discrete point in the grid and at a discrete point in time is denoted by *u,* then:

$$\frac{\partial u^{n}(i,j,k)}{\partial x} = \frac{u^{n}(i+\frac{1}{2},j,k) - u^{n}(i-\frac{1}{2},j,k)}{\Delta x} \tag{1}$$

$$\frac{\partial u^{n}(i,j,k)}{\partial t} = \frac{u^{n+\frac{1}{2}}(i,j,k) - u^{n-\frac{1}{2}}(i,j,k)}{\Delta t} \tag{2}$$

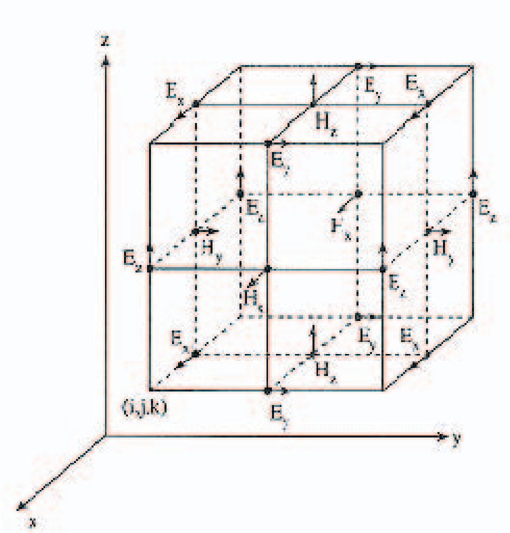

Fig.1 Position of the electric and magnetic fields in Yee's scheme.

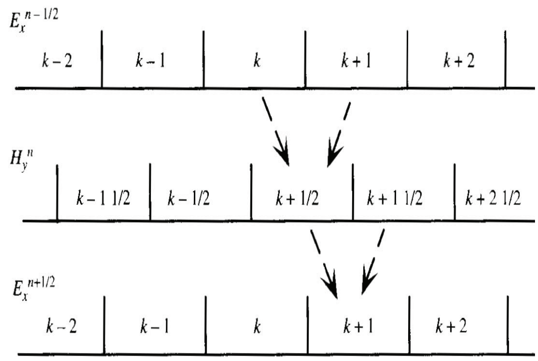

Fig. 2 Diagram of the calculation of E and H fields in FDTD

The price to pay for obtaining a solution directly in the time domain using the FDTD method is that the values of the spatial discretization Δx, Δy, Δz and the temporal discretization step Δt can not be chosen arbitrarily. FDTD is a conditionally stable numerical process. The stability condition is known as the CFL condition [2][3][4].

$$\Delta t \le \Delta t_{max} = \frac{1}{C\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}} \tag{3}$$

Using the D-H-formulation of the Maxwell equations [4], we obtain the system of equations for the time-dependent fields. These are first order partial differential equations that express mathematically the relations between the fundamental electromagnetic field quantities and their dependence on their sources with the electromagnetic properties of media.

$$\frac{\partial D}{\partial t} = \nabla \times H \tag{4}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0}\nabla \times E \tag{5}$$

$$D(\omega) = \varepsilon_0 \varepsilon_r(\omega)\cdot E(\omega) \tag{6}$$

where

E: electric field [V/m],
H: magnetic field [A/m],
D: electric flux density [C/m^2],
μ_0: magnetic permeability of free space [H/m],
ε_0: dielectric permittivity of free space [F/m],
ε_r: relative dielectric permittivity [F/m],
t: time (s).

In this formulation, the generic D-E-relationship in (6) is used to describe the dispersive dielectric properties. For the purpose of convenience, ***E*** and ***D*** are normalized as :

$$\tilde{E} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\cdot E \tag{7}$$

$$\tilde{D} = \sqrt{\frac{1}{\varepsilon_0 \cdot \mu_0}}\cdot D \tag{8}$$

Therefore, the material parameters are omitted in ***D*** and ***H*** computation, and there are some other advantages for constructing the absorbing boundary condition. Then, Maxwell's equations become:

$$\frac{\partial \tilde{D}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 . \mu_0}}\nabla \times H \tag{9}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 . \mu_0}}\nabla \times \tilde{E} \tag{10}$$

$$\tilde{D}(\omega) = \varepsilon_r(\omega)\cdot \tilde{E}(\omega) \tag{11}$$

In the cartesian coordinate system, for TM wave propagation in the *z*-direction we can rewrite (9) and (10) as the following partial differential equations:

$$\frac{\partial \tilde{D}_z}{\partial t} = c_0\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \tag{12}$$

$$\frac{\partial H_x}{\partial t} = c_0\left(-\frac{\partial \tilde{E}_z}{\partial y}\right) \tag{13}$$

$$\frac{\partial H_y}{\partial t} = c_0\left(\frac{\partial \tilde{E}_z}{\partial x}\right) \tag{14}$$

where

$c_0 = \frac{1}{\sqrt{\varepsilon_0 . \mu_0}}$ is the speed of light.

Using the central difference scheme in both the time and spatial domains, (12) – (14) are discretized in FDTD scheme as:

$$D_z^{n+1}(i,j) - D_z^n(i,j) = \Delta t c_0 \left[\left(\frac{H_y^{n+\frac{1}{2}}\left(i+\frac{1}{2},j\right) - H_y^{n+\frac{1}{2}}\left(i-\frac{1}{2},j\right)}{\Delta x} \right) - \left(\frac{H_x^{n+\frac{1}{2}}\left(i,j+\frac{1}{2}\right) - H_x^{n+\frac{1}{2}}\left(i,j-\frac{1}{2}\right)}{\Delta y} \right) \right] \quad (15)$$

$$H_x^{n+1/2}\left(i,j+\frac{1}{2}\right) = H_x^{n-1/2}\left(i,j+\frac{1}{2}\right) - \frac{\Delta t c_0}{\Delta y}\left[E_z^n(i,j+1) - E_z^n(i,j)\right] \quad (16)$$

$$H_y^{n+1/2}\left(i+\frac{1}{2},j\right) = H_y^{n-1/2}\left(i+\frac{1}{2},j\right) - \frac{\Delta t c_0}{\Delta y}\left[E_z^n(i+1,j) - E_z^n(i,j)\right] \quad (17)$$

III. IMPLEMENTAION OF DISPERSIVE MEDIA

In dispersive media, the electrical permittivity is a function of frequency this material can be adequately represented by:

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\sigma}{j\omega\varepsilon_0} + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega t_0} \quad (18)$$

σ : the electrical conductivity [S/m],
ε_∞: the relative permittivity at infinite frequency,
ε_s: the zero-frequency relative permittivity,
t_0: the relaxation time.
Equation (18) is referred to as the Debye formulation (simple pole)[2][3][5].

$$\tilde{D}(\omega) = \varepsilon_\infty \tilde{E}(\omega) + \frac{\sigma}{j\omega\varepsilon_0}\tilde{E}(\omega) + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega t_0}\tilde{E}(\omega) \quad (19)$$

In order to simulate this medium in FDTD, we must put the equation (19) in the sampled time.

$$\tilde{D}^n = \varepsilon_\infty \tilde{E}^n + \left(\varepsilon_s - \varepsilon_\infty\right)\cdot\frac{\Delta t}{t_0}\left(E^n + \sum_{i=0}^{n-1} e^{-\Delta t(n-i)/t_0}\cdot E^i\right) + \frac{\sigma\cdot\Delta t}{\varepsilon_0}\left(\tilde{E}^n + \sum_{i=0}^{n-1} E^i\right) \quad (20)$$

The update equation for E_z is:

$$E^n = \frac{D^n - \dfrac{\sigma\cdot\Delta t}{\varepsilon_0}\sum_{i=0}^{n-1} E^i - \left(\varepsilon_s - \varepsilon_\infty\right)\cdot\dfrac{\Delta t}{t_0}\sum_{i=0}^{n-1} e^{-\Delta t(n-i)/t_0} E^i}{\varepsilon_\infty + \dfrac{\sigma\cdot\Delta t}{\varepsilon_0} + \left(\varepsilon_s - \varepsilon_\infty\right)\cdot\dfrac{\Delta t}{t_0}} \quad (21)$$

IV. FDTD IMPLIMENTATION OF UPML

The UPML uses anisotropic tensors for the permittivity and permeability [6][7]. From Maxwell's equations, the coupled equations of electro- magnetic wave can be expressed, in the UPML medium as follow [2][3][6][7]:

$$\begin{cases} j\omega\overline{s}\tilde{D} = c_0 \nabla\times H \\ j\omega\overline{s}H = -c_0 \nabla\times\tilde{E} \end{cases} \quad (22)$$

Where S is matrix diagonal which is written as:

$$S = \begin{pmatrix} \frac{SySz}{Sx} & 0 & 0 \\ 0 & \frac{SxSz}{Sy} & 0 \\ 0 & 0 & \frac{SxSy}{Sz} \end{pmatrix} \quad (23)$$

with:

$$s_m = 1 + \frac{\sigma_D(m)}{j\omega\varepsilon_0} \quad (24)$$

For $m = x, y$ and z

We can then write UPML Maxwell's equations of 2D TM as:

$$j\omega H_y\cdot\left(1+\frac{\sigma_D(x)}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_D(y)}{j\omega\varepsilon_0}\right)^{-1} = c_0\left(\frac{\partial\tilde{E}_z}{\partial x}\right) \quad (25)$$

$$j\omega H_x\cdot\left(1+\frac{\sigma_D(x)}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_D(y)}{j\omega\varepsilon_0}\right) = c_0\left(-\frac{\partial\tilde{E}_z}{\partial y}\right) \quad (26)$$

$$j\omega\tilde{D}_z\cdot\left(1+\frac{\sigma_D(x)}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_D(y)}{j\omega\varepsilon_0}\right) = c_0\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \quad (27)$$

Expanding equation (21) and taking its inverse Fourier transform leads to:

$$\frac{\partial H_x}{\partial t} + \frac{\sigma_D(y)}{\varepsilon_0}H_x = -c_0\frac{\partial\tilde{E}_z}{\partial y} - \frac{\sigma_D(x).c_0}{\varepsilon_0}\int_0^t \frac{\partial\tilde{E}_z}{\partial y}dt \quad (28)$$

After some mathematical arrangements, it can be expressed as :

$$H_x^{n+1/2}(i,j+\frac{1}{2}) = f_1(j)H_x^{n-1/2}(i,j+\frac{1}{2}) - f_2(j)\left(E_z^n(i,j+1)-E_z^n(i,j)\right)$$
$$-f_3(i,j)\sum_0^n\left(E_z^n(i,j+1)-E_z^n(i,j)\right) \quad (29)$$

Where:

$$f_1(j) = \left(\frac{2\varepsilon_0 - \sigma_D(j)}{2\varepsilon_0 + \sigma_D(j)}\right) \quad (30)$$

$$f_2(j) = \frac{2c_0\varepsilon_0\Delta t}{\Delta y\left(2\varepsilon_0 + \sigma_D(j)\right)} \quad (31)$$

$$f_3(i,j) = \frac{2c_0\sigma_D(i)\Delta t^2}{\Delta y\left(2\varepsilon_0 + \sigma_D(j)\right)} \quad (32)$$

the D_z and H_y can be obtained by analogy.

V. NUMERICAL EXAMPLES

To validate the FDTD modelling, a number of models were designed to simulate. The source is excited by Gaussian pulse.

First, a point source radiating in free-space is studied. A uniform mesh with 1cm is used. The thickness of the UPML layers is 8cells in the total mesh dimension of 60x 60. The evaluating wave propagation at different time steps is shown in figure 3.

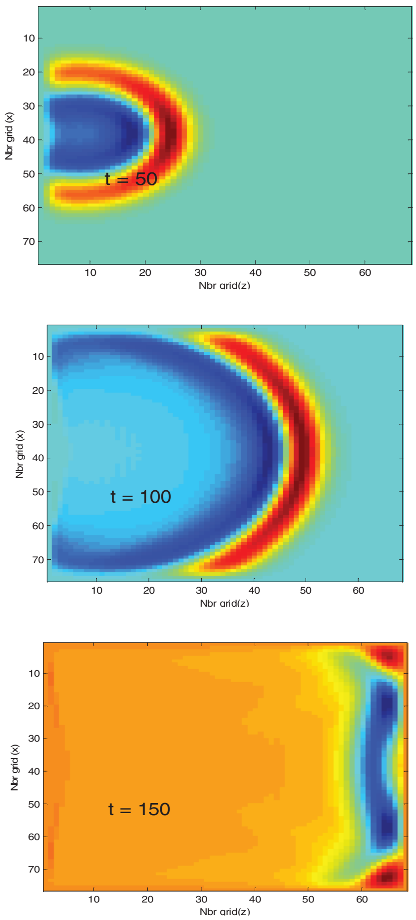

Fig3. Snapshot of wave propagation in free space at time steps 50, 100 and 150.

The second simulation given in fig. 4, presents the propagation of a pulse in a dispersive medium with the parameters: ε_s =25, ε_∞ =5, σ = 0and t_0 = 10ns.

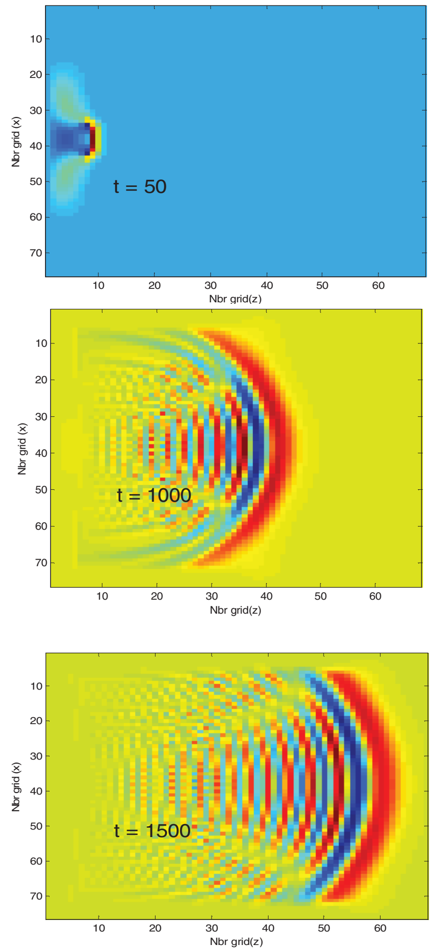

Fig4. Snapshot of wave propagation in dispersive medium at time steps 50, 1000, 1250 and 1500.

Finally, the propagation of a Gaussian pulse is simulated in a medium composed of two media, the first one is a free space, while the second part is a dispersive and conductive medium with the electric parameters: ε_s =30, ε_∞ =5, σ = 0.01S and t_0 = 10 ns. Snapshot of wave propagation at different time steps are presented in figure 5.

The numerical simulation with FDTD, for the multiple media show the propagation in free space is faster than in the dielectric (fig 4). In the dispersive and conductive medium, the velocity of propagation and the wave form are affected by the electric parameters (permittivity and conductivity). In figure 5, the pulse has struck the medium, and part of it has been transmitted and part reflected.

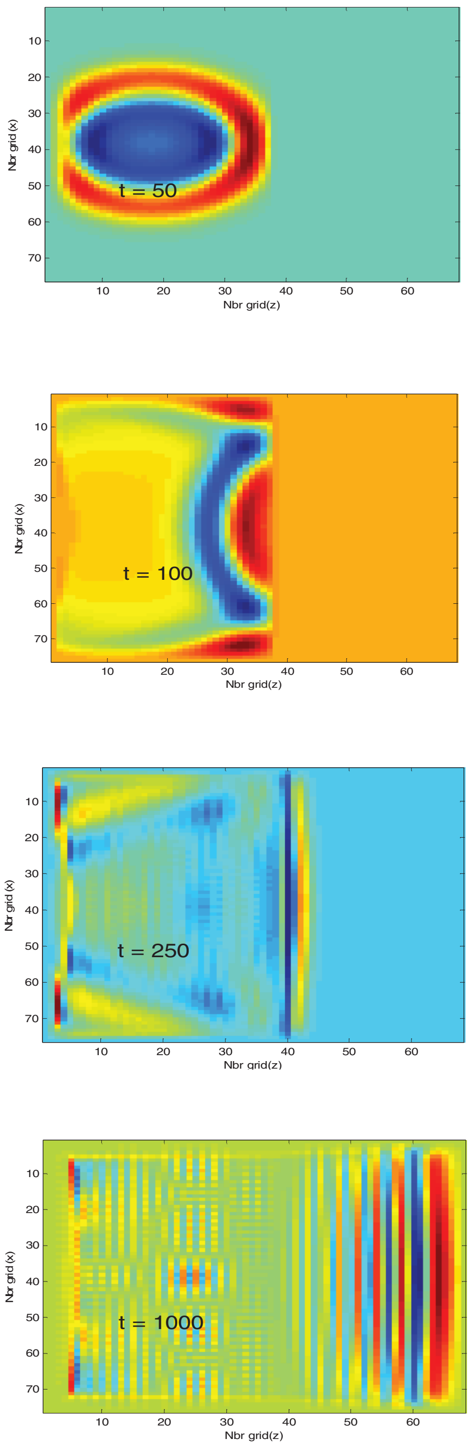

Fig5. Snapshot of wave propagation in dispersive and inhomogeneous medium at time steps 50, 100, 250 and 1000.

V. CONCLUSIONS

A 2D- FDTD- UPML numerical simulation of EM wave propagation is described. The dispersive media with conductive loss are introduced in this model by the ADE method and the dispersive effect is modeled by the Debye model. Numerical simulations on dispersive media show that the pulse can be affected by dispersive effects. Also, a reflected pulse is detected because of the difference in relative permittivity between two media in the inhomogeneous medium.

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