

# Representation Theory

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### Abstract

This document gathers my personal notes on representation theory of finite groups. It is mostly based on two references : The Symmetric Group by Bruce Sagan and Harmonic Analysis on Finite Groups by Ceccherini-Silberstein, Scarabotti and Tolli.

## 1 Basic Definitions and Motivations

The notion of representation arises quite frequently in mathematics. Morally, a representation is a way of saying that whatever the abstract object we are studying really is, it can be thought of as a concrete object of that said structure. In other words, our abstract definition probably catches the essence of the object we are studying. Representation theorems are usually great theorems. For example, Cayley's theorem says that every group can be thought of as a subgroup of the permutation group.

Here is an informal definition of what we call a representation in mathematics. However, representation theory is not that general and focuses mainly on the structure of group actions.

**Definition 1.1.** Let  $G$  be a group and  $X$  be a set. We say that  $G$  acts on  $X$  if there is a map

$$G \times X \ni (g, x) \mapsto g \cdot x \in X$$

such that

1.  $e \cdot x = x$  for all  $x \in X$ ,
2.  $g \cdot (h \cdot x) = (gh) \cdot x$  for all  $g, h \in G$  and  $x \in X$ .

**Example 1.2.** 1. The group  $\mathfrak{S}_n$  acts on the set  $\{x_1, \dots, x_n\}$  with the action defined by

$$\sigma \cdot x_i = x_{\sigma(i)}$$

2. The group  $\mathcal{GL}_n(\mathbb{C})$  acts on a vector space  $V$  with the action defined by

$$\mathcal{GL}_n(\mathbb{C}) \times V \ni (A, v) \mapsto Av \in V$$

The second example should bring a whole new intuition to the notion of group actions as a generalization of the notion of linear transformations. In particular, it is easier to see that  $X$  somehow inherits a structure from  $G$ . Representation theory is all about studying the action structure through concrete objects from linear algebra.

**Definition 1.3.** A representation of  $G$  over a vector space  $V$  is an action of  $G$  on  $V$  defined by

$$G \times V \ni (g, v) \mapsto \rho(g)v \in V \tag{1}$$

where  $G \ni g \mapsto \rho(g) \in \mathcal{GL}_d$  is a group homomorphism.

## 2 Gelfand Pair