Question 7: Solve the following questions from the Discrete Math zyBook:
a) Exercise 6.1.5, sections b-d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(b) What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, {4♠, 4♦, 4♣, J♠, 8♥} is a three of a kind.

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}4^2}{\binom{52}{5}}$$

We first need to choose a card of the rank, so 13 choose 1. Of the four suits, we need three, so 4 choose 3. The remaining two cards can be taken from twelve possibilities, then eleven possibilities, so 12 choose 2. They can be any of the 4 remaining suits, so $\binom{4}{1}^2$ or 4^2 simplified. Divide all of that by the sample space!

(c) What is the probability that all 5 cards have the same suit?

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

We must first choose the suit, which is 4 choose 1. They must be a different rank, so 13 choose 5. Divide that by the sample space.

(d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}}$$

Similar to the first question, we need to first determine the rank, so 13 choose 1. Then choose the suits, 4 choose 2. However, there are three remaining cards this time, so 12 choose 3, and the remaining cards can be any of the 4 remaining suits, so 4^3, all divided by the sample space.

b) Exercise 6.2.4, sections a-d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(a) The hand has at least one club.

The odds of drawing a non-club at any turn is $\binom{39}{5}$. p(non-club) = $\frac{\binom{39}{5}}{\binom{52}{5}} \approx 0.2215$, so 1 - 0.2215 = .7785 or roughly a 78% chance the hand has at least one club.

(b) The hand has at least two cards with the same rank.

First, we determine the cards of different ranks, which is $\binom{13}{5}$. $\binom{39}{5}$ have at least two cards of the same rank. To draw the actual card, we pull from any of the suits $\binom{4}{1}$. They can be any of the 5 cards in the hand which gives us would make it $\binom{4}{1}^5$. We need to get the number of possible hands and divide that by the sample space. $\binom{52}{5}$ - $\binom{13}{5}4^5$ all divided by $\binom{52}{5}$.

$\frac{\binom{52}{5} - \binom{13}{5}4^5}{\binom{52}{5}} \approx$ .4929 chance that the cards are not of the same rank. 1 - .4929 = .5071. There is a 50.71% change that the hand has at least two cards with the same rank.

(c) The hand has exactly one club or exactly one spade.

To choose a club of any rank, that is $\binom{13}{1}$. No other cards can be of the same rank, so that's $\binom{39}{4}$ The same for a spade. They intersect at exactly one of each, so $\binom{13}{1}\binom{13}{1}$ and the three remaining cards can only be hearts or diamonds, so that leaves 3 cards from 26 possibilities $\binom{26}{3}$.

p(Ec ∪ Es) = $\frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} + \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} - \frac{\binom{13}{1}\binom{13}{1}\binom{26}{3}}{\binom{52}{5}} = \frac{82251 * 13}{2598960} + \frac{82251 * 13}{2598960} - \frac{13 * 13 * 2600}{2598960}$

$=\frac{1069263}{2598960} + \frac{1069263}{2598960} - \frac{439400}{2598960} = \frac{1699126}{2598960} \approx$ 0.65377, or a 65.38% chance

(d) The hand has at least one club or at least one spade.

We can use the calculations from part a for this. The odds of drawing any non-club and non-spade are $\frac{\binom{26}{5}}{\binom{52}{5}}$. Pulling a club or spade is not mutually exclusive so 1 - $\frac{\binom{26}{5}}{\binom{52}{5}} \approx$ 0.9746 or a 97.5% chance that the hand has one club or one spade.

Question 8: Solve the following questions from the Discrete Math zyBook:
a) Exercise 6.3.2, sections a-e

The letters {a, b, c, d, e, f, g} are put in a random order. Each permutation is equally likely. Define the following events:

A: The letter b falls in the middle (with three before it and three after it)
B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "agbdcef" would be an outcome in this event.
C: The letters "def" occur together in that order (e.g. "gdefbca")

(a) Calculate the probability of each individual event. That is, calculate p(A), p(B), and p(C).

p(A) has b in a fixed position and 6 letters in any order, so 6!. |S| = 7!, so $\frac{6!}{7!}$ or $\frac{1}{7}$.
p(B) has c to the right of b. However, every possibility has c to the left of b an equal number of times, so with $\frac{7!}{2}$ for the 50% of the time c is to the right of be, $\frac{\frac{7!}{2}}{7!} = \frac{1}{2}$.
p(C) has def in a group, so we can treat that as a single object, so that would be 5!. $\frac{5!}{7!} = \frac{1}{42}$

(b) What is p(A|C)?

"def" can be treated as a single item, so if it's on one side of b, there are 3! ways to make that set. Multiply that by 2 for the other side all over the given event.

p(A|C) = $\frac{|A \cap C|}{|C|} = \frac{2 * 3!}{5!} = \frac{2 * 6}{120} = \frac{12}{120} = \frac{1}{10}$

(c) What is p(B|C)?

I believe this follows the same principle as finding p(B).

p(B|C) = $\frac{|B \cap C|}{|C|} = \frac{\frac{5!}{2}}{5!} = \frac{1}{2}$

(d) What is p(A|B)?

Since "b" is in the middle, there are only three positions for "c" to be in, which is 3. That also means that there are 5! options for the rest of the letters

p(A|B) = $\frac{|A \cap B|}{|B|} = \frac{3 * 5!}{\frac{7!}{2}} = \frac{3 * 5!}{\frac{7!}{2}} = \frac{2 * 3 * 5!}{7!} = \frac{6 * 120}{5040} = \frac{720}{5040} = \frac{1}{7}$

(e) Which pairs of events among A, B, and C are independent?

p(A|C) = $\frac{1}{10} \neq \frac{1}{7}$ = p(A) A and C are not independent.

p(B|C) = $\frac{1}{2}$ = p(B) B and C are independent.

p(A|B) = $\frac{1}{7}$ = p(A) A and B are independent.

b) Exercise 6.3.6, sections b, c

A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is 1/3 and the probability of tails is 2/3. The outcomes of the coin flips are mutually independent. What is the probability of each event?

(b) The first 5 flips come up heads. The last 5 flips come up tails.

$\frac{1}{3} * \frac{1}{3} * \frac{1}{3} * \frac{1}{3} * \frac{1}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} = (\frac{1}{3})^5 * (\frac{2}{3})^5$

(c) The first flip comes up heads. The rest of the flips come up tails.

$\frac{1}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} = (\frac{1}{3}) * (\frac{2}{3})^9$

c) Exercise 6.4.2, section a

Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

We will define p(F) as the probability of picking the fair die. p($\overline{F}$), or p of fancy F, is the probability of picking the loaded die. p(X) will be the probability that we get the rolls 4, 3, 6, 6, 5, 5.

p(F) = $\frac{1}{2}$

p(X|F) = $(\frac{1}{6})^6$

p(X|$\overline{F}$) = $(\frac{3}{20})^4 * (\frac{1}{4})^2$

p(F|X) = $\dfrac{(\frac{1}{6})^6 * \frac{1}{2}}{((\frac{1}{6})^6 * \frac{1}{2}) + ((\frac{3}{20})^4 * (\frac{1}{4})^2 * \frac{1}{2})}$ $\approx$ 0.4038 or

Question 9: Solve the following questions from the Discrete Math zyBook:
a) Exercise 6.5.2, sections a, b

A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

(a) What is the range of A?

There are 4 aces, so the range can be from 0 to 4, {0, 1, 2, 3, 4}

(b) Give the distribution over the random variable A.

There are $\binom{48}{5}$ cards for 0 aces, $\binom{4}{1}\binom{48}{4}$ for 1 ace, $\binom{4}{2}\binom{48}{3}$ for 2 aces, $\binom{4}{3}\binom{48}{2}$ for 3, and $\binom{4}{4}\binom{48}{1}$ for one.

$(0, \frac{\binom{26}{5}}{\binom{52}{5}}), (1, \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}), (2, \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}), (3, \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}), (4, \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}})$

b) Exercise 6.6.1, section a

(a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?

E[G] = 2·(21/45) + 1·(21/45) + 0·(3/45)
2·(7/15) + 1·(7/15) + 0·(1/15) = 21/15 = 7/5 = 1.4

c) Exercise 6.6.4, sections a, b

(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then X = 25. What is E[X]?

$E[X] = (1^2 * \frac{1}{6}) + (2^2 * \frac{1}{6}) + (3^2 * \frac{1}{6}) + (4^2 * \frac{1}{6}) + (5^2 * \frac{1}{6}) + (6^2 * \frac{1}{6}) = 15 + \frac{1}{6}$

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and Y = 4. What is E[Y]?

$E[X] = (0^2 * \frac{1}{8}) + (1^2 * \frac{3}{8}) + (2^2 * \frac{3}{8}) + (3^2 * \frac{1}{8}) = 3$

d) Exercise 6.7.4, section a

(a) A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

We will use E[C] as the expected value that a child will receive their own coat. If a child receives their coat, the value will be 1 and 0 otherwise. There are 10! possibilities for distribution of coats. If the first student gets their coat, there are 9! possibilities for the remaining students.

$E[C] = 1 * \frac{9!}{10!} = 1 * \frac{9*8*7*6*5*4*3*2*1}{10*9*8*7*6*5*4*3*2*1} = \frac{1}{10}$

Using the Linearity of Expectations, there are 10 children who have a $\frac{1}{10}$ chance to receive their coat, so $E[C] = 10 * \frac{1}{10} = \frac{10}{10} = 1$

Question 10: Solve the following questions from the Discrete Math zyBook:
a) Exercise 6.8.1, sections a-d

The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

(a) What is the probability that out of 100 circuit boards made *exactly* 2 have defects?

In this case, there are two failures, so

$\binom{100}{98}p^{98}q^{100 - 98} = 4950 * {\frac{99}{100}}^{98} * {\frac{1}{100}}^{2} = 4950 * {\frac{99}{100}}^{99} * \frac{1}{10000} \approx 0.18486$

(b) What is the probability that out of 100 circuit boards made *at least* 2 have defects?

b(k; n, p) with no defects would be b(100; 100, 0.99)

$\binom{100}{100}p^{100}q^{100 - 100} = 1 * {\frac{99}{100}}^{100} * {\frac{1}{100}}^{0} = {\frac{99}{100}}^{100}$

One defect would be b(99; 100, 0.99)

$\binom{100}{99}p^{99}q^{100 - 99} = 100 * {\frac{99}{100}}^{99} * {\frac{1}{100}}^{1} = {\frac{99}{100}}^{99} * \frac{100}{100} = {\frac{99}{100}}^{99}$

Anything greater than that amount would count, so we can subtract the possibility of 0 or 1 defects from 1

$p(D) = 1 - ({\frac{99}{100}}^{100}) - ({\frac{99}{100}}^{99}) \approx 0.2642$

(c) What is the expected number of circuit boards with defects out of the 100 made?

There is a failure rate of 1%, so out of 100, we expected 1 defect. 100 * .01 = 1

(d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compare to the situation in which each circuit board is made separately?

We can use a similar calculation as in part b to find out how many have at least two defects

0 defects $\binom{50}{50}p^{50}q^{50 - 50} = 1 * {\frac{99}{100}}^{50} * {\frac{1}{100}}^{0} = {\frac{99}{100}}^{50}$

1 defect $\binom{50}{49}p^{49}q^{50 - 49} = 50 * {\frac{99}{100}}^{49} * {\frac{1}{100}}^{1} = {\frac{99}{100}}^{49} * \frac{50}{100} = {\frac{99}{100}}^{49} * \frac{1}{2}$

p(D) = 1 - ($\frac{99}{100}^{50}$) - ($\frac{99}{100}^{49}$ * $\frac{1}{2}$) ≈ 0.08944

The failure rate is still 1% so with 50 batches of 2 that gives us

2 * 50 * 0.01 = 100 * 0.01 = 1

It's the same rate, it doesn't seem to matter if they're made separately or not.

Expected number of batches with defects: 50·(0.01) = 0.5. There are two circuit boards in each batch, so the expected number of boards with defects is (0.5)·2 = 1. The expectation is the same as the case in which the boards are made separately.

There are at least two circuit boards with defects unless none of the batches have defects. The probability that at least two circuit boards have defects is $1-(0.99)^{50} \approx 0.395$. It is much more likely (probability 0.395 compared to probability 0.264) that there are at least two boards with defects in the situation in which boards are made in batches.

b) Exercise 6.8.3, section b

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

(b) What is the probability that you reach an incorrect conclusion if the coin is biased?

Let X be the probability that the coin is heads. We'll need to find the probability of all flips with heads being less than 4.

Incorrect conclusions are that the result is heads less than 4 times out of 10 flips with a fair coin or that the result is heads more than 3 times out of 10 flips if the coin is biased. We're concerned with reaching the latter.

$E[X] = 1 - ((0.7^{10}) + (\binom{10}{1} * 0.3^1 * 0.7^9) + (\binom{10}{2} * 0.3^2 * 0.7^8) + (\binom{10}{3} * 0.3^3 * 0.7^7)) \approx 0.35$