Question 3:
a. Solve Exercise 8.2.2, section b from the Discrete Math zyBook.

Give complete proofs for the growth rates of the polynomials below. You should provide specific values for c and n_0 and prove algebraically that the functions satisfy the definitions for O and Ω.

(b) $f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$.

$n^3 + 3n^2 + 4 \leq n^3 + 3n^3 \Rightarrow n^3 + 3n^2 + 4 \leq 4n^3$ implies $T(N) \leq c(f(N))$

$n^3 \leq n^3 + 3n^2 + 4$ implies $T(N) \geq c(f(N))$

$0 \leq 3n^2 + 4 \quad \Rightarrow \quad 4 \leq 3n^2$ when $n \geq 2$

$c_1 = 4$
$c_2 = 1$
$n_0 = 2$

$\Omega(n^3) = O(n^3)$

b. Solve Exercise 8.3.5, sections a-e from the Discrete Math zyBook

The algorithm below makes some changes to an input sequence of numbers.

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MysteryAlgorithm

Input: a_1, a_2,...,a_n
       n, the length of the sequence.
       p, a number.
Output: ??

i := 1
j := n

While (i < j)
        While (i < j and a_i < p)
                i := i + 1
        End-while
        While (i < j and a_j ≥ p)
                j := j - 1
        End-while
        If (i < j), swap a_i and a_j
End-while

Return( a_1, a_2,...,a_n )
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(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with $p = 0$)

After watching several videos on the subject, the algorithm sorts numbers based on the quick sort algorithm. The pivot is p and any numbers less than p will be pushed into lower order positions of the array and numbers greater than p will be pushed into higher order positions.

(b) What is the total number of times that the lines "i := i + 1" or "j := j - 1" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

The number of times i is incremented or j is decremented is exactly n-1, regardless of the values in the input sequence.

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

The swap depends on where you put the pivot. It could be conceivably executed 0 times if it just so happens the numbers are already sorted around the pivot, but assuming the exact opposite, p would be the middle point and the swap would execute $\frac{n}{2}$ times.

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

If we pick the best possible case for the pivot point (a value actually in the middle of all other values), it will execute $\frac{n}{2}$ times, so the best possible case is $\Omega(\frac{n}{2}) = \Omega(\frac{1}{2}n) = \Omega(n)$.

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

If we pick the worst possible case, we can set the iterations for i or the iterations for j to the maximum value if p = n or p = 1. If p = 1, then the number of iterations for i is 0 and the number of iterations for j is n times, once for every element in the array. If p = n, then j will iterate once and i will iterate n - 1 times. In both cases, the max iteration time is O(n).

Question 4: Solve the following questions from the Discrete Math zyBook:

a) Exercise 5.1.2, sections b, c

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

|Digits| + |Letters| + |Special characters| = 10 + 26 + 4 = 40.

Length 7 = 40^7
Length 8 = 40^8
Length 9 = 40^9

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

|Digits| + |Letters| + |Special Characters| = 40

40 - |Letters| = 40 - 26 = 14

Length 7 = 14 + 40^6
Length 8 = 14 + 40^7
Length 9 = 14 + 40^8

$14(40^6 + 40^7 + 40^8) = 9.41 \times 10^{13}$

b) Exercise 5.3.2, section a

(a) How many strings are there over the set {a, b, c} that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

We first have 3 possibilities for the first entry, and subsequently only 2 after that. For instance, if the first entry was a, then the only allowable second entries are b or c. Presume the second entry is b, then the only allowable entries would be a or c, etc. That would give us

$3*2*2*2*2*2*2*2*2*2 = 3 * 2^9$

c) Exercise 5.3.3, sections b, c

License plate numbers in a certain state consist of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

(b) How many license plate numbers are possible if no digit appears more than once?

26 letters can appear for any of their entries, but the digits can be 10 for the first entry, 9 for the second digit entry, and 8 for the third digit entry.

$10*26*26*26*26*9*8 = 26^4 * 10 * 9 * 8 = 329{,}022{,}720$

(c) How many license plate numbers are possible if no digit or letter appears more than once?

We can use the same reasoning as before for the digits and also apply it to the letters, so we get instead

$10 * 26 * 25 * 24 * 23 * 9 * 8 = 258{,}336{,}000$

d) Exercise 5.2.3, sections a, b

Let $B = \{0, 1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

(Let $x \in B9$. If the number of 1's in x is even, then $f(x) = x0$. If the number of 1's in x is odd, then $f(x) = x1$.
For $x, y \in B9$, if $x \neq y$, then $f(x) \neq f(y)$ because two different binary strings can not be made equal by adding a bit to the end of the strings. Therefore f is one-to-one.
Let y be a string in E10. Let x be the string consisting of the first 9 bits of y. If the last bit of y is 0, then the number of 1's in x must be even and $f(x) = x0 = y$. If the last bit of y is 1, then the number of 1's in x must be odd and $f(x) = x1 = y$. Therefore, f is onto.

(b) What is $|E_{10}|$?

E_{10} is bijection with B^9, so 2^9.

Question 5: Solve the following questions from the Discrete Math zyBook:
a) Exercise 5.4.2, sections a, b

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

(a) How many different phone numbers are possible?

We have seven digits and there are 2 options for the first three digits. There are 10^4 options for the remaining digits, so there are $2 * 10^4$ possibilities.

(b) How many different phone numbers are there in which the last four digits are all different?

We know that because of the answer to (a) that there are 2 options for the first three digits. We know we have 4 digits that must be different, so after each choice we'll have one less digit to choose from. That would give us 10*9*8*7. Multiply that by 2 as previously mentioned and we get 2*10*9*8*7.

b) Exercise 5.5.3, sections a-g

How many 10-bit strings are there subject to each of the following restrictions?

(a) No restrictions.

2^{10}

(b) The string starts with 001.

$1*2^7$

(c) The string starts with 001 or 10.

From above, $1*2^7$, and using the same principle to calculate $1*2^8$, that gives us $2^7 + 2^8$.

(d) The first two bits are the same as the last two bits.

Once the first 8 bits are determined, the last two bits must match the first two bits, so there are no remaining choices for the string. Thus, the number of strings in which the first two bits are the same as the last two bits is $2^6*2^2= 2^8 = 256$

(e) The string has exactly six 0's.

$$\binom{10}{6} = \frac{10!}{6!(10-6)!} = \frac{10*9*8*7*6!}{6!4!} = \frac{10*9*8*7}{4*3*2*1} = \frac{10*9*8*7}{8*3} = \frac{10*9*7}{3} = \frac{630}{3} = 210$$

(f) The string has exactly six 0's and the first bit is 1.

1 * $\binom{9}{6}$ = $\binom{9}{6}$ and I do the same calculations again = 84

(g) There is exactly one 1 in the first half and exactly three 1's in the second half.

$\binom{5}{1}$ = 5 and $\binom{5}{3}$ = 10, so 5 * 10 = 50 possibilities.

c) Exercise 5.5.5, section a

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

$\binom{35}{10}$ * $\binom{30}{10}$

d) Exercise 5.5.8, sections c-f

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(c) How many five-card hands are made entirely of hearts and diamonds?

13 hearts, 13 diamonds = 26. $\binom{26}{5}$ = 65780

(d) How many five-card hands have four cards of the same rank?

$\binom{13}{1}$ = 13 for the card rank in each suit and $\binom{4}{4}$ for that rank. $\binom{13}{1}$ * $\binom{4}{4}$ = 13 * 1 = 13. That leaves 52 - 4 choices for the remaining card, so $\binom{48}{1}$ = 48. 48 * 13 = 624

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?

$\binom{13}{1}$ for the first card rank and $\binom{4}{3}$ possibilities. That rank is no longer an option, so $\binom{12}{1}$ for the second card rank and $\binom{4}{2}$ cards to round out the house. $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ = 13 * 4 * 12 * 6 = 3744

(f) How many five-card hands do not have any two cards of the same rank?

There are 13 different possible ranks. The number of ways to select 5 distinct ranks from 13 possible ranks is (13 choose 5). For each rank chosen, there are four possible cards with that rank that can be selected. Therefore once the ranks have been determined, there are 4^5 ways to select the cards in the hand. The total number of ways to select a five-card hand in which no two cards have the same rank is

$4^5\binom{13}{5}$ = 1,317,888

e) Exercise 5.6.6, sections a, b

A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

There must be 5 from each in order to be equal, so
$\binom{44}{5} * \binom{56}{5}$

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

$(\binom{44}{1} * \binom{43}{1}) * (\binom{56}{1} * \binom{55}{1})$

Question 6: Solve the following questions from the Discrete Math zyBook:
a) Exercise 5.7.2, sections a, b

A 5-card hand is drawn from a deck of standard playing cards.

(a) How many 5-card hands have at least one club?

There are 39 cards that aren't clubs, so to get 0 clubs, that is $\binom{39}{5}$. There are $\binom{52}{5}$ possible five-card hands, so to get 1-5 clubs, $\binom{52}{5}$ - $\binom{39}{5}$

(b) How many 5-card hands have at least two cards with the same rank?

13 ranks, so to get them all different, we'd need $\binom{13}{5}$. That means $\binom{39}{5}$ for cards of the same rank. We also need another card that's the same rank, so that would be $\binom{4}{1}$. It can be any of the other 5 cards, however, which would make it $\binom{4}{1}^5$.
That ultimately leaves us with $\binom{52}{5}$ - $\binom{13}{5}\binom{4}{1}^5$
or
$\binom{52}{5}$ - $\binom{13}{5}$*4^5

b) Exercise 5.8.4, sections a, b

20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

Since there are no restrictions whatsoever, then the possibilities are 5^{20}.

(b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

The first child would get $\binom{20}{4}$. There are 16 comics left, so the second child would get $\binom{16}{4}$, etc.
$\binom{20}{4}*\binom{16}{4}*\binom{12}{4}*\binom{8}{4}*\binom{4}{4}$

Question 7: How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 0. Either not all elements would match or two would go to the same number

b) 5 First choice, 5. One of the choices is gone, so 4 for the second, etc., so 5*4*3*2*1 or 5!.

c) 6 Applying the product rule, we'd have 5! possible functions 6 times, or 720

d) 7 Applying the product rule again 5! possible functions 7 times, or 5040. However, the unmapped elements are interchangeable, so we need to divide by 2. 2520