Question 5: a) Solve the following questions from the Discrete Math zyBook:
1. Exercise 1.12.2, sections b, e

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

(b) $p \rightarrow (q \land r)$
$\neg q$
$\therefore \neg p$

Line #	Statement	Justification
1	$\neg q$	Hypothesis2
2	$\neg q \lor \neg r$	Addition 1
3	$\neg(q \land r)$	De Morgan's Law
4	$p \rightarrow (q \land r)$	Hypothesis1
5	$\neg p$	Modus Tollens 3,4

(e) $p \lor q$
$\neg p \lor r$
$\neg q$
$\therefore r$

Line #	Statement	Justification
1	$p \lor q$	Hypothesis1
2	$\neg q$	Hypothesis3
3	p	Disjunctive Syllogism 1, 2
4	$\neg p \lor r$	Hypothesis2
5	r	Disjunctive Syllogism 4

2. Exercise 1.12.3, section c

Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

(c) One of the rules of inference is Disjunctive syllogism :
$p \lor q$
$\neg p$
$\therefore q$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

Line #	Statement	Justification
1	$p \lor q$	Hypothesis1
2	$\neg\neg p \lor q$	Double negation 1
3	$\neg p \to q$	Conditional Identities 2
4	$\neg p$	Hypothesis2
5	q	Modus ponens 3, 4

3. Exercise 1.12.5, sections c, d:
Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

(c) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
∴ I will not buy a new car.

$(p \land q) \to r$
$\neg r$
$\therefore \neg p$

p	q	r	$(p \land q) \to r$	$\neg r$	$\neg p$
T	F	F	T	T	F

The argument is invalid.

(d) I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.
∴ I will not buy a new car.

$(p \land q) \to r$
$\neg r$
q
$\therefore \neg p$

Line #	Statement	Justification
1	$(p \land q) \to r$	Hypothesis1
2	$\neg r$	Hypothesis2
3	$\neg(p \land q)$	Modus tollens 1, 2
4	$\neg p \lor \neg q$	De Morgan's Law 3
5	$\neg q \lor \neg p$	Commutative laws 4
6	$q \to \neg p$	Conditional Identities 5
7	q	Hypothesis3
8	$\neg p$	Modus ponens 6, 7

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

(b) $\exists x\,(P(x) \lor Q(x))$

$\exists x\,\neg Q(x)$

$\therefore\ \exists x\,P(x)$

	P(x)	Q(x)
a	F	T
b	F	F

2. Exercise 1.13.5, sections d, e

Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid. The domain for each problem is the set of students in a class.

(d) Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.
Penelope did not get a detention.

M(x), where x is a student who missed class.
D(x), where x is a student who got detention.

$\forall x\ M(x) \to D(x)$
Penelope is a student in the class
$\neg M(\text{Penelope})$
$\therefore\ \neg D(\text{Penelope})$

	M(x)	D(x)
Penelope	F	T

That would make the hypotheses True and the conclusion False. The argument is invalid.

(e) Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.
Penelope did not get a detention.

M(x), where x is a student who missed class.
D(x), where x is a student who got detention.
A(x), where x is a student who got an A.

$\forall x\ (M(x) \lor D(x)) \rightarrow \neg A(x)$
Penelope is a student in the class.
A(Penelope)
$\therefore$ Penelope did not get a detention.

	M(x)	D(x)	A(x)	¬A(x)	M(x) ∨ D(x)	(M(x) ∨ D(x)) → ¬A(x)
Penelope	T	T	T	F	T	F
Penelope	T	F	T	F	T	F
Penelope	F	T	T	F	T	F
Penelope	F	F	T	F	F	T

The argument is valid. ∃P(b) ∨ Q(b)

Line #	Statement	Justification
1	$\forall x\ (M(x) \lor D(x)) \rightarrow \neg A(x)$	Hypothesis1
2	Penelope is a student in the class.	Element Definition
3	A(Penelope)	Hypothesis3
4	(M(Penelope) ∨ D(Penelope)) → ¬A(Penelope)	Universal Instantiation 1
5	¬(M(Penelope) ∨ D(Penelope))	Modus Tollens 3, 4
6	¬M(Penelope) ∧ ¬D(Penelope)	De Morgan's Law 5
7	¬D(Penelope)	Simplification 6

Question 6: Solve Exercise 2.4.1, section d;

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

Prove each of the following statements using a direct proof.

(d)The product of two odd integers is an odd integer.

Proof: Let x and y be odd integers. Since x is odd, there is an integer k such that $x = 2k + 1$. Since y is odd, there is an integer j such that $x = 2j + 1$.

$xy = (2k + 1)*(2j + 1) = 4jk + 2k + 2j + 1 = 2(2jk + k + j) + 1$

Since k and j are integers, then $2jk + k + j$ is also an integer.

Since xy is equal to $2m + 1$, where $m = 2jk + k + j$, xy is odd.

Exercise 2.4.3, section b

Prove each of the following statements using a direct proof.

(b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Assume that for real number x, $x \leq 3$. We will show that $12 - 7x + x^2 \geq 0$.

Subtract x from both sides of the inequality $3 \geq x$ to get $3 -x \geq 0$. Since $4 - x$ is one larger than

$3 - x$, then $4 - x > 3 -x \geq 0$.

Since $3 -x \geq 0$ and $4 -x \geq 0$, the product of $(3 - x)$ and $(4 - x)$ is also at least 0:

$(3 - x)(4 - x) \geq 0$.

Multiplying out the two terms gives:

$12 - 7x + x^2 \geq 0$

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Question 7: Solve Exercise 2.5.1, section d;

Prove each statement by contrapositive

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Assume n is an even number. We will prove $n^2 - 2n + 7$ is odd.

$n = 2k$ for some integer k

$n^2 - 2n + 7 = (2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7 = 2(2k^2 - 2k) + 7$

Since k is an integer, $2k^2 - 2k$ is also an integer.

Therefore n can be expressed as 2m, where $m = 2k^2 - 2k$, thus n is even

We can conclude that $n^2 - 2n + 7$ is odd.

Exercise 2.5.4, sections a, b;

Prove each statement by contrapositive

(a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Let x and y be real numbers. Suppose that $x > y$. We will prove that $x^3 + xy^2 > x^2y + y^3$.
Assume that $(x^2 + y^2) > 0$.
$x^3 + xy^2 = x(x^2 + y^2)$.
$x^2y + y^3 = y(x^2 + y^2)$.
$x(x^2 + y^2) > y(x^2 + y^2)$
Dividing both sides by $(x^2 + y^2)$ yields $x > y$.
Therefore $x > y$.

(b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Let x and y be real numbers. Suppose that $x \leq 10$ and $y \leq 10$. We will prove that $x + y \leq 20$.

We will choose the maximum values to verify the proof. If $x = 10$ and $y = 10$, then $x + y = 20$

Thus we can prove $x + y \leq 20$

Exercise 2.5.5, section c

Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other.

(c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Let x be a not irrational non-zero real number. We will prove that $\frac{1}{x}$ is also not irrational.

Since x is not irrational, non-zero, and real, $\frac{1}{x}$ is not irrational, non-zero, and real.

Therefore, $\frac{1}{x}$ = 1/($\frac{y}{z}$) for some integers y and z where y ≠ 0 and z ≠ 0.

$\frac{1}{x}$ = 1/($\frac{y}{z}$) = 1 * $\frac{z}{y}$ = $\frac{1}{x}$ = $\frac{z}{y}$

Multiplying by y gives us $\frac{y}{x}$ = z.

Multiplying by x gives us y = xz.

Dividing by y gives us $\frac{y}{z}$ = x.

Thus, x is equal to the ratio of two integers with a non-zero denominator, so x is a rational number and not irrational.

Question 8: Solve Exercise 2.6.6, sections c, d

Give a proof for each statement.

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

Proof: We assume the average of three real numbers is not greater than or equal to all of the numbers included in the average. Let x, y, and z stand in place of these numbers. We will assume x = 1, y = 2, and z = 45.

$\neg\exists x,y,z\ (\frac{x+y+z}{3}) \geq x$

Applying De Morgan's Law yields

$\forall x,y,z\ \neg(\frac{x+y+z}{3}) \geq x$

Applying the negation yields

$\forall x,y,z\ (\frac{x+y+z}{3}) < x$

Plugging in the previously assumed values for x, y, and z, we get

$(\frac{1+2+45}{3}) < 1$

$\frac{1+2+45}{3} = \frac{48}{3} = 16 < 1$

The fact that 16 is not less than 1 means it is greater than or equal to at least one of the numbers, violating the assumption that it is not greater than or equal to all of the numbers included in the average.

(d) There is no smallest integer.

Proof: We assume that an integer x is the smallest integer an integer can be. So tiny. I cannae belie'e how tineh it is. Vsmol.

$x - 1 < x$

Thus x is not the smallest integer.

Question 9: Solve Exercise 2.7.2, section b

Prove each statement.

(b) If integers x and y have the same parity, then x + y is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof: Case 1: x and y have even parity.

Since x is even, $x = 2k$ for some integer k. Since y is even, $y = 2j$ for some integer j.
Plugging in 2k for x and 2j for y, we get $2k + 2j = 2(k + j)$.
Since k and j are integers, $k + j$ is also an integer. Therefore, $x + y$ is even.

Case 2: x and y have odd parity.

Since x is odd, $x = 2k + 1$ for some integer k. Since y is odd, $y = 2j +1$ for some integer j.
Plugging in $2k + 1$ for x and $2j + 1$ for y, we get $2k + 2j + 2 = 2(k + j + 1)$.
Since k and j are integers, $k + j + 1$ is also an integer. Therefore, $x + y$ is even.