Question 5:

a. Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder). Hint: you may want to use the formula: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Theorem: For any positive value of n, $n^3 + 2n$ is evenly divisible by 3.

Base Case: for n = 1, $1^3 + 2(1) = 1 + 2 = 3$, which is divisible by 3.

Proof: By induction on n. Assume that for any positive integer k, $k^3 + 2k$ is divisible by 3. We will prove that $(k+1)^3 + 2(k+1)$ is divisible by 3.

$(k+1)^3 + 2(k+1)$

$= (k^3 + 3k^2 + 3k + 1) + 2k+2$
$= k^3 + 2k + 3k^2 + 3k + 3$ From induction hypothesis, we know that $k^3 + 2k$ is divisible by 3
$= k^3 + 2k + 3(k^2 + k + 1)$ 3 times and number is divisible by 3
Thus, $(k+1)^3 + 2(k+1)$ is divisible by 3.

b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

Theorem: Any positive integer n, such that n ≥ 2, can be written as a product of primes.

Proof: By strong induction

Base case: n = 2: 2 * 1 = 2. Thus it is true for the base case.

Let the theorem that n ≥ 2 can be written as a product of primes be expressed P(n). Assume that for any k ≥ 2, P(j) through P(k) is true. We will prove P(k+1). Since it is a product of prime numbers P(k+1) can be expressed as $ab = k + 1$. Using algebra, we can find that $a = \frac{k+1}{b}$ and $b = \frac{k+1}{a}$. Since k ≥ a ≥ b ≥ 2, $a = \frac{k+1}{b} < k + 1$ and $b = \frac{k+1}{a} < k + 1$.

a and b can be expressed as j_a and j_b as they call in the range P(j) through P(k).

Thus $k+1 = P(j_a) * P(j_b)$

Question 6: Solve the following questions from the Discrete Math zyBook:

a) Exercise 7.4.1, sections a-g

Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that P(3) is true.

P(3) = 3^2 = $1^2 + 2^2 + 3^2$ = 14 = $\frac{3(3+1)(2*3+1)}{6} = \frac{3(4)(7)}{6} = \frac{84}{6}$ = 14

(b) Express P(k).

P(k) = $1^2 + 2^2 + ... + k^2$ = $\frac{k(k+1)(2k+1)}{6}$

(c) Express P(k + 1).

P(k + 1) = $1^2 + 2^2 + ... + (k+1)^2$ = $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

(d)In an inductive proof that for every positive integer n,

$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ **what must be proven in the base case?**

P(1) must be proven. P(1) = 1^2 = $\frac{1(1+1)(2*1+1)}{6} = \frac{6}{6}$ = 1

(e) In an inductive proof that for every positive integer n,

$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ **what must be proven in the inductive step?**

For all k, P(k) = $1^2 + 2^2 + ... + k^2$ = $\frac{k(k+1)(2k+1)}{6}$

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

P(k)

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Theorem: $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

Proof: By induction on n.

Base case: P(1) = 1^2 = $\frac{1(1+1)(2*1+1)}{6}$ = $\frac{6}{6}$ = 1

Inductive Hypothesis: Assume that for all k ≥ 1, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$. We will prove $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Inductive Step:

$j^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$j^2 = \frac{k(k+1)(2k+1)}{6}$ By Inductive Hypothesis

$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$

$= \frac{(k+1)(2k^2+k) + 6(k+1)^2}{6}$

$= \frac{(k+1)(2k^2+k) + 6(k^2+2k+1)}{6}$

$= \frac{(k+1)(2k^2+k) + 6k^2+12k+6}{6}$

$= \frac{2k^3+k^2+2k^2+k+6k^2+12k+6}{6}$

$= \frac{2k^3+9k^2+13k+6}{6}$

$= \frac{(k^2+3k+2)(2k+3)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

Thus the theorem is proven.

b) Exercise 7.4.3, section c Hint: you may want to use the following fact:

$$\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$$

(c) Prove that for $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Theorem: For any integer n ≥ 1, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Base Case: n = 1. $\sum_{j=1}^{1} \frac{1}{1^2} \leq 2 - \frac{1}{1}$ = 1 $\leq$ 1

Thus for n = 1, the theorem is proven.

Induction Hypothesis: Let $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$ be expressed as P(n). For any k $\geq$ 1, P(k) = $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$. We will prove $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$

Induction Step:

$\sum_{j=1}^{k+1} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$

= $2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ By induction hypothesis

= $\frac{1}{k(k+1)} - \frac{1}{k} + 2$ From hint

= $\frac{1}{k}\left(\frac{1}{k+1} - 1\right) + 2$

= $\frac{1}{k}\left(\frac{1}{k+1} - \frac{k+1}{k+1}\right) + 2$

= $\frac{1}{k}\left(\frac{-k}{k+1}\right) + 2$

= $-\frac{1}{k+1} + 2$

= $2 - \frac{1}{k+1}$

Thus the theorem is proven.

c) Exercise 7.5.1, section a

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides 3^{2n}-1.

Theorem: For any positive integer n $\geq$ 1, 3^{2n} - 1 can be divided by 4 with no remainder.
Base Case: n = 1. $3^{2(1)}$ - 1 = 3^2 - 1 = 9 - 1 = 8. 8/4 = 2, and no remainder.
Thus for n = 1, the theorem is proven.
Induction Hypothesis: Let 3^{2n} - 1 divided by 4 with no remainder be expressed as P(n)
Assume that P(k) = 3^{2k} - 1. We will prove P(k + 1) = $3^{2k + 1}$ - 1
Induction Step:
$3^{2k + 1}$ - 1 = 3*3^{2k} - 1
3*3^{2k} - 1 By induction hypothesis

We know from the induction hypothesis that 3^{2k} - 1 is divisible by 4, so P(k + 1) is true.