INFO 6205
Program Structure and Algorithms

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Proofs

Topics

- Counterexamples
- Proof by contradiction
- Proof by induction

Counter Examples

You can disprove often with counter examples. For example, if you could show an instance of Gale-Shapley that terminated but had an unstable pairing then you could prove that Gale-Shapley doesn't always generate stable pairings.

Proof by Contradiction

Let r be a proposition.
A proof of r by contradiction consists of
proving that not(r) implies a contradiction,
thus concluding that not(r) is false,
which implies that r is true.

Proof by Contradiction

- For all integers n, if n^2 is odd, then n must be odd.
  Proof: Suppose not. Then $\exists n$ with n even and n^2 odd.
  So $n = 2k$ for some integer k.
  So $n^2 = 2 \cdot 2 \cdot k \cdot k$.
  $= 2(2k^2)$, which is even.

- There are an infinite number of prime numbers.
  Proof: Suppose not. Then $\exists k$ (finite integer) many primes, $p_1 \ldots p_k$.
  Define $x = \Pi p_i + 1$
  If x is prime, then we have a contradiction already.
  If not, then $\exists$ prime $q > 1$ that divides x evenly.
  and since q is in the set of primes, this divides Πp_i evenly.
  So it divides their difference evenly.
  That difference is 1. So $q = 1$ and $q > 1$. So contradiction.

Proof by Contradiction

Prove that the sum of an even integer and a non-even integer is non-even.
(Note: a non-even integer is an integer that is not even.)

This is the same as proving that For all integers a,b, if [a is even and b is non-even] then [a+b is non-even].

We prove that by contradiction.

Assume that
[a is even and b is non-even],
and that [a+b is even]. So for some integers m,n, a=2m and a+b=2n.
Since b=(a+b)-a, b=2n-2m=2(n-m).
We conclude that b is even. This leads to a contradiction, since we assumed that b is non-even.

Contradiction Example $\sqrt{2}$ is irrational

- Let p be the proposition '$\sqrt{2}$ is an irrational number'
- Assume $\neg p$ holds, and show that it yields a contradiction
- $\sqrt{2}$ is rational

$\rightarrow \sqrt{2} = a/b$, $a, b \in R$ <u>and</u> a, b have <u>no common factor</u> (proposition r)
Definition of rational numbers

$\rightarrow 2 = a^2/b^2$ *Squarring the equation*

$\rightarrow (2b^2 = a^2) \rightarrow (a^2 \text{ is even}) \rightarrow (a = 2c)$ *Algebra*

$\rightarrow (2b^2 = 4c^2) \rightarrow (b^2 = 2c^2) \rightarrow (b^2 \text{ is even}) \rightarrow (b \text{ is even})$ *Algebra*

$\rightarrow$ (a, b are even) $\rightarrow$ (a, b have a common factor 2) $\rightarrow \neg r$

$\rightarrow (\neg p \rightarrow (r \wedge \neg r))$, which is a contradiction

So, ($\neg p$ is false) $\rightarrow$ (p is true), which means $\sqrt{2}$ is irrational

Contradiction Example No smallest positive real number.

Result: There is no smallest positive real number.

Proof: Assume, to the contrary, that there is a smallest positive real number, say r. Since $0<r/2<r$, it follows that r/2 is a positive real number that is smaller than r. This, however, is a contradiction

Contradiction Example – The sum of a rational number and an irrational number is irrational

Result: The sum of a rational number and an irrational number is irrational.

Proof: Assume, to the contrary, that there exist a rational number x and an irrational number y whose sum is a rational number z. Thus $x+y=z$, where $x=a/b$ and $z=c/d$ for some integers a, b, c, d $\in$ Z and b, d $\neq 0$. This implies that

$$y=c/d-a/b=(bc-ad)/bd.$$

Since bc-ad and bd are integers and bd $\neq 0$ it follows that y is rational, which is a contradiction.

What is induction?

Three parts:

- Base case(s): show it is true for one element
- Inductive hypothesis: assume it is true for any given element
- Show that if it true for the next highest element

Induction

- Suppose
  - S(k) is true for fixed constant k
    - Often k = 0
  - S(n) ➠ S(n+1) for all n >= k
- Then S(n) is true for all n >= k

Induction

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

P_1 is true, and

the truth of P_k implies the truth of P_{k+1}, for every positive integer k,

then P_n must be true for all integers n

Principle of Mathematical Induction

- Hypothesis: $P(n)$ is true for all integers $n \geq b$
- To prove that $P(n)$ is true for all integers $n \geq b$ (*), where $P(n)$ is a propositional function, follow the steps:
- Basic Step or *Base Case:* Verify that $P(b)$ is true;
- Inductive Hypothesis: assume $P(k)$ is true for some $k \geq b$;
- Inductive Step: Show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers $k \geq b$. *This can be done by showing that under the inductive hypothesis that $P(k)$ is true, $P(k+1)$ must also be true.*

Proof by Induction

- Claim:S(n) is true for all n >= k
- Basis:
  - Show formula is true when n = k
- Inductive hypothesis:
  - Assume formula is true for an arbitrary n
- Step:
  - Show that formula is then true for n+1

Example $n < 2^n$ for all positive integers n

1. P(1) is true, because $1 < 2^1 = 2$. (Base Step)
2. Show that if P(n) is true, then P(n + 1) is true. (inductive step)

Assume that $n < 2^n$ is true. We need to show that P(n + 1) is true, i.e.
$n + 1 < 2^{n+1}$

We start from $n < 2^n$: $n + 1 < 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$
(i.e.) $n + 1 < 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$
Therefore, if $n < 2^n$ then $n + 1 < 2^{n+1}$

Therefore, if $n < 2^n$ then $n + 1 < 2^{n+1}$
3. $n < 2^n$ is true for any positive integer (Conclusion)

Induction Example: Gaussian Closed Form

- Prove $1 + 2 + 3 + \ldots + n = n(n+1) / 2$
  - Basis:
    - If $n = 0$, then $0 = 0(0+1) / 2$
  - Inductive hypothesis:
    - Assume $1 + 2 + 3 + \ldots + n = n(n+1) / 2$
  - Step (show true for n+1):

    $1 + 2 + \ldots + n + n+1 = (1 + 2 + \ldots + n) + (n+1)$

    $= n(n+1)/2 + n+1 = [n(n+1) + 2(n+1)]/2$

    $= (n+1)(n+2)/2 = (n+1)(n+1 + 1) / 2$

Inductive hypothesis:

Suppose that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ for some *k>=1*.

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

Inductive step:

We will show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

For n = 1 1=1 (Base case)

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$ by the inductive hypothesis

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

It follows that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all *n>=1*. ▯

Example $S_n = 2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$

Use mathematical induction to prove

$$S_n = 2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$$

for every positive integer n.

1. Show that the formula is true when $n = 1$. (Base Case)

$S_1 = n(n + 1) = 1(1 + 1) = 2$ True

2. Assume the formula is valid for some integer k. Use this assumption to prove the formula is valid for the next integer, $k + 1$ and show that the formula $S_{k+1} = (k + 1)(k + 2)$ is true.

$S_k = 2 + 4 + 6 + 8 + \cdots + 2k = k(k + 1)$ Assumption

$S_{k+1} = 2 + 4 + 6 + 8 + \cdots + 2k + [2(k + 1)] = 2 + 4 + 6 + 8 + \cdots + 2k + (2k + 2)$

$= S_k + (2k + 2) = k(k + 1) + (2k + 2) = k^2 + k + 2k + 2 = k^2 + 3k + 2$

$= (k + 1)(k + 2) = (k + 1)((k + 1)+1)$

THEREFORE The formula $S_n = n(n + 1)$ is valid for all positive integer values of n.

Example $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

- Use induction to prove that the $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all non-negative integers n.

$P(n) = 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

for all non-negative integers n.

- 1 – Hypothesis?

2 - Base case?

$\boxed{n = 0 \quad 1^0 = 2^1 - 1.}$ ⟵ not n=1! The base case can be negative, zero, or positive

3 – Inductive Hypothesis

Assume $P(k) = 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

Inductive hypothesis

Example $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

4 – Inductive Step: show that $\forall(k)\ P(k) \rightarrow P(k+1)$, assuming P(k).

How?

$$P(k+1) = \underbrace{1 + 2 + 2^2 + \ldots + 2^k}_{p(k)} + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1}$$

By inductive hypothesis

$$= 2\ 2^{k+1} - 1$$

$$P(k+1) = 2^{k+2} - 1$$

$$= 2^{(k+1)+1} - 1$$

QED

Proof by Contradiction

- Khan Academy by contradiction
  https://www.khanacademy.org/math/geometry/geometry-worked-examples/v/ca-geometry--proof-by-contradiction

Khan Academy by contradiction 2 https://www.youtube.com/watch?v=u6O0YHyarlI

Proof by Contradiction: Arithmetic Mean & Geometric Mean:

http://youtu.be/yEFCHrsn2n4

Maths Skills: Proof by Contradiction: http://youtu.be/qZ736F8ljYU

CA Geometry: Proof by Contradiction: http://youtu.be/u6O0YHyarlI

Proof by Induction

- Khan Academy Proof by induction https://www.khanacademy.org/math/precalculus/seq_induction/proof_by_induction/v/proof-by-induction
- Khan Academy Proof by induction 2 https://www.youtube.com/watch?v=wblW_M_HVQ8
- Mathematical Induction - Proof by Maths Induction - Year 12 HSC Maths ...: http://youtu.be/ruBnYcLzVlU
- Proof by Induction - Example 1: http://youtu.be/IFqna5F0kW8
- Introduction to Proof by Induction: http://youtu.be/iSaRqVkImfw
- Proving with Induction: http://youtu.be/CuZJmf3XrTo