PREFACE

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Chapter 1

Mathematical Preparation for General Relativity—Differential Manifolds and Differential Geometry

1.1 Differential Manifolds

Differential manifolds (manifolds in short) is the most fundamental mathematical object in the field of general relativity (GR), for example, the 4-dimensional spacetime is a kind of manifolds. The definition in this class for manifolds will begin with the concept of **open ball**, the definition is listed as follow,

Definition 1.1. *An open ball with radius r centered at point $y = (y^1, y^2, \cdots, y^n)$ in $\mathbb{R}^n$ is defined by*

$$B := \left\{ x \in \mathbb{R}^n \middle| |x - y| = \left[\sum_{\mu=1}^{n} (x^\mu - y^\mu)^2 \right]^{1/2} < r \right\}. \tag{1.1}$$

After this definition we can define the open set for our system and the definition is

Definition 1.2. *An open set in $\mathbb{R}^n$ is a set of points expressed as the union of open balls.*

Open set is the most fundamental concept to build our system, an open set we will use the notation O_α, that is also to say if we use the symbol O_α, we means this set is an open set. With the definition of open set we can give the definition of differential manifolds.

Definition 1.3. *A n-dim, C^∞, real manifold $\mathcal{M}$ is a set together with a collection of subsets $\{O_\alpha\}$ satisfying*

a.Each $p \in \mathcal{M}$ must lie in at least one O_α.

b.For each α, there is an one-to-one, onto, map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$.

c.For two sets $O_\alpha \cap O_\beta \neq \varnothing$, then the map $\psi_\beta \circ \psi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞.

♦NOTE:

This is the precise definition of manifolds, but in Physics, we always talk about a simple version, which says the manifold is a set made up of pieces 'looks like open set of $\mathbb{R}^n$', such that these pieces are 'sewn/glued' together smoothly.

And an important thing is that if you want to get a manifold, you not only need to give the background set $\mathcal{M}$ but also the subsets $\{O_\alpha\}$ and the map ψ_α, and we always say $\{(O_\alpha, \psi_\alpha)\}$ is an atlas, and we say the map, ψ, is a chart. But in physics, we say this is a coordinate system.

There are three conditions in our definition, cond.1 asks that $\{O_\alpha\}$ gives an open covering for our background set. Cond.2 tells us if we know a point of $\mathcal{M}$ and by the map ψ we can get its coordinate and if we know the coordinate and the map ψ, the origin point is also given. So this map gives us the approach to get the coordinate, this is the reason why we say ψ is actually a coordinate frame system. Condition 3 in fact tells us we can make coordinate transformation and this transformation is smooth, this is easy to understand by what we have known in classic mechanics and classic field theory.

By the way, here $\{O_\alpha\}$ is the collection of open sets of $\mathcal{M}$, but we just define the open set in $\mathbb{R}^n$, as the matter of fact, the open set defined on any sets depends on the topological space, here we needn't to talk about it. ♦

Now we can see some examples of manifolds.

Example 1.1. $\mathbb{R}^n$ *itself is a trivial manifold.*

Proof. In order to prove a set can be a manifold, we only need to give the atlas. For this case, we choose the atlas is $\{(\mathbb{R}^n, i)\}$, i.e. we let $\mathbb{R}^n$ be the only open subset, and the chart is the identity map $i(x) = x$. There is no doubt that this atlas satisfies three conditions. In fact, cond.1 and cond.2 hold obviously. As for cond.3, we know there is only one set in the atlas, we needn't to consider it.Or we know $\psi_\beta \circ \psi_\alpha^{-1} = i \circ i^{-1} = i$, this map is C^∞. □

♦NOTE:

Similarly, we know that any open subset of $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, is also a manifold and we only need let the atlas be $\{(A, i)\}$. ♦

Example 1.2. *A 2-sphere* $S^2 := \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ *forms a manifold.*

Proof. We try to give an atlas of S^2. We can use the projection to get the open covering. We divide S^2 into 6 parts: left hemisphere $O_2^1 = \{x \in S^2 \mid x^2 < 0\}$, right hemisphere $O_2^2 = \{x \in S^2 \mid x^2 > 0\}$, upper hemisphere $O_3^2 = \{x \in S^2 \mid x^3 > 0\}$, lower hemisphere $O_3^1 = \{x \in S^2 \mid x^3 < 0\}$, forward hemisphere $O_1^2 = \{x \in S^2 \mid x^1 > 0\}$, rear hemisphere $O_1^1 = \{x \in S^2 \mid x^1 < 0\}$.The map is given by $\psi_{ij} : O_i^j \to S^1$, where S^1 is the circle $\{(y^1, y^2) \in \mathbb{R}^2 \mid (y^1)^2 + (y^2)^2 = 1\}$. Now $\{(O_i^j, \psi_{ij})\}(i = 1, 2, 3; j = 1, 2)$ gives an atlas.

There is no doubt that all the points of S^2 can be found in one O_i^j and our map ψ_{ij} is truly an 1-1 corresponding map. Specifically, for the set $O_i^j = \{x \in S^2 \mid x^i \lesseqgtr 0\}$, assume another two coordinates are x^m and x^n, then we let $y^1 = x^m, y^2 = x^n$, the map ψ_{ij} is given in this way. This is 1-1 and C^∞ for the intersetion part (we don't prove it because it is intuitionistic and the precise proof is the task for mathematicians). □

♦NOTE:

Just as what we have said, for most of the propositions we won't give them a precise proof, we just give some formalism proofs and for these intuitive proposition we don't show the full proof since it is always difficult. We only show the full proof if it is very important and not very hard.

As a example, we need to know that in string theory, the spacetime is 10 or 11 dimension and the 10-d or 11-d spacetime also form manifolds. ♦

Now let we gives a new definition which is important for our course, i.e.

Definition 1.4. *Given two manifolds,* $\mathcal{M}$ *and* $\mathcal{M}'$*, with dimension* d *and* d'*, respectively.* $\mathcal{N} := \mathcal{M} \times \mathcal{M}'$ *is the Cartesian production of these two manifolds, the map defined on these two are* $\psi_\alpha : O_\alpha \to U_\alpha$ *and* $\psi'_\beta : O'_\beta \to U'_\beta$*, let* $O_{\alpha\beta} := O_\alpha \times O'_\beta$ *and* $U_{\alpha\beta} := U_\alpha \times U'_\beta$*, let* $\psi_{\alpha\beta}(p, p') = [\psi_\alpha(p), \psi'_\beta(p')]$ *with* $p \in \mathcal{M}, p' \in \mathcal{M}'$*, we say* $\mathcal{N} = \mathcal{M} \times \mathcal{M}'$ *is the product manifold of* $\mathcal{M}$ *and* $\mathcal{M}'$*.*

Then we have

Theorem 1.1. *The product manifold of* $\mathcal{M}$ *and* $\mathcal{M}'$*,* $\mathcal{M} \times \mathcal{M}'$*, is truly a manifold.*

Proof. We verify that $\{(O_{\alpha\beta}, \psi_{\alpha\beta})\}$ satisfies three conditions of manifold one by one.

Condition 1: by definition, we know for any point of $\mathcal{M}$, there exists at least one open set O_α such that $\forall p \in \mathcal{M}$, $p \in O_\alpha$. Similarly, $\forall p' \in \mathcal{M}'$, there exists O'_β such that $p' \in \mathcal{M}'$, so for any point (p, p') in $\mathcal{M} \times \mathcal{M}'$, we just need to let $O_{\alpha\beta} = O_\alpha \times O'_\beta$, then for any $(p, p') \in \mathcal{M} \times \mathcal{M}'$, we have at least one open set $O_{\alpha\beta}$ such that $(p, p') \in O_{\alpha\beta}$.

Condition 2: since $\mathcal{M}, \mathcal{M}'$ are manifolds, so $\forall \alpha, \forall \beta$, there exists 1-1 corresponding map $\psi_\alpha : O_\alpha \to U_\alpha, \psi'_\beta : O'_\beta \to U'_\beta$, so any α, β give a 1-1 corresponding map $\psi_{\alpha\beta} : O_{\alpha\beta} \to U_{\alpha\beta}$.

Condition 3: this is an intuitive result. Since $\psi_{\alpha\beta}(p, p')$ is defined by $\{\psi_\alpha(p)\} \times \{\psi'_\beta(p')\}$, let $\psi_{\alpha'\beta'}(p, p') = (q^i, q'^j)$, then

$$\begin{aligned}
\psi_{\alpha\beta} \circ \psi^{-1}_{\alpha'\beta'}(q^i, q'^j) &= \psi_{\alpha\beta}(p, p') = (\psi_\alpha(p), \psi_\beta(p')) \\
&= (\psi_\alpha(\psi_{\alpha'}(q^i)), \psi_\beta(\psi_{\beta'}(q^{j'}))) \\
&= (\psi_\alpha \circ \psi_{\alpha'}) \times (\psi_\beta \circ \psi_{\beta'})(q^i, q'^j)
\end{aligned}$$

is C^∞ by definition. □

Since the Cartesian product of manifolds still forms a manifold, we say it is product manifold.

♦**NOTE:**

BTW, most of the manifolds in this course is not the trival manifold but always the product manifold of $\mathbb{R}^n$ and S^m with $n + m = 4$, that is also to say the space-time manifold in GR is 4-d manifold and some place it seems like flat, some place seems like curved, the general case is $\mathbb{R}^n \times S^m$ with $m + n = 4$. ♦

1.2 Vectors

1.2.1 Tangent Vector And Tangent Space

Vector is one of the most important concept in classic mechanics and we want to give a similar object to manifolds. In $\mathbb{R}^3$ we can use an arrow to denote the vector but for manifold we can't always do such thing which asks us to find the essential attributes of the vector. Intuitively, when a manifold embedded in $\mathbb{R}^n$, for example S^2 is embedded in $\mathbb{R}^3$, we can find the tangent vector, but if the embedding is not explicit or possible, we can't define such tangent vector. Of course we know one of the property is the operation property for vector such as addition and scalar-multiplication, however they are not the essential attributes. Go back the geometry, we know a vector in 2-d space can be get by the direction vector of a line $y = kx + b$ and the direction vector is $(1, k)$ or $(\frac{\mathrm{d}x}{\mathrm{d}x}, \frac{\mathrm{d}y}{\mathrm{d}x})$ in general, so may be the derivative is a good choice, but unfortunately, we don't define the derivative on manifolds up to now, but why don't we choose the property of derivative? Based on this idea, we have

Definition 1.5. *Let $\mathcal{F}$ be the set of all smooth function from the manifold $\mathcal{M}$ to $\mathbb{R}^1$,i.e.*

$$\mathcal{F} := \{all\ C^\infty\ function\ from\ \mathcal{M} \to \mathbb{R}^1\}. \tag{1.2}$$

Then a tangent vector v at point $p \in \mathcal{M}$ is a map from $\mathcal{F}$ to $\mathbb{R}$, $v : \mathcal{F} \to \mathbb{R}^1$, satisfies

a. linear property, $v(af + bg) = av(f) + bv(g)$, $\forall a, b \in \mathbb{R}, \forall f, g \in \mathcal{F}$.

b. Leibnitz's law, $v(fg)|_p = f(p)v(g)|_p + g(p)v(f)|_p, \forall f, g \in \mathcal{F}$.

The set of all the tangent vector at point p will be denoted by V_p, which is called the tangent space at point p.

Now that we say v is a vector, then we must prove V_p forms a vector space to ensure that this definition is harmonious with the general definition. In fact, it is true, i.e. we have

Theorem 1.2. *V_p is a vector space.*

Proof. We know if a set is a vector space, we need to prove the addition and scalar-multiplication defined on it satisfy the eight laws, i.e. $(V_p, +)$ forms an Abelian group (4 conditions) and for scalar-multiplication, we need to prove another four conditions. So we introduce the addition and the scalar multiplication on V_p as

Addition: $(v_1 + v_2)(f) := v_1(f) + v_2(f), \forall v_1, v_2 \in V_p, \forall f \in \mathcal{F}$.

Scalar-multiplication:$(a \cdot v)(f) := a \cdot v(f), \forall a \in \mathbb{R}, f \in \mathcal{F}$.

BTW, we need to prove there exists the zero element in the set V_p, this is easy, we assume θ satisfies $\theta(f) = 0, \forall f \in \mathcal{F}$, then we know $0 = \theta(af + bg) = a\theta(f) + b\theta(g) = a \cdot 0 + b \cdot 0 = 0$ and $0 = \theta(fg)|_p = f(p)\theta(g)|_p + g(p)\theta(f)|_p = 0$, this means θ is a tangent vector at point p, i.e. this map is the zero element of V_p.

$v_1 + v_2 = v_2 + v_1$, $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$, $a(b \cdot v) = (ab) \cdot v$ are obviously, and now we need to prove θ is also the identity of addition. In fact, we have $(v + \theta)(f) = v(f) + \theta(f) = v(f), \forall f \in \mathcal{F}$, i.e. $v + \theta = v$, so θ is the identity of the addition. For the law $(a+b)v = av + bv$ we can proved by the linear property, we just need let $f = g$ and then we have $v(af + bf) = v((a + b)f) = av(f) + bv(f)$ and let $g = 0$ we have $v(af) = av(f)$, so $v((a + b)f) = av(f) + bv(f) = (a + b)v(f), \forall f \in \mathcal{F}$, then $(a + b)v = av + bv$ is proved. For $a(v_1 + v_2) = av_1 + av_2$, we can verify it in this way: $av_1(f) + av_2(f) = v_1(af) + v_2(af) = (v_1 + v_2)(af) = a(v_1 + v_2)(f), \forall f \in \mathcal{F}$. As for $1v = v$, we have $v(f) = v(1f) = 1v(f), \forall f \in \mathbb{R}$. For the inverse of addition, we know if $v(f)$ is in V_p, then $v(-f)$ is also in V_p, then $v(f) + v(-f) = v(f) - v(f) = 0 = \theta(f), \forall f \in \mathcal{F}$, thus $-v$ is the inverse of v in V_p.

Above all, we showed that V_p is a vector space. □

Now that V_p is a vector space, we need to find the basis of this vector space and we need to show the coordinate component for a tangent vector under the basis, and we also need to give the dimension of V_p, it will be show in the next theorem, but before this theorem, we need to introduce some lemmas.

Lemma 1.1. *If f is a constant function, then $v(f) = 0$ or $v(C) = 0$.*

Proof. We need to use a property we have proved $v(af) = av(f)$, then by Leibnitz's law we have

$$v(Cf)|_p = Cv(f)|_p + f(p)v(C)|_p, \forall p \in \mathcal{M}, \forall f \in \mathcal{F}.$$

In the same time we have $v(Cf)|_p = Cv(f)|_p$, combine these two results and we have

$$f(p)v(C)|_p = 0, \forall p \in \mathcal{M}, \forall f \in \mathcal{F}.$$

Since $f(p) \not\equiv 0$, we must have

$$v(C)|_p = 0, \forall p \in \mathcal{M}$$

i.e. $v(C) = 0$ □

Lemma 1.2 (Taylor's theorem for smooth function with integral remainder term)**.** *If f is a C^∞ function defined on the star-shaped neighbourhood U*[①] *at $p = (p^1, p^2, \cdots, p^n)$, then $\forall x \in U$, there exists functions $g_1, \cdots, g_n \in C^\infty(U)$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}\Big|_p \tag{1.3}$$

Proof. Since U is the star-shaped neighbourhood at p, then $\forall x \in U$, any point in the straight line l connecting x and p is given by $q(t) = p + t(x - p), 0 \le t \le 1, \forall q \in l \subset U$ with $q(0) = p, q(1) = x$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f(q) = \sum_{i=1}^{n}(x^i - p^i)\frac{\partial f}{\partial x^i}\Big|_q.$$

[①] A star-shaped neighbourhood U at p is the neighbourhood satisfies (1) $p \in U$; (2)$\forall x \in U$, the straight line connecting x and p, l, is the subset of U, or $l \subset U$.

Integrate the equation above from 0 to 1 for t and then

$$f(q)|_0^1 = f(q(1)) - f(q(0)) = \sum_{i=1}^{n}(x^i - p^i)\int_0^1 \frac{\partial f}{\partial x^i}\Big|_{q(t)} \mathrm{d}t = f(x) - f(p).$$

Let

$$\frac{\partial f}{\partial x^i}\Big|_q = g_i(x)$$

and then we have

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^1)g_i(x).$$

Take the derivative for x^i at $x = p$ and we have

$$g_i(p) = \frac{\partial f}{\partial x^i}\Big|_p$$

□

Now let we show the most important theorem in this part, which tells us the basis of V_p and the coordinate components for arbitray vector.

Theorem 1.3. *Let $\mathcal{M}$ be a n-dimension manifold, $p \in \mathcal{M}$ is a point of $\mathcal{M}$ and V_p denote the tangent space at p.Then we have* $\dim(V_p) = n = \dim(\mathcal{M})$. *And the operator (or vector) $X_\mu : \mathcal{F} \to \mathbb{R}^1$ defined as*

$$X_\mu \circ f := \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}), \mu = 1, 2, \cdots, n, \forall f \in \mathcal{F} \tag{1.4}$$

spans the tangent space V_p, i.e.$\{X_\mu\}_1^n$ are the basis of V_p, here $(x^1, x_2, \cdots, x^n)$ are Cartesian coordinates of $\mathbb{R}^n$ and ψ is the chart from $\mathcal{M}$ to $\mathbb{R}^n$.

Proof. Let $\psi : O \to U \subset \mathbb{R}^n$ be one of the charts, for $p \in O$, we know $f \circ \psi^{-1}$ is C^∞ because f and $\psi^{-1} : U \to \mathbb{R}^1$ are both C^∞ and the coordinate of point p is given by $\psi(p)$, then for $\mu = 1, 2, \cdots, n$ we know the operator X_μ at p is

$$X_\mu(f) := \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\Big|_{\psi(p)}$$

by the definition of X_μ. The we can see

$$X_\mu(af + bg) = \frac{\partial}{\partial x^\mu}[(af + bg) \circ \psi^{-1}]\Big|_{\psi(p)} = \frac{\partial}{\partial x^\mu}(af \circ \psi^{-1} + bg \circ \psi^{-1})\Big|_{\psi(p)} = aX_\mu(f) + bX_\mu(g)$$

and

$$\begin{aligned} X_\mu(fg) &= \frac{\partial}{\partial x^\mu}(fg \circ \psi^{-1})\Big|_{\psi(p)} = \frac{\partial}{\partial x^\mu}[(f \circ \psi^{-1})(g \circ \psi^{-1})]\Big|_{\psi(p)} \\ &= g(p)\frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\Big|_{\psi(p)} + f(p)\frac{\partial}{\partial x^\mu}(g \circ \psi^{-1})\Big|_{\psi(p)} \\ &= f(p)X_\mu(g) + g(p)X_\mu(f) \end{aligned}$$

So we know $\{X_\mu\}_1^n$ are tangent vectors, now we try to prove

$$V_p = \mathrm{Span}\{X_\mu\}.$$

It includes 2 sub-propositions: 1. $\{X_\mu\}_1^n$ are linear independent and 2. all the other vectors $v \in V_p$ is the linear combination of $\{X_\mu\}_1^n$. We verify them respectively.

(A) $\{X_\mu\}_1^n$ are linear independent.

Proof. We just need to prove the linear combination of $\{X_\mu\}_1^n$

$$\sum_{\mu=1}^{n} a^\mu X_\mu = a^\mu X_\mu = \theta$$

holds if and only if $a^\mu = 0$, $\forall \mu$. And from now on, we are going to use the Einstein's summation convention. Act θ on $\hat{x}^\nu$ (here $\hat{x}^\nu$ is a map to get the ν-th coordinate of point p and $\hat{x}^\nu : \mathcal{M} \to \mathbb{R}^1$ is also a function in $\mathcal{F}$, by the way, according to this definition, we have $\hat{x}^\nu(p) = x_p^\nu$ and we don't write the subscript index p from now on and we don't distinguish $\hat{x}^\nu$ and x^ν, we will use the same symbol x^ν), then we have

$$0 = \theta(\hat{x}^\nu) = a^\mu X_\mu(\hat{x}^\nu) = a^\mu \frac{\partial}{\partial x^\mu}(\hat{x}^\nu \circ \psi^{-1})\Big|_{\psi(p)} = a^\mu \frac{\partial x^\nu}{\partial x^\mu} = a^\mu \delta_{\mu\nu} = a^\nu, \forall \nu$$

i.e. $a^\mu X_\mu = 0$ iff $\forall \mu$, $a^\mu = 0$. **QED.**

(B)$\forall v \in V_p$, we have $v = v^\mu X_\mu$.

Proof. Assume $F : \mathbb{R}^n \to \mathbb{R}^1$ is an arbitrary smooth function and by lemma 1.2 (Taylor's theorem) we know it can be wrote as

$$F(x) = f(a) + (x^\mu - a^\mu)H_\mu(x)$$

in the star-shaped neighbourhood O at $a = (a^1, \cdots, a^n)$ for any point $x = (x^1, \cdots, x^n) \in O$ with

$$H_\mu(a) = \frac{\partial F}{\partial x^\mu}\Big|_{x=a}.$$

Now let $F = f \circ \psi^{-1}$, $a = \psi(p)$, then for any $q \in O$, F on $\psi(q) \equiv x$ yields

$$\begin{aligned} LHS &= F(x) = f \circ \psi^{-1}(\psi(q)) = f \circ \psi^{-1} \circ \psi(q) = f(q), \\ RHS &= F(a) + (x^\mu - a^\mu)H_\mu(x) = f(p) + (\hat{x}^\mu \circ \psi(q) - \hat{x}^\mu \circ \psi(p))H_\mu(\psi(q)). \end{aligned}$$

i.e.

$$f(q) = f(p) + [x^\mu \circ \psi(q) - x^\mu \circ \psi(p)](H_\mu \circ \psi)(q).$$

Assume $v : \mathcal{F} \to \mathbb{R}^1$ is a vector and we evaluate this v on f at q, it yields

$$v(f)|_q = v(f)|_p + v[(x^\mu \circ \psi|_q - x^\mu \circ \psi|_p)(H_\mu \circ \psi)]|_q.$$

Notice that the first term in this situation is a constant, by lemma 1.1, it yields $v(f)|_p = 0$, the second term can be decomposed by the Leibnitz'law, which gives

$$\begin{aligned} v(f)|_q &= (x^\mu \circ \psi|_q - x^\mu \circ \psi|_p)v(H_\mu \circ \psi)|_q + (H_\mu \circ \psi|_q)v(x^\mu \circ \psi|_q - x^\mu \circ \psi|_p) \\ &= (x^\mu \circ \psi|_q - x^\mu \circ \psi|_p)v(H_\mu \circ \psi)|_q + (H_\mu \circ \psi|_q)[v(x^\mu \circ \psi|_q) - v(x^\mu \circ \psi|_p)] \\ &= (x^\mu \circ \psi|_q - x^\mu \circ \psi|_p)v(H_\mu \circ \psi)|_q + (H_\mu \circ \psi|_q)v(x^\mu \circ \psi|_q). \end{aligned}$$

Then we have

$$\begin{aligned} v(f(p)) &= \lim_{q\to p} v(f)|_q = \lim_{q\to p} x^\mu \circ (\psi(q) - \psi(p))v(H_\mu \circ \psi_q) + \lim_{q\to p}(H_\mu \circ \psi|_q)v(x^\mu \circ \psi|_q) \\ &= (H_\mu \circ \psi|_p)v(x^\mu \circ \psi|_p). \end{aligned}$$

Notice that

$$H_\mu \circ \psi(p) = H_\mu(a) = \frac{\partial f \circ \psi^{-1}}{\partial x^\mu}\Big|_{\psi(p)} = X_\mu \circ f|_{\psi(p)}$$

Let $v^\mu = v(x^\mu \circ \psi)|_p$ and then we have

$$v(f)|_p = v^\mu X_\mu(f)|_p, \forall f \in \mathcal{F}$$

i.e. $\forall v \in V_p, v = v^\mu X_\mu$.**QED**

Above all, we show that that $\{X_\mu\}_1^n$ are the basis of V_p and the dimension of V_p is n or the dimension of the manifold. $\square$

Up to now, we show the structure of V_p, and by the definition of X_μ we know that X_μ only depends on ψ and now we have another question, if $\psi' : O' \to U'$ is another coordinate system and the basis are X'_μ, what is the relation of X_μ and X'_μ? The answer is

Theorem 1.4. *If $\psi : O \to U$ and $\psi' : O' \to U'$ are two coordinate system and $O \cap O' \neq \varnothing$, $\{X_\mu\}$ and $\{X'_\mu\}$ are basis at $p \in O \cap O'$, then*

$$X_\mu = \frac{\partial x'^\nu}{\partial x^\mu} X'_\nu \tag{1.5}$$

Proof. Assume $f : \mathcal{M} \to \mathbb{R}$ is one of the function of $\mathcal{F}$, then we have

$$X_\mu(F) = \frac{\partial}{\partial x^\mu} F \circ \psi^{-1}\Big|_p, X'_\nu(F) = \frac{\partial}{\partial x'^\nu} F \circ \psi'^{-1}\Big|_p.$$

Since ψ is reversible, if $\psi(p) = x, \psi'(p) = x'$ then $p = \psi^{-1}(x)$ and $x' = \psi'(p) = \psi' \circ \psi(x) \equiv x'(x)$. At the same time, we know $F|_p = F \circ \psi^{-1}(x) = F \circ \psi'^{-1}(x')$, denote $F \circ \psi^{-1} \equiv f, F \circ \psi'^{-1} = f'$, then we have $f(x) = f'(x') = F|_p$, so we have

$$X_\mu(F) = \frac{\partial f(x)}{\partial x^\mu} = \frac{\partial f'(x'(x))}{\partial x^\mu} = \frac{\partial f'(x')}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} F \circ \psi'^{-1} = \frac{\partial x'^\nu}{\partial x^\mu} X'_\nu(F), \forall F \in \mathcal{F}$$

So we proved the coordinate basis transformation relation. □

Now that the basis transformation relation is given, then the coordinate components transformation relation is also given, in fact we have

Corollary 1.4.1. *If v'^ν is the coordinate components of $v \in V_p$ under the basis X'_ν and v^μ is the coordinate components of v under the basis X_μ, then we have*

$$v'^\nu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \tag{1.6}$$

Proof. We know any vector can be expressed by the linear combination of the basis, so $\forall v \in V_p$

$$v = v'^\nu X'_\nu = v^\mu X_\mu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} X'_\nu$$

so v'^ν and $v^\mu \frac{\partial x'^\nu}{\partial x^\mu}$ are both the coordinate components of $v \in V_p$ under the basis X'_ν, we know the result has only one, so we have

$$v'^\nu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu}$$

□

1.2.2 Smooth Curve And Concepts Based On It

This section we are going to define the curve on a manifold, the idea is based on the motion in classic mechanics. It is known to all, in classic mechanics, a motion is a curve in Euclidian space $\mathbb{R}^3$ or in specific, E^3. A curve in E^3 has the parameter equation, and any one-parameter vector equation $\boldsymbol{f} = (x(t), y(t), z(t))$ gives a curve. So in general, a curve can be regarded as a map for the open subset (i.e. open interval) to E^3, notice that E^3 is one of the manifold, so we can define the curve (in particular, the smooth curve) on the manifold as a map. i.e.

Smooth curve

Definition 1.6. *A smooth curve on a manifold $\mathcal{M}$ is a C^∞ map*

$$C : A \to \mathcal{M} \quad \text{or } C : t \in (a, b) \mapsto p,$$

where $A = (a, b) \subset \mathbb{R}$ and $p \in \mathcal{M}$.

Now we know, for any point in $\mathcal{M}$, which also lies on C, $\forall f \in \mathcal{F}$, we can associate a tangent vector T_p defined as

$$T_p(f) := \frac{\mathrm{d}}{\mathrm{d}t}(f \circ C)\Big|_p \tag{1.7}$$

Now we will take a convection, **we won't write the subscript p everytime and all the derivative should be regarded as the result evaluated at p.** Notice that by a chart we can give any point in the curve coordinates, i.e. $\psi : p \in M \mapsto x^\mu$, so we have $p = \psi^{-1}x^\mu$, notice $C : t \mapsto p$, then $f \circ C(t) = f(p(t)) = [f \circ \psi^{-1}]x^\mu(t)$, thus we have

$$T_p(f) = \frac{\mathrm{d}}{\mathrm{d}t}[f \circ \psi^{-1}]x^\mu = \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}X_\mu(f).$$

Notice: we can exchange the order because here $\mathrm{d}x^\mu/\mathrm{d}t$ is a number by our convention. And X_μ is the basis of V_p, here $T_p \in V_p$ and $\mathrm{d}x^\mu/\mathrm{d}t$ is the coordinate components of T_p in the basis X_μ. We can see that though $T_p(f)$ depends on the choice of f but $\mathrm{d}x^\mu/\mathrm{d}t$ independents on f, it shows the properties of C it self to some extent. Let we introduce the notation $T^\mu = \mathrm{d}x^\mu/\mathrm{d}t$ and then we have

$$T_p = T^\mu X_\mu, T^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}. \tag{1.8}$$

The result only depends on the chart ψ we choose.

Tangent Field

If $p, q \in \mathcal{M}$ are two difference points in the manifold, then the vector space at two points V_p and V_q are always independent. But we can choose one tangent vector $v|_p \in V_p$ and when p traverses all the points of $\mathcal{M}$, $v|_p$ forms what we say "tangent field". i.e.

Definition 1.7. *A tangent field, v, on a manifold is an association of a tangent vector $v|_p \in V_p$ at each point $p \in \mathcal{M}$.*

We know every tangent vector $v|_p \in V_p$ can be wrote as

$$v|_p = v^\mu|_p X_\mu,$$

if we choose one of the basis vector at each point, then then tangent field we constructed is called coordinate basis field. For any $f \in \mathcal{F}$, for each point of $\mathcal{M}$, $v|_p(f)$ is a number, when p changes, $v|_p(f)$ also changes, which means $v(f)$ is a function from $\mathcal{M}$ to $\mathbb{R}$. So we can define the smooth tangent field as

Definition 1.8. *A tangent field v is smooth if and only if $v(f)$ is a smooth function for any $f \in \mathcal{F}$. i.e.*

$$v \text{ is } C^\infty \Longleftrightarrow \forall f \in \mathcal{F}, v(f) \in \mathcal{F}$$

Then we have a simple theorem for judge whether a tangent field is smooth.

Theorem 1.5. *A tangent field v on the manifold $\mathcal{M}$ is smooth if and only if its components v^μ for any point is smooth.*

Proof. We know that for any vector in the tangent field at point p yields

$$v|_p(f) = v^\mu|_p X_\mu(f)$$

and here

$$X_\mu(f) = \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}$$

is always smooth since ψ and f are both smooth by the definition. So $v|_p(f)$ is smooth if and only if v^μ is smooth. □

1.3 Tensor

1.3.1 The Definition Of Tensor

Dual Vector Space

Definition 1.9. *If V is a n-dimensional vector space, $f : V \to \mathbb{R}$ is the linear map, i.e.*

$$f(av_1 + bv_2) = af(v_1) + bf(v_2)$$

for any $a, v \in \mathbb{R}$ and $v_1, v_2 \in V$. Then V^*, the collection of all linear maps is called the dual space of V, i.e.

$$V^* := \{f | f : V \to \mathbb{R} \text{ s.t. } f(av_1 + bv_2) = af(v_1) + bf(v_2), \forall a, b \in \mathbb{R}, \forall v_1, v_2 \in V.\}$$

Theorem 1.6. *If V is a vector space, V^* is its dual space, and the addition and scalar multiplication of V^* is defined as $[f + g](v) = f(v) + g(v)$ and $[\alpha f](v) = \alpha \cdot f(v)$. Then V^* is also a vector space.*

Proof. If we define the zero map $\underline{0}$ s.t. $\underline{0}v = 0, \forall v \in V$, then $\underline{0} \in V^*$. And also we notice if $f : v \mapsto x$ is the element of V^*, then $\bar{f} : v \mapsto -x$ is also the element of V^*. The Result is now obvious. □

Since the dual space of a vector space is also a vector space, $f \in V^*$ is also called dual vector. So now, we need to find the dimension of V^* and the basis of V^*. As the result, we have

Theorem 1.7. *If $\{v_\mu\}_1^n$ is the basis of V, then $\{v^{\mu*}\}_1^n$ satisfying*

$$v^{\mu*}(v_\nu) = \delta^\mu{}_\nu$$

is the basis of V^*, where $\delta^\mu{}_\nu$ is the Kronecker symbol, i.e.

$$\delta^\mu{}_\nu = \begin{cases} 1 & , \mu = \nu \\ 0 & , \mu \neq \nu \end{cases}.$$

Proof. Since v_μ are basis of V, so any vector in V can be expressed as

$$\xi = \xi^\mu v_\mu.$$

An arbitrary dual vector ξ^* acts on ξ shows

$$\xi^*(\xi) = \xi^*(\xi^\mu v_\mu) = \xi^\mu \xi^*(v_\mu)$$

Notice that

$$v^{\mu*}(\xi) = v^{\mu*}(\xi^\mu v_\mu) = \xi^\mu v^{\mu*}(v_\mu) = \xi^\mu,$$

thus

$$\xi^*(\xi) = \xi^\mu \xi^*(v_\mu) = \xi^*(v_\mu) v^{\mu*}(\xi), \forall \xi \in V,$$

which means $\xi^*(v_\mu)$ are the coordinates of ξ^* under the basis $v^{\mu *}$. Now we prove that $v^{\mu *}$ is linear independent. Let

$$a_\mu v^{\mu *} = \underline{0},$$

then for arbitrary vector $\xi = \xi^\nu v_\nu$ we have

$$a_\mu v^{\mu *}(\xi) = a_\mu \xi^\nu \delta^\mu{}_\nu = a_\mu \xi^\mu = \underline{0}(\xi) = 0$$

Since $\xi^\mu \not\equiv 0$, so this equation holds if and only if $a_\mu = 0, \forall \mu$, which tells us that $\{v^{\mu *}\}_1^n$ are linear independent. □

In order to make the problem clear, let we define the isomorphism as the preparation.

Definition 1.10. *Two sets A and B are isomorphic to each other if and only if there exists a corresponding map $f : A \leftrightarrow B$ such that for any operator $*$ in A, $a * b = c \Rightarrow f(a) \star f(b) = f(c)$, where $a, b, c \in A$ and $f(a), f(b), f(c) \in B$ or $f(a * b) = f(a) \star f(b)$, $\star$ is the corresponding operator of $*$ defined on B.*

Now we have a simple theorem to judge whether two finite-dimensional vector spaces are isomorphic to each other or not. It says

Theorem 1.8. *Two vector spaces V and W are isomorphic to each other if and only if* $\dim V = \dim W$ *without other operators, i.e. only consider the addition and scalar multiplication.*

Proof. Obvious. Observe the map $f : v_i \mapsto w_i$, if it is isomorphic map, this map should be corresponding map, that is also to say it is a one-to-one and onto map, so $\dim V = \dim W$. And if $\dim V = \dim W$, then the map f must be isomorphism, because the only operator on them are addition and scalar multiplication. □

There is no doubt that $f : v_\mu \leftrightarrow v^{\mu *}$ is an isomorphic map, however, this isomorphism depends on the choice of the basis, so there is no special isomorphism on them. :et we consider V^{**}, the dual space of V^* or double dual space of V since V^* is also a vector space. The element of V^{**} is the linear map $f : V^* \to \mathbb{R}$, if ω^{**} is one of the elements of V^{**}, ω^* is one of the elements of V^* and v is a vector of V, there is a special isomorphism $g : \omega^{**} \leftrightarrow v$ which asks $\omega^{**}(\omega^*) = \omega^*(v)$, it is said natural isomorphism.[①] So we say V^{**} and V is same under the meaning of natural isomorphism, Only V and V^* are enough for us, V^{**}, V^{***} and so on couldn't gives us more thing, unless we define new structure on them.

Tensor

Definition 1.11. *Let V be a n-dimensional vector space, V^* is its dual space, a tensor, T, of type (k, l) is a multilinear map*

$$T : \underbrace{V^* \times V^* \times \cdots V^*}_{k} \times \underbrace{V \times V \times \cdots \times V}_{l} \to \mathbb{R}.$$

The set collecting all tensor of type (k, l) is denoted by $\mathcal{T}(k, l)$.

If we defined the usual addition and scalar multiplication, the set $\mathcal{T}(k, l)$ is a vector space with dimension n^{k+l}. We can choose one vector from each set, each set has n basis vector, assume $v^{i_1 *}, \cdots, v^{i_k *}, v_{j_1}, \cdots, v_{j_l}$ are the basis of V^* and V, here each index has n choices, i.e. $i_1, \cdots, i_k$, $j_1, \cdots, j_l = 1, \cdots, n$. If we know the result of $T \in \mathcal{T}(k, l)$ acting on these basis, then T is completely known.

♦NOTE:

①Natural isomorphism means that the isomorphism is basis independent. It means we needn't to write all the dual basis for each vector in V.

The element of $\mathcal{T}(0,1)$ is $T : V \to \mathbb{R}$, which is similar to V^*, in fact, this is the definition of V^* and so we have $V^* = \mathcal{T}(0,1)$. Similarly, $T : V^* \to \mathbb{R}$ is the element of $\mathcal{T}(1,0)$, and this is the definition of double duel space, so $\mathcal{T}(1,0) = V^{**}$, but notice there exists a natural isomorphism between V^{**} and V, so V^{**} and V are equivalent, so we can regard they are same, i.e., $\mathcal{T}(1,0) = V$.

What's more, for $\mathcal{T}(1,1)$, we can see that $T(\omega^*, \cdot)$ for fixed ω gives us a linear function on V, i.e. $T(\omega^*, \cdot) \in V^*$, similarly, we have $T(\cdot, \omega)$ for fixed ω^* gives us a linear function on V^*, thus $T(\cdot, \omega) \in V^{**}$, or as what we said, $T(\cdot, \omega) \in V$. ⧫

1.3.2 Two Operations Of Tensor

Definition

Definition 1.12. *If T is a tensor of $\mathcal{T}(k,l)$, the map $C : \mathcal{T}(k,l) \to \mathcal{T}(k-1,l-1)$ defined as*

$$\begin{aligned} CT &:= T(\cdots, v^{1*}, \cdots, v_1, \cdots) + \cdots + T(\cdots, v^{n*}, \cdots, v_n, \cdots) \\ &= \sum_{\sigma=1}^{n} T(\cdots, v^{\sigma*}, \cdots, v_\sigma, \cdots). \end{aligned}$$

This is named the contraction of the tensor.

⧫NOTE:

For clearly, we can denote C as C^i_j, it means we do the sum with the i-th dual vector and j-th ordinary vector. i.e.

$$C^i_j T := \sum_{\sigma=1}^{n} T(\cdots, \underbrace{v^{\sigma*}}_{i-\text{th}}, \cdots, \underbrace{v_\sigma}_{j-\text{th}}, \cdots)$$

We can prove that CT is basis independent, i.e. if $\{v^{\mu*}, v_\mu\}_1^n$ and $\{w^{\mu*}, w_\mu\}$ are two different basis of V^* and V, then we have

$$\sum_{\sigma=1}^{n} T(\cdots, \underbrace{v^{\sigma*}}_{i-\text{th}}, \cdots, \underbrace{v_\sigma}_{j-\text{th}}, \cdots) = \sum_{\sigma=1}^{n} T(\cdots, \underbrace{w^{\sigma*}}_{i-\text{th}}, \cdots, \underbrace{w_\sigma}_{j-\text{th}}, \cdots)$$

⧫

Definition 1.13. *The outer product (or tensor product) of (k,l)-type tensor T and (k',k')-type tensor T' is a $(k+k', l+l')$-type tensor $T \otimes T'$ and the result $T \otimes T'$ acting on $v^{i_1*}, \cdots, v^{i_k*}, v'^{i_1*}, v'^{i_{k'}}$, $v_{j_1}, \cdots, v_{j_l}, v'_{j_1}, \cdots, v'_{j_{l'}}$ is defined as*

$$\begin{aligned} T \otimes T'(v^{i_1*}, \cdots, v^{i_k*}, v'^{i_1*}, v'^{i_{k'}}, v'_{j_1}, \cdots, v_{j_l}, v'_{j_1}, \cdots, v'_{j_{l'}}) &:= \\ T(v^{i_1*}, \cdots, v^{i_k*}, v_{j_1}, \cdots, v_{j_l}) T'(v'^{i_1*}, v'^{i_{k'}}, v'_{j_1}, \cdots, v'_{j_{l'}}) \end{aligned}$$

⧫NOTE:

Based on the definition of outer product of tensor, we can see that any (k,l)-type tensor could be regarded as the tensor product of $(1,0)$-type tensor and $(0,1)$-type tensor. In fact, we just need to prove that $v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1*} \otimes \cdots \otimes v^{\nu_l*}$ are basis of $\mathcal{T}(k,l)$. And since so, any $T \in \mathcal{T}(k,l)$ can be expressed as

$$\begin{aligned} T &= \sum_{\mu_1=1}^{n} \cdots \sum_{\mu_k=1}^{n} \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_l=1}^{n} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1*} \otimes \cdots \otimes v^{\nu_l*} \\ &\equiv T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1*} \otimes \cdots \otimes v^{\nu_l*} \end{aligned}$$

here $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$ is the components of T under the basis $\{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1*} \otimes \cdots \otimes v^{\nu_l*}\}_1^n$.

The proof of this proportion is so complex, so we just ignore its proof. ⧫

The Components Representation For Contraction And Outer Production

From now on, all the proportions that we don't want to give a proof we will use **claim** to denote them. It is a theorem without proof.

Claim 1.1. *Let $T \in \mathcal{T}(k,l)$, its components under the basis $\{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes v^{\nu_1 *} \otimes \cdots \otimes v^{\nu_l *}\}$ are ${T^{\mu_1 \cdots \mu_k}}_{\nu_1 \cdots \nu_l}$, then*

$$(C^i_j T)^{\mu_1 \cdots \mu_{k-1}}{}_{\nu_1 \cdots \nu_{l-1}} = \sum_{\sigma=1}^{n} T^{\mu_1 \cdots \sigma \cdots \mu_k}{}_{\nu_1 \cdots \sigma \cdots \nu_l},$$

here superscript index σ is the i-th superscript index μ_i and the subscript index σ is the j-th subscript index ν_j.

Claim 1.2. *The outer product represented by the components of two tensor $T \in \mathcal{T}(k,l)$ and $T \in \mathcal{T}(k',l')$ is*

$$(T \otimes T')^{\mu_1 \cdots \mu_{k+k'}}{}_{\nu_1 \cdots \nu_{l+l'}} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} T'^{\mu_{k+1} \cdots \mu_{k+k'}}{}_{\nu_{l+1} \cdots \nu_{l+l'}}$$

More Discussion. Contravariant And Covariant Vectors

Definition 1.14. *If V_p is the vector space at p on the manifold $\mathcal{M}$, its dual space is V_p^*, the basis of V_p,or*

$$X_\mu = \frac{\partial}{\partial x^\mu}$$

are named contravariant vectors and the basis of V_p^* are named covariant vectors with notation $\mathrm{d}x^\mu$. V_p is also named tangent space and V_p^* is named cotangent space, the elements in them are named tangent vector and cotangent vector.

♦NOTE:

As what we have said before, this definition shows that

$$\mathrm{d}x^\mu X_\nu = \mathrm{d}x^\mu \frac{\partial}{\partial x^\nu} = \delta^\mu{}_\nu$$

♦

Definition 1.15. *Let v be a vector field, we choose one of the dual vector from dual space V_p^* for each point p, which is named the dual vector field.*

Claim 1.3. *If f is one of the linear map of $\mathcal{F}$, then $\mathrm{d}f$ acting on any vector of the vector field defined as*

$$\mathrm{d}f|_p(v) := v(f)$$

induce a dual vector field. And it can be expressed as

$$\mathrm{d}f = \frac{\partial f \circ \psi^{-1}}{\partial x^\mu} \mathrm{d}x^\mu$$

Claim 1.4. *If ω_μ denotes the components of a dual vector ω with respect to the dual basis $\{\mathrm{d}x^\mu\}$, it follow that under the coordinate transformation the components become*

$$\omega'_{\mu'} = \sum_{\mu=1}^{n} \omega_\mu \frac{\partial x^\mu}{\partial x'^{\mu'}}$$

Claim 1.5. *The tensor transformation law is*

$$T'^{\mu'_1 \cdots \mu'_n}{}_{\nu'_1 \cdots \nu'_n} = \sum_{\mu_1 \cdots \mu_n = 1}^{n} T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n} \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\nu'_l}}{\partial x^{\nu_l}}$$

Definition 1.16. *The collection of choosing one tensor over V_p at each point of manifold $\mathcal{M}$ of same type is called a tensor field with notation $\mathscr{T}$. A tensor field $\mathscr{T}$ is said smooth if and only if $\forall T \in \mathscr{T}$, $T(\omega^1, \cdots, \omega^k; v_1, \cdots, v_l)$ is smooth where $\{\omega^i\}_1^k, \{v_j\}_1^l$ are arbitrary smooth maps on V^* and V.*

Definition 1.17. *The tangent vector field v is called contravariant vector field and the cotangent vector field ω is said covariant vector field.*

1.3.3 Metric Tensor

Definition 1.18. *A metric tensor, g, on a manifold $\mathcal{M}$ is a symmetric, non-degenerated tensor of type $(0, 2)$.*

♦NOTE:

Here symmetric means $g(v_1, v_2) = g(v_2, v_1)$ for any $v_1, v_2 \in V$, and non-degenerated means $g(v, v_1) = 0$ for all $v \in V$, then $v_1 = 0$. According to the components representation of any tensor, we know that g can be expressed as

$$g = \sum_{\mu,\nu} g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

it can be also wrote as

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

which seems like a quadratic form as

$$\mathrm{d}s^2 = \begin{bmatrix} \mathrm{d}x^1 & \cdots & \mathrm{d}x^n \end{bmatrix} \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^n \end{bmatrix}$$

Here the matrix $(g_{\mu\nu})$ is called metric matrix. In our course, the metric matrix wrote as

$$(g_{\mu\nu}) = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$$

and index 0 means time, 1,2,3 means space. Since any metric matrix is non-degenerated, we can use similar transformation

$$(g_{\mu\nu}) = P\Lambda Q$$

where P and Q are invertible matrix or we say $\mathrm{rank}(P) = \mathrm{rank}(Q) = 4$, and Λ is a diagonal matrix with diagonal elements ± 1, $+$ and $-$ is named signature. We always say the metric with signature $\mathrm{diag}(+, +, \cdots, +)$ (all plus) is Euclidian and Riemannian otherwise, specially, the metric with signature $\mathrm{diag}(-, +, +, +)$ or $\mathrm{diag}(+, +, +, -)$ is said Lorentzian. ♦