Sudoku is Hard

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Abstract

sudoku overview

Contents

1 Introduction **2**
- 1.1 History 2
- 1.2 Defining Sudoku Notation 2

2 Classic solving techniques **3**

3 Sudoku is Hard **4**
- 3.1 Computational Complexity 4
- 3.2 Verification is Easy 5
- 3.3 Existance is Hard 5
  - 3.3.1 Proof Outline 6
  - 3.3.2 Sudoku $\geq_p$ Latin Square 6
  - 3.3.3 Latin Square $\geq_p$ Triangulate A Tripartite Graph 7
  - 3.3.4 Triangulated Tripartite $\geq_p$ 3SAT 9
  - 3.3.5 3SAT is NP-Complete 10
  - 3.3.6 Sudoku $\geq_p$ Graph Colouring 10
  - 3.3.7 Example, dimension analysis 10
- 3.4 Determining Uniqueness is Hard 10

4 Solving Techniques **11**
- 4.1 Backtracking 11
- 4.2 Simulated Annealing 11
  - 4.2.1 Convergence 11
  - 4.2.2 Speed of Convergence 11

5 Group theory **13**
- 5.1 Starting Simple 4×4 13
- 5.2 Equivalence Classes 13
- 5.3 6×6 13
- 5.4 8×8 13
- 5.5 9×9 13

6 Other **14**
- 6.1 Other Related Problems 14
  - 6.1.1 Latin Squares 14
- 6.2 Magic Squares 14
  - 6.2.1 Greco-Latin Squares 15
- 6.3 Generating Techniques 15
- 6.4 17 is the Magic Number 15
  - 6.4.1 Sparsity - information theory 15
- 6.5 Topology 15
  - 6.5.1 Torus 15
- 6.6 Polynomials & Constraint Programming 15

Chapter 1

Introduction

Sudoku is a simple logic game, in the standard 9×9 (or $3 \times 3 \times 3 \times 3$) one must complete the grid such that every row, column and box contains the numbers 1 to 9, that is all, yet it is filled with mathematics. Through sudoku we can explore the connections between various areas of maths: complexity theory, graph theory, group theory and information theory.

1.1 History

1.2 Defining Sudoku Notation

Def^n: A valid sudoku puzzle is a function $S : i, j \to x$ for values i,j $\in \{1, ..., D^2\}$ and $x \in \{0, ..., D^2\}$ satisfying the following:

- for all $a, b, c \in \{1, ..., D^2\}$ with $S(a, b) \neq 0$ and $S(a, c) \neq 0$, then $S(a, b) \neq$ S(a,c)
- for all $a, b, c \in \{1, ..., D^2\}$ with $S(a, b) \neq 0$ and$S(c, b) \neq 0$, then $S(a, b) \neq$ S(c,b)
- for all $a, b, c, d \in \{1, ..., D^2\}$ with a mod $D = c$ mod D, b mod $D = d$ mod D, $S(a, b) \neq 0$ and $S(c, d) \neq 0$, then $S(a, b) \neq S(a, c)$

Def^n: A completed sudoku puzzle is a function $S : i, j \to x$ as above but with the added condition that $x \neq 0$.

Chapter 2

Classic solving techniques

Def^n: A forced cell is a value pair (a, b) such that $S(a, b)$ can only be a single value call this x as $\{1, ..., D^2\}/\{x\}$ are already present in $S(a, j)$ for $j \in \{1, ..., D^2\}/\{b\}$ or $S(i, b)$ for $i \in \{1, ..., D^2\}/\{a\}$ or $S(i, j)$ where $a \bmod D = i \bmod D$ and $b \bmod D = j \bmod D$.

define x wing define y wing

Chapter 3

Sudoku is Hard

Let's imagine a sudoku of size $D^2 \times D^2$. How big does D have to be for you to need more than a day to solve it? Maybe 6 or 10 or even just 4. Don't worry if you said a smaller number than your friends, this has nothing to do with your problem solving skills, even a computer finds sudoku hard. In fact just incrementing D by 1 leads to an exponential increase in compute time and the most optimal algorithms for solving sudoku are infeasible for 100×100.

We prove sudoku's hardness by transforming it into a known 'difficult' problem; we will use SAT, a problem that has plagued computer scientists for decades.

3.1 Computational Complexity

For those with a mathematical mind, outraged by the lack of definitions for 'difficulty' and 'hardness', let's take a detour into complexity theory.

Def$^{\mathbf{n}}$: Let f be a function indicating the execution time for an algorithm and g a strictly positive function. $f(x) = O(g(x))$ if $\exists$ positive M and x_0 such that $|f(x)| \leq Mg(x)\ \forall\ x \geq x_0$. This is coined **Big O Notation**.

Example of a linear time algorithm. Given a sudoku board and a square to check it takes a linear amount of time to validate this. Example of a polynomial time algorithm, checking a whole sudoku board is polynomial. Example of an exponential time algorithm. Brute Force Alg?

Def$^{\mathbf{n}}$: A **Reduction**, $A \leq_p B$, is a transformation in polynomial time ($O(x^c)$) from problem A to B.

Def$^{\mathbf{n}}$: A **Turing Machine** is the mathematical model of a CPU.

Def$^{\mathbf{n}}$: A **non-deterministic** Turing Machine is the mathematical model of a CPU that can undertake any possible action all at once, for example with a sudoku it would be able to explore solutions with a cell taking all values 1 to n^2 all at once.

Sets of Difficulty: We care about decision problems, these are problems that given an input produce a 'yes' or 'no' answer. We will discuss three sets of these problems:

- P is the the class of problems that can be solved in polynomial time (if the input is order n then the program halts in order n^2 steps) by a Turing machine;
- NP is the class of problems that can be verified in polynomial time and solved in polynomial time by a non-deterministic Turing machine;
- the NP-complete set has problems that any NP problem can be reduced to in polynomial time.

Problems in P are considered feasible and those in NP-complete are infeasible as their complexity scales exponentially with respect to the input size and as it is assumed they cannot be solved in polynomial time ($P \neq NP$) and are therefore infeesible for large inputs. [1]

[1] We can only assume that $P \neq NP$ as this problem is yet to be proven, it is in fact one of the Millennium Prize problems.

So when we state sudoku is hard we are actually saying sudoku belongs to NP-complete. We cannot just prove sudoku belongs to NP as this also includes problems in P. [2]

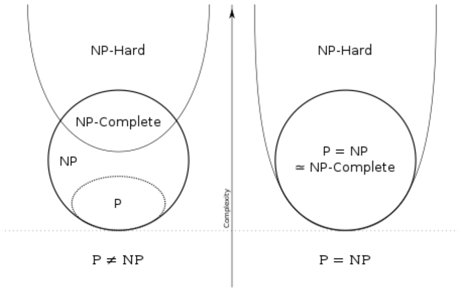

Figure 3.1: P, NP, NP-complete & NP-hard sets [1]

How to prove NP completeness generally? Call the problem we wish to prove is NP-complete x. First show there exists a verifier for x with a polynomial or less runtime, this is a algorithm that decides if a proposed solution to problem x is correct. Then take a known NP-complete problem call this y, and reduce it to x, one does this by transforming the input of y to the input of x in polynomial time, we call this function $g(y) = x$. Assume there exists a polynomial time algorithm to solve x, $f()$ we could solve y in polynomial time too, $f(g(y))$, this implies P=NP, a contradiction. Therefore a polynomial time algorithm does not exist for x.

Our base NP-complete problem. If, as the above suggests, we require a NP-complete problem to prove a problem is NP-complete then we seem to have reached a paradox. Luckily we have the Cook-Levin Theorem.

Cook-Levin Theorem: SAT is NP-Complete *CITE*

Def^n: SAT is the following decision problem. Given a set of boolean variables B and a collection of clauses C does a valid truth assignment exist that satisfies C?

We now have a NP-complete problem to reduce other problems to.

Let's make this more intuitive with examples.

3.2 Verification is Easy

Verificaiton decision problem, "is the sudoku puzzle complete?":

$$\Psi(S(,)) = \begin{cases} \text{True if the puzzle is complete} \\ \text{False if the puzzle is not complete.} \end{cases} \tag{3.1}$$

There exists an algorithm to do this in polynomial time with respect to the dimensions of the grid.

1. For each row in the grid check there exists no repeated numbers. $O(n^2)$
2. For each column in the grid check there exists no repeated numbers. $O(n^2)$
3. For each box in the grid check there exists no repeated numbers. $O(n^2)$

If all tests pass return True else return False. This algorithm has complexity of $O(n^2 + n^2 + n^2) = O(3n^2) = O(n^2)$, this is polynomial and therefore $\Psi(S(,)) \in P$.

3.3 Existance is Hard

Checking if a solution to sudoku exists is NP-complete, let us define the decision problem:

[2]Due to Ladner's Theorem there exists problem $\in$ NP but $\notin$ NP-complete and $\notin$ P iff $P \neq NP$.

$$\Phi(S(,)) = \begin{cases} \text{True if a completion exists} \\ \text{False if a completion does not exist.} \end{cases} \tag{3.2}$$

Our question is does there exist a function Φ that when given an instance of the problem will, in polynomial time or less, return True if it can be solved and False otherwise.

3.3.1 Proof Outline

The verifier is $O(n^2)$, as will be seen in the above subsection 'Validation is Easy', this shows the Sudoku decision problem belongs to the set NP.

Now we need a reduction from sudoku to a known NP-complete problem to prove sudoku is also NP-hard. We will be creating a chain of reductions: **Sudoku $\geq_p$ Latin Square $\geq_p$ Triangulated Tripartite $\geq_p$ 3SAT $\geq_p$ SAT**.

As the Sudoku decision problem is a member of NP and NP-hard it is NP-complete by definition.

Note: Theoretically any problem in the set NP-complete can be reduced to Sudoku and therefore this reduction is not unique, however, it is the most intuitive way. Some readers may question why we are not looking at a reduction to a Graph n^2-Colouring problem but in section **cite** we explore this is the wrong direction of reduction.

3.3.2 Sudoku $\geq_p$ Latin Square

Def^n: A valid Latin Square puzzle is a function $L : i, j \to x$ for values $i, j \in \{1, .., D\}$ and $x \in \{0, ..., D\}$ satisfying the following:

- for all $a, b, c \in \{1, ..., D\}$ with $L(a, b) \neq 0$ and $L(a, c) \neq 0$ then $L(a, b) \neq L(a, c)$
- for all $a, b, c \in \{1, ..., D\}$ with $L(a, b) \neq 0$ and $L(c, b) \neq 0$ then $L(a, b) \neq L(c, b)$

It is complete or solved if for all $i, j \in \{1, ..., D\}$, $L(i, j) \neq 0$.

By observation we see this is a superset of the sudoku puzzle, we just add the restrictions that the dimension must be a square number and also add the third property of the sudoku puzzle defintion.

What is the Latin Square decision problem? Given a latin square puzzle $L(,)$, can the function be augmented, by changing only the value of the function for value pairs i, j that previously gave $L(i, j) = 0$, to get a complete latin square puzzle?

Proof idea: We must reduce a given latin square grid of size $D \times D$ to a sudoku grid size $D^2 \times D^2$ that is solvable iff the Latin square is.

Lemma: Let S_l be a Sudoku problem with the following construction

$$S_l(i, j) = \begin{cases} 0 & \text{when } (i, j) \in L_s \\ ((i - 1 \bmod n)n + \lfloor i - 1/n \rfloor + j - 1) \bmod n^2 + 1 & \text{otherwise} \end{cases} \tag{3.3}$$

where $L_s = \{(i, j) | \lfloor i - 1/n \rfloor = 0 \text{ and } (j \bmod n) = 1\}$. Then there exists an augmentation S_l' to complete the sudoku puzzle if and only if the square L such that $L(i, j/n) = S_l'(i, j) - 1/n + 1$ for all $(i, j) \in L_s$ is a Latin square.

Note: The fact we have a formula to generate a valid sudoku for any size D^2 is interesting and we should explore if this can be done for a $M \times N$ sudoku too. (explored in section 3). Figure 3.2 gives examples of generated sudokus from this formula.

Proof:

First we must show $S_l(i, j) = ((i - 1 \bmod n)n + \lfloor i - 1/n \rfloor + j - 1) \bmod n^2 + 1$ forms a complete and valid sudoku puzzle.

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

$n = 2$

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

$n = 4$

Figure 3.2: Formula Generation of Valid Sudoku

When $i = [1, ..., n^2]$ then:

$$0 < \lfloor i - 1/n \rfloor < n - 1 \tag{3.4}$$
$$0 < i - 1 \bmod n < n - 1 \tag{3.5}$$
$$0 < (i - 1 \bmod n)n + \lfloor i - 1/n \rfloor < n^2 - n \tag{3.6}$$
$$0 < (i - 1 \bmod n)n + \lfloor i - 1/n \rfloor + j - 1 < n^2 - 1 \tag{3.7}$$
$$1 < ((i - 1 \bmod n)n + \lfloor i - 1/n \rfloor + j - 1) \bmod n^2 + 1 < n^2 \tag{3.8}$$
$$1 < S_l(i, j) < n^2 \tag{3.9}$$

Note $\lfloor i - 1/n \rfloor$ gives the row coordinate when indexed at 0 in which the larger box that (i,j) belongs to starts and $i - 1 \bmod n$ gives the row within that box when indexed at 0. Therefore $(\lfloor i - 1/n \rfloor, i - 1 \bmod n)$ will take all value pairs of integers between 0 and $n - 1$.

When j is fixed (particular column), assume two cells have the same value, that is $S_l(i, j) = S_l(i', j)$ then

$$(i - 1 \bmod n)n + \lfloor i - 1/n \rfloor + j - 1 = (i' - 1 \bmod n)n + \lfloor i' - 1/n \rfloor + j - 1 \tag{3.10}$$
$$(i - 1 \bmod n)n + \lfloor i - 1/n \rfloor = (i' - 1 \bmod n)n + \lfloor i' - 1/n \rfloor \tag{3.11}$$

from the above $i = i'$. No cell on a column has the same value.

When i is fixed (particular column) assume two cells have the same value, that is $S_l(i, j) = S_l(i, j')$ implies $j - 1 = j' - 1(modn)$ therefore $j = j'$.

For the third condition fix $\lfloor i - 1/n \rfloor$. (i-1 mod n,j) takes all value pairs of integers 0 to n-1 so if a cell has the same value as another within the n by n square $S_l(i, j) = S_l(i', j')$ implying $(i - 1 \bmod n, j) = (i' - 1 \bmod n, j')$ which means $i = i'$ and $j = j'$. Therefore S_l is a valid and complete sudoku puzzle.

Now consider which integers fill the blanks in L_s. For $(i, j) \in L_s$, $S_l(i, j) - 1 = ((i - 1 \bmod n)n + j - 1) \bmod n^2$ as $jmodn = 1$, $j - 1modn = 0$ therfore $S_l(i, j) - 1$ is divisible by n so $S_l - 1/n + 1$ gives integers between $[1, ..., n]$. Therefore $L(i, j) \in [0, ..., n]$.

We must validate the Latin square conditions. The row constraint in S_l ensures $S'(i, j) = S'(i, j') \implies j = j'$, $S'(i, j) - 1/n + 1 = S'(i, j') - 1/n + 1 \implies j = j'$, $L(i, j/n) = S'(i, j'/n) \implies j = j'$ is equivalent to the row constraint of L. The column constraint of S_l is equivalent to the column constraint of L. The small square constraint of S_l is equivalent to the column constraint of L. $\square$

3.3.3 Latin Square $\geq_p$ Triangulate A Tripartite Graph

Def^n: A graph $G = (V, E)$ is tripartite if a partition V_1, V_2, V_3 exists such that the vertices are split into three sets with no edges between vertices that belong to the same set, i.e for all $(v_i, v_j) \in E$ if $v_i \in V_i$ then $v_j \notin V_i$.

Def^n: A triangulation T of a graph is a way to divide edges into disjoint subsets T_i, each forming a triangle ($T_i = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$).

If a tripartite graph can be triangulated it must be uniform, that is: every vertex in V_1 (or V_2 or V_3) has the same number of neighbour in V_2 and V_3 (or the respective sets).

What is the Triangulated Tripartite decision problem? Given a graph G that is tripartite (can be split into 3 subgroup, within these subgroups vertices should not share edges) can it be triangulated ?

Theorem: Completing a Latin square with dimensions n by n is equivalent to triangulating a tripartite graph $G = V_1, V_2, V_3$.

Proof:

Intuitively, we map a graph to a Latin square L through the following: given tripartite graph G=(V,E) label vertices in V_1 with distinct lables $\{r_1, ...r_n\}$, label vertices in V_2 with distinct lables $\{c_1, ...c_n\}$ and label vertices in V_3 with distinct lables $\{e_1, ...e_n\}$. Add edges such that:

- If $L(i, j) = 0$ then add the edge (r_i, c_j)
- If for all $i \in [0, ..., n]$ and constant j, $L(i, j) \neq k$ then add the edge (r_i, e_k)
- If for all $j \in [0, ..., n]$ and constant i, $L(i, j) \neq k$ then add the edge (c_j, e_k)

This graph has a triangulation iff $L(i, j)$ can be solved.

EXAMPLE

Let us show every uniform tripartite graph can be transformed to the above formulation of a Latin square.

First we need an intermediate that is a generalisation of a latin square

Def^n: A Latin framework LF for tripartite graph G, size (r,s,t) is a r by s array with values [1,...,t]. With constraints:

- Each row/column contain each element only once.
- If $(r_i, c_j) \in E$ then LF(i,j)=0 else LF(i,j)= k, $k \in [1, ..., t]$
- If $(r_i, e_k) \in E$ then $\forall j \; LF(i, j) \neq k$
- If $(c_j, e_k) \in E$ then $\forall i \; LF(i, j) \neq k$

If r=s=t then LF is a latin square (formulation above) which can be completed iff G has a triangle partition.

Lemma: For tripartite graph G=(V,E) with $|V_1| = |V_2| = |V_3| = n$ (uniform), there's a Latin framework of (n,n,2n).

Define LF an n by n array. For $(r_i, c_j) \in E$ $LF(i, j) = 0$ else $LF(i, j) = 1 + n + ((i + j) mod n)$. LF is a latin framework as the first two bullet points of the definition hold by construction and as $1 + n \leq LF(i, j) \leq 2n$ LF will never equal a value in $1, ...n$ and therefore the last two bullet points hold. The size is trivial. $\square$

Lemma: Given latin framework LF(n,n,2n) for uniform tripartite graph G, we can extend the latin framework to have size (n,2n,2n).

First we have a few denotions: R(k) = the number of times k appears in L plus half $|e_k|$; $S_i = \{k | k \notin LF(i, j) \forall j \cap (r_i, e_k) \notin E\}$; $M = \{k | R(k) = r + s - t\}$. We show sets $S_1, ...S_r$ have a system distinct representative (**DEFINE**) containing all elements of M, we then add this system as the $(s + 1)$st column and repeat until we have $2n$ columns.

Using Hoffman and Kuhn's theorem **CITE** we need only show that $S_1, ...S_r$ have a system distinct representative and that for every $M' \subseteq M$ at least $|M'|$ of sets $S_1, ...S_r$ have non empty subsections with M'.

First choose any m sets such that $1 \leq m \leq r$. As G is uniform each set has t-s elements, so m sets together have $m(t - s)$ cardinality.Each value $1, ..., t$ appears at least r+s-t times in LF, so note each value appears in at most $t - s$ of the sets S_i. Consider the union of the m sets, this contains some p elements so we have $p(t - s) \geq m(t - s)$ therfore $p \geq m$. So any m sets have at least m elements in their union and by the P Hall theorem **CITE** a system distinct representative exists.

Next take $M' \subseteq M$ and assume there are p sets in $S_1, ...S_r$ that have a nonempty intersection with M'. Each set has $t - s$ elements and together have caridinality $p(t - s)$, each element of M appears in exactly $r-(r+s-t)=t-s$ of the s_is, therefore $|M'|(t-s) \leq p(t-s)$ so $|M'| \leq p$. At least $|M'|$ sets have nonempty intersections with M'.

The Hoffman and Kuhn theorem holds and therefore a system distinct representative exists and can be added to the end. We repeat this n times. □

Lemma: Latin framework (n,2n,2n) for grpah G, can be extended to (2n,2n,2n).

We can transpose the array and do the same as the previous lemma. □

Note: we can find a system distinct representative using the Hopcroft-Karp **CITE** algorithm which solves bipartite matching in polynomial time.

Given a tripartite graph G, if it is not uniform then no triangulation exists, else we apply above to produce a latin framework of size (2n,2n,2n) in polynomial time. This is a Latin square which can be completed iff G has a triangulation. The latin square problem has been reduced to the triangulating a tripartite graph problem. □

3.3.4 Triangulated Tripartite $\geq_p$ 3SAT

What is 3SAT? With a set of boolean variables B and a collection of clauses C, with at most 3 literals (a literal is any $b \in B$ or its negation $\bar{b}$) in each, does a valid truth assignment exist that satisfies C?

$$\phi(C,B) = \begin{cases} \text{True if a truth assignment exists} \\ \text{False if a truth assignment does not exist.} \end{cases} \tag{3.12}$$

This decision problem is therefore an enforced limitation of SAT as defined in the section Computational Complexity.

Proof:

This reduction is a little trickier as we need to introduce the Holyer graph H.

Def^n: The Holyer graph $H_{3,p}$ is the set of vertices $V = \{(x_1, x_2, x_3) \in \mathbb{Z}_p^3 \| x_1 + x_2 + x_3 \equiv 0(modp)\}$ and an edge exists between vertices (x_1, x_2, x_3) and (y_1, y_2, y_3) if distinct i,j and k exist such that:

- $x_i \equiv y_i (\text{mod } p)$
- $x_j \equiv y_j + 1 (\text{mod } p)$
- $x_k \equiv y_k - 1 (\text{mod } p)$

This is much easier to grasp with a diagram see figure **FIGURE**. This graph is tripartite if and only if $p \equiv 0$ (mod 3), this is demonstrated by a 3-colouring (a graph is tripartite if and only if it is 3-colourable) in figure **FIGURE**.

Def^n: $H_{3,p}$ has only two triangulations, termed a true and a false triangulation **FIGURE**.

Note: We connect graphs together by taking a set of vertices in G_1 and making them the 'same' as a set of vertices in G_2, sets are the same size.

Def^n: We will connect out graph with F-patches and T-patches **FIGURE**.

Lemma: Connecting two $H_{3,p}$ by two A-patches then our triangulations of these graphs can be of the form (T,T), (T,F) or (F,T). Then by removing the centre triangles from both graphs we get only the triangulations (T,F) or (F,T). If we expand this to x $H_{3,p}$ we get only one false triangulation and the rest true.

Transformation process: (select p large enough to prevent patch overlap and $p \equiv 0$ (mod 3))

- For $b_i \in B$ create $H_{3,p}$ called G_{b_i}.
- For all $c_j \in C$, for each literal $l_{i,j}$ $j \in [1,2,3]$ create $H_{3,p}$ called $G_{i,j}$.
- If $l_{i,j} = b_k$ connect an F-patch in G_{b_k} to an F-patch in $G_{i,j}$, else if $l_{i,j} = \neg b_i$ connect a F-patch in $G_{i,j}$ to a T-patch in G_{b_k}.
- For each i connect one F-patch from each $G_{i,1}$, $G_{i,2}$ and $G_{i,3}$ then delete the centre triangle.
- $G = \{G_{b_i} \mid b_i \in B\} \cup \{G_{i,j} \mid c_j \in C \text{ and } i \in [1,..,3]\}$

$(a \vee b) \wedge (\neg a \vee \neg b)$

a	b		$(a \vee b)$	$(\neg a \vee \neg b)$	$(a \vee b) \wedge (\neg a \vee \neg b)$
F	F		F	T	F
F	T		T	T	T
T	F		T	T	T
T	T		T	F	F

Figure 3.3: Truth Assignment Example with Highlighted Valid Assignment

We now need to prove the graph produced by the above transformation can be triangulated if and only if there is a truth assignment satisfying the 3SAT formula.

Assume a triangulation of G exists, consider a H within the construction of G. H is eitehr a true triangulation or a false triangulation. Now assume $l_{i,j}$ is b_k and consider the join between $G_{i,j}$ and G_{b_k} as this joins two F-patches we get at least one true triangulation. (if $G_{i,j}$ is a true triangulation this acounts for all edges near the joining patch but the actual patch can be attributed to G_{b_k} which can be triangulated either way, if both are false triangulations the connecting patch is forced to belong to both $G_{i,j}$ and G_{b_k} which is a contradiction.)

Similarly if $l_{i,j} = \neg b_i$ then $G_{i,j}$ is a false triangulatio or G_{b_k} is a true triangulation.

Next the join between clause graphs allow for one false triangulation and the rest are true triangulations. As the centre of the patch is missing a single $G_{i,j}$ must take the outer edges of the patch by being a false triangulation.

If G can be triangulated a truth assignment exists such that variable b_k is true if G_{b_k} has a true partition otherwise it is false.

If there exists a truth assignment we can triangulate G_{b_k} according to this truth assignment and this will allow for the whole graph to be triangulated.

This transformation takes place in polynomial time and therefore Triangulated Tripartite $\geq_p$ 3SAT. □

3.3.5 3SAT is NP-Complete

Proof:

Given a truth assignment t check each clause is satisfied, if all are satisfied return True else False, this algorithm is at most the length of C multiplied by the length of B. $O(BC)$ is polynomial, a polynomial verifier exists.

Given a SAT instance with the input sets of B and C. C is in conjunctive normal form (every clause set can be converted to an equivalent set in CNF form [2]) such that $\forall c \in C$ and for some $b_1, ..., b_n \in B$, $c = b_1 \vee b_2 \vee ... \vee b_n$. For each $c \in C$ with more than 3 literals we can transform these to a new set of clauses of length 3.

For $c = b_1 \vee b_2 \vee ... \vee b_n$ we introduce a new literal: a_1 to give $b_1 \vee b_2 \vee a_1$, $\bar{b}_1 \vee a_1$, $\bar{b}_2 \vee a_1$ and $a_1 \vee b_3 \vee ... \vee b_n$. Then $a_1 \vee b_3 \vee ... \vee b_n$ becomes $b_3 \vee b_4 \vee a_2$, $\bar{b}_3 \vee a_2$, $\bar{b}_4 \vee a_2$ and $a_1 \vee a_2 \vee b_5 \vee ... \vee b_n$. This continues at most $n/2$ times to give $a_1 \vee ... \vee a_{n/2}$ or $a_1 \vee ... \vee a_{n/2} \vee b_n$ if n is odd.

Because we can convert a clause larger than 3 into multiple clauses of at most 3 literals in linear time ($O(n/2 + n/4 + ...) = O(n)$) this means we can reduce SAT to 3SAT in polynomial time.

As SAT is NP-complete by the Cook-Levin Theorem, this proves 3SAT is NP-Complete. □

3.3.6 Sudoku $\geq_p$ Graph Colouring

3.3.7 Example, dimension analysis

3.4 Determining Uniqueness is Hard

Def^n: The Sudoku Uniqueness problem is: Given a partially completed sudoku grid S does only a single completion exist?

$$\Gamma(S) = \begin{cases} \text{True if only a single completion exists} \\ \text{False if multiple completions or none exist.} \end{cases} \tag{3.13}$$

This is NP-hard (no polynomial verifier exists), NP-complete reduction exists.

It is hard to determine if a puzzle has a unique solution. *TO COMPLETE: FIND PAPER WITH PROOF*

Chapter 4

Solving Techniques

4.1 Backtracking

The standard way to solve a 9×9 sudoku puzzle is by the backtracking algorithm. This is a brute force method with a few optimisations. One can expect to find this algorithm in a computer science course introduction to recursion, that is to say it is not a complex concept and while useful for the usual sizes, as soon as we increase to 16×16 this becomes infeesible.

Algorithm 1 Backtracking

```
procedure BACKTRACKING(grid)
    for row do
        for column do
            if grid(row,column) = 0 then
                try a value in this position
                Backtracking(grid with new value)
                if successful then
                    return grid
                else:
                    try another value
                end if
                if no values left to try then
                    return False
                end if
            end if
        end for
    end for
    return grid
end procedure
```

Why does brute force not work for larger examples? It will work *TO DO: PROVE ALG CORRECTNESS* but due to the complexity of the problem (point back to sudoku is hard chapter) it is infeesible.

4.2 Simulated Annealing

Based on metalurgy

4.2.1 Convergence

4.2.2 Speed of Convergence

[**?**] one of 100 most cited papers, one of the first AI algs

Algorithm 2 Simulated Annealing

```
procedure SimAnnealing(grid, schedule, f)
    current = initialise state
    for t = 1 to inf do
        T=schedule[t]
        if T ≤ ε then
            return current
        else
            choose successor at random
            ΔE = f(successor) - f(current)
            if ΔE ≥ 0 then
                current = succ
            else choose with probability e^(ΔE/T)
                current = successor
            end if
        end if
    end for
end procedure
```

Chapter 5

Group theory

5.1 Starting Simple 4×4

Let us analyse Shidoku which is a specific set of sudokus with dimensions 4 by 4 the smallest non trivial sudoku puzzle. Only 2 fundamentally different. One has 96 identical, other has 192. Why not the same amount?

5.2 Equivalence Classes

5.3 6×6

Define Rodoku Define which sudoku sizes can exist

812

5.4 8×8

5.5 9×9

5,472,730,538

Chapter 6

Other

6.1 Other Related Problems

6.1.1 Latin Squares

- A latin square is an n by n matrix filled with n characters that must not repeat along columns or rows.
- Reduced Form - f first row and column is in the natural order
- Equivalence classes
- Number of n by n latin squares is bounded
- Latin squares can be considered a bipartite graph
- Agronomic Research
- Latin hypercube

6.2 Magic Squares

- A magic square is a matrix of numbers with each column, row and diagonal summing to the same value, this value is known as a magic constant and the degree is the number of columns/rows.
- A normal magic square is one containing the integers 1 to n^2.
- Magic Squares with repeating digits are considered trivial.
- Semimagic squares omit the diagnonal sums also summing to the magic constant.
- Truly thought to be magic Shams Al-ma'arif.
- Generation, there exists not completely general techniques. Diamond Method
- Associative Magic Squares
- Pandiagonal Magic Squares
- Most-Perfect Magic Squares
- Equivalence classes for $n <= 5$ but not for higher orders.
- The enumeration of most perfect magic squares of any order.
- 880 distinct magic squares of order four
- Normal magic squares can be constructed for all values except 2

- Preserving the magic property when transformed
- Methods of construction
- Multiplicative magic squares - produce infinite
- Sator square
- magic square of squares - Parker Square is a failed example of this

6.2.1 Greco-Latin Squares

- Two orthogonal latin squares super imposed, such that the pairs of values are unique.
- Group based greco latin squares
- Eulers interest came from construction of magic squares
- Exists for all but 2 and 6.

6.3 Generating Techniques

A polynomial generation algorithm without requiring a uniqueness checker which we have proven to be np-complete and therefore infeesible for large n.

6.4 17 is the Magic Number

4 for shidoku

6.4.1 Sparsity - information theory

Bomb sudoku/latin squares - Additional rule: the same number can not occur in adjacent or diagonally adjacent squares.

6.5 Topology

6.5.1 Torus

6.6 Polynomials & Constraint Programming

Use of polynomials Roots of unity Grobner Basis

Bibliography

[1] https://en.wikipedia.org/wiki/NP_(complexity)

[2] Artificial Intelligence: A modern Approach Archived 2017-08-31 at the Wayback Machine [1995...] Russell and Norvig

[3] https://www.researchgate.net/publication/251863893_A_New_Algorithm_for_Generating_Unique-Solution_Sudoku

[4] https://fse.studenttheses.ub.rug.nl/22745/1/bMATH_2020_HoexumES.pdf.pdf

[5] http://web.math.ucsb.edu/p̃adraic/mathcamp_2014/np_and_ls/mc2014_np_and_ls_lecture3.pdf, http://web.math.ucsb.edu/p̃adraic/mathcamp_2014/np_and_ls/mc2014_np_and_ls_lecture4.pdf

[6] https://scholar.rose-hulman.edu/cgi/viewcontent.cgi?article=1398&context=rhumj

[7] https://onlinelibrary.wiley.com/doi/10.1002/(SICI)1520-6610(1996)4:6¡405::AID-JCD3¿3.0.CO;2-J

[8] http://joas.agrif.bg.ac.rs/archive/article/59

[9] https://www.semanticscholar.org/paper/Permutation-arrays-for-powerline-communication-and-Colbourn-Kløve/7e69cfdbd2082463c66de698da1e326f0556d1d4

[10] http://www.multimagie.com/English/SquaresOfSquaresSearch.htm

[11] https://plus.maths.org/content/anything-square-magic-squares-sudoku

[12] https://link.springer.com/book/10.1007/978-1-4302-0138-0

[13] an Laarhoven, P.J.M., Aarts, E.H.L. (1987). Simulated annealing. In: Simulated Annealing: Theory and Applications. Mathematics and Its Applications, vol 37. Springer, Dordrecht. https://doi.org/10.1007/978-94-015-7744-1_2