CSYS 300 PoCS Assignment 4

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Code is located at

https://github.com/Evelios/PoCS_Assignment_04

1 Problem 1

Adding some content to problem 1

2 Problem 2

Normalized number of groups inthe long time limit, n_k satisfies the following difference equation with $k \geq 2$,

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} \tag{1}$$

For $k = 1$,

$$n_1 = \rho - (1-\rho)n_1 \tag{2}$$

The following steps are to derive the exact solution for n_k. The following derivation will use both the gamma function and the beta function with the following definitions,

$$\Gamma(x) = (k-1)! \tag{3}$$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4}$$

So now simplifying for n_k

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} \tag{5}$$

$$n_k = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} n_{k-1} \tag{6}$$

$$= \left[\frac{(k-1)(1-\rho)}{1+(1-\rho)k}\right]\left[\frac{(k-2)(1-\rho)}{1+(1-\rho)(k-2)}\right] n_{k-2} \tag{7}$$

$$= \left[\frac{(k-1)(1-\rho)}{1+(1-\rho)k}\right]\left[\frac{(k-2)(1-\rho)}{1+(1-\rho)(k-2)}\right] ...n_2 n_1 \tag{8}$$

$$= \left[\frac{(k-1)(1-\rho)}{1+(1-\rho)k}\right]\left[\frac{(k-2)(1-\rho)}{1+(1-\rho)(k-1)}\right] \cdots \left[\frac{(2-1)(1-\rho)}{1+(1-\rho)(2)}\right] n_1 \tag{9}$$

$$= \frac{(1-\rho)^{k-1}}{(1-\rho)^{k-1}}\left[\frac{(k-1)}{\frac{1}{1-\rho}+k}\right]\left[\frac{(k-2)}{\frac{1}{1-\rho}+(k-1)}\right] \cdots \left[\frac{(2-1)}{\frac{1}{1-\rho}+(2)}\right] n_1 \tag{10}$$

$$= \Gamma(k)\left[\frac{1}{\frac{1}{1-\rho}+k}\right]\left[\frac{1}{\frac{1}{1-\rho}+(k-1)}\right] \cdots \left[\frac{1}{\frac{1}{1-\rho}+(2)}\right] n_1 \tag{11}$$

$$n_k = \frac{\Gamma\left(k\right)\Gamma\left(1-\rho+1\right)}{\Gamma\left(1-\rho+1+k\right)} n_1 \tag{12}$$

$$n_k = \frac{\Gamma\left(k\right)\Gamma\left(\rho\right)}{\Gamma\left(\rho+k\right)} n_1 \tag{13}$$

$$n_k = \beta\left(k,\rho\right) n_1 \tag{14}$$

3 Problem 3

From the lectures we arrived at the following,

$$\gamma = 1 + \frac{1}{1-\rho} \tag{15}$$

For $\rho \to 0$,

$$\gamma = \lim_{\rho \to 0} 1 + \frac{1}{1-\rho} = 2 \tag{16}$$

For $\rho \to 1$,

$$\gamma = \lim_{\rho \to 1} 1 + \frac{1}{1-\rho} = \infty \tag{17}$$

For the case of $\gamma \to 2$, there is no mutation rate and the entire population is in the same grounp, this would cause the slope of the distribution to become infinite because there would be nothing to scale against. In the case where $\gamma \to \infty$, we are guarenteed to make a new group every time. This would cause the frequency of all the groups to be one, causing the power law slope to go to zero because each group occurs as often as every other one.

4 Problem 4

$$n_1 = \frac{\rho}{2-\rho} \tag{18}$$

$$n_k^{(g)} = \frac{1}{\rho t} N_{kt} = \frac{1}{\rho} n_k \tag{19}$$

Solving for $n_2^{(g)}$,

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} \tag{20}$$

$$n_k = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} n_{k-1} \tag{21}$$

$$n_2 = \frac{(2-1)(1-\rho)}{1+(1-\rho)2} n_1 \tag{22}$$

$$n_2 = \frac{(1-\rho)}{1+2(1-\rho)} \left[\frac{\rho}{2-\rho}\right] \tag{23}$$

$$n_2^{(g)} = \frac{1}{\rho} n_2 = \frac{1}{\rho} \left[\frac{1-\rho}{3-2\rho}\right] \left[\frac{\rho}{2-\rho}\right] \tag{24}$$

$$n_2^{(g)} = \frac{(1-\rho)}{(2-\rho)(3-2\rho)} \tag{25}$$

Now for $n_3^{(g)}$,

$$n_k = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} n_{k-1} \tag{26}$$

$$n_3 = \frac{(3-1)(1-\rho)}{1+(1-\rho)3} n_2 \tag{27}$$

$$n_3 = \frac{2(1-\rho)}{4-3\rho} \left[\frac{(1-\rho)}{3-2\rho}\right] \left[\frac{\rho}{2-\rho}\right] \tag{28}$$

$$n_3^{(g)} = \frac{1}{\rho t} N_{3t} = \frac{1}{\rho} n_3 \tag{29}$$

$$n_3^{(g)} = \frac{1}{\rho} \left[\frac{2(1-\rho)}{4-3\rho}\right] \left[\frac{(1-\rho)}{3-2\rho}\right] \left[\frac{\rho}{2-\rho}\right] \tag{30}$$

$$n_3^{(g)} = \frac{2(1-\rho)^2}{(4-3\rho)(3-2\rho)(2-\rho)}$$

5 Problem 5

6 Problem 6