# CSYS 300 PoCS Assignment 4
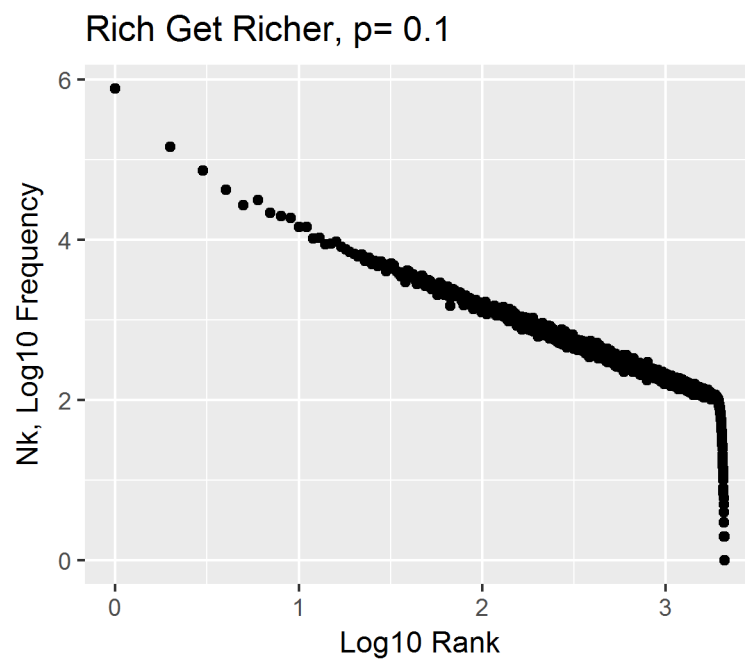
Thomas Waters

September 28th 2018
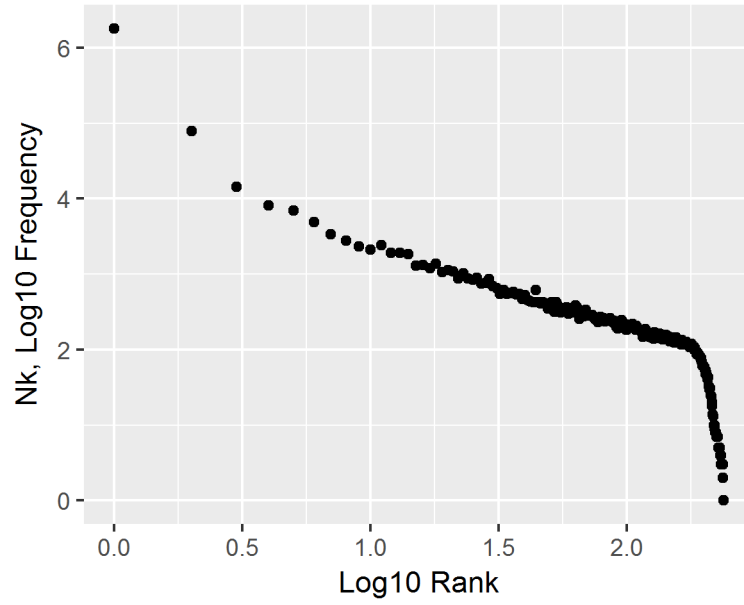
Code is located at

https://github.com/Evelios/PoCS_Assignment_04
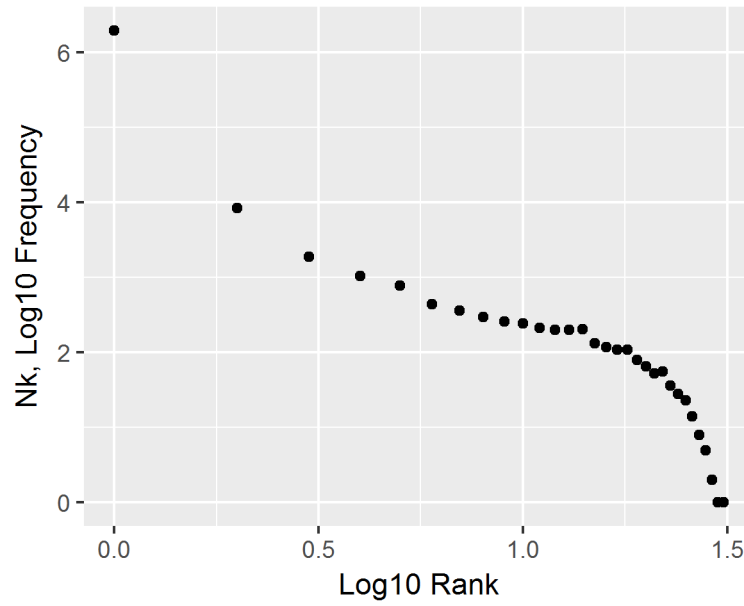
# 1  Problem 1

Rich Get Richer, p= 0.1

# Rich Get Richer, p= 0.01



# Rich Get Richer, p= 0.001

# 2   Problem 2

Normalized number of groups in the long time limit, $n_k$ satisfies the following difference equation with $k \geq 2$,

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} \tag{1}$$

For $k = 1$,

$$n_1 = \rho - (1-\rho)n_1 \tag{2}$$

The following steps are to derive the exact solution for $n_k$. The following derivation will use both the gamma function and the beta function with the following definitions,

$$\Gamma(x) = (k-1)! \tag{3}$$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4}$$

So now simplifying for $n_k$

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1+(1-\rho)k} \tag{5}$$

$$n_k = \frac{(k-1)(1-\rho)}{1+(1-\rho)k}n_{k-1} \tag{6}$$

$$= \left[\frac{(k-1)(1-\rho)}{1+(1-\rho)k}\right]\left[\frac{(k-2)(1-\rho)}{1+(1-\rho)(k-2)}\right]n_{k-2} \tag{7}$$

$$= \left[\frac{(k-1)(1-\rho)}{1+(1-\rho)k}\right]\left[\frac{(k-2)(1-\rho)}{1+(1-\rho)(k-2)}\right]...n_2n_1 \tag{8}$$

$$= \left[\frac{(k-1)(1-\rho)}{1+(1-\rho)k}\right]\left[\frac{(k-2)(1-\rho)}{1+(1-\rho)(k-1)}\right]\cdots\left[\frac{(2-1)(1-\rho)}{1+(1-\rho)(2)}\right]n_1 \tag{9}$$

$$= \frac{(1-\rho)^{k-1}}{(1-\rho)^{k-1}}\left[\frac{(k-1)}{\frac{1}{1-\rho}+k}\right]\left[\frac{(k-2)}{\frac{1}{1-\rho}+(k-1)}\right]\cdots\left[\frac{(2-1)}{\frac{1}{1-\rho}+(2)}\right]n_1 \tag{10}$$

$$= \Gamma(k) \left[ \cfrac{1}{\frac{1}{1-\rho} + k} \right] \left[ \cfrac{1}{\frac{1}{1-\rho} + (k-1)} \right] \cdots \left[ \cfrac{1}{\frac{1}{1-\rho} + (2)} \right] n_1 \tag{11}$$

$$n_k = \frac{\Gamma(k)\,\Gamma(1-\rho+1)}{\Gamma(1-\rho+1+k)} n_1 \tag{12}$$

$$n_k = \frac{\Gamma(k)\,\Gamma(\rho)}{\Gamma(\rho+k)} n_1 \tag{13}$$

$$n_k = \beta(k,\rho)\, n_1 \tag{14}$$

## 3   Problem 3

From the lectures we arrived at the following,

$$\gamma = 1 + \frac{1}{1-\rho} \tag{15}$$

For $\rho \to 0$,
$$\gamma = \lim_{\rho \to 0} 1 + \frac{1}{1-\rho} = 2 \tag{16}$$

For $\rho \to 1$,
$$\gamma = \lim_{\rho \to 1} 1 + \frac{1}{1-\rho} = \infty \tag{17}$$

For the case of $\gamma \to 2$, there is no mutation rate and the entire population is in the same grounp, this would cause the slope of the distribution to become infinite because there would be nothing to scale against. In the case where $\gamma \to \infty$, we are guarenteed to make a new group every time. This would cause the frequency of all the groups to be one, causing the power law slope to go to zero because each group occurs as often as every other one.

## 4   Problem 4

$$n_1 = \frac{\rho}{2-\rho} \tag{18}$$

$$n_k^{(g)} = \frac{1}{\rho t} N_{kt} = \frac{1}{\rho} n_k \tag{19}$$

4

Solving for $n_2^{(g)}$,

$$\frac{n_k}{n_{k-1}} = \frac{(k-1)(1-\rho)}{1 + (1-\rho)k} \tag{20}$$

$$n_k = \frac{(k-1)(1-\rho)}{1 + (1-\rho)k} n_{k-1} \tag{21}$$

$$n_2 = \frac{(2-1)(1-\rho)}{1 + (1-\rho)2} n_1 \tag{22}$$

$$n_2 = \frac{(1-\rho)}{1 + 2(1-\rho)} \left[\frac{\rho}{2-\rho}\right] \tag{23}$$

$$n_2^{(g)} = \frac{1}{\rho} n_2 = \frac{1}{\rho} \left[\frac{1-\rho}{3-2\rho}\right] \left[\frac{\rho}{2-\rho}\right] \tag{24}$$

$$n_2^{(g)} = \frac{(1-\rho)}{(2-\rho)(3-2\rho)} \tag{25}$$

Now for $n_3^{(g)}$,

$$n_k = \frac{(k-1)(1-\rho)}{1 + (1-\rho)k} n_{k-1} \tag{26}$$

$$n_3 = \frac{(3-1)(1-\rho)}{1 + (1-\rho)3} n_2 \tag{27}$$

$$n_3 = \frac{2(1-\rho)}{4-3\rho} \left[\frac{(1-\rho)}{3-2\rho}\right] \left[\frac{\rho}{2-\rho}\right] \tag{28}$$

$$n_3^{(g)} = \frac{1}{\rho t} N_{3t} = \frac{1}{\rho} n_3 \tag{29}$$

$$n_3^{(g)} = \frac{1}{\rho} \left[\frac{2(1-\rho)}{4-3\rho}\right] \left[\frac{(1-\rho)}{3-2\rho}\right] \left[\frac{\rho}{2-\rho}\right] \tag{30}$$

$$n_3^{(g)} = \frac{2(1-\rho)^2}{(4-3\rho)(3-2\rho)(2-\rho)}$$

# 5 Problem 5

# 6 Problem 6

### Kmax, N= 10^1



### Kmax, N= 10^2



### Kmax, N= 10^3



### Kmax, N= 10^4



### Kmax, N= 10^5



### Kmax, N= 10^6

Average Kmax for n=1000 sample sets