Zombie Model

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Abstract

Imagine a biochemical laboratory explodes and leaks out a zombie virus. We will be facing an inevitable zombie outbreak in the short future and how do we know whether we, human beings, are going to survive? In this project, we apply continuous mathematical modeling skills to analyze the population dynamics of human and zombie – which specie can survive in the end, which specie will extinct or will there be a coexisting situation? This project utilizes both the population model and the infectious disease model separately to simulate the interaction between zombies and human. We will discuss the effect of logistic growth, birth rate, death rate and combat conditions on our model. Given particular parameters, our model can be utilized to analyze the effect of zombie attacks in specific backgrounds such as the TV series Walking Dead.

1 Introduction

In this project, we simulate the interaction between human and zombie population. The idea of this problem is inspired by the original zombie problem raised in Munz's paper "When Zombie Attack!: Mathematical Modelling of an Outbreak of Zombie Infection"[1]. We modified the model by introducing killing rate, natural death rate and used logistic growth for human birth rate. In the first section of this report, we will construct and explain our zombie model in detail. We proceed to analyze our model using non-dimensionalization strategy. Followed by that, we numerically solve for stable points by carrying out the *Jacobian* matrices for each fixed points. After that, three conditions are specifically discussed in the computational results section. Finally we will finish up this report with conclusions and future development.

2 Model

2.1 Variables

Table 1: Variables

Human		Unit	Zombie		Unit
Human Population	H	N	Zombie Population	Z	N
birth rate	a	1/T	birth rate (human infection)	α	1/TN
natural death rate	b	1/T	death rate (human killing)	β	1/TN
death rate by zombie killing	δ	1/TN	natural death rate	c	1/T
death rate by zombie infection	α	1/TN			
carrying capacity	k	N			

2.2 Assumption

We base our model on the following assumptions:

- Human population grows and dies at constant rates
- There is environmental pressure on the growth of human population

- Zombies don't reproduce naturally, rather zombie's population will grow by infecting human
- Zombie will die out at a constant rate, so zombie is not immortal. This constant rate is introduced based on accidents or starvation.
- The rates at which human kills zombie, zombie kills human and zombie infects human are all constant

2.3 Equations

$$\frac{dH}{dt} = aH(1 - \frac{H}{k}) - bH - \alpha HZ - \sigma HZ \tag{1}$$

$$\frac{dZ}{dt} = \alpha HZ - \beta HZ - cZ \tag{2}$$

$aH(1 - \frac{H}{k})$: human birth; bH: natural death; αHZ: infectious;
σHZ: zombie kill human; βHZ: human kill zombie; cZ: natural death.

3 Non-dimensionalizion

We define three dimensionless variables as follows

$$\begin{aligned} h &= H/k \\ z &= Z/k \\ \tau &= a \cdot t \end{aligned}$$

Then we substitute the above three definitions into equation (1) and (2)

$$\begin{aligned} dH/dt &= d(h \cdot k)/d(\tau/a) \\ &= dh/d\tau \cdot (k \cdot a) \\ &= a \cdot h \cdot k(1-h) - b \cdot h \cdot k - \alpha \cdot h \cdot z \cdot k^2 - \delta \cdot h \cdot z \cdot k^2 \\ dZ/dt &= d(z \cdot k)/d(\tau/a) \\ &= dz/d\tau \cdot (k \cdot a) \\ &= \alpha \cdot h \cdot z \cdot k^2 - \beta \cdot h \cdot z \cdot k^2 - c \cdot z \cdot k \end{aligned}$$

After simplification, we got

$$dh/d\tau = h(1-h) - b \cdot h/a - \alpha \cdot k/a \cdot h \cdot z - \delta \cdot k/a \cdot h \cdot z \tag{3}$$

$$dz/d\tau = \alpha \cdot k/a \cdot h \cdot z - \beta \cdot k/a \cdot h \cdot z - c/a \cdot z \tag{4}$$

For simplicity we define the four variables below

$$\begin{aligned}\gamma &= b/a \\ \mu &= \alpha \cdot k/a \\ \nu &= \delta \cdot k/a \\ \epsilon &= \beta \cdot k/a \\ \omega &= c/a\end{aligned}$$

The our dimensionless equation becomes

$$dh/d\tau = h(1-h) - \gamma h - \mu \cdot h \cdot z - \nu \cdot h \cdot z \tag{5}$$

$$dz/d\tau = \mu \cdot h \cdot z - \epsilon \cdot h \cdot z - \omega \cdot z \tag{6}$$

3.1 Meaning of the dimensionless variables

k: Let k be the unit for the population size of H and Z.
a: Let a-1 be the unit for time.
h: Non-dimensionalized population of human.
z: Non-dimensionalized population of zombie.
τ: Non-dimensionalized time.
γ: ratio between human natural death rate and birth rate.
μ: infectious rate per capita per unit time.
ν: zombie killing human rate per capita per unit time.
ϵ: human killing zombie rate per capita per unit time.
ω: ratio between zombie death rate and human natural birth rate.

4 Fixed Points Condition

The following part shows all the fixed points and their corresponding Jacobian matrices as well as eigenvalues:

4.1 Fixed Point 1

$h = 0$, $z = 0$

Jacobian Matrix:

$$J = \begin{bmatrix} -\gamma + 1 & 0 \\ 0 & -\omega \end{bmatrix}$$

Eigenvalues:

$$\lambda_1 = -\gamma + 1$$
$$\lambda_2 = -\omega$$

When $\gamma > 1$, $\lambda_1, \lambda_2 < 0$. $(0, 0)$ is a stable node.

4.2 Fixed Point 2

$h = 1 - \gamma$, $z = 0$

Jacobian Matrix:

$$J = \begin{bmatrix} \gamma - 1 & (\gamma - 1)(\mu + \nu) \\ 0 & (1 - \gamma)(\mu - \epsilon) - \omega \end{bmatrix}$$

Eigenvalues:

$$\lambda_1 = \gamma - 1$$
$$\lambda_2 = (1 - \gamma)(\mu - \epsilon) - \omega$$

When $\gamma < 1$ and $\mu < \epsilon, \lambda_1, \lambda_2 < 0$. $(1 - \gamma, 0)$ is a stable node.
When $\gamma < 1$, and $\mu > \epsilon$ and $(1 - \gamma)(\mu - \epsilon) < \omega$,$\lambda_1, \lambda_2 < 0$. $(1 - \gamma, 0)$ is a stable node.

4.3 Fixed Point 3

$h = -\frac{\omega}{\epsilon - \mu}$, $z = \frac{\mu(\gamma - 1) - \epsilon\gamma + \omega + \epsilon}{(\epsilon - \mu)(\mu + \nu)}$

Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\omega}{\epsilon-\mu} & \frac{\omega}{\epsilon-\mu}(\mu+\nu) \\ -\frac{\mu(\gamma-1)-\epsilon\gamma+\omega+\epsilon}{\mu+\nu} & 0 \end{bmatrix}$$

Eigenvalues:

$$\lambda_1 = \frac{\omega + \sqrt{\omega^2 - 4(\mu - \epsilon)(-\gamma\mu\omega + \gamma\epsilon\omega + \mu\omega - \epsilon\omega - \omega^2)}}{2(\mu - \nu)}$$

$$\lambda 2 = \frac{\omega - \sqrt{\omega^2 - 4(\mu - \epsilon)(-\gamma\mu\omega + \gamma\epsilon\omega + \mu\omega - \epsilon\omega - \omega^2)}}{2(\mu - \nu)}$$

The eigenvalues for this fixed point are too complicated to analyze, hence its $Trace$ and $Determinant$ are used to determine stability:

$$Trace = \frac{\omega}{\epsilon - \mu}$$

$$Determinant = \frac{\omega((\gamma - 1)(\mu + \epsilon) - \omega)}{\epsilon - \mu}$$

When $\epsilon < \mu$ and $\gamma > 1$, $Tr(J) < 0$ and $Det(J) > 0$. However $z < 0$ under this condition, which indicates non-existence.
When $\epsilon < \mu$, $\gamma < 1$ and $(\gamma - 1)(\mu - \epsilon) < -\omega$, $Tr(J) < 0$, $Det(J) > 0$ and $z > 0$. Under the above conditions, the fixed point is a stable spiral.

4.4 Overall Analysis of Fixed Points

- When $\gamma < 1$, which means that Human natural death rate < Human birth rate, Fixed point 1 is always unstable, which indicates that either human or zombie will survive.

  - If $\epsilon > \mu$, which means Human killing zombie rate > Infectious rate:

    - * Fixed point 2 is stable, which indicates human survival;
      * Fixed point 3 is non-existent because $h < 0$ and $z < 0$;
  - If $\epsilon < \mu$, which means Human killing zombie rate < Infectious rate:
    - * When $(1-\gamma)(\mu-\epsilon) < \omega$,
        - · Fixed point 2 is stable;
        - · Fixed point 3 does not exist.
    - * When $(\gamma-1)(\mu-\epsilon) < -\omega$, i.e. $(1-\gamma)(\mu-\epsilon) > \omega$,
        - · Fixed point 3 exists and it is a stable spiral. Under this condition, human and zombies will co-exist and the populations will stay unchanged at the equilibrium.
        - · Fixed Point 2 is unstable.

  Note that there is a flip point at which the stability of fixed point 2 and fixed point 3 exchange. The physical meaning of the equation $\frac{(1-\gamma)(\mu-\epsilon)}{\omega}$ will be elaborated in the next section.

- When $\gamma > 1$, which means Human natural death rate > Birth rate. Under this condition:
  - Fixed point 1 is always stable, zombie and human will both die out;
  - Fixed point 2 is always unstable;
  - Fixed point 3 is non-existent.

4.5 Explanation of the physical meaning of the fixed points

For sake of simplicity, we represents a fixed point $(h*, z*)$ as an ordered pair of numbers (a, b), which means $h* = a$ and $z* = b$

- Fixed point 1: $(0, 0)$ indicates there is no dynamics between human and zombie because both population is zero. So then their population will always stay zero. It is stable if and only if human natural death rate is greater than human birth rate. This makes sense because if human birth rate exceeds natural death rate, then at (0, 0), the $\frac{dh}{dt} > 0$, so human population would show an logistic growth rather than die out.

- Fixed point 2: (1- γ, 0). The prerequisite of this fixed point being stable is 1) human birth rate is greater than human natural death rate because otherwise human can't have survived even with no zombies. And 2) from previous analysis, we know the condition is $(1-\gamma)(\mu-\epsilon) < \omega$, then we substitute our original variables with dimensions into this inequality

$$(1 - \frac{b}{a}) \cdot (\frac{\alpha \cdot k}{a} - \frac{\beta \cdot k}{a}) < \frac{c}{a}$$
$$(a - b) \cdot (\alpha - \beta) < \frac{c}{k}$$

  One possible view of this inequality is to fix c and k, then we know the difference between human natual birth and death rate is inversely proportional to the difference between zombie birth rate and death rate. This means if either human or zombie grows too fast, then human will not eventually prevail. This makes sense because 1) zombie feeds on human, so more zombie is obviously bad for human's eventual prevail and 2) more human leads to more zombie which then lead to the eventual extinction of human. That is to say, for human to prevail eventually, first human must survive in the environment (natural birth rate > natural death rate) and second needs to either

  - kill zombie faster than zombie infects human
  - or not reproduce too quickly to let zombies reproduce enough to eventually kill or infect all human

- Fixed point 3 being stable requires a more subtle condition. Using an analysis similar to the analysis we did to fixed point 2, we know that human and zombie population must be kept within a nice range to make neither go to distinction. In this situation human will reproduce but not enough to kill all zombies, and zombie will reproduce by infecting human and their population will eventually become stable and non-zero, which is the so-called coexist state.

5 Computational Results

Condition 1: Human Survival

Population Diagram (Condition 1)

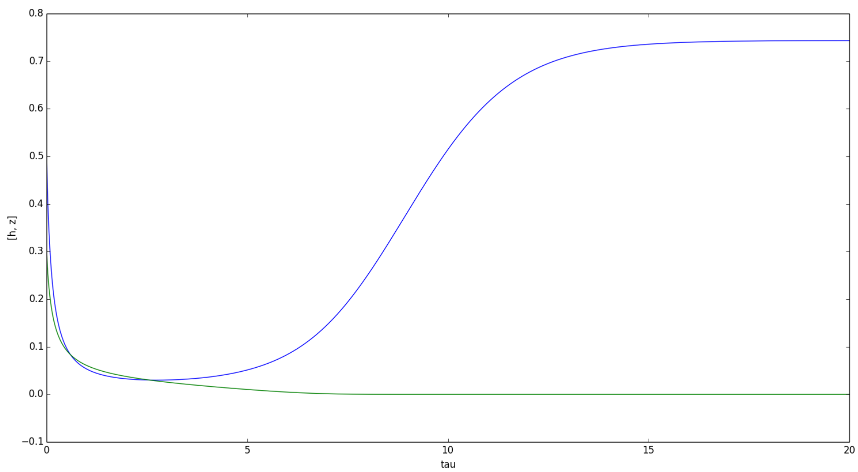

Figure 1. The population ratio with respect to the carrying capacity of human (blue) and zombie (green). Zombie population converges to 0 as t $\to \infty$.

Assumptions for this model:

(h_0, z_0): Initial Population of human and zombie are (450, 270);
(a, b): Human birth rate and natural death rate are (0.39, 0.1);
(α, σ): Zombie infectious and killing human rate are (0.01, 0.001);
(β, c): Zombie natural death rate and human killing zombie rate are (0.015, 0.01).

We used python to simulate our model. Given initial conditions and bounding equations, we obtain the computational result and plot the population flow. We can see from the model outcome in figure 1 that zombie die faster than human do and they eventually extinct before human's population drop to 0. Human population starts to increase after zombie's extinction and finally converges to a stable number.

Based on the previous assumptions, we can see that $b < a$ and therefore $\gamma < 1$. Also since $\alpha < \beta$, we have $\mu < \epsilon$. From our analysis of the fixed point in section 3.4, this condition corresponds to the case when fixed point 1 and 3 are unstable while fixed point 2 is stable. Figrue 2 is the phase plot for the solution of this model. We can see that the phase line converges to a point on the x axis. This indicates that the final condition is human survive with a stable population and zombies extinct.

We have shown that our calculation result corresponds to our simulation outcome.

Phase Plot (Condition 1)

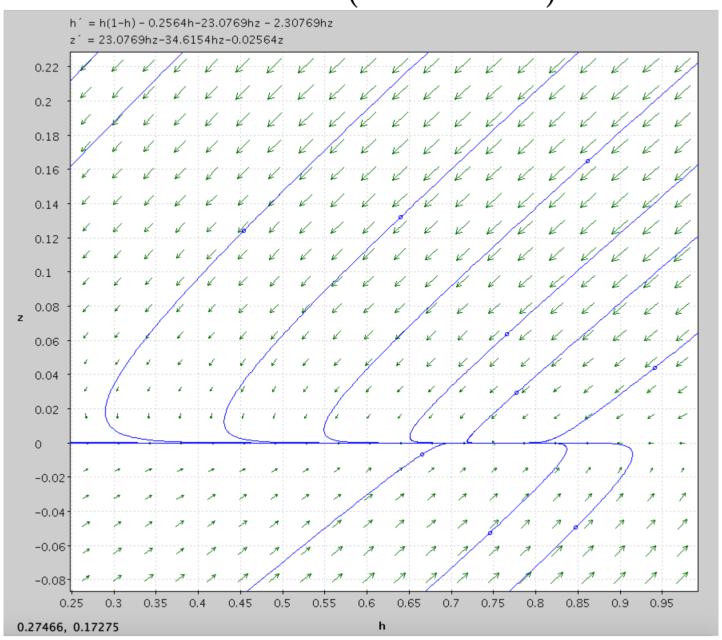

Figure 2. The phase plot for the solution. X axis is the population of human, Y axis is the population of zombies. Lines converge to approximately (0.74, 0).

Condition 2: Co-Existence

Population Diagram (Condition 2)

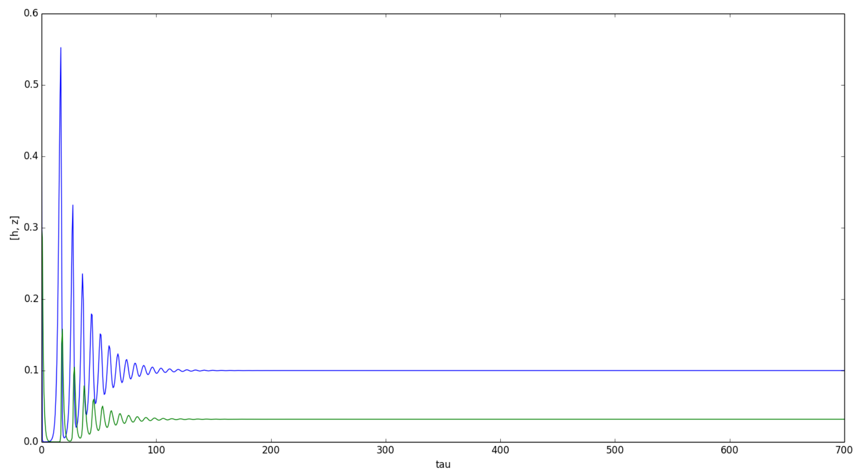

Figure 3. The population ratio with respect to the carrying capacity of human (blue) and zombie (green). Zombie and human population both converge to a stable equilibrium.

Assumptions for this model:

(h_0, z_0): Initial Population of human and zombie are (450, 270);
(a, b): Human birth rate and natural death rate are (0.5, 0.1);
(α, σ): Zombie infectious and killing human rate are (0.01, 0.001);
(β, c): Zombie natural death rate and human killing zombie rate are (0.005, 0.5).

Compared to the last model, we raised the human birth rate and human killing zombie rate, so basically humans become more powerful. At the mean time, we decreased zombie's natural death rate so that they do not die out too fast. We can see from the model outcome in figure 3 that after the initial shifting, the populations finally converged to a equilibrium where human out numbers zombies.

From calculation, we find that $b < a$ and therefore $\gamma < 1$. Also since $\alpha > \beta$, we have $\mu > \epsilon$ and $(1 - \gamma)(\mu - \epsilon) > \omega$. From our analysis of the fixed point in section 3.4, this condition corresponds to the case when fixed point 1 and 2 are unstable while fixed point 3 is a stable spiral. Figrue 4 is the phase plot for the solution of this model. We can see that the phase line converges to a point where both z and h are non-zero. This indicates that after dramatic changes, both population will converge to a co-existence equilibrium.

Phase Plot (Condition 2)

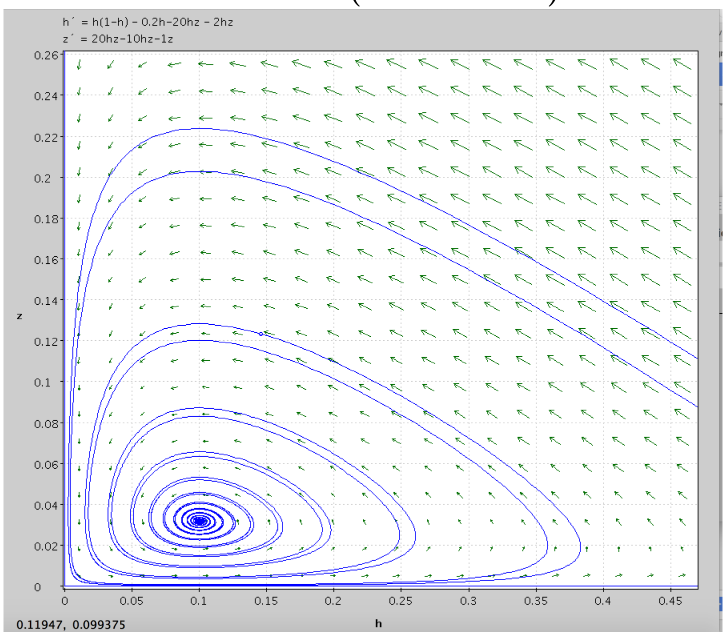

Figure 4. The phase plot for the solution. Fixed point is stable, phase lines are spirals and converge to non-zero equilibrium.

Condition 3: Game Over

Population Diagram (Condition 3)

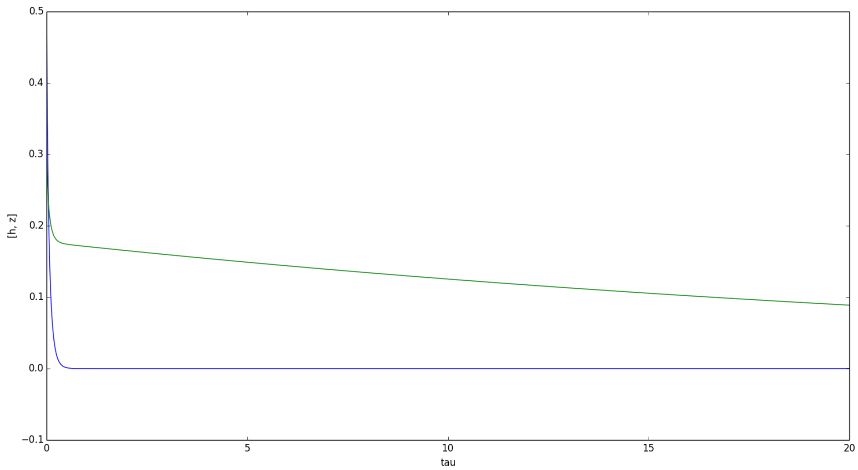

Figure 5. The population ratio with respect to the carrying capacity of human (blue) and zombie (green). Zombie dies out after human do.

Assumptions for this model:

(h_0, z_0): Initial Population of human and zombie are (450, 270);
(a, b): Human birth rate and natural death rate are (0.29, 0.3);
(α, σ): Zombie infectious and killing human rate are (0.01, 0.01);
(β, c): Zombie natural death rate and human killing zombie rate are (0.015, 0.01).

Compared to the first model, we raised human's natural death rate and zombie infectious rate. Generally speaking, humans are weaker while zombies become more powerful. We can see from the model outcome in figure 5 that human population drop rapidly and zombie population decreases gradually.

For calculation, first we can see that $b > a$ and therefore $\gamma > 1$. From our analysis of the fixed point in section 3.4, this condition corresponds to the case when fixed point 1 is the only stable node. Figrue 6 is the phase plot for the solution of this model. We can see that no matter what the initial conditions are, the phase lines all converge to (0, 0). This means that either human or zombie die first, then the other population follows the pattern.

This model is pretty straight forward to interpret. Human death rate is lower than human birth rate, so with or without the influence of zombie, human will extinct. If human extinct before zombie do, there will be no human to infect so no birth rate for zombie. However since zombie has a natural death rate due to accidents, zombies will always die out after human do. Note, when the phase line approaches the horizontal axis, it curves because human follows logistic growth. When the phase line approaches the vertical axis, it doesn't curve because zombie dies at a constant rate. This is also why there is no condition when zombies survive without human. We have another set of initial conditions which will result in the condition that zombie's population becomes 0, then followed by human population decreasing to 0. Code for this condition can be found in the Appendix.

Phase Plot (Condition 3)

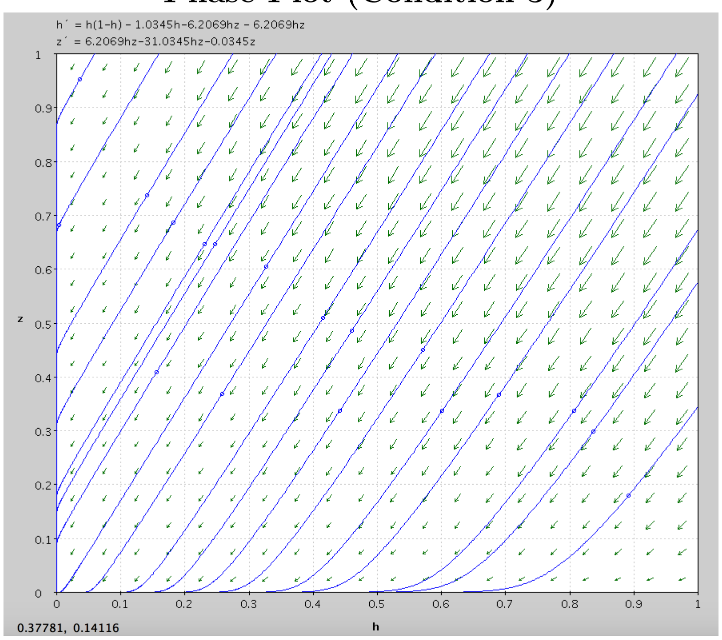

Figure 6. The phase plot for the solution. X axis is the population of human, Y axis is the population of zombies. All lines converge to (0, 0).

Stalemate

Under this condition, it is possible that both human and zombie die out at the same time. Using iterations of population pairs and an tolerance range of $8 \cdot e^{-4}$, we find out that when the initial populations of human and zombie are approximately 4 : 1, our model achieves stalemate. One possible initial condition is $(h_0, z_0) = (0.59, 0.14)$. Figure 7 shows that human population has greater magnitude, as well as decrease rate than zombie population. Both population converge to 0 at the same time unit: $\tau \approx 27.427$. Back in figure 6, the corresponding phase line which goes through (0.59, 0.14) experiences a exponential decay as it converges to the original point.

Population Diagram - Stalemate

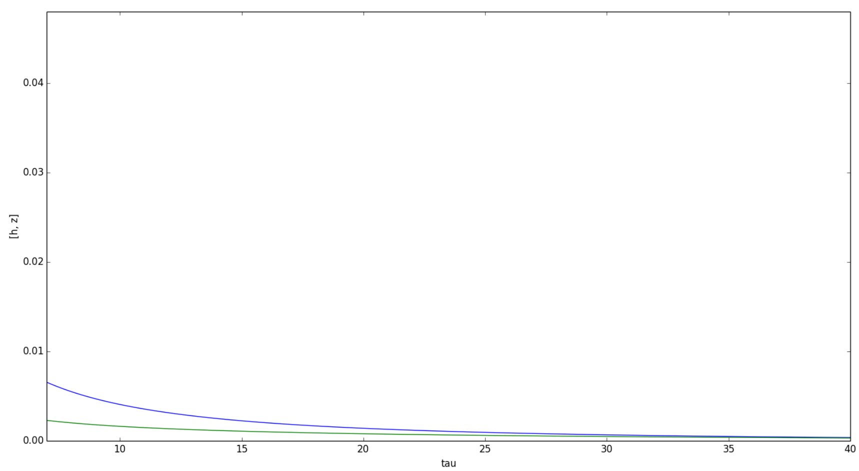

Figure 7. Population of human (blue) and zombie (green). Both population decrease and converge to 0 at approximately 27.43.

6 Conclusions and Future Development

Conclusion

- When $\gamma < 1$ and $\mu < \epsilon$, Fixed point 1 and 3 are unstable while Fixed point 2 is stable, indicating that human will survive with stable population while zombie will die out.

- When $\gamma < 1$ and $\mu > \epsilon$, Fixed point 1 and 2 are unstable while Fixed point 3 is a stable spiral, indicating that both population of human and zombies will converge to a equilibrium for co-existence.

- When $\gamma > 1$, Fixed point 1 is the only stable node and all the phase lines converge to (0, 0), indicating that the population of either human or zombies will converge to zero first then the population of the other one will also converge to zero in the end.

Thus if our human beings want to survive successfully under situation of "Zombie Virus" outbreak, we need to ensure that human natural birth rate is greater than human natural death rate ($a > b$). And we also expect that human beings are able to kill zombies at a rate faster than that of zombies infect/kill human or human beings are able to prevent reproducing too quickly since zombie may get a potential chance to thrive then infect and kill all our human beings. Although the human still get an opportunity to survive, requirements for survival are tough to achieve and we always hope that such Virus Outbreak will never occur in future.

Discussion and Future Development

In our project, we build a fairly realistic zombie model which includes human birth rate and zombie natural death rate. After computation, we have three reasonable possible outcomes in this zombie world. However, there are some more factors we can include to improve our model.

First, including baby factor will make the model even more realistic. In our model, all three possible results only depend on human birth rate, human natural death rate, infectious rate, which is equivalent to zombie birth rate, and zombie natural death rate. It is surprised to notice that all outcomes are not influenced by initial condition. This tells us that even there is only ten human survivors and they are facing ten million zombies, human beings still have chance to win under certain conditions. One reason this may happen is that we do not consider baby factor. In our model, we assume all new born babies have the same killing power and fertility as adults so that as long as human beings have high birth rate, human beings will not die out. Since we all know this will not happen in reality, adding a baby factor into the model will make our solution more practical.

Moreover, adding weapon factor will make the model more interesting. Zombie death rate, which is also human beings' killing power, is an important element in our model. As we can imagine, if human beings have access to more advanced weapon, such as nuclear weapon, people can easily win this battle. Thus, if we divide human into different groups of people with different weapon and change the initial condition of human population into the populations of new divided groups of human, the initial condition will be very important in this new version of model.

Considering the possibility of mutation among zombies may introduce another outcome. One interesting thing about our results is that zombies can never truly survive on this planet even all human beings die out. The reason is that we assume zombies have low intelligence so that they cannot prevent accidents or avoid natural disasters. If we include the possibility that zombies evolve high intelligence in our model, we can have two new factors in our model. On the one hand, we have a new result that after human die out, zombies avoid their death rate and become the new master of this world. On the other hand, zombies may evolve during the war and it makes the battle becomes more complicate and more existing.

All in all, zombie outbreak is one of the most popular topics now. We have relative video games, TV series, and horror movies. It is even a realistic topic because of some biochemical weapon experiments. In our project, we analyze one possible zombie model and its outcomes, but as we discussed above, a lot of variations may be applied on our model and make it more realistic and more interesting.

References

[1] Munz P, Hudea I, Imad J, Smith RJ. *When Zombies Attack! - Mathematical Modelling of an Outbreak of Zombie Infection*. In Infectious Disease Modeling Research Progress, edited by J.M. Tchuenche and C. Chiyaka. Hauppage, NY: Nova Science Publishers. 2009.

[2] Thomas E. Woolley, Ruth E. Baker, Eamonn A. Gaffney, Philip K. Maini. *How Long Can We Survive*. University of Ottawa Press. 2014.

Appendix: Algorithm codes

Solve differential equations for $\gamma < 1$

```
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint

y0 = [0.5, 0.3] # the first entry is h and second entry is z
tau = np.linspace(0, 70, num=1000)

# the original params
a = 0.5 # human birth rate
b = 0.1 # human natural death rate
delta = 0.001 # human death rate caused by zombie killing
alpha = 0.01 # human death rate caused by zombie infection
k = 100.0 # human carrying capacity
beta = 0.005 # zombie death rate caused by human killing
c = 0.5 # zombie natural death rate

# dimensionless variables
gamma = b / a # 0.2
mu = alpha * k / a # 2
nu = delta * k / a # 0.2
epsilon = beta * k / a # 1
omega = c / a # 1

params = [gamma, mu, nu, epsilon, omega]
```

```
def f(y, t0, args):
    h = y[0]
    z = y[1]
    gamma = args[0]
    mu = args[1]
    nu = args[2]
    epsilon = args[3]
    omega = args[4]
    return [h * (1 - h) - gamma * h - mu * h * z - nu * h
    * z, mu * h * z - epsilon * h * z - omega * z]


y = odeint(f, y0, tau, args=(params,))
line = plt.plot(tau, y)
plt.xlabel("tau")
plt.ylabel("[h, z]")
plt.show()
```

Search for stalemate condition

```
import math
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint
import itertools

ep = 8e-4
y00 = np.linspace(0, 1, num=150)
y01 = np.linspace(0, 1, num=150)
allys = [] # record all ys such that stalemate happens

tau = np.linspace(0, 200, num=1000)

# the original params
a = 0.29 # human birth rate
b = 0.3 # human natural death rate
delta = 0.01 # human death rate caused by zombie killing
```

```
alpha = 0.01 # human death rate caused by zombie infection
k = 900.0 # human carrying capacity
beta = 0.015 # zombie death rate caused by human killing
c = 0.01 # zombie natural death rate

# dimensionless variables
gamma = b / a
mu = alpha * k / a
nu = delta * k / a
epsilon = beta * k / a
omega = c / a

params = [gamma, mu, nu, epsilon, omega]

def f(y, t0, args):
    h = y[0]
    z = y[1]
    gamma = args[0]
    mu = args[1]
    nu = args[2]
    epsilon = args[3]
    omega = args[4]
    return [h * (1 - h) - gamma * h - mu * h * z - nu * h
    * z, mu * h * z - epsilon * h * z - omega * z]

def allZero(y):
    return cmpz(y[0], 0, ep) and cmpz(y[1], 0, ep)

def cmpz(num1, num2, ep):
    return abs(num1 - num2) < ep

for y00, y01 in itertools.product(y00, y01):
    y0 = [y00, y01] # the first entry is h and second entry is z


    y = odeint(f, y0, tau, args=(params,))
    for i in range(len(y)):
        ysol = y[i]
```

```
        if (i != 0) and cmpz(ysol[0], ysol[1], ep) and cmpz(ysol[1], 0, ep):
            allys.append([y00, y01])
            print "stalemate is h={}, z={}".format(y00, y01)
            break
        elif cmpz(ysol[0], 0, ep) or cmpz(ysol[1], 0, ep):
            break

y = odeint(f, [allys[len(allys)/2][0],
allys[len(allys)/2][1]], tau, args=(params,)) # choose one stalemate to plot
# find first point where h and z are both zero
stalemateIdx = [x for x in y if allZero(x)][0]

# find first tau when h and z are both zero
for i in range(len(y)):
    if y[i][0] == stalemateIdx[0] and y[i][1] == stalemateIdx[1]:
        print "stalemate tau is {}".format(tau[i])
plt.plot(tau, y)
plt.xlabel("tau")
plt.ylabel("[h, z]")
plt.show()
```