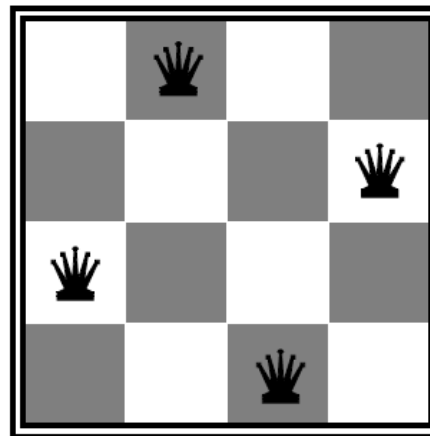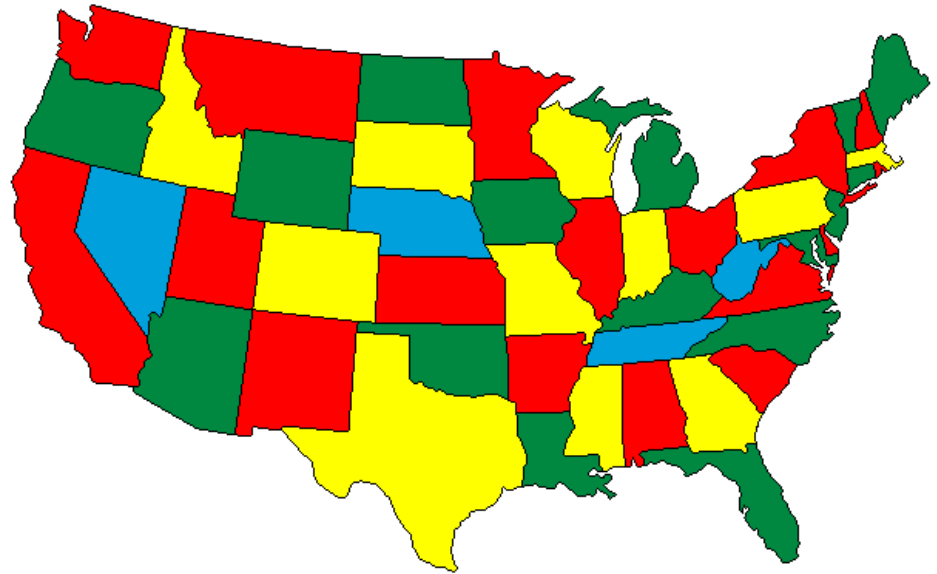# CS 5/7320
Artificial Intelligence

Constraint Satisfaction Problems
AIMA Chapter 6

Slides by Michael Hahsler
based on Slides by Svetlana Lazepnik
with figures from the AIMA textbook

Constraint satisfaction problems (CSPs)

Definition:
- **State** is defined by a set of variables X_i
  - Each variable can be unassigned (partial solution) or have a value from domain $D_{i.}$
- **Constraints** are a set of rules specifying allowable combinations of values for subsets of variables (e.g., $X_1 \neq X_7$ or $X_2 > X_9 + 3$)
- **Solution**: a state that is a
  a) Consistent assignment: satisfies all constraints
  b) Complete assignment: assigns value to each variable

How does this compare to the "generic" tree search formulation?
- A **structured representation for states**, expressed by a set of variable/value combinations and constraints represented in a formal language.
- General-purpose algorithms with more power than standard search algorithms exit.

(a) Atomic   (b) Factored   (c) Structured
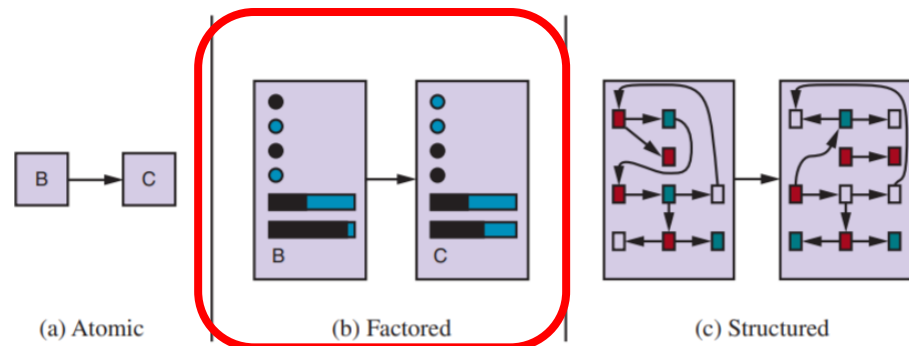
Example: Map Coloring (Graph coloring)

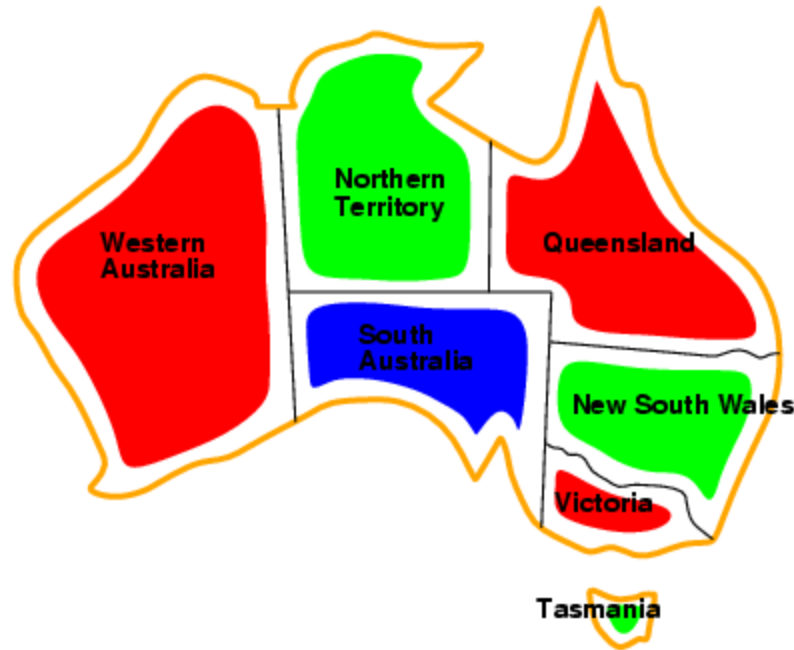Problem → Constraint graph
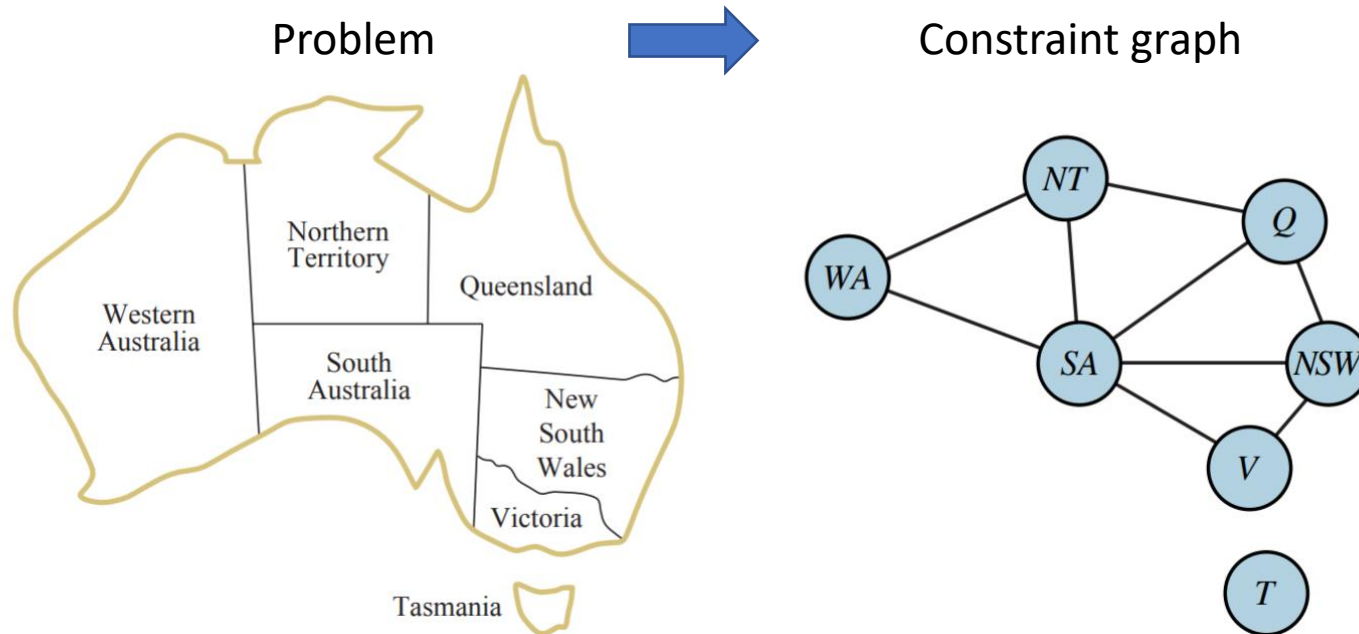
- **Variables representing state:** WA, NT, Q, NSW, V, SA, T
- **Variable Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors e.g.,

WA ≠ NT ⟺ (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

Example: Map Coloring

Solutions are *complete* and **consistent** assignments, e.g.,

WA = red, NT = green, Q = red, NSW = green,
V = red, SA = blue, T = green

Example: N-Queens

- **Variables:** X_{ij} for $i, j \in \{1, 2, \dots, N\}$

- **Domains:** $\{0, 1\}$ # Queen: no/yes

- **Constraints:**
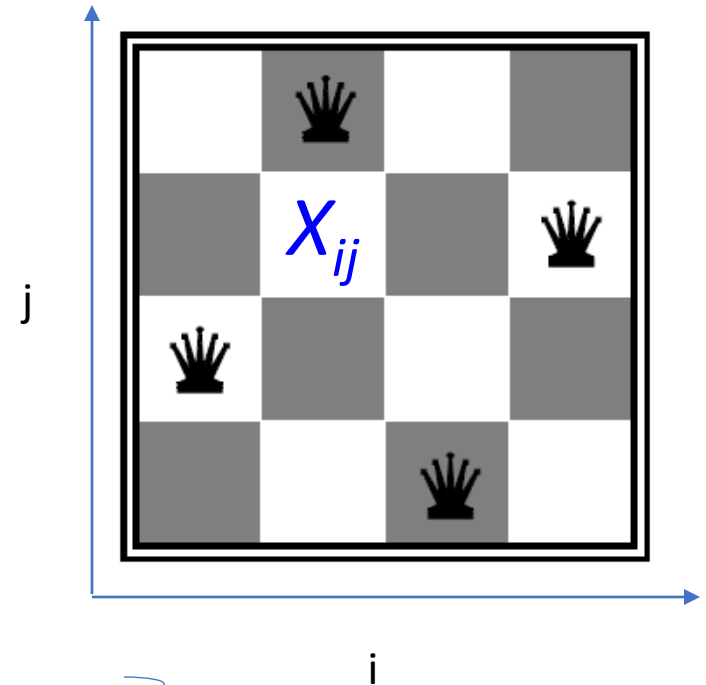
  $\Sigma_{i,j}\, X_{ij} = N$

  $(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be in same col.

  $(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be in same row.

  $(X_{ij}, X_{i+k,\, j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be diagonal
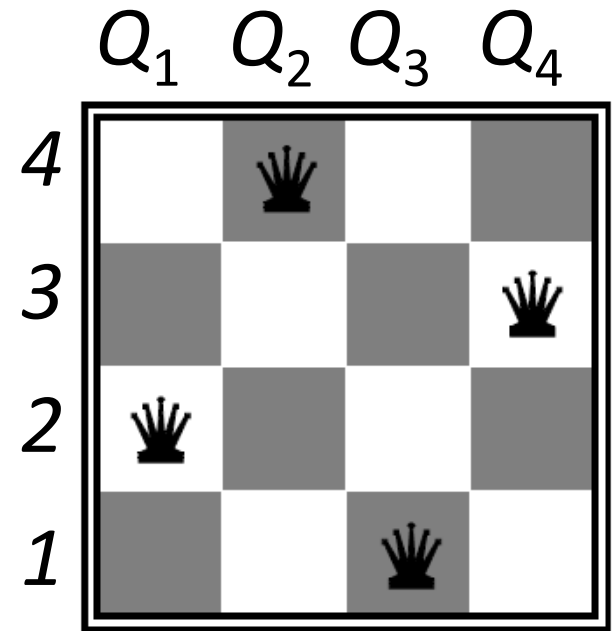
  $(X_{ij}, X_{i+k,\, j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$ # cannot be diagonal

  for $i, j, k \in \{1, 2, \dots, N\}$

N-Queens: Alternative formulation

- **Variables:** $Q_1, Q_2, \dots, Q_N$

- **Domains:** $\{1, 2, \dots, N\}$ # row for each col.

- **Constraints:**

  $\forall\ i, j$ non-threatening (Q_i, Q_j)

Example:
Q1 = 2, Q2 = 4, Q3 = 1, Q4 = 3

Example: Cryptarithmetic Puzzle

- **Variables:** T, W, O, F, U, R
  $$X_1, X_2$$

- **Domains**: {0, 1, 2, …, 9}

- **Constraints:**

  Alldiff(T, W, O, F, U, R)

  $O + O = R + 10 * X_1$

  $W + W + X_1 = U + 10 * X_2$

  $T + T + X_2 = O + 10 * F$

  $T \neq 0, F \neq 0$

Given Puzzle:
Find values for the letters.
Each letter stands for a
different digit.

$$X_2 \; X_1$$

$$
\begin{array}{cccc}
 & T & W & O \\
+ & T & W & O \\
\hline
F & O & U & R \\
\end{array}
$$

Example: Sudoku

- **Variables:** X_{ij}

- **Domains:** {1, 2, ..., 9}

- **Constraints:**

    Alldiff(X_{ij} in the same *unit)*

    Alldiff(X_{ij} in the same *row)*

    Alldiff(X_{ij} in the same *column*)

Some Popular Types of CSPs

- **Boolean Satisfiability Problem (SAT)**
  Find variable assignments that makes a Boolean expression (often expressed in conjunctive normal form) evaluate as true.

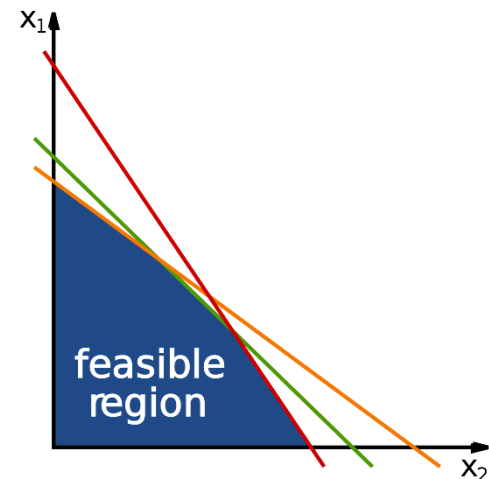$$(x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2 \lor x_3) \land \neg x_1 = \text{True}$$

- **Integer Programming**
  Variables are restricted to integers. Find a feasible solution that satisfies all constraints. The traveling salesman problem can be expressed as an integer program.

- **Linear Programming**
  Variables are continuous and constraints are linear (in)equalities.
  Find a feasible solution.

Real-world CSPs

- Assignment problems

    e.g., who teaches what class for a fixed schedule. Teacher cannot be in two classes at the same time!

- Timetable problems

    e.g., which class is offered when and where? No two classes in the same room at the same problem.

- Scheduling in transportation and production (e.g., order of production steps).

- Many problems can naturally also be formulated as CSPs.

- More examples of CSPs: http://www.csplib.org/

CSP as a Standard Search Formulation

State:
- Values assigned so far

Initial state:
- The empty assignment { } (all variables are unassigned)

Successor function:
- Choose an unassigned variable and assign it a value that does not violate any constraints
- Fail if no legal assignment is found

Goal state:
- Any complete and consistent assignment.

Backtracking search

- In CSP's, variable assignments are **commutative**
  For example,
  [WA = *red* then NT = *green*] is the same as
  [NT = *green* then WA = *red*]. → Order is not important

- We can build a search tree that assigns the value to one variable per level.
  - Tree depth n (number of variables)
  - Number of leaves: d^n (d is the number of values per variable)

- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

Example: Backtracking search (DFS)

backtracking

...

fail
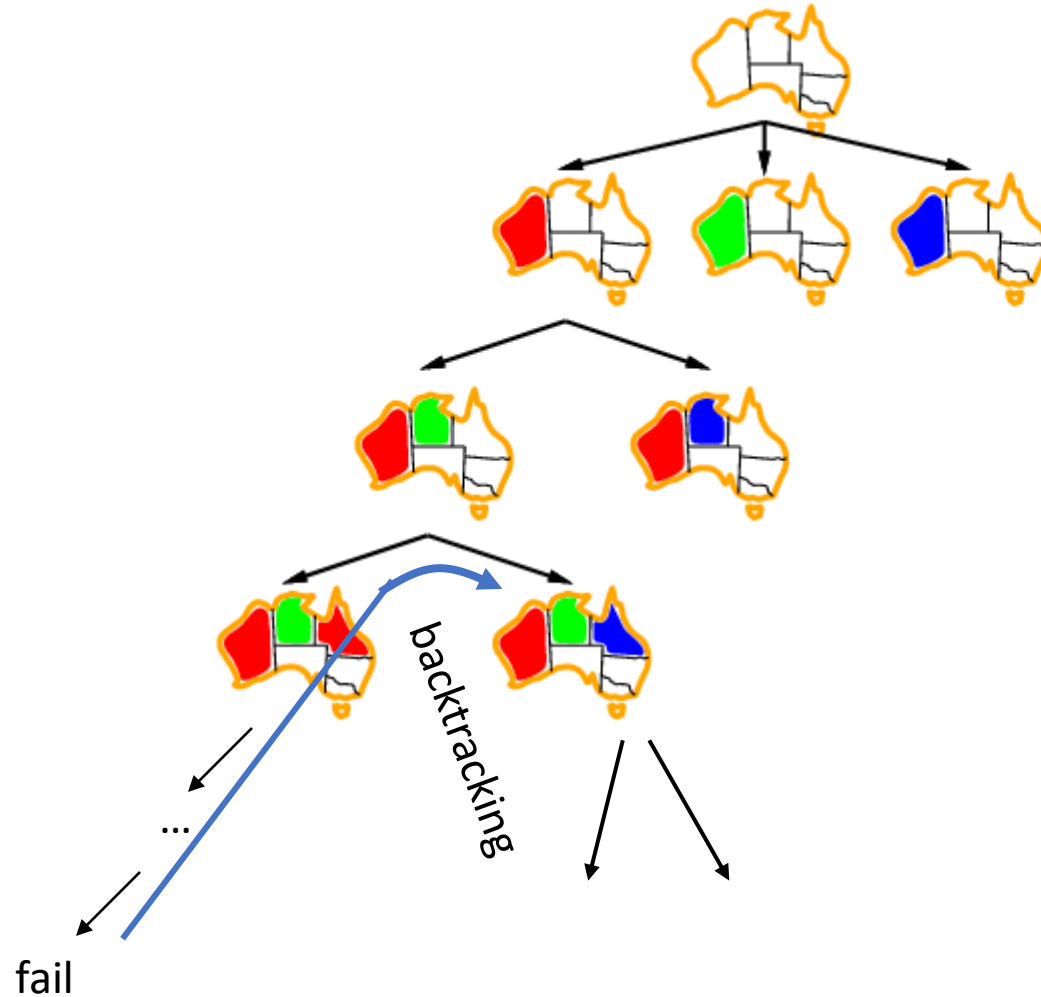
Backtracking search algorithm

```
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
        if value is consistent with assignment given CONSTRAINTS[csp]
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Call: Recursive-Backtracking({}, csp)

Improving backtracking efficiency:
- Which variable should be assigned next?
- In what order should its values be tried?
- Can we detect inevitable failure early?

Which variable should be assigned next? In which order should its values be tried?

- **Most constrained variable:**
  - Keep track of remaining legal values for unassigned variables (using constraints)
  - Choose the variable with the fewest legal values left
 - A.k.a. **minimum remaining values** (MRV) heuristic

- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables

Early detection of failure – Forward checking Node consistency

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values (i.e., minimum remaining values = 0)
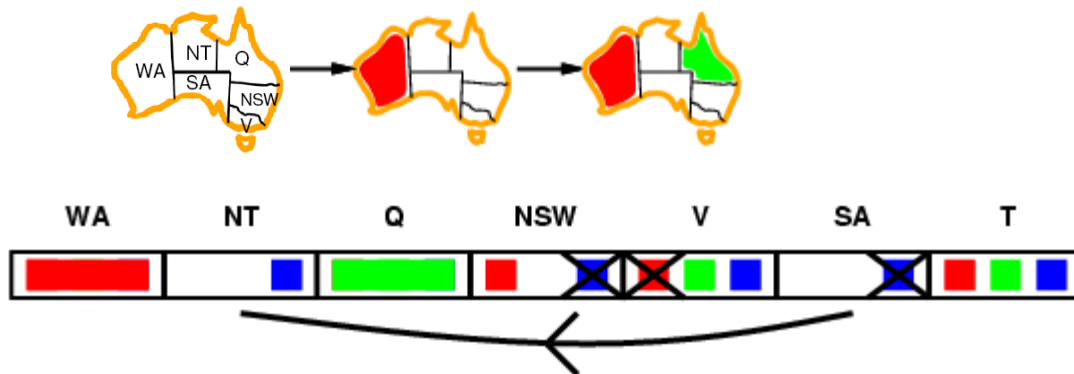

Stop and backtrack
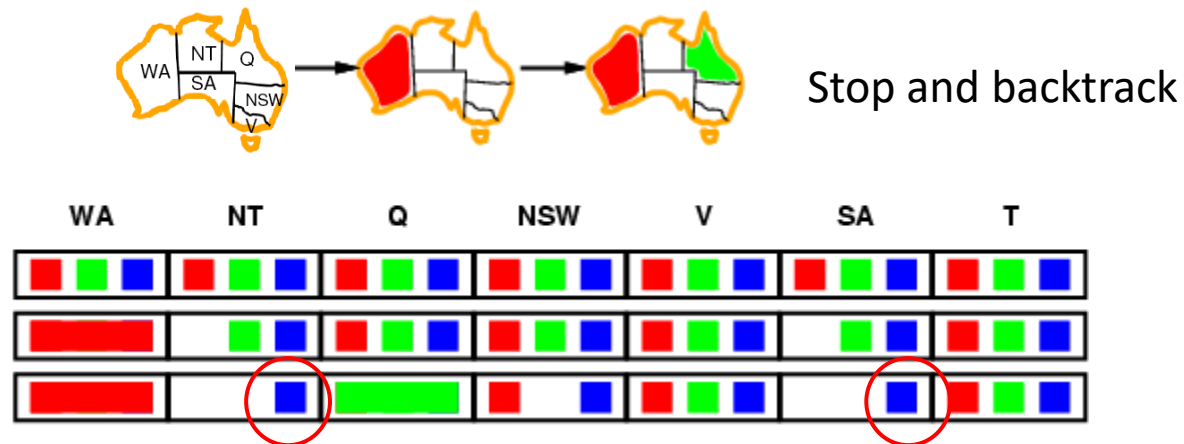
- NT and SA cannot both be blue! This violates the constraint.

Early detection of failure – Forward checking Arc consistency

- *X* is arc consistent wrt *Y* iff for every value of *X* there is some allowed value of *Y*.

- Make *X* arc consistent wrt *Y* by throwing out any values of *X* for which there is no allowed value of *Y*.

1. NWS cannot be blue because SA has to be blue.
2. V cannot be red because NSW has to be red.
3. SA cannot be blue because NT is blue.
4. Fail and backtrack

- Arc consistency detects failure earlier than node consistency

- There are more consistency checks (path consistency, K-consistency)

Backtracking search with inference

```
function RECURSIVE-BACKTRACKING(assignment, csp)
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
        if value is consistent with assignment given CONSTRAINTS[csp]
            add {var = value} to assignment
                result ← RECURSIVE-BACKTRACKING(assignment, csp)
                if result ≠ failure then return result
                remove {var = value} from assignment
    return failure
```

Call: Recursive-Backtracking({}, csp)

```
If (inference(csp, var, assignment) == failure)
        return failure
```
# Check consistency here (called "inference") and backtrack if we know that the branch will lead to failure.

Local search for CSPs

Local search is often very effective for CSPs.

Hill-climbing and simulated annealing work only with "complete" states, i.e., all variables assigned:

- Allow states with unsatisfied constraints
- Attempt to improve states by the **min-conflicts** heuristic:
    1. Select a conflicted variable and
    2. Choose a new value that produces violates the fewest constraints (local improvement step)
    3. Repeat till all constraints are met.

h = 5   h = 2   h = 0
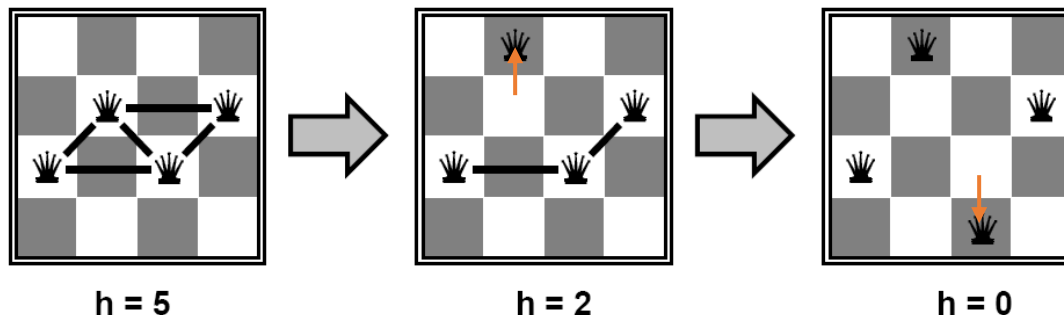
Summary

- CSPs are a special type of search problem:
  - States are **structured** and defined by a set of variables and values assignments
  - Variables can be unassigned
  - Goal test defined by
    - **Consistency** with constraints
    - **Completeness** of assignment

- **Backtracking search** = depth-first search where a successor state is generated by a consistent value assignment to a single unassigned variable
  - Starts with {} and only considers consistent assignments.
  - **Variable ordering** and **value selection** heuristics can help significantly
  - **Forward checking** prevents assignments that guarantee later failure

- Local search can be used to search the space of all complete assignments for consistent assignments = **min-conflicts heuristic.**