GATE 2023(ST) Question -55

RAMBHA SATVIK - EE22BTECH11043

EE2102 - IITH

October 11, 2023

Question

Suppose that U and V are two independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0 \\ 0 & \textit{Otherwise}, \end{cases} \tag{1}$$

where $\lambda > 0$. Which of the following statements is/are true?

1. The distribution of $U - V$ is symmetric about 0
2. The distribution of UV does not depend on λ
3. The distribution of $\frac{U}{V}$ does not depend on λ
4. The distribution of $\frac{U}{V}$ is symmetric about 1

Solution

A continuous random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, shown as $X \sim Gamma(\alpha, \lambda)$, if its PDF is given by

$$p_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & Otherwise, \end{cases} \tag{2}$$

$$E(X) = \frac{\alpha}{\lambda} \tag{3}$$

It is clear that both U and V are of Gamma distribution i.e, $U, V \sim Gamma(2, \lambda)$. Both U and V are symmetric ditributions about mean,

$$E(U) = E(V) = \frac{2}{\lambda} \tag{4}$$

Option 1

As U and V are of same distribution,

$$E(U - V) = E(U) - E(V) = 0 \tag{5}$$

Hence, the distribution of U-V is symmetric about 0.
Option 1 is correct.

Option 2

As U and V are independent and identically distributed,

$$E(U.V) = E(U) \times E(V) \tag{6}$$
$$= \frac{4}{\lambda^2} \tag{7}$$

The distribution of U.V depends on λ.
Option 2 is incorrect

Option 3 I

When $X \sim Gamma(\alpha, \lambda)$,

$$E(X) = \frac{\alpha}{\lambda} \tag{8}$$

In that case, $X^{-1} \sim inv.Gamma(\alpha, \beta)$,

$$E(X^{-1}) = \frac{\beta}{\alpha - 1} \tag{9}$$

Using (9) we can evaluate the distribution for $\frac{1}{V}$.

$$E(V^{-1}) = \frac{1}{\lambda} \tag{10}$$

Option 3 II

Using (10) and (4),

$$E\frac{U}{V} = E(U).E(V^{-1}) \tag{11}$$
$$= \frac{2}{\lambda}.\lambda \tag{12}$$
$$= 2 \tag{13}$$

$E\frac{U}{V}$ does not depend on λ.
Option 3 is correct.

Option 4

From (13), distribution of $\frac{U}{V}$ is symmetric about 2 (since it is its mean) and not 1.
Option 4 is not correct.