55.2023

Rambha Satvik*

Suppose that U and V are two independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0 \\ 0 & Otherwise, \end{cases} \tag{1}$$

where $\lambda > 0$. Which of the following statements is/are true?

1) The distribution of $U - V$ is symmetric about 0
2) The distribution of UV does not depend on λ
3) The distribution of $\frac{U}{V}$ does not depend on λ
4) The distribution of $\frac{U}{V}$ is symmetric about 1

Solution: A continuous random variable X is said to have a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, shown as $X \sim Gamma(\alpha, \lambda)$, if its PDF is given by

$$p_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & Otherwise, \end{cases} \tag{2}$$

$$E(X) = \frac{\alpha}{\lambda} \tag{3}$$

It is clear that both U and V are of Gamma distribution i.e, $U, V \sim Gamma(2, \lambda)$. Both U and V are symmetric ditributions about mean,

$$E(U) = E(V) = \frac{2}{\lambda} \tag{4}$$

1) As U and V are of same distribution,

$$E(U - V) = E(U) - E(V) = 0 \tag{5}$$

Hence, the distribution of U-V is symmetric about 0.
Option 1 is correct.
2) As U and V are independent and identically distributed,

$$E(U.V) = E(U) \times E(V) \tag{6}$$

$$= \frac{4}{\lambda^2} \tag{7}$$

The distribution of U.V depends on λ.
Option 2 is incorrect
3) When $X \sim Gamma(\alpha, \lambda)$,

$$E(X) = \frac{\alpha}{\lambda} \tag{8}$$

In that case, $X^{-1} \sim inv.Gamma(\alpha,\beta)$,

$$E(X^{-1}) = \frac{\beta}{\alpha - 1} \tag{9}$$

Using (9) we can evaluate the distribution for $\frac{1}{V}$.

$$E(V^{-1}) = \frac{1}{\lambda} \tag{10}$$

Using (10) and (4),

$$E\left(\frac{U}{V}\right) = E(U).E(V^{-1}) \tag{11}$$

$$= \frac{2}{\lambda}.\lambda \tag{12}$$

$$= 2 \tag{13}$$

$E\left(\frac{U}{V}\right)$ does not depend on λ.
Option 3 is correct.
4) From (13), distribution of $\frac{U}{V}$ is symmetric about 2 (since it is its mean) and not 1.
Option 4 is not correct.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: ee22btech11043@iith.ac.in.