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Suppose that U and V are two independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{1}$$

where $\lambda > 0$. Which of the following statements is/are true?

1) The distribution of $U - V$ is symmetric about 0
2) The distribution of UV does not depend on λ
3) The distribution of $\frac{U}{V}$ does not depend on λ
4) The distribution of $\frac{U}{V}$ is symmetric about 1

Solution: The give distribution is of the following form, with, $\alpha = 2$.

$$Gamma(\alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{2}$$

Where,

$$\Gamma(\alpha) = (\alpha - 1)! \text{ when } \alpha \in 1, 2, 3, ... \tag{3}$$

Hence, U and V are Gamma distributions. Let us find out the CDF of the U,

$$f_U(x) = \int_0^x \lambda^2 y e^{-\lambda y}\, dy \tag{4}$$

$$\implies = \lambda^2 \int_0^x y e^{-\lambda y}\, dy \tag{5}$$

$$\tag{6}$$

Substituting $t = \lambda y$, i.e. $y = \frac{t}{\lambda}$,this becomes:

$$f_U(x) = \lambda^2 \int_{0.\lambda}^{x.\lambda} \left(\frac{t}{\lambda}\right) e^{-\lambda.\left(\frac{t}{\lambda}\right)}\, d\left(\frac{t}{\lambda}\right) \tag{7}$$

$$= \int_0^{bx} t e^{-t}\, dt \tag{8}$$

$$\tag{9}$$

With the definition of the lower incomplete gamma function,

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t}\, dt \tag{10}$$

$$\tag{11}$$

we can say that,

$$f_U(x) = \gamma(2, bx) \tag{12}$$

Which upon simplification would lead to the following,

$$f_U(x) = e^{-\lambda x}(\lambda x + 1) - 1 \tag{13}$$

1) Let us consider a random variable Z such that

$$Z = U - V \tag{14}$$

$$\tag{15}$$

By the definition of Laplace transform,

$$M_Z(s) = E(e^{-sZ}) \tag{16}$$

$$= E(e^{-s(U-V)}) \tag{17}$$

$$= E(e^{-s(U)})E(e^{s(V)}) \tag{18}$$

$$= \int_0^\infty e^{-sx} \lambda^2 x e^{-\lambda x}\, dx \int_{-\infty}^0 e^{sx} \lambda^2 x e^{-\lambda x}\, dx \tag{19}$$

$$= \lambda^4 \left[\frac{x.e^{-(s+\lambda)x}}{-(s+\lambda)} - \frac{e^{-(s+\lambda)x}}{(s+\lambda)^2}\right]_0^\infty . \left[\frac{x.e^{-(\lambda-s)x}}{-(\lambda-s)} - \frac{e^{-(\lambda-s)x}}{(\lambda-s)^2} \right. \tag{20}$$

$$= \lambda^4 \frac{-1}{(s+\lambda)^2} \frac{-1}{(\lambda-s)^2} \tag{21}$$

$$= \lambda^4 \frac{1}{(s^2 - \lambda^2)^2} \tag{22}$$

$$= \frac{\lambda^3}{4}\left[\frac{1}{s(s-\lambda)^2} - \frac{1}{s(s+\lambda)^2}\right] \tag{23}$$

$$\tag{24}$$

$$\implies M_Z(s) = \frac{\lambda^3}{4}\left[\left(\frac{1}{\lambda^2 s} - \frac{1}{\lambda^2(s-\lambda)} + \frac{1}{\lambda(s-\lambda)^2}\right)\right] - \frac{\lambda^3}{4}\left[\left(\frac{1}{\lambda^2 s} - \frac{1}{\lambda^2(s+\lambda)} - \frac{1}{\lambda(s+\lambda)^2}\right)\right] \tag{25}$$

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Now, we can find the inverse lapalce of $M_Z(s)$ to obtain the pdf of the random variable Z.

$$f_Z(x) = L^{-1}(M_Z(s)) \tag{26}$$

$$\tag{27}$$

$$f_Z(x) = \frac{\lambda^3}{4}\left[\frac{1}{\lambda^2} - \frac{1}{\lambda^2}e^{\lambda x} + \frac{1}{\lambda}xe^{\lambda x}\right]u(x) - \frac{\lambda^3}{4}\left[\frac{1}{\lambda^2} - \frac{1}{\lambda^2}e^{-\lambda x} + \frac{1}{\lambda}xe^{-\lambda x}\right]u(x) \tag{28}$$

$$f_Z(x) = \left[\frac{\lambda}{4}\left(e^{-\lambda x} - e^{\lambda x}\right) + \frac{\lambda^2}{4}\left(e^{-\lambda x} + e^{\lambda x}\right)\right]u(x) \tag{29}$$

Looking at (28), we can say that distribution of $U - V$ is not symmetric about 0.

2)

$$Pr(UV < x) = E\left[Pr(U < \frac{x}{V})\right] \tag{30}$$

$$Pr(UV < x) = E\left[Pr(f_U\left(\frac{x}{V}\right))\right] \tag{31}$$

$$Pr(UV < x) = \int_{-\infty}^{x} Pr(y)f_U\left(\frac{x}{y}\right)\,dy \tag{32}$$

$$\tag{33}$$

$$Pr(UV < x) = \int_0^x \lambda^2 ye^{-\lambda y}\left(1 - e^{-\lambda\frac{x}{y}}\left(\frac{x}{y}\lambda + 1\right)\right)dy \tag{34}$$

$$Pr(UV < x) = \int_0^x \left(\lambda^2 ye^{-\lambda y} - \lambda^3 xe^{-\lambda\left(y+\frac{x}{y}\right)} + \lambda^2 ye^{-\lambda\left(y+\frac{x}{y}\right)}\right)dy \tag{35}$$

$$\tag{36}$$

3)

$$Pr(\frac{U}{V} < x) = E\left[Pr(U < x.V)\right] \tag{37}$$

$$Pr(\frac{U}{V} < x) = E\left[Pr(f_U\,(x.V)\right]) \tag{38}$$

$$Pr(\frac{U}{V} < x) = \int_{-\infty}^{x} Pr(y)f_U\,(x.y)\;dy \tag{39}$$

$$Pr(\frac{U}{V} < x) = \int_0^x \lambda^2 ye^{-\lambda y}\left(1 - e^{-\lambda xy}\,(\lambda xy + 1)\right)dy \tag{40}$$

$$Pr(\frac{U}{V} < x) = \int_0^x \left(\lambda^2 ye^{-\lambda y} - \lambda^3 xy^2e^{-\lambda y(1+x)} - \lambda^2 ye^{-\lambda y(1+x)}\right) d \tag{41}$$

$$= \lambda xe^{-\lambda x} - e^{-\lambda x} + 1 + \frac{\lambda^2 x^3 e^{-\lambda x(1+x)}}{1+x} + \tag{42}$$

$$\frac{\lambda x^2 e^{-\lambda x(1+x)}}{(1+x)^2} + \frac{xe^{-\lambda x(1+x)}}{(1+x)^3} - \lambda^3 x \tag{43}$$