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Rambha Satvik*

Suppose that U and V are two independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{1}$$

where $\lambda > 0$. Which of the following statements is/are true?

1) The distribution of $U - V$ is symmetric about 0
2) The distribution of UV does not depend on λ
3) The distribution of $\frac{U}{V}$ does not depend on λ
4) The distribution of $\frac{U}{V}$ is symmetric about 1

Solution: The give distribution is of the following form, with, $\alpha = 2$.

$$Gamma(\alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{2}$$

Where,

$$\Gamma(\alpha) = (\alpha - 1)! \text{ when } \alpha \in 1, 2, 3, ... \tag{3}$$

Hence, U and V are Gamma distributions. Let us find out the CDF of the U,

$$f_U(x) = \int_0^x \lambda^2 y e^{-\lambda y} \, dy \tag{4}$$

$$\implies = \lambda^2 \int_0^x y e^{-\lambda y} \, dy \tag{5}$$

$$\tag{6}$$

Substituting $t = \lambda y$, i.e. $y = \frac{t}{\lambda}$,this becomes:

$$f_U(x) = \lambda^2 \int_{0.\lambda}^{x.\lambda} \left(\frac{t}{\lambda}\right) e^{-\lambda.(\frac{t}{\lambda})} \, d\left(\frac{t}{\lambda}\right) \tag{7}$$

$$= \int_0^{bx} t e^{-t} \, dt \tag{8}$$

$$\tag{9}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: ee22btech11043@iith.ac.in.

With the definition of the lower incomplete gamma function,

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \, dt \tag{10}$$

$$\tag{11}$$

Using, (2), we can say that,

$$f_U(x) = \gamma(2, bx) \tag{12}$$

1) Let us consider a random variable Z such that

$$Z = U - V \tag{13}$$

$$\tag{14}$$

By the definition of Laplace transform,

$$L_Z(s) = E(e^{-sZ}) \tag{15}$$

$$\implies L_Z(s) = E(e^{-s(U-V)}) \tag{16}$$

$$\implies L_Z(s) = \frac{E(e^{-s(U)})}{E(e^{-s(V)})} \tag{17}$$

$$\tag{18}$$

As U and V are independent and identically distributed random variables,

$$\implies L_Z(s) = \frac{E(e^{-s(U)})}{E(e^{-s(U)})} \tag{19}$$

$$\implies L_Z(s) = 1 \tag{20}$$

Inverse transform of $L_Z(s)$, would give us,

$$p_{U-V}(x) = L^{-1}(1) \tag{21}$$

$$p_{U-V}(x) = \delta(x) \tag{22}$$

where,

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{23}$$

The CDF of U-V would be,

$$f_{U-V}(x) = \int_{-\infty}^x p_{U-V}(x) \, dx \tag{24}$$

$$\implies f_{U-V}(x) = u(x) \tag{25}$$

where,

$$u(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{26}$$

Hence, as per (22), we can say that The distribution of $U - V$ is symmetric about 0.

2)

$$Pr(UV < x) = E\left[Pr(U < \frac{x}{V})\right] \tag{27}$$

$$Pr(UV < x) = E\left[Pr(f_U\left(\frac{x}{V}\right))\right] \tag{28}$$

$$Pr(UV < x) = \int_{-\infty}^{x} Pr(y) f_U\left(\frac{x}{y}\right) dy \tag{29}$$

$$\tag{30}$$

From (**??**) and (1),

$$Pr(UV < x) = \int_{0}^{x} \lambda^2 y e^{-\lambda y} \left(1 - e^{-\lambda \frac{x}{y}} \left(\frac{x}{y}\lambda + 1\right)\right) dy \tag{31}$$

$$Pr(UV < x) = \int_{0}^{x} \left(\lambda^2 y e^{-\lambda y} - \lambda^3 x e^{-\lambda\left(y + \frac{x}{y}\right)} + \lambda^2 y e^{-\lambda\left(y + \frac{x}{y}\right)}\right) dy \tag{32}$$

$$\tag{33}$$