Exemplar - 10.13.3.39

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A die has its face marked 0,1,1,1,6,6. Two such dice are thrown together and their score is recorded.

1) How many different scores are possible ?
2) What is the probability of getting a total 7 ?

Solution: Let the random variables be defined as:

Random Variable	Values	Description
X	X = {0,1,6}	First Dice Roll
Y	Y = {0,1,6}	Second Dice Roll

1) **Possible outcomes:** The following data can be interpreted from the data given in the question,

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k = 0 \\ \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{3} & \text{if } k = 6 \\ 0 & \text{Otherwise} \end{cases} \tag{1}$$

$$p_Y(k) = \begin{cases} \frac{1}{6} & \text{if } k = 0 \\ \frac{1}{2} & \text{if } k = 1 \\ \frac{1}{3} & \text{if } k = 6 \\ 0 & \text{Otherwise} \end{cases} \tag{2}$$

$$\tag{3}$$

The Z-transform of a die is defined as

$$M_X(z) = z^{-X} = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{4}$$

The Z-transform of the first die X_1 is given by

$$M_{X_1}(z) = \frac{1}{6}z^0 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-6} \tag{5}$$

$$= \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \tag{6}$$

$$\tag{7}$$

The Z-transform of the second die X_2 is given by

$$M_{X_2}(z) = \frac{1}{6}z^0 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-6} \tag{8}$$

$$= \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \tag{9}$$

$$\tag{10}$$

$$M_{X_1+X_2}(z) = \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \times \frac{z^0 + 3z^{-1} + 2z^{-6}}{6} \tag{11}$$

$$M_{X_1+X_2}(z) = \frac{\left(1 + 3z^{-1} + 2z^{-6}\right)^2 \left(1 - z^{-1}\right)^2}{36\left(1 - z^{-1}\right)^2} \tag{12}$$

We also know that,

$$p_{X_1+X_2}(n-k) \overset{Z}{\longleftrightarrow} M_{X_1+X_2}(z) z^{-k}; \tag{13}$$

$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2} \tag{14}$$

Hence, after some algebra, it can be shown that,

$$\frac{1}{36}[(n+1)u(n+1) + 4nu(n) - 2n - 1u(n-1) \\ -12(n-2)u(n-2) + 9(n-3)u(n-3) + 4(n-5)u(n-5) \\ -20(n-7)u(n-7) + 12(n-8)u(n-8) \\ +4(n-11)u(n-11) - 8(n-12)u(n-12) \\ +4(n-13)u(n-13)] \overset{Z}{\longleftrightarrow} \\ \frac{\left(1 + 3z^{-1} + 2z^{-6}\right)^2 \left(1 - z^{-1}\right)^2}{36\left(1 - z^{-1}\right)^2} \tag{15}$$

The probability mass function for the case where total score of both the dice is 'k' is,

$$p_{X+Y}(k) = \Pr(X + Y = k) \tag{16}$$

$$= \Pr(X = k - Y) \tag{17}$$

$$= E\left(p_X(k - Y)\right) \tag{18}$$

$$= \sum_{i=0}^{6} \left(p_X(k - i)\right)\left(p_Y(i)\right) \tag{19}$$

$$p_X(n) = \begin{cases} \frac{1}{36} & n = 0 \\ \frac{1}{6} & n = 1 \\ \frac{1}{4} & n = 2 \\ \frac{1}{9} & n = 6, 12 \\ \frac{1}{3} & n = 7 \\ 0 & else \end{cases} \tag{20}$$

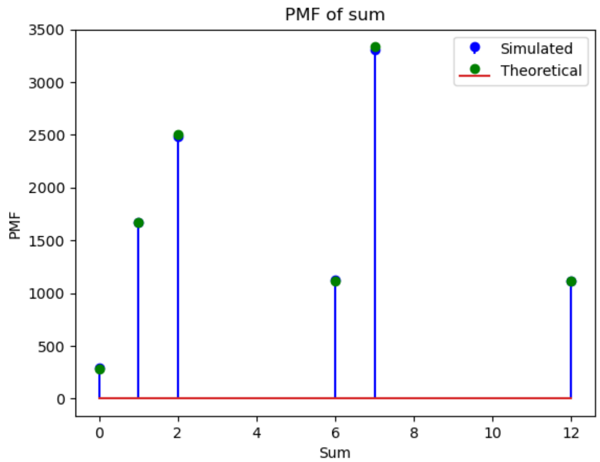

Fig. 1. Sketch of Probability Mass Function for Sum

The possible outcomes: 0,1,2,6,7&12

2) **Probability of getting a 7 :**

$$p_{X+Y}(7) = \sum_{i=0}^{6} (p_X(7-i))\,(p_Y(i)) \tag{21}$$

$$= p_X(6)p_Y(1) + p_X(1)p_Y(6) \tag{22}$$

$$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \tag{23}$$

$$= \frac{1}{3} \tag{24}$$

The probability of getting a 7 that we obtained by using Z-transform and PMF are the same, i.e $\frac{1}{3}$.