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Suppose that U and V are two independent and identically distributed random variables each having probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \qquad (1)$$

where $\lambda > 0$. Which of the following statements is/are true?

1) The distribution of $U - V$ is symmetric about 0
2) The distribution of UV does not depend on λ
3) The distribution of $\frac{U}{V}$ does not depend on λ
4) The distribution of $\frac{U}{V}$ is symmetric about 1

Solution: The give distribution is of the following form, with, $\alpha = 2$.

$$Gamma(\alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & \text{Otherwise,} \end{cases} \qquad (2)$$

Where,

$$\Gamma(\alpha) = (\alpha - 1)! \text{ when } \alpha \in 1, 2, 3, \dots \qquad (3)$$

Hence, U and V are Gamma distributions. Let us find out the CDF of the U,

$$f_U(x) = \int_0^x \lambda^2 y e^{-\lambda y} \, dy \qquad (4)$$

$$\implies = \lambda^2 \int_0^x y e^{-\lambda y} \, dy \qquad (5)$$

$$\qquad (6)$$

Substituting $t = \lambda y$, i.e. $y = \frac{t}{\lambda}$,this becomes:

$$f_U(x) = \lambda^2 \int_{0.\lambda}^{x.\lambda} \left(\frac{t}{\lambda}\right) e^{-\lambda.\left(\frac{t}{\lambda}\right)} d\left(\frac{t}{\lambda}\right) \qquad (7)$$

$$= \int_0^{bx} t e^{-t} \, dt \qquad (8)$$

$$\qquad (9)$$

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With the definition of the lower incomplete gamma function,

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \, dt \qquad (10)$$

$$\qquad (11)$$

we can say that,

$$f_U(x) = \gamma(2, bx) \qquad (12)$$

1) Let us consider a random variable Z such that

$$Z = U - V \qquad (13)$$

$$\qquad (14)$$

By the definition of Laplace transform,

$$M_Z(s) = E(e^{-sZ}) \qquad (15)$$

$$\implies = E(e^{-s(U-V)}) \qquad (16)$$

$$\implies = E(e^{-s(U)})E(e^{s(V)}) \qquad (17)$$

$$\implies = \int_0^\infty e^{-sx} \lambda^2 x e^{-\lambda x} \, dx \int_0^\infty e^{sx} \lambda^2 x e^{-\lambda x} \, dx \qquad (18)$$

$$\implies = \lambda^4 \left[\frac{x.e^{-(s+\lambda)x}}{-(s+\lambda)} - \frac{e^{-(s+\lambda)x}}{(s+\lambda)^2} \right]_0^\infty \cdot \left[\frac{x.e^{-(\lambda-s)x}}{-(\lambda-s)} - \frac{e^{-(\lambda-s)x}}{(\lambda-s)^2} \right] \qquad (19)$$

$$\implies = \lambda^4 \frac{-1}{(s+\lambda)^2} \frac{-1}{(\lambda-s)^2} \qquad (20)$$

$$\implies = \lambda^4 \frac{1}{(s^2 - \lambda^2)^2} \qquad (21)$$

$$\implies = \frac{\lambda^3}{4} \left[\frac{1}{s(s-\lambda)^2} - \frac{1}{s(s+\lambda)^2} \right] \qquad (22)$$

$$\qquad (23)$$

$$\implies M_Z(s) = \\ \frac{\lambda^3}{4} \left[\left(\frac{1}{\lambda^2 s} - \frac{1}{\lambda^2(s-\lambda)} + \frac{1}{\lambda(s-\lambda)^2} \right) \right] \\ - \frac{\lambda^3}{4} \left[\left(\frac{1}{\lambda^2 s} - \frac{1}{\lambda^2(s+\lambda)} - \frac{1}{\lambda(s+\lambda)^2} \right) \right] \qquad (24)$$

Now, we can find the inverse lapalce of $M_Z(s)$ to obtain the pdf of the random variable Z.

$$f_Z(x) = L^{-1}(M_Z(s)) \tag{25}$$

$$\tag{26}$$

$$f_Z(x) = \frac{\lambda^3}{4}\left[\frac{1}{\lambda^2} - \frac{1}{\lambda^2}e^{\lambda x} + \frac{1}{\lambda}xe^{\lambda x}\right]u(x) -$$

$$\frac{\lambda^3}{4}\left[\frac{1}{\lambda^2} - \frac{1}{\lambda^2}e^{-\lambda x} + \frac{1}{\lambda}xe^{-\lambda x}\right]u(x) \tag{27}$$

$$f_Z(x) = \left[\frac{\lambda}{4}\left(e^{-\lambda x} - e^{\lambda x}\right) + \frac{\lambda^2}{4}\left(e^{-\lambda x} + e^{\lambda x}\right)\right]u(x) \tag{28}$$

Looking at (28), we can say that distribution of $U - V$ is not symmetric about 0.

2)

$$Pr(UV < x) = E\left[Pr(U < \frac{x}{V})\right] \tag{29}$$

$$Pr(UV < x) = E\left[Pr(f_U\left(\frac{x}{V}\right))\right] \tag{30}$$

$$Pr(UV < x) = \int_{-\infty}^{x} Pr(y)f_U\left(\frac{x}{y}\right) dy \tag{31}$$

$$\tag{32}$$

$$Pr(UV < x) = \int_0^x \lambda^2 y e^{-\lambda y}\left(1 - e^{-\lambda \frac{x}{y}}\left(\frac{x}{y}\lambda + 1\right)\right) dy \tag{33}$$

$$Pr(UV < x) = \int_0^x \left(\lambda^2 y e^{-\lambda y} - \lambda^3 x e^{-\lambda\left(y + \frac{x}{y}\right)} + \lambda^2 y e^{-\lambda\left(y + \frac{x}{y}\right)}\right) dy \tag{34}$$

$$\tag{35}$$