2015 12th Set-3

February 27, 2024

Questions

1 Vectors

1. Find a vector of magnitude $\sqrt{171}$ which is to both of the vectors $\overrightarrow{a} = \left(\hat{i} - 2\hat{j} - 3\hat{k}\right)$ and $\overrightarrow{b} = \left(3\hat{i} - \hat{j} - 2\hat{k}\right)$.

2. In a triangle OAC, if B is the mid-point of side AC and $\overrightarrow{OA} = \overrightarrow{a}$, $\overrightarrow{OA} = \overrightarrow{b}$ then what is $\overrightarrow{OC}$?

3. If $\left|\overrightarrow{a}\right| = a$, then find the value of the following:

$$\left|\overrightarrow{a} \times \hat{i}\right|^2 + \left|\overrightarrow{a} \times \hat{j}\right|^2 + \left|\overrightarrow{a} \times \hat{k}\right|^2$$

4. The vectors $\overrightarrow{a} = \left(3\hat{i} + x\hat{j}\right)$ and $\overrightarrow{b} = \left(2\hat{i} + \hat{j} + y\hat{k}\right)$ are mutually perpendicular. If $\left|\overrightarrow{a}\right| = \left|\overrightarrow{b}\right|$, find the value of y.

5. Let $\overrightarrow{a} = \left(\hat{i} + 4\hat{j} + 2\hat{k}\right)$, $\overrightarrow{b} = \left(3\hat{i} - 2\hat{j} + 7\hat{k}\right)$ and $\overrightarrow{c} = \left(2\hat{i} - \hat{j} + 4\hat{k}\right)$. Find a vector $\overrightarrow{d}$ which is perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$ and $\overrightarrow{c}.\overrightarrow{d} = 27$.

6. Find the shortest distance between the following lines:

$$\overrightarrow{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)$$
$$\overrightarrow{r} = \left(2\hat{i} + 4\hat{j} + 5\hat{k}\right) + \mu\left(4\hat{i} + 6\hat{j} + 8\hat{k}\right)$$

7. Find a unit vector perpendicular tot he plane of triangle ABC where the coordinates of its vertices are $A(3, -1, 2)$, $B(1, -1, 3)$ and $C(4, -3, 1)$

2 Linear Forms

8. Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$.

9. Find the angle θ between the line $\frac{x-2}{3} = \frac{y-3}{5} = \frac{z-4}{4}$ and the plane $2x - 2y + z - 5 = 0$

10. Find the equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and is parallel to the line $\frac{x-1}{2} = \frac{y-3}{-4} = \frac{5-z}{-5}$

11. Find the equation of a plane passing through the point $P(6, 5, 9)$ and parallel to the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$. Also find the distance of this plane from the point A.

12. Find the equation of a plane passing through the point $P(6, 5, 9)$and parallel to the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$. Also find the distance of this plane from the point A.

13. Find the shortest distance between the lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$.

14. From the point $P(a, b, c)$, perpendiculars PL and PM are drawn to YZ and ZX planes respectively. Find the equation of the plane OLM.

15. Find the coordinates of the point where the line through the points $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the plane determined by the points $P(2, 1, 2)$, $Q(3, 1, 0)$ and $R(4, -2, 1)$.

3 Differential Equations

16. Find the sum of the *order* and the *degree* pf the following differential equation:

$$y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$$

17. If $(ax + b)e^{y/x} = x$, then show that

$$x^3\left(\frac{d^2y}{dx^2}\right) = \left(x\frac{dx}{dy} - y\right)^2$$

18. Find the solution of the following differential equation:

$$x\sqrt{1 + y^2}dx + y\sqrt{1 + x^2}dy = 0$$

19. Find the sum of the *order* and the *degree* pf the following differential equation:

$$\frac{d}{dx}\left[\left(\frac{d^2y}{dx^2}\right)^4\right]$$

20. Find the integrating factor of the following differential equation:

$$x\log x\frac{dy}{dx} + y = 2\log x$$

21. Find the differential equation of the family of curves $(x - h)^2 + (y - k)^2 = r^2$, where h and k are arbitrary constants.

22. Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it also.

23. $\left(x^2 + y^2\right)dy = (xy)\,dx$. If $y(x_0) = e$, then find the value of x_0.

24. Find the particular solution of the differential equation $\frac{dy}{dx} + y.tan(x) = 3x^2 + x^3tan(x)$, $x \neq \frac{pi}{2}$,given that $y = 0$ when $x = \frac{\pi}{3}$.

25. If $y = \log\left(\frac{x}{a+bx}\right)^x$, prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

4 Differentiation

26. If $x = a(cos(2t) + 2t.sin(2t))$ and $y = a(sin(2t) - 2t.cos(2t))$, then find $\frac{d^2y}{dx^2}$.

27. Find the derivative of $sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t $\sqrt{1-x^2}$ at $x = \frac{1}{2}$.

5 Matrices

28. If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$, then for any natural number n, find the value of $Det(A^n)$.

29. Find the value of $(x + y)$ from the following matrix equation:

$$2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$$

30. Using elementary row operations (transformations), find the inverse of the following matrix:

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix}$$

31. If $A = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$, then calculate $AC, BC, (A+B)C$. Also verify that $(A + B)C = AC + BC$.

32. There are 2 families A and B. There are 4 men, 6 women and 2 children in family A, and 2 men, 2 women and 4 children in family B. The recommended daily amount of calories is 2400 for men, 1900 for women, 1800 for children and 45 grams of proteins for men, 55 grams for women and 33 grams for children. Represent the above information using matrices. Using matrix

multiplication, calculate the total requirement of calories and proteins for each of the 2 families. What awareness can you create among people about the balanced diet from this question?

33. Using the properties of determinants, prove that:

$$\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix} = 2(a-b)(b-c)(c-a)(a+b+c).$$

34. Using elementary row operations, find the inverse of the following matrix :

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{pmatrix}$$

35. If $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{pmatrix}$, find A^{-1} using elementary row transformations.

36. If $a+b+c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then using the properties of determinants, prove that $a = b = c$.

37. A trust caring for handicapped children gets 30,000 every month from its donors. The trust spends half of the funds received for medical and educational care of the children and for that it charges 2% of the spent amount from them, and deposits the balance amount in a private bank to get the money multiplied so that in future the trust goes on functioning regularly. What percent of interest should the trust get from the bank to get a total of ₹1800 every month? Use matrix method, to find the rate of interest. Do you think people should donate to such trusts?

6 Integration

38. Evaluate:

$$\int_0^{\pi/2} \frac{\cos^2 x}{(1 + 3\sin^2 x)} dx$$

39. Evaluate::

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x} dx$$

40. Evaluate :

$$\int \frac{\sin x - x\cos x}{x(x + \sin x)} dx$$

41. Evaluate:

$$\int \frac{x^3}{(x-1)(x^2+1)} dx$$

42. Find:

$$\int \frac{x dx}{1 + x\tan x}$$

43. Find:

$$\int \frac{x^4}{(x-1)(x^2+1)} dx$$

44. Evaluate :

45. If the area bounded by the parabola $y^2 = 16ax$ and the line $y = 4mx$ is $\frac{a^2}{2}$ sq. units, then using integration, find the value of m.

46. Find the area of the region $\left\{(x, y) : x^2 + y^2 \leq 4, x + y \geq 2\right\}$, using the method of integration.

7 Function

47. Discuss the continuity and differentiability for he function $f(x) = |x| + |x-1|$ in the region $(-1, 2)$.

48. Determine whether the relation R defined on the set $\mathbb{R}$ of all real numbers as R $= \Big\{(a, b) : a, b \in \mathbb{R}$ and $a - b + \sqrt{3} \in S$, where S is the set all irrational numbers$\Big\}$, is reflexive, symmetric and transitive.

49. Let $A = \mathbb{R} \times \mathbb{R}$ and "$*$" be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Prove that "$*$" is commutative and associative. Find the identity element for "$*$" on A. Also write the inverse element of the element $(3, -5)$ in A.

50. On the set $\{0, 1, 2, 3, 4, 5, 6\}$, a binary operation "*" is defined as:

$$a * b = \begin{cases} a + b & \text{if } a + b < 7, \\ -a + b - 7 & \text{if } a + b \geq 7. \end{cases}$$

    Write the operation table of the operation "*" and prove that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $7 - a$ being the inverse of a.

51. Let $f(x) = x - \left|x - x^2\right|, x \in [-1, 1]$. Find the point of discontinuity,(if any), of this function on $[-1, 1]$.

8 Probability

52. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of 5 steps, he is one step away from the starting point.

53. Suppose a girl throws a die. If she gets a 1 or 2, she tosses a times and notes the number of 'tails'. If she gets $3, 4, 5$ or 6 coin once more and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw $3, 4, 5$ or 6 with the die.

54. An urn contains 5 red and 2 black balls. Two balls are randomly drawn, without replacement. Let X represent the number of black balls drawn. What are the

possible values of X? Is X a random variable? If yes, find the mean and variance of X.

55. In 3 trials of a binomial distribution, the probability of exactly 2 successes is 9 times the probability of 3 successes. Find the probability of success in each trial.

56. An urn contains 3 red and 5 black balls. A ball is drawn at random, its colour is noted and returned to the urn. Moreover, 2 additional balls of the colour noted down, are put in the urn and then two balls are drawn at random (without replacement) from the urn. Find the probability that both the balls drawn are of red colour.

57. A man is known to speak truth 3 out of 5 times. He throws a die and reports that it is 4. Find the probability that it is actually 4.

9 Optimization

58. Solve the following linear programming problem graphically. Minimise $z = 3x + 5y$ subject to constraints

$$\begin{aligned} x + 2y &\geq 10 \\ x + y &\geq 6 \\ 3x + y &\geq 8 \\ x, y &\geq 0 \end{aligned}$$

59. A dealer in a rural area wishes to purchase some sewing machines. He has only ₹57,600 to invest and has space for at most 20 items. An electronic machine costs him ₹3,600 and a manually operated machine costs ₹2,400. He can sell an electronic machine at a profit of 220 and a manually operated machine at a profit of ₹180. Assuming that he can sell all the machines that he buys, how should he invest his money in order to maximize his profit ? Make it as a LPP and solve it graphically.

10 Algebra

60. Evaluate:

$$\tan\left\{2\tan^{-}1\left(\frac{1}{5}\right)+\frac{\pi}{4}\right\}$$

61. Find the value of 'x', if

$$\sin\cot^{-1}x+1=\cos\tan^{-1}x$$

62. Prove the following

$$2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$$

11 Co-ordinate Geometry

63. Tangent to the circle $x^2+y^2=4$ at any point on it in the first quadrant makes intercepts OA and OB on x and y axes respectively, O being the centre of the circle. Find the minimum value of $(OA+OB)$.

64. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ with its vertex at one end of the major axis.