Matrix Analysis

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CONTENTS

***Abstract*—This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE; JEE and JNTU.**

1 Definitions

1.1 2×1 *vectors*

1.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1.1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{1.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1.1.1.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{1.1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{1.1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{1.1.1.6}$$

1.1.2. The value of the cross product of two vectors is given by (1.1.1.5).

1.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2}\left|\mathbf{A} - \mathbf{B} \quad \mathbf{A} - \mathbf{C}\right| \tag{1.1.3.1}$$

1.1.4. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{1.1.4.1}$$

1.1.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{1.1.5.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{1.1.5.2}$$

1.1.6. *norm* of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{1.1.6.1}$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{1.1.6.2}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left|\lambda \overrightarrow{A}\right| \tag{1.1.6.3}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{1.1.6.4}$$

1.1.7. The distance betwen the points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{1.1.7.1}$$

1.1.8. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.1.8.1}$$

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Solution:

$$\|\mathbf{x}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{B}\| \tag{1.1.8.2}$$
$$\implies \|\mathbf{x}-\mathbf{A}\|^2 = \|\mathbf{x}-\mathbf{B}\|^2 \tag{1.1.8.3}$$

which can be expressed as

$$\left(\mathbf{x}-\mathbf{A}\right)^\top \left(\mathbf{x}-\mathbf{A}\right) = \left(\mathbf{x}-\mathbf{B}\right)^\top \left(\mathbf{x}-\mathbf{B}\right)$$
$$\implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{A} + \|\mathbf{A}\|^2$$
$$= \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{B} + \|\mathbf{B}\|^2 \tag{1.1.8.4}$$

which can be simplified to obtain (1.1.8.1).

1.1.9. If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{1.1.9.1}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A}-\mathbf{B}\right)^\top \mathbf{e}_1} \tag{1.1.9.2}$$

Solution: From (1.1.8.1).

$$x\left(\mathbf{A}-\mathbf{B}\right)^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.1.9.3}$$

yielding (1.1.9.2).

1.1.10. The angle between two vectors is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}^\top\mathbf{B}}{\|A\|\,\|B\|} \tag{1.1.10.1}$$

1.1.11. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top\mathbf{B} = 0 \tag{1.1.11.1}$$

1.1.12. The *direction vector* of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A}-\mathbf{B} \tag{1.1.12.1}$$

1.1.13. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{1.1.13.1}$$

1.1.14. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1.1.14.1}$$

the m is defined to be the slope of the line.

1.1.15. The *normal vector* to $\mathbf{m}$ is defined by

$$\mathbf{m}^\top\mathbf{n} = 0 \tag{1.1.15.1}$$

1.1.16. The point $\mathbf{P}$ that divides the line segment AB in the ratio $k:1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B}+\mathbf{A}}{k+1} \tag{1.1.16.1}$$

1.1.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.1.17.1}$$

1.2 3×1 vectors

1.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1\overrightarrow{i} + a_2\overrightarrow{j} + a_3\overrightarrow{j}, \tag{1.2.1.1}$$
$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{1.2.1.2}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}, \tag{1.2.1.3}$$

1.2.2. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A}\times\mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{1.2.2.1}$$

1.2.3. Verify that

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{1.2.3.1}$$

1.2.4. The area of a triangle is given by

$$\frac{1}{2}\|\mathbf{A}\times\mathbf{B}\| \tag{1.2.4.1}$$

1.3 Eigenvalues and Eigenvectors

1.3.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1.3.1.1}$$

1.3.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I}-\mathbf{A}\right| = 0 \tag{1.3.2.1}$$

The above equation is known as the characteristic equation.

1.3.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{1.3.3.1}$$

1.3.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{1.3.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

1.3.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{1.3.5.1}$$

1.4 Determinants

1.4.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{1.4.1.1}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{1.4.1.2}$$

1.4.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{1.4.2.1}$$

1.4.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right| \tag{1.4.3.1}$$

1.4.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{1.4.4.1}$$

1.5 Rank of a Matrix

1.5.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

1.5.2. Row rank = Column rank.

1.5.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

1.5.4. An $n \times n$ matrix is invertible if and only if its rank is n.

1.6 Inverse of a Matrix

1.6.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{1.6.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{1.6.1.2}$$

1.6.2. For higher order matrices, the inverse should be calculated using row operations.

2 Linear Forms

2.1 Two Dimensions

2.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

2.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{2.1.2.1}$$

2.1.3. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{2.1.3.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

2.1.4. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If p_2 is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{2.1.4.1}$$

Solution: The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{2.1.4.2}$$
$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{2.1.4.3}$$
$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{2.1.4.4}$$
$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{2.1.4.5}$$
$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \tag{2.1.4.6}$$

yielding (2.1.4.1).

2.1.5. The distance from a point $\mathbf{P}$ to the line in (2.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\left\|\mathbf{n}\right\|} \tag{2.1.5.1}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (2.1.3.1). The equation of the normal to (2.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{n} \tag{2.1.5.2}$$
$$\implies \mathbf{P} - \mathbf{A} = \lambda\mathbf{n} \tag{2.1.5.3}$$

$\because$ $\mathbf{P}$ lies on (2.1.5.2). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{2.1.5.4}$$

From (2.1.5.3),

$$\mathbf{n}^\top\left(\mathbf{P} - \mathbf{A}\right) = \lambda\mathbf{n}^\top\mathbf{n} = \lambda\left\|\mathbf{n}\right\|^2 \tag{2.1.5.5}$$
$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top\left(\mathbf{P} - \mathbf{A}\right)\right|}{\left\|\mathbf{n}\right\|^2} \tag{2.1.5.6}$$

Substituting the above in (2.1.5.4) and using the fact that

$$\mathbf{n}^\top\mathbf{A} = c \tag{2.1.5.7}$$

from (2.1.1.1), yields (2.1.5.1).

2.1.6. The distance from the origin to the line in (2.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{2.1.6.1}$$

2.1.7. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top\mathbf{x} &= c_1 \\ \mathbf{n}^\top\mathbf{x} &= c_2 \end{aligned} \tag{2.1.7.1}$$

is given by

$$d = \frac{\left|c_1 - c_2\right|}{\left\|\mathbf{n}\right\|} \tag{2.1.7.2}$$

2.1.8. The equation of the line perpendicular to (2.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top\left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{2.1.8.1}$$

2.1.9. The foot of the perpendicular from $\mathbf{P}$ to the line in (2.1.1.1) is given by

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^\top \mathbf{x} = \begin{pmatrix}\mathbf{m}^\top\mathbf{P} \\ c\end{pmatrix} \tag{2.1.9.1}$$

Solution: From (2.1.1.1) and (2.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top\mathbf{x} = c \tag{2.1.9.2}$$
$$\mathbf{m}^\top\left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{2.1.9.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (2.1.9.1).

2.2 *Three Dimensions*

2.2.1. The area of a triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\left\|\left(\mathbf{A} - \mathbf{B}\right) \times \left(\mathbf{A} - \mathbf{C}\right)\right\| \tag{2.2.1.1}$$

2.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\text{rank}\begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C}\end{pmatrix} = 1 \tag{2.2.2.1}$$

2.2.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\text{rank}\begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D}\end{pmatrix} = 1, \text{rank}\begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C}\end{pmatrix} = 2 \tag{2.2.3.1}$$

2.2.4. The equation of a line is given by (2.1.3.1)

2.2.5. The equation of a plane is given by (2.1.1.1)

2.2.6. The distance from the origin to the line in (2.1.1.1) is given by (2.1.6.1)

2.2.7. The distance from a point $\mathbf{P}$ to the line in (2.1.3.1) is given by

$$d = \left\|\mathbf{A} - \mathbf{P}\right\|^2 - \frac{\left\{\mathbf{m}^\top\left(\mathbf{A} - \mathbf{P}\right)\right\}^2}{\left\|\mathbf{m}\right\|^2} \tag{2.2.7.1}$$

Solution:

$$d\left(\lambda\right) = \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\right\| \tag{2.2.7.2}$$
$$\implies d^2\left(\lambda\right) = \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\right\|^2 \tag{2.2.7.3}$$

which can be simplified to obtain

$$\begin{aligned} d^2\left(\lambda\right) = \lambda^2\left\|\mathbf{m}\right\|^2 &+ 2\lambda\mathbf{m}^\top\left(\mathbf{A} - \mathbf{P}\right) \\ &+ \left\|\mathbf{A} - \mathbf{P}\right\|^2 \end{aligned} \tag{2.2.7.4}$$

which is of the form

$$d^2\left(\lambda\right) = a\lambda^2 + 2b\lambda + c \tag{2.2.7.5}$$
$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{2.2.7.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{2.2.7.7}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (2.2.7.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{2.2.7.8}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{2.2.7.9}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{2.2.7.10}$$

$$= c - \frac{b^2}{a} \tag{2.2.7.11}$$

yielding (2.2.7.1) after substituting from (2.2.7.7).

2.2.8. The distance between the parallel planes (2.1.7.1) is given by (2.1.7.2).

2.2.9. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.2.9.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{2.2.9.2}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{2.2.9.3}$$

Solution: Any point on the line (2.2.9.2) should also satisfy (2.2.9.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \tag{2.2.9.4}$$

which can be simplified to obtain (2.2.9.3)

2.2.10. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.2.10.1}$$

is given by
Solution: The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \tag{2.2.10.2}$$

From (2.2.13.1), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2}\mathbf{n} \tag{2.2.10.3}$$

2.2.11. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.2.11.1}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{2.2.11.2}$$

Solution: Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (2.2.10.3),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{2.2.11.3}$$

2.2.12. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{2.2.12.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{2.2.12.2}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.2.12.3}$$

Solution: From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{2.2.12.4}$$

$$\mathbf{p}^\top \mathbf{B} = d \tag{2.2.12.5}$$

$$\mathbf{p}^\top \mathbf{n} = 0 \tag{2.2.12.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (2.2.12.6) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{2.2.12.7}$$

which yields (2.2.12.3) upon normalising with d.

2.2.13. The intersection of the line represented by (2.1.3.1) with the plane represented by (2.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{2.2.13.1}$$

Solution: From (2.1.3.1) and (2.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{2.2.13.2}$$
$$\mathbf{n}^\top \mathbf{x} = c \tag{2.2.13.3}$$
$$\implies \mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = c \tag{2.2.13.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{2.2.13.5}$$
$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{2.2.13.6}$$

Substituting the above in (2.2.13.4) yields (2.2.13.1).

2.2.14. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (2.1.3.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \tag{2.2.14.1}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{2.2.14.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{2.2.14.3}$$
$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{2.2.14.4}$$

The desired foot of the perpendicular is the intersection of (2.2.14.2) with (2.2.14.3) which can be obtained from (2.2.13.1) as (2.2.14.1)

2.2.15. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \tag{2.2.15.1}$$

Solution: The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \tag{2.2.15.2}$$

Hence, the equation of the plane is (2.2.15.1).

2.2.16. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.2.16.1}$$

Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{2.2.16.2}$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \tag{2.2.16.3}$$
$$\mathbf{n}^\top \mathbf{B} = 1 \tag{2.2.16.4}$$
$$\mathbf{n}^\top \mathbf{C} = 1 \tag{2.2.16.5}$$

which can be combined to obtain (2.2.16.1).

2.2.17. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{2.2.17.1}$$

Solution: Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{2.2.17.2}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \tag{2.2.17.3}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \tag{2.2.17.4}$$
$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \tag{2.2.17.5}$$

2.2.18. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{P} \mathbf{e}_1 + \mathbf{A} \tag{2.2.18.1}$$
$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{P} \mathbf{e}_2 + \mathbf{A} \tag{2.2.18.2}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \tag{2.2.18.3}$$

and

$$\cos\theta = \frac{(\mathbf{C} - \mathbf{A})^\top \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{e}_1\|} \tag{2.2.18.4}$$

3 Quadratic Forms

3.1 Conic Sections

3.1.1. Let $\mathbf{P}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line $L : \mathbf{n}^\top\mathbf{x} = c$ is constant, given by

$$\frac{\|\mathbf{P}-\mathbf{F}\|}{d} = e \tag{3.1.1.1}$$

The locus of $\mathbf{P}$ such is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

a) For $e = 1$, the conic is a parabola
b) For $e < 1$, the conic is an ellipse
c) For $e > 1$, the conic is a hyperbola

3.1.2. The equation of a conic with directrix $\mathbf{n}^\top\mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{3.1.2.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top, \tag{3.1.2.2}$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2\mathbf{F}, \tag{3.1.2.3}$$
$$f = \|\mathbf{n}\|^2\|\mathbf{F}\|^2 - c^2e^2 \tag{3.1.2.4}$$

Solution: From (3.1.1.1) and (2.1.5.1), for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x}-\mathbf{F}\|^2 = e^2\frac{\left(\mathbf{n}^\top\mathbf{x}-c\right)^2}{\|\mathbf{n}\|^2} \tag{3.1.2.5}$$

$$\implies \|\mathbf{n}\|^2(\mathbf{x}-\mathbf{F})^\top(\mathbf{x}-\mathbf{F}) = e^2\left(\mathbf{n}^\top\mathbf{x}-c\right)^2 \tag{3.1.2.6}$$

yielding

$$\begin{aligned}\|\mathbf{n}\|^2&\left(\mathbf{x}^\top\mathbf{x} - 2\mathbf{F}^\top\mathbf{x} + \|\mathbf{F}\|^2\right)\\ &= e^2\left(c^2 + \left(\mathbf{n}^\top\mathbf{x}\right)^2 - 2c\mathbf{n}^\top\mathbf{x}\right)\\ &= e^2\left(c^2 + \left(\mathbf{x}^\top\mathbf{n}\mathbf{n}^\top\mathbf{x}\right) - 2c\mathbf{n}^\top\mathbf{x}\right)\end{aligned} \tag{3.1.2.7}$$

which can be expressed as (3.1.2.1) after simplification.

3.1.3. (3.1.2.1) represents

a) a pair of straight lines if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^\top & f\end{vmatrix} = 0, \quad \left|\mathbf{V}\right| < 0 \tag{3.1.3.1}$$

else, it represents

b) a parabola for $\left|\mathbf{V}\right| = 0$,
c) ellipse for $\left|\mathbf{V}\right| > 0$ and
d) hyperbola for $\left|\mathbf{V}\right| < 0$.

3.2 Conic Parameters

3.2.1. The conic in (3.1.2.1) can be expressed in standard form (centre/vertex at the origin, major axis - x axis) as

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \quad |V| \neq 0 \tag{3.2.1.1}$$
$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = -2\eta\mathbf{e}_1^\top\mathbf{y} \quad |V| = 0 \tag{3.2.1.2}$$

where

$$\mathbf{P}^\top\mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{3.2.1.3}$$

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \tag{3.2.1.4}$$
$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{3.2.1.5}$$
$$\eta = \mathbf{u}^\top\mathbf{p}_1 \tag{3.2.1.6}$$
$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix} \tag{3.2.1.7}$$

Solution: Using

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{3.2.1.8}$$

(3.1.2.1) can be expressed as

$$\begin{aligned}(\mathbf{P}\mathbf{y}+\mathbf{c})^T\mathbf{V}(\mathbf{P}\mathbf{y}+\mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y}+\mathbf{c}) + f\\ = 0,\end{aligned} \tag{3.2.1.9}$$

yielding

$$\begin{aligned}\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c}+\mathbf{u})^T\mathbf{P}\mathbf{y}\\ + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0\end{aligned} \tag{3.2.1.10}$$

From (3.2.1.10) and (3.2.1.3),

$$\begin{aligned}\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c}+\mathbf{u})^T\mathbf{P}\mathbf{y}\\ + \mathbf{c}^T(\mathbf{V}\mathbf{c}+\mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0\end{aligned} \tag{3.2.1.11}$$

When $\mathbf{V}^{-1}$ exists,

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or,}\, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{3.2.1.12}$$

and substituting (3.2.1.12) in (3.2.1.11) yields (3.2.1.1). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{3.2.1.13}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (3.2.1.3)

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \tag{3.2.1.14}$$

Substituting (3.2.1.14) in (3.2.1.11),

$$\begin{aligned}\mathbf{y}^T\mathbf{D}\mathbf{y} &+ 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y} \\ &+ \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\ + 2&\left(\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y} \\ &+ \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\ &+ 2\left(\mathbf{u}^T\mathbf{p}_1 \quad \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y} \\ &+ \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \text{ from (3.2.1.13)} \\ &\implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2\left(\lambda_2\mathbf{c} + \mathbf{u}\right)^T\mathbf{p}_2 \\ &\qquad + \mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0\end{aligned}$$

which is the equation of a parabola. Thus, (3.2.1.15) can be expressed as (3.2.1.2) by choosing

$$\eta = \mathbf{u}^T\mathbf{p}_1 \tag{3.2.1.15}$$

and $\mathbf{c}$ in (3.2.1.11) such that

$$\mathbf{P}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta\begin{pmatrix}1\\0\end{pmatrix} \tag{3.2.1.16}$$

$$\mathbf{c}^T\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.2.1.17}$$

Multiplying (3.2.1.16) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta\mathbf{p}_1, \tag{3.2.1.18}$$

which, upon substituting in (3.2.1.17) results in

$$\eta\mathbf{c}^T\mathbf{p}_1 + \mathbf{u}^T\mathbf{c} + f = 0 \tag{3.2.1.19}$$

(3.2.1.18) and (3.2.1.19) can be clubbed together to obtain (3.2.2.2).

3.2.2. The centre/vertex of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{3.2.2.1}$$

$$\begin{pmatrix}\mathbf{u}^\top + \eta\mathbf{p}_1^\top \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \eta\mathbf{p}_1 - \mathbf{u}\end{pmatrix} \qquad |V| = 0 \tag{3.2.2.2}$$

Solution: From (3.2.1.8),

$$\mathbf{y} = \mathbf{P}^\top\left(\mathbf{x} - \mathbf{c}\right) \tag{3.2.2.3}$$

For the standard conic, $\mathbf{y} = \mathbf{0}$ is the centre/vertex and in (3.2.2.3),

$$\mathbf{y} = \mathbf{0} \implies \mathbf{x} = \mathbf{c} \tag{3.2.2.4}$$

3.2.3. The focal length of the parabola in (3.2.1.2) is given by

$$\left|\frac{2\eta}{\lambda_2}\right| \tag{3.2.3.1}$$

where λ_2 is the nonzero eigenvalue of $\mathbf{V}$ and η is defined in (3.2.1.6).

3.2.4. For $\left|V\right| \neq 0$, the lengths of the semi-major and semi-minor axes of the conic in (3.1.2.1) are given by

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}, \sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}}. \quad \text{(ellipse)} \tag{3.2.4.1}$$

$$\sqrt{\frac{\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}}, \sqrt{\frac{f - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}}, \quad \text{(hyperbola)} \tag{3.2.4.2}$$

Solution: For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{3.2.4.3}$$

and (3.2.1.1) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \tag{3.2.4.4}$$

yielding (3.2.4.1). Similarly, (3.2.4.2) can be obtained for

$$|\mathbf{V}| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \tag{3.2.4.5}$$

3.2.5. The equation of the minor and major axes are respectively given by

$$\mathbf{p}_i^\top\left(\mathbf{x} - \mathbf{c}\right) = 0, i = 1, 2 \tag{3.2.5.1}$$

3.2.6. The eccentricity, directrices and foci of (3.1.2.1) are given by (3.2.6.1) - (3.2.6.4) **Solution:** From (3.1.2.2),

$$\begin{aligned}\mathbf{V}^\top\mathbf{V} &= \left(\left\|\mathbf{n}\right\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \\ &\quad \left(\left\|\mathbf{n}\right\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \\ \implies \mathbf{V}^2 &= \left\|\mathbf{n}\right\|^4\mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top \\ &\quad - 2e^2\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top \\ &= \left\|\mathbf{n}\right\|^4\mathbf{I} + e^4\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top - 2e^2\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top \\ &= \left\|\mathbf{n}\right\|^4\mathbf{I} + e^2\left(e^2 - 2\right)\left\|\mathbf{n}\right\|^2\mathbf{n}\mathbf{n}^\top \\ &= \left\|\mathbf{n}\right\|^4\mathbf{I} + \left(e^2 - 2\right)\left\|\mathbf{n}\right\|^2\left(\left\|\mathbf{n}\right\|^2\mathbf{I} - \mathbf{V}\right)\end{aligned} \tag{3.2.6.5}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \left\|\mathbf{n}\right\|^2 \mathbf{V} - \left(e^2 - 1\right) \left\|\mathbf{n}\right\|^4 \mathbf{I} = 0 \tag{3.2.6.6}$$

Using the Cayley-Hamilton theorem, (3.2.6.6) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \left\|\mathbf{n}\right\|^2 \lambda + \left(1 - e^2\right) \left\|\mathbf{n}\right\|^4 = 0 \tag{3.2.6.7}$$

which can be expressed as

$$\left(\frac{\lambda}{\left\|\mathbf{n}\right\|^2}\right)^2 - \left(2 - e^2\right) \left(\frac{\lambda}{\left\|\mathbf{n}\right\|^2}\right) + \left(1 - e^2\right) = 0 \tag{3.2.6.8}$$

$$\implies \frac{\lambda}{\left\|\mathbf{n}\right\|^2} = 1 - e^2, 1 \tag{3.2.6.9}$$

$$\text{or, } \lambda_2 = \left\|\mathbf{n}\right\|^2, \lambda_1 = \left(1 - e^2\right) \lambda_2 \tag{3.2.6.10}$$

From (3.2.6.10), the eccentricity of (3.1.2.1) is given by (3.2.6.1). Multiplying both sides of (3.1.2.2) by $\mathbf{n}$,

$$\begin{aligned} \mathbf{V}\mathbf{n} &= \left\|\mathbf{n}\right\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} && (3.2.6.11) \\ &= \left\|\mathbf{n}\right\|^2 \left(1 - e^2\right) \mathbf{n} && (3.2.6.12) \\ &= \lambda_1 \mathbf{n} && (3.2.6.13) \end{aligned}$$

from (3.2.6.10) Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (3.2.1.5), (3.2.6.10) and (3.2.6.13),

$$\mathbf{n} = \left\|\mathbf{n}\right\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \tag{3.2.6.14}$$

From (3.1.2.3) and (3.2.6.10),

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \tag{3.2.6.15}$$

$$\implies \left\|\mathbf{F}\right\|^2 = \frac{\left(ce^2 \mathbf{n} - \mathbf{u}\right)^\top \left(ce^2 \mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{3.2.6.16}$$

$$\implies \lambda_2^2 \left\|\mathbf{F}\right\|^2 = c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \left\|\mathbf{u}\right\|^2 \tag{3.2.6.17}$$

Also, (3.1.2.4) can be expressed as

$$\lambda_2 \left\|\mathbf{F}\right\|^2 = f + c^2 e^2 \tag{3.2.6.18}$$

From (3.2.6.17) and (3.2.6.18),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \left\|\mathbf{u}\right\|^2 = \lambda_2 \left(f + c^2 e^2\right) \tag{3.2.6.19}$$

$$\implies \lambda_2 e^2 \left(e^2 - 1\right) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \left\|\mathbf{u}\right\|^2 - \lambda_2 f = 0 \tag{3.2.6.20}$$

yielding (3.2.6.4).

3.2.7. For

$$\mathbf{V} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \mathbf{u} = 0, f = -1 \tag{3.2.7.1}$$

in (3.1.2.1),

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 + \rho & 0 \\ 0 & 1 - \rho \end{pmatrix} \tag{3.2.7.2}$$

and

$$\rho = \frac{1 - \left\|\mathbf{x}\right\|^2}{\mathbf{x}^\top \mathbf{R} \mathbf{x}} \tag{3.2.7.3}$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2.7.4}$$

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{3.2.6.1}$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1, \tag{3.2.6.2}$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 \left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2 (e^2 - 1) \left(\left\|\mathbf{u}\right\|^2 - \lambda_2 f\right)}}{\lambda_2 e (e^2 - 1)} & e \neq 1 \\ \frac{\left\|\mathbf{u}\right\|^2 - \lambda_2 f}{2e^2 \mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \tag{3.2.6.3}$$

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \tag{3.2.6.4}$$

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.2.7.5}$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = 1 \tag{3.2.7.6}$$

Since

$$\left|\mathbf{V}\right| = 1 - \rho^2, 0 < \left|\mathbf{V}\right| < 1, \tag{3.2.7.7}$$

(3.2.7.6) represents the equation of an ellipse. Using eigenvalue decomposition,

$$\mathbf{V} = \mathbf{P}^\top \mathbf{D} \mathbf{P} \tag{3.2.7.8}$$

where Using the affine transformation,

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{3.2.7.9}$$

(3.2.7.6) can be expressed as

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 1 \implies y_1^2 \left(1 + \rho\right) + y_2^2 \left(1 - \rho\right) \quad = 1 \tag{3.2.7.10}$$

which can be simplified to obtain

$$\rho = \frac{1 - y_1^2 - y_2^2}{y_1^2 - y_2^2} \tag{3.2.7.11}$$

$$= \frac{1 - \left\|\mathbf{y}\right\|^2}{\mathbf{y}^\top \mathbf{Q} \mathbf{y}} \tag{3.2.7.12}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2.7.13}$$

From (3.2.7.9), (3.2.7.12) can be expressed as (3.2.7.3)

3.3 Tangent and Normal

3.3.1. The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{3.3.1.1}$$

with the conic section in (3.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{3.3.1.2}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right) \left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)}\right) \tag{3.3.1.3}$$

Solution: Substituting (3.3.1.1) in (3.1.2.1),

$$\left(\mathbf{q} + \mu \mathbf{m}\right)^T \mathbf{V} \left(\mathbf{q} + \mu \mathbf{m}\right) \tag{3.3.1.4}$$

$$+2\mathbf{u}^T \left(\mathbf{q} + \mu \mathbf{m}\right) + f = 0 \tag{3.3.1.5}$$

$$\implies \mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \tag{3.3.1.6}$$

$$+\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \tag{3.3.1.7}$$

Solving the above quadratic in (3.3.1.7) yields (3.3.1.3).

3.3.2. If L in (3.3.1.1) touches (3.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) = 0 \tag{3.3.2.1}$$

Solution: In this case, (3.3.1.7) has exactly one root. Hence, in (3.3.1.3)

$$\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right) \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right) = 0 \tag{3.3.2.2}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (3.1.2.1) and

$$\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \tag{3.3.2.3}$$

Substituting (3.3.2.3) in (3.3.2.2) and simplifying, we obtain (3.3.2.1).

3.3.3. Given the point of contact $\mathbf{q}$, the equation of a tangent to (3.1.2.1) is

$$\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)^T \mathbf{x} + \mathbf{u}^T \mathbf{q} + f = 0 \tag{3.3.3.1}$$

Solution: The normal vector is obtained from (3.3.2.1) and (2.1.1.1) as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{3.3.3.2}$$

From (3.3.3.2) and (2.1.2.1), the equation of the tangent is

$$\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)^T \left(\mathbf{x} - \mathbf{q}\right) = 0 \tag{3.3.3.3}$$

$$\implies \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)^T \mathbf{x} - \mathbf{q}^T \mathbf{V} \mathbf{q} - \mathbf{u}^T \mathbf{q} = 0 \tag{3.3.3.4}$$

which, upon substituting from (3.3.2.3) and simplifying yields (3.3.1.1).

3.3.4. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (3.1.2.1) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1}\left(\kappa_i \mathbf{n} - \mathbf{u}\right), i = 1, 2$$
$$\text{where } \kappa_i = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{3.3.4.1}$$

Solution: From (3.3.3.2),

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right), \quad \kappa \in \mathbb{R} \tag{3.3.4.2}$$

Substituting (3.3.4.2) in (3.3.2.3),

$$\left(\kappa\mathbf{n} - \mathbf{u}\right)^T \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) + 2\mathbf{u}^T\mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) + f = 0 \tag{3.3.4.3}$$

$$\implies \kappa^2\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{3.3.4.4}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{3.3.4.5}$$

Substituting (3.3.4.5) in (3.3.4.2) yields (3.3.4.1).

3.3.5. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (3.1.2.1) is given by the matrix equation

$$\begin{pmatrix}\mathbf{u}^\top + \kappa\mathbf{n}^\top \\ \mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}-f \\ \kappa\mathbf{n} - \mathbf{u}\end{pmatrix} \tag{3.3.5.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^T\mathbf{u}}{\mathbf{p}_1^T\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{3.3.5.2}$$

Solution: If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{3.3.5.3}$$

From (3.3.3.2),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{3.3.5.4}$$
$$\implies \kappa\mathbf{p}_1^T\mathbf{n} = \mathbf{p}_1^T\mathbf{V}\mathbf{q} + \mathbf{p}_1^T\mathbf{u} \tag{3.3.5.5}$$
$$\text{or, } \kappa\mathbf{p}_1^T\mathbf{n} = \mathbf{p}_1^T\mathbf{u}, \quad \because \mathbf{p}_1^T\mathbf{V} = 0, \tag{3.3.5.6}$$
$$(\text{ from (3.3.5.3)}) \tag{3.3.5.7}$$

yielding κ in (3.3.5.2). From (3.3.5.4),

$$\kappa\mathbf{q}^T\mathbf{n} = \mathbf{q}^T\mathbf{V}\mathbf{q} + \mathbf{q}^T\mathbf{u} \tag{3.3.5.8}$$
$$\implies \kappa\mathbf{q}^T\mathbf{n} = -f - \mathbf{q}^T\mathbf{u} \quad \text{from (3.3.2.3),} \tag{3.3.5.9}$$
$$\text{or, } \left(\kappa\mathbf{n} + \mathbf{u}\right)^\top\mathbf{q} = -f \tag{3.3.5.10}$$

(3.3.5.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{3.3.5.11}$$

(3.3.5.10) and (3.3.5.11) clubbed together result in (3.3.5.1).

3.4 Pair of Straight Lines

3.4.1. When (3.1.2.1) is a hyperbola, its *asymptotes* are defined as the pair of intersecting straight lines

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} = 0, \quad \left|\mathbf{V}\right| < 0 \tag{3.4.1.1}$$

Solution: From (3.2.4.4)

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \tag{3.4.1.2}$$

which represents a pair of straight lines for

$$f = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}, \left|\mathbf{V}\right| < 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 < 0 \tag{3.4.1.3}$$

Thus, (3.4.1.1) are the asymptotes of all hyperbolas defined by (3.1.2.1). Also, any pair of straight lines has the form in (3.4.1.1).

3.4.2. (3.4.1.1) can be expressed as the lines

$$\begin{pmatrix}\sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|}\end{pmatrix}\mathbf{P}^\top\left(\mathbf{x} - \mathbf{c}\right) = 0 \tag{3.4.2.1}$$

Solution: Reducing (3.4.1.1) to standard form using the *affine transformation* yields

$$\lambda_1 y_1^2 - \left(-\lambda_2\right) y_1^2 = 0 \tag{3.4.2.2}$$

From (3.4.1.1), the equation of the asymptotes for (3.4.2.2) is

$$\begin{pmatrix}\sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|}\end{pmatrix}\mathbf{y} = 0 \tag{3.4.2.3}$$

from which (3.4.2.1) is obtained using (3.2.1.8).

3.4.3. The angle between the asymptotes is then given by using the inner product

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{3.4.3.1}$$

Solution: The normal vectors of the lines in (3.4.2.1) are

$$\mathbf{n}_1 = \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|}\end{pmatrix}, \quad \mathbf{n}_2 = \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|}\end{pmatrix} \tag{3.4.3.2}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\left\|\mathbf{n_1}\right\| \left\|\mathbf{n_2}\right\|} \tag{3.4.3.3}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\left\|\mathbf{n_1}\right\| = \left\|\mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| \tag{3.4.3.4}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \left\|\mathbf{n_2}\right\| \tag{3.4.3.5}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{3.4.3.6}$$

Thus, the angle between the asymptotes is obtained from (3.4.3.3) as (3.4.3.1).

3.4.4. The *conjugate* hyperbola with the same asymptotes as (3.4.2.1) can be obtained from (3.1.2.1) and (3.4.1.1) as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + 2\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{3.4.4.1}$$

3.5 Latus Rectum

3.5.1. The latus rectum of a conic section is the chord that passes through the focus, is perpendicular to the major axis and has both endpoints on the curve.

3.5.2. The latus rectum is parallel to the directrix.

3.5.3. The equation of the latus rectum is given by

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{F}\right) = 0 \tag{3.5.3.1}$$

$$\text{or, } \mathbf{x} = \mathbf{F} + \mu\mathbf{m} \tag{3.5.3.2}$$

where $\mathbf{F}$ is the focus and $\mathbf{m}$ is the normal to the directrix, i.e.

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{3.5.3.3}$$

3.5.4. The affine transform preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.
Solution: Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{3.5.4.1}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\|\mathbf{x}_1 - \mathbf{x}_2\right\| = \left\|\mathbf{P}\left(\mathbf{y}_1 - \mathbf{y}_2\right)\right\| \tag{3.5.4.2}$$

which can be expressed as

$$\begin{aligned}\left\|\mathbf{x}_1 - \mathbf{x}_2\right\|^2 &= \left(\mathbf{y}_1 - \mathbf{y}_2\right)^\top \mathbf{P}^\top \mathbf{P}\left(\mathbf{y}_1 - \mathbf{y}_2\right) \\ &= \left\|\mathbf{y}_1 - \mathbf{y}_2\right\|^2\end{aligned} \tag{3.5.4.3}$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{3.5.4.4}$$

3.5.5. The length of the latus rectum is given by

$$\frac{2\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{F} + \mathbf{u}\right)\right]^2 - \left(\mathbf{F}^T\mathbf{V}\mathbf{F} + 2\mathbf{u}^T\mathbf{F} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left\|\mathbf{m}\right\| \tag{3.5.5.1}$$

Solution:
From (3.3.1.2) and (3.3.1.3), substituting $\mathbf{q} = \mathbf{F}$, the end points of the latus rectum on the conic section can be obtained. Thus, the distance between these points is given by

$$\left\|\mathbf{x}_1 - \mathbf{x}_2\right\| = \left|\mu_1 - \mu_2\right| \left\|\mathbf{m}\right\| \tag{3.5.5.2}$$

which can be used to obtain the length of the latus rectum as (3.5.5.1).

a) From (3.5.4.3), we may consider the standard ellipse/hyperbola given by (3.2.1.1) as

$$\begin{aligned}&\mathbf{y}^\top \mathbf{D}\mathbf{y} = -f, \mathbf{V} = \mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \\ &f = -1, \mathbf{u} = 0, \mathbf{p}_1 = \mathbf{e}_1, \mathbf{p}_2 = \mathbf{e}_2\end{aligned} \tag{3.5.5.3}$$

for computing the length of the latus rectum in (3.5.5.1). Note that $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors and $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors. Substituting from (3.5.5.3) in (3.2.6.2), the parameters of the directrix are obtained as

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1, \tag{3.5.5.4}$$

$$c = \pm\frac{1}{e\sqrt{e^2 - 1}} \tag{3.5.5.5}$$

and the focus is

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{3.5.5.6}$$

$$\mathbf{F} = \frac{e}{\sqrt{\lambda_2}\sqrt{e^2 - 1}}\mathbf{e}_1 \tag{3.5.5.7}$$

From (3.5.5.4),

$$\mathbf{m} = \mathbf{e}_2 \tag{3.5.5.8}$$

Substituting the above in (3.5.5.1) along with (3.5.5.6) and (3.5.5.7), the length of the latus rectum for an ellipse and hyperbola is obtained from (3.5.5.1) as

$$2\frac{\sqrt{\lambda_1}}{\lambda_2} \tag{3.5.5.9}$$

3.6 Intersection of Conics

3.6.1. Let the intersection of the conics

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{3.6.1.1}$$

be

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu\mathbf{V}_2\right) \mathbf{x} + 2\left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top \mathbf{x} + f_1 + \mu f_2 = 0 \tag{3.6.1.2}$$

From (3.1.3.1), the above equation will represent a pair of straight lines if and only if

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ \left(\mathbf{u}_1 + \mu\mathbf{u}_2\right)^\top & f \end{vmatrix} = 0, \left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| < 0 \tag{3.6.1.3}$$

3.6.2. Once μ is obtained, (3.6.1.2) can be used to obtain the pair of straight lines passing throught the intersection of the conics in (3.6.1.1).

3.6.3. The intersection of (3.6.1.2) with (3.6.1.1) can be obtained using (3.3.1.2) to obtain the points of intersection of the conics.

3.6.4. Note that there could be several cases to be considered, since the conics can intersect in 1 to 4 points.

3.7 Miscellaneous

3.1. Given unit basis vectors $\mathbf{a}, \mathbf{b}$, with angle θ between them, the locus of the coordinates of a unit vector $\mathbf{c}$ in the space spanned by $\mathbf{a}, \mathbf{b}$ is given by (3.2.7.6), with $\rho = \cos\theta$.
Solution: Let

$$\mathbf{c} = x_1\mathbf{a} + x_2\mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{3.1.1}$$

Then,

$$\left\|\mathbf{c}\right\|^2 = \mathbf{x}^\top \begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{3.1.2}$$

$$= \mathbf{x}^\top \begin{pmatrix} 1 & \mathbf{a}^\top\mathbf{b} \\ \mathbf{a}^\top\mathbf{b} & 1 \end{pmatrix} \mathbf{x} \tag{3.1.3}$$

which can be expressed as (3.2.7.6),

$$\because \left\|\mathbf{a}\right\| = \left\|\mathbf{b}\right\| = 1. \tag{3.1.4}$$

3.2. Given the coordinates of $\mathbf{c}$, the angle θ between the basis vectors is given by $\rho = \cos\theta$ in (3.2.7.3).