CBSE Maths 12, 2007

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1 Boolean Algebra

1.1. Write the boolean expressions representing the following circuit and simplify the Boolean expression.

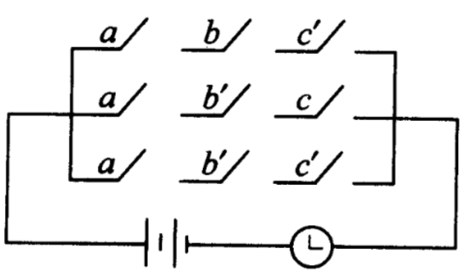

Fig. 1.1

Solution:
Truth table of the circuit 1.1 is given in table 1.1
Therefore the boolean expression representing the circuit 1.1

$$out = ab'c' + ab'c + abc' \tag{1.1.1}$$

a	b	c	out
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

TABLE 1.1

1.2. Show that the following argument is valid:

$$S_1 : p \vee q$$
$$S_2 : \sim q$$
$$S : p \wedge \sim q$$

Solution:
Truth table is shown in table 1.2
Therefore from table 1.2 we can conclude that

p	q	$p \vee q$	$\sim q$	$p \wedge \sim q$
0	0	0	1	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

TABLE 1.2

S is 1, if and only if s_1 and s_2 are 1.

2 Algebra

2.1. A particle starting with initial velocity of 30 m/sec moves with a uniform acceleration of 9 m/sec^2. Find :

a) the velocity of the particle after 6 seconds.
b) how far it will go in 9 seconds.
c) its velocity when is has traversed 150 m.

Solution:
Given,

$$u = \text{initial velocity} = 30m/sec \tag{2.1.1}$$
$$a = \text{acceleration} = 9m/sec^2 \tag{2.1.2}$$

a)

$$v = u + at \tag{2.1.3}$$
$$v = 30 + 9(6) = 84m/sec \tag{2.1.4}$$

b)

$$s = ut + \frac{1}{2}at^2 \tag{2.1.5}$$
$$s = 30(9) + 0.5(9)(9)^2 = 634.5m \tag{2.1.6}$$

c)

$$v = \sqrt{u^2 + 2as} \tag{2.1.7}$$
$$v = \sqrt{900 + 2(9)(150)} = 60m/sec \tag{2.1.8}$$

2.2. A ball projected with a velocity of 28 m/sec has a horizontal range 40 m. Find the two angles of projection.
Solution:
Given,

$$R = \text{Range} = 40m \tag{2.2.1}$$
$$u = \text{Initial velocity} = 28m/sec \tag{2.2.2}$$
$$g = \text{acceleration due to gravity} = 9.8m/sec^2 \tag{2.2.3}$$

Let Angle of projection are θ and $90° - \theta$.

$$R = \frac{u^2 \sin 2\theta}{g} \tag{2.2.4}$$
$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{Rg}{u^2}\right) \tag{2.2.5}$$
$$\theta = 15° \tag{2.2.6}$$

Therefore angle of projection are 15°and 75°.

2.3. The resultant of two unlike parallel forces of 18 N and 10 N act along a line at a distance of 12 cm from the line of action of the smaller force. Find the distance between the lines of action of two forces.
Solution:
Given, Two unlike forces 18N and 10N acting on a line. The distance of resultant from the smaller force 10N is 12cm and from 18N is x cm
Therefore,

$$18x = (12)(10) \tag{2.3.1}$$
$$x = 6.66cm \tag{2.3.2}$$
$$\tag{2.3.3}$$

The distance between two forces is 12+x = 18.66cm

2.4. Find the face value of a bill, discounted at 6% per annum 146 days before the legal due date, if the banker's gain is Rs. 36.
Solution:
Given,

Symbol	Description	Value
G	Banker's gain	36
D	Discount	6% per annum
n	Period of interest	146 days
F	Face value	?

TABLE 2.4

$$F = G\left(1 + \frac{Dn}{100}\right) \tag{2.4.1}$$
$$F = 36\left(1 + \frac{(6)(146)}{(100)(365)}\right) \tag{2.4.2}$$
$$F = 36.84 \tag{2.4.3}$$

2.5. A bill for Rs. 7650 was drawn on 8th March 2005 at 7 months. It was discounted on 18 May 2005 and the holder of the bill received Rs. 7497. What rate of interest did the banker charge ?
Solution:
Given,

Symbol	Description	Value
G	Banker's gain	7497
D	Discount	?
n	Period of interest	146 days
F	Face value	7650

TABLE 2.5

$$I = \frac{F.D.n}{100} \tag{2.5.1}$$

$$7650 - 7497 = \frac{(7650)(D)(146)}{(100)(365)} \tag{2.5.2}$$

$$D = 5\% \tag{2.5.3}$$

2.6. A, B, C entered into a partnership investing Rs. 12000, Rs. 16000 and Rs. 20000 respectively. A as working partner gets 10% of the annual profit for the same. After 5 months, B invested Rs. 2000 more while C withdrew Rs. 2000 after 8 months from the start of the business. Find the share of each in an annual profit of Rs. 97,000.
Solution:
Amount A invests per annum = 12000 x 12 = 1,44,000
Amount B invests per annum = 16000 x 5 + 18000 x 7 = 2,06,000
Amount C invests per annum = 20000 x 8 + 18000 x 4 = 2,32,000
Share of A

$$\frac{10}{100} x97000 = 9,700 \tag{2.6.1}$$

Total left = 97000-9700 = 87,300
Share of B and C are in the ratio = 206000 : 232000 = 103:116
Share of B

$$\frac{103}{219} x87300 = 41,031 \tag{2.6.2}$$

Share of C

$$\frac{116}{219} x87300 = 46,269 \tag{2.6.3}$$

2.7. Find the present value of an annuity due of Rs. 700 per annum payable at the beginning of each year for 2 years allowing interest 6% per annum, compounded annually.[Take $(1.06)^{-1}$ = 0.943]
Solution:
Given,

Symbol	Description	Value
P	Principal	?
R	Rate	6%
n	Period of interest	2 years
T	Total	700

TABLE 2.7

$$T = P\left(1 + \frac{R}{100}\right)\left(1 + \frac{R}{100}\right) \tag{2.7.1}$$

$$P = \frac{700}{\left(1 + \frac{6}{100}\right)\left(1 + \frac{6}{100}\right)} \tag{2.7.2}$$

$$P = 622.47 \tag{2.7.3}$$

2.8. The total cost C(x), associated with the production and making x units of an item is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Find:

a) the average cost function.
b) the average cost of output of 10 units.
c) the marginal cost function.
d) the marginal cost when 3 units are produced.

Solution:
Given,

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000 \tag{2.8.1}$$

a)

$$\text{average cost} = \frac{C(x)}{x} \tag{2.8.2}$$

$$= 0.005x^2 - 0.02x + 30 + \frac{5000}{x} \tag{2.8.3}$$

$$\text{average cost}|_{x=10} = 530.3 \tag{2.8.4}$$

b)

$$\text{marginal cost} = \frac{dC(x)}{dx} \tag{2.8.5}$$

$$= 0.015x^2 - 0.04x + 30 \tag{2.8.6}$$

$$\text{marginal cost}|_{x=3} = 31.015 \tag{2.8.7}$$

3 Calculus

3.1. Evaluate the following integral:

$$\int \frac{1 + x^2}{1 + x^4}\, dx$$

Solution:

$$\int \frac{1+x^2}{1+x^4}\,dx \tag{3.1.1}$$

$$= \int \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}\,dx \tag{3.1.2}$$

$$= \int \frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+2}\,dx \tag{3.1.3}$$

let $x - \frac{1}{x} = t$ then, $\frac{1}{x^2} + 1 = dt$. substituting in (3.1.3).

$$= \int \frac{1}{t^2+2}\,dt \tag{3.1.4}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c \tag{3.1.5}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + c \tag{3.1.6}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c \tag{3.1.7}$$

3.2. Solve the following differential equation:

$$x\cos y dy = (xe^x \log x + e^x)dx$$

Solution:

$$x\cos y dy = (xe^x \log x + e^x)dx \tag{3.2.1}$$

$$\cos y dy = \frac{(xe^x \log x + e^x)}{x}dx \tag{3.2.2}$$

$$\int \cos y\,dy = \int \frac{(xe^x \log x + e^x)}{x}\,dx \tag{3.2.3}$$

$$\sin y = \int \left(\frac{xe^x \log x}{x} + \frac{e^x}{x}\right)dx \tag{3.2.4}$$

$$\sin y = e^x \log x + \int \frac{e^x}{x}\,dx - \int \frac{e^x}{x}\,dx \tag{3.2.5}$$

$$\sin y = e^x \log x \tag{3.2.6}$$

3.3. Form the differential equations of the family of the curves $y = A\cos 2x + B\sin 2x$, where A and B are constants.

Solution:

$$y = A\cos 2x + B\sin 2x \tag{3.3.1}$$

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x \tag{3.3.2}$$

$$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x \tag{3.3.3}$$

$$\frac{d^2y}{dx^2} = -4(A\cos 2x + B\sin 2x) \tag{3.3.4}$$

$$\frac{d^2y}{dx^2} = -4y \tag{3.3.5}$$

3.4. Solve the following differential equation:

$$\frac{dy}{dx} + 2y = 6e^x$$

Solution:

$$\frac{dy}{dx} + 2y = 6e^x \tag{3.4.1}$$

Linear Differential Equation of form

$$\frac{dy}{dx} + P(X)y = Q(x) \tag{3.4.2}$$

. The Integrating factor is defined as,

$$\text{IF} = e^{\int P(x)dx} \tag{3.4.3}$$

$$\text{IF} = e^{\int 2dx} \tag{3.4.4}$$

$$\text{IF} = e^{2x} \tag{3.4.5}$$

Solution is of the form,

$$y.IF = \int (Q(x).IFdx) \tag{3.4.6}$$

$$y.e^{2x} = \int (6e^x.e^{2x})dx \tag{3.4.7}$$

$$y.e^{2x} = 2e^{3x} + c \tag{3.4.8}$$

3.5. Evaluate:

$$\int \cos 4x \cos 3x\,dx \tag{3.5.1}$$

Solution:

$$\int \cos 4x \cos 3x\,dx \tag{3.5.2}$$

$$\int \cos(7x) + \cos(x)dx \tag{3.5.3}$$

$$\frac{-\sin 7x}{7} + \frac{-\sin x}{1} \tag{3.5.4}$$

3.6. Using the properties of definite integrals, prove the following:

$$\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}\, dx = \frac{\pi^2}{4}$$

Solution:

$$I(x) = \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}\, dx \tag{3.6.1}$$

$$I(\pi - x) = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x)\operatorname{cosec}(\pi - x)}\, dx \tag{3.6.2}$$

$$I(\pi - x) = \int_0^{\pi} \frac{(\pi - x) - \tan x)}{-\sec(x)\operatorname{cosec}(x)}\, dx \tag{3.6.3}$$

$$I(x) = I(\pi - x) \tag{3.6.4}$$

$$2I = \int_0^{\pi} \frac{\tan x}{\sec x \operatorname{cosec} x}\, dx \tag{3.6.5}$$

$$2I = \int_0^{\pi} \sin^2 x\, dx \tag{3.6.6}$$

$$2I = \int_0^{\pi} \frac{1 - \cos 2x}{2}\, dx \tag{3.6.7}$$

$$2I = \frac{\pi^2}{2} \tag{3.6.8}$$

$$I = \frac{\pi^2}{4} \tag{3.6.9}$$

3.7. Evaluate:

$$\int \frac{\sin x}{(1 - \cos x)(2 - \cos x)}\, dx$$

Solution:

$$\int \frac{\sin x}{(1 - \cos x)(2 - \cos x)}\, dx \tag{3.7.1}$$

$\cos x = t, \sin x = dt,$ substituting in (3.7.1)

$$= \int \frac{dt}{(1 - t)(2 - t)} \tag{3.7.2}$$

$$= \int \frac{1}{1 - t} + \frac{-1}{2 - t}\, dt \tag{3.7.3}$$

$$= -\log 1 - t + \log 2 - t + c \tag{3.7.4}$$

$$= \log \frac{2 - t}{1 - t} + c \tag{3.7.5}$$

$$= \log \frac{2 - \cos x}{1 - \cos x} + c \tag{3.7.6}$$

3.8. Find the value of k if the function

$f(x) = \begin{cases} kx^2, & x \geq 1 \\ 4, & x < 1 \end{cases}$ is continuous at x = 1

Solution:
For continuity,

$$f(a) = f(x^-)|_a = f(x^+)|_a \tag{3.8.1}$$

$$f(1) = k(1)^2 = k \tag{3.8.2}$$

$$f(1^-) = 4 \tag{3.8.3}$$

$$f(1^+) = k \tag{3.8.4}$$

From equation (3.8.1) we get k=4.

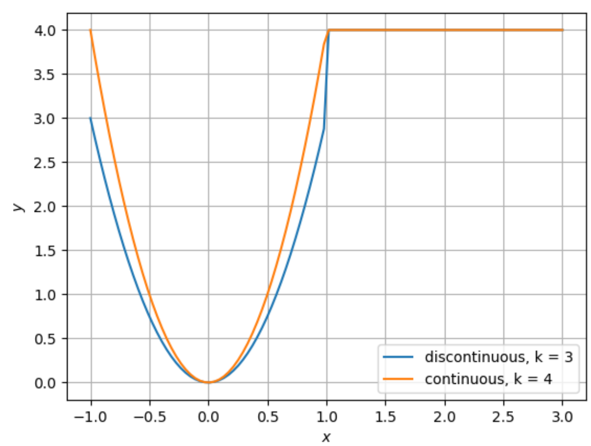

Fig. 3.8

3.9. Evalutate:

$$\lim_{x \to \frac{\pi}{4}} \left(\frac{\sin x - \cos x}{x - \frac{\pi}{4}} \right)$$

Solution:

$$\lim_{x\to\frac{\pi}{4}}\left(\frac{\sin x-\cos x}{x-\frac{\pi}{4}}\right) \tag{3.9.1}$$

$$f\left(\frac{\pi}{4}^{+}\right)=\lim_{\delta\to0}\left(\frac{\sin(\frac{\pi}{4}+\delta)-\cos(\frac{\pi}{4}+\delta)}{\frac{\pi}{4}+\delta-\frac{\pi}{4}}\right) \tag{3.9.2}$$

$$f\left(\frac{\pi}{4}^{+}\right)=\lim_{\delta\to0}\left(\frac{\sqrt{2}\sin\delta}{\delta}\right) \tag{3.9.3}$$

$$f\left(\frac{\pi}{4}^{+}\right)=\sqrt{2} \tag{3.9.4}$$

$$f\left(\frac{\pi}{4}^{-}\right)=\lim_{\delta\to0}\left(\frac{\sin(\frac{\pi}{4}-\delta)-\cos(\frac{\pi}{4}-\delta)}{\frac{\pi}{4}-\delta-\frac{\pi}{4}}\right) \tag{3.9.5}$$

$$f\left(\frac{\pi}{4}^{-}\right)=\lim_{\delta\to0}\left(-\frac{\sqrt{2}\sin\delta}{-\delta}\right) \tag{3.9.6}$$

$$f\left(\frac{\pi}{4}^{-}\right)=\sqrt{2} \tag{3.9.7}$$

For the limit to exist

$$f\left(\frac{\pi}{4}^{+}\right)=f\left(\frac{\pi}{4}^{-}\right)=\lim_{x\to\frac{\pi}{4}}f(x) \tag{3.9.8}$$

Therefore,

$$\lim_{x\to\frac{\pi}{4}}f(x)=\sqrt{2} \tag{3.9.9}$$

3.10. Differentiate $\sin(x^2+1)$ with respect to x from first principle.
Solution:
First principle of differentiation,

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h} \tag{3.10.1}$$

$$=\lim_{h\to0}\frac{\sin((x+h)^2+1)-\sin(x^2+1)}{h} \tag{3.10.2}$$

$$=\lim_{h\to0}\frac{\sin((x+h)^2+1)-\sin(x^2+1)}{h} \tag{3.10.3}$$

$$=\lim_{h\to0}2\cos\left(\frac{2x^2+h^2+2xh+2}{2}\right).\frac{sin\left(\frac{h^2+2xh}{2}\right)}{h} \tag{3.10.4}$$

$$=2\cos\left(x^2+1\right)\lim_{h\to0}\frac{\sin\left(\frac{h^2+2xh}{2}\right)}{h\left(\frac{h^2+2xh}{2}\right)}\left(\frac{h^2+2xh}{2}\right) \tag{3.10.5}$$

$$=2\cos\left(x^2+1\right) \tag{3.10.6}$$

3.11. If y = $\sin(\log x)$, prove that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Solution:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0 \tag{3.11.1}$$

$$y=\sin(\log x) \tag{3.11.2}$$

$$\frac{dy}{dx}=\frac{\cos(\log x)}{x} \tag{3.11.3}$$

$$\frac{d^2y}{dx^2}=\frac{-\sin(\log x)-\cos(\log x)}{x^2} \tag{3.11.4}$$

Substituting (3.11.2), (3.11.3) and (3.11.4) in (3.11.1).

$$=-\sin(\log x)-\cos(\log x)+\sin(\log x)+\cos(\log x) \tag{3.11.5}$$

$$=0 \tag{3.11.6}$$

3.12. Verify Rolle's theorem for the function $f(x)=x^2-5x+4$ on [1,4].
Solution:
Rolle's theorem states that if a function f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b) such that f(a) = f(b), then f'(x) = 0 for some $x\in a\le x\le b$.

Given,

$$f(x) = x^2 - 5x + 4 \tag{3.12.1}$$

a) f(x) is continuous on [1, 4] being algebraic function.

b) f'(x) = 2x-5 and differentiable on [1,4].

c)

$$f(a) = f(1) = (1)^2 - 5(1) + 4 = 0 \tag{3.12.2}$$
$$f(b) = f(4) = (4)^2 - 5(4) + 4 = 0 \tag{3.12.3}$$
$$f(a) = f(b) \tag{3.12.4}$$

Rolle's is valid for f(x). Therefore f'(c) = 0

$$2c - 5 = 0 \tag{3.12.5}$$
$$c = 5/2 \tag{3.12.6}$$

3.13. Evaluate $\int_0^2 (x^2 + 2x + 1)\, dx$ as limit of sum.
Solution:
Limit of the sum is defined as,

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + . + f(a + (n-1)h)] \tag{3.13.1}$$

where,

$$h = \frac{b-a}{n} \tag{3.13.2}$$
$$h = \frac{2}{n} \tag{3.13.3}$$
$$\int_0^2 (x^2 + 2x + 1)dx \tag{3.13.4}$$
$$= \lim_{h\to 0} h[f(0) + f(h) + f(2h) + ... + f((n-1)h)] \tag{3.13.5}$$
$$= \lim_{h\to 0} h[1 + ... + ((n-1)h)^2 + 2((n-1)h) + 1] \tag{3.13.6}$$
$$= \lim_{h\to 0} h[n + ... + ((n-1)h)^2 + 2((n-1)h) + 1] \tag{3.13.7}$$
$$= \lim_{h\to 0} h[n + h^2(1^2 + .. + (n-1)^2 + 2h(1 + .. + (n-1)))] \tag{3.13.8}$$
$$= \lim_{h\to 0} h[n + h^2\left(\frac{n(n-1)(2n-1)}{6}\right) + 2h\left(\frac{n(n-1)}{2}\right) \tag{3.13.9}$$
$$h \to 0; n \to \infty \tag{3.13.10}$$
$$= \lim_{n\to\infty} \frac{2}{n}[n + \left(\frac{2}{n}^2\right)\left(\frac{n(n-1)(2n-1)}{6}\right) + 2\left(\frac{2}{n}\right)\left(\frac{n(n-1)}{2}\right) \tag{3.13.11}$$
$$= \lim_{n\to\infty}[2 + \frac{4}{3}(1 - \frac{1}{n})(2 - \frac{1}{n}) + 4(1 - \frac{1}{n})] \tag{3.13.12}$$
$$= 2 + 4 + \frac{8}{3} = \frac{26}{3} \tag{3.13.13}$$

4 Probability

4.1. An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting

a) 2 red balls
b) 2 blue balls
c) one red and one blue ball

Solution:
Let X be a random variable such that $X \in \{0, 1\}$.
X represent type of ball.
All the red balls are mapped to $X = 0$
All the blue balls are mapped to $X = 1$

$$P(\text{Red}) = P(X = 0) = \frac{7}{11} \tag{4.1.1}$$
$$P(\text{Blue}) = P(X = 1) = \frac{4}{11} \tag{4.1.2}$$

a) 2 red balls

$$P(X = 0).P(X = 0) = \frac{7}{11}.\frac{7}{11} = \frac{49}{121} \tag{4.1.3}$$

b) 2 blue balls

$$P(X = 1).P(X = 1) = \frac{4}{11}.\frac{4}{11} = \frac{16}{121} \tag{4.1.4}$$

c) 1 red and 1 blue ball

$$P(X = 1).P(X = 0) = \frac{4}{11}.\frac{7}{11} = \frac{28}{121} \tag{4.1.5}$$

4.2. A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that it is neither a ace nor a king.
Solution:
Let X be a random variable such that $X \in \{0, 1\}$.
X represent type of card.
Ace is mapped to $X = 0$

King is mapped to $X = 1$

$$P(\text{Ace}) = P(X = 0) = \frac{4}{52} = \frac{1}{13} \quad (4.2.1)$$

$$P(\text{King}) = P(X = 1) = \frac{4}{52} = \frac{1}{13} \quad (4.2.2)$$

$$P(\overline{\text{Ace}}.\overline{\text{king}}) = 1 - P(\text{Ace} + \text{king}) \quad (4.2.3)$$

$$1 - P(\text{Ace} + \text{king}) = 1 - P(X = 0 + X = 1) \quad (4.2.4)$$

$$P(X = 0 + X = 1) = P(X = 0) + P(X = 1) = \frac{2}{13} \quad (4.2.5)$$

$$1 - P(X = 0 + X = 1) = 1 - \frac{2}{13} = \frac{11}{13} \quad (4.2.6)$$

4.3. There are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag II.

Solution:

Let X be a random variable such that $X \in \{0, 1\}$.
X represent a Bag.
Bag1 is mapped to $X = 0$
Bag2 is mapped to $X = 1$
Let Y be a random variable such that $Y \in \{0, 1\}$.
Y represent type of Color.
All the red balls are mapped to $Y = 0$
All the white balls are mapped to $Y = 1$
Given,

$$P(Bag1) = P(X = 0) = \frac{1}{2} \quad (4.3.1)$$

$$P(Bag2) = P(X = 1) = \frac{1}{2} \quad (4.3.2)$$

$$P\left(\frac{\text{Red}}{\text{Bag1}}\right) = P\left(\frac{Y = 0}{X = 0}\right) = \frac{3}{5} \quad (4.3.3)$$

$$P\left(\frac{\text{White}}{\text{Bag1}}\right) = P\left(\frac{Y = 1}{X = 0}\right) = \frac{2}{5} \quad (4.3.4)$$

$$P\left(\frac{\text{Red}}{\text{Bag2}}\right) = P\left(\frac{Y = 0}{X = 1}\right) = \frac{5}{9} \quad (4.3.5)$$

$$P\left(\frac{\text{White}}{\text{Bag2}}\right) = P\left(\frac{Y = 1}{X = 1}\right) = \frac{4}{9} \quad (4.3.6)$$

Applying the Bayes theorem

$$P\left(\frac{\text{Bag2}}{\text{Red}}\right) = P\left(\frac{X = 1}{Y = 0}\right) \quad (4.3.7)$$

$$P\left(\frac{X = 1}{Y = 0}\right) = \frac{P\left(\frac{Y = 0}{X = 1}\right).P(X = 1)}{P\left(\frac{Y = 0}{X = 1}\right).P(X = 1) + P\left(\frac{Y = 0}{X = 0}\right).P(X = 0)} \quad (4.3.8)$$

$$= \frac{\frac{5}{9}.\frac{1}{2}}{\frac{5}{9}.\frac{1}{2} + \frac{3}{5}.\frac{1}{2}} \quad (4.3.9)$$

$$= \frac{25}{52} \quad (4.3.10)$$

4.4. Find the mean μ, variance σ^2 for the following probability distribution:

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

TABLE 4.4

Solution:

Mean or Expectation is defined as

$$E[X] = \Sigma_{i=0}^{\infty} x_i p(X = x) \quad (4.4.1)$$

Mean square is defined as

$$E[X^2] = \Sigma_{i=0}^{\infty} x_i^2 p(X = x) \quad (4.4.2)$$

Variance is defined as

$$var(X) = E[X^2] - E[X]^2 \quad (4.4.3)$$

$$E[X] = \Sigma_{i=0}^{\infty} x_i p(X = x) \tag{4.4.4}$$
$$= 0.\frac{1}{8} + 1.\frac{3}{8} + 2.\frac{3}{8} + 3.\frac{1}{8} \tag{4.4.5}$$
$$= \frac{3}{2} \tag{4.4.6}$$
$$E[X^2] = \Sigma_{i=0}^{\infty} x_i^2 p(X = x) \tag{4.4.7}$$
$$= 0^2.\frac{1}{8} + 1^2.\frac{3}{8} + 2^2.\frac{3}{8} + 3^2.\frac{1}{8} \tag{4.4.8}$$
$$= 3 \tag{4.4.9}$$
$$var(X) = E[X^2] - E[X]^2 \tag{4.4.10}$$
$$= 3 - \left(\frac{3}{2}\right) \tag{4.4.11}$$
$$= \frac{3}{4} \tag{4.4.12}$$

4.5. Find the binomial distribution for which the mean is 4 and variance 3.
Solution:
The PMF of Binomial distribution is defined as,

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \tag{4.5.1}$$

Variance and Expectation of binomial distribution are given by,

$$E[X] = np \tag{4.5.2}$$
$$var[X] = npq \tag{4.5.3}$$

Given,

$$E[X] = 4 \tag{4.5.4}$$
$$var[X] = 3 \tag{4.5.5}$$
$$q = \frac{npq}{np} = \frac{3}{4} \tag{4.5.6}$$
$$p = 1 - q = \frac{1}{4} \tag{4.5.7}$$
$$n = 16 \tag{4.5.8}$$

Therefore, the PMF is defined as

$$P(X = x) = \binom{16}{x} \frac{1}{4}^x \frac{3}{4}^{n-x} \tag{4.5.9}$$

5 Linear Algebra

5.1. If $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$, show that $\mathbf{A}^2 - 6\mathbf{A} + 17\mathbf{I} = 0$. Hence find $\mathbf{A}^{-1}$. **Solution:** Consider the matrix given in the problem statement.

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix} \tag{5.1.1}$$

Considering the characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{5.1.2}$$

From (5.1.2) we get,

$$\begin{vmatrix} 2 - \lambda & -3 \\ 3 & 4 - \lambda \end{vmatrix} = 0 \tag{5.1.3}$$
$$(2 - \lambda)(4 - \lambda) + 9 = 0 \tag{5.1.4}$$
$$\lambda^2 - 6\lambda + 17 = 0 \tag{5.1.5}$$

From the Cayley-Hamilton theorem (5.1.5) can be written as

$$\mathbf{A}^2 - 6\mathbf{A} + 17\mathbf{I} = 0 \tag{5.1.6}$$

Multiplying with $\mathbf{A}^{-1}$ on both sides of equation (5.1.6) We get,

$$\mathbf{A} - 6\mathbf{I} + 17\mathbf{A}^{-1} = 0 \tag{5.1.7}$$
$$\mathbf{A}^{-1} = \frac{6\mathbf{I} - \mathbf{A}}{17} \tag{5.1.8}$$
$$\mathbf{A}^{-1} = \begin{pmatrix} 4/17 & 3/17 \\ -3/17 & 2/17 \end{pmatrix} \tag{5.1.9}$$

5.2. Using the properties of determinants, prove the following:

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution:

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \tag{5.2.1}$$

$$\xleftrightarrow{R_1 \leftarrow R_1+R_2+R_3}$$

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \tag{5.2.2}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \tag{5.2.3}$$

$$\xleftrightarrow{C_2 \leftarrow C_2-C_1, C_3 \leftarrow C_3-C_1}$$

$$(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & a+b+c & 0 \\ 2c & 0 & a+b+c \end{vmatrix} \tag{5.2.4}$$

$$= (a+b+c)(a+b+c)^2 \tag{5.2.5}$$

$$= (a+b+c)^3 \tag{5.2.6}$$

5.3. Using matrices, solve the following system of equation:

$$x+2y-3z=6 \tag{5.3.1}$$

$$3x+2y-2z=3 \tag{5.3.2}$$

$$2x-y+z=2 \tag{5.3.3}$$

Solution: Consider the equations given in the problem statement. The solution can be found by solving the above system of linear equations.
System of linear equations are defined as

$$\mathbf{Ax} = \mathbf{B} \tag{5.3.4}$$

From the equations (5.3.1) ,(5.3.2) and (5.3.3),

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix} \tag{5.3.5}$$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5.3.6}$$

$$\mathbf{B} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{5.3.7}$$

Substituting the values of **A**, **x** and **B** in the equation (5.3.4) We get,

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \tag{5.3.8}$$

Considering the augmented matrix

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 3 & 2 & -2 & 3 \\ 2 & -1 & 1 & 2 \end{pmatrix} \tag{5.3.9}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-3R_1, R_3 \leftarrow R_3-2R_1} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & -5 & 7 & -10 \end{pmatrix} \tag{5.3.10}$$

$$\xleftrightarrow{R_3 \leftarrow 4R_3-5R_2} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & 0 & -7 & 35 \end{pmatrix} \tag{5.3.11}$$

$$\xleftrightarrow{R_3 \leftarrow R_3/-7} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 7 & -15 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{5.3.12}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-7R_3} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -4 & 0 & 20 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{5.3.13}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/-4} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{5.3.14}$$

$$\xleftrightarrow{R_1 \leftarrow R_1-2R_2, R_1 \leftarrow R_1+3R_2} \begin{pmatrix} 1 & 0 & -0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{pmatrix} \tag{5.3.15}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -5 \end{pmatrix} \tag{5.3.16}$$

By solving equation (5.3.16) we get,

$$x = 1 \tag{5.3.17}$$

$$y = -5 \tag{5.3.18}$$

$$z = -5 \tag{5.3.19}$$

Therefore, x=1, y= -5 and z=-5 are solutions to the given equations.

5.4. Find the projection of $\overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{c}}$ on $\overrightarrow{\mathbf{a}}$ where $\overrightarrow{\mathbf{a}} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\overrightarrow{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\overrightarrow{\mathbf{c}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

Solution: Consider the given vectors,

$$\mathbf{A} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{5.4.1}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \tag{5.4.2}$$

$$\mathbf{C} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \tag{5.4.3}$$

Projection of $(\mathbf{B} + \mathbf{C})$ on $\mathbf{A}$ is given by

$$\frac{(\mathbf{B} + \mathbf{C})^T \mathbf{A}}{\|\mathbf{A}\|} \tag{5.4.4}$$

By substituting $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in (5.4.4) we get,

$$\frac{(\mathbf{B} + \mathbf{C})^T \mathbf{A}}{\|\mathbf{A}\|} = \frac{\begin{pmatrix} 3 & 1 & 2 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{4+4+1}} = 2 \tag{5.4.5}$$

5.5. Find the value of λ which makes the vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ coplanar, where $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}, \vec{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\vec{\mathbf{c}} = 3\hat{\mathbf{i}} - \lambda\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

Solution: Consider the given vectors,

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{5.5.1}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \tag{5.5.2}$$

$$\mathbf{C} = \begin{pmatrix} 3 \\ -\lambda \\ 5 \end{pmatrix} \tag{5.5.3}$$

For the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ to be coplanlar, the three vectors are linearly dependent. Therfore,

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & -\lambda & 5 \end{vmatrix} = 0 \tag{5.5.4}$$

$$= 2(10 - 3\lambda) + 1(5 + 9) + 1(-\lambda - 6) = 0 \tag{5.5.5}$$

$$\lambda = 4 \tag{5.5.6}$$

5.6. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

Solution: consider the given planes as

$$\mathbf{A}^T\mathbf{x} = c_1$$

$$= \begin{pmatrix} 5 & 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -8 \tag{5.6.1}$$

$$\mathbf{B}^T\mathbf{x} = c_2$$

$$= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 \tag{5.6.2}$$

$$\mathbf{C}^T\mathbf{x} = c_3$$

$$= \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -5 \tag{5.6.3}$$

Plane $\perp$ to $5x + 3y + 6z + 8 = 0$ is given by,

$$(\mathbf{B} + k\mathbf{C})\mathbf{x} = c \tag{5.6.4}$$

$$\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right)^T . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 - 5k \tag{5.6.5}$$

The plane in equation (5.6.5) is $\perp$ to plane in equation (5.6.1). Therefore,

$$\mathbf{A}^T . (\mathbf{B} + k\mathbf{C}) = 0 \tag{5.6.6}$$

$$\begin{pmatrix} 5 & 3 & 6 \end{pmatrix} . \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right] = 0 \tag{5.6.7}$$

$$k = \frac{-29}{7} \tag{5.6.8}$$

The required plane is

$$\begin{pmatrix} 51 & 15 & -50 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -173 \tag{5.6.9}$$

5.7. Find the resultant of two velocities 4 m/sec and 6 m/sec inclined to one another at an angle of 120°.

Solution:

$$\mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{5.7.1}$$

$$\mathbf{B} = \begin{pmatrix} 6\cos 120^\circ \\ 6\sin 120^\circ \end{pmatrix} = \begin{pmatrix} -3 \\ 5.196 \end{pmatrix} \tag{5.7.2}$$

$$\text{Resultant} = \mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 \\ 5.196 \end{pmatrix} \tag{5.7.3}$$

5.8. A body of weight 70 N is suspended by two

strings of length 27 cm and 36 cm, fastened to two points in the same horizontal line 45 cm apart and is in equilibrium. Find the tensions in the strings.
Solution:
All the forces are shown in the figure.
Resolving the Tension along horizontal and vertical directions.

$$\cos\theta_1 = \sin\theta_2 = \frac{27}{45} \tag{5.8.1}$$

$$\cos\theta_2 = \sin\theta_1 = \frac{36}{45} \tag{5.8.2}$$

Equating the Horizontal forces

$$T_1\cos\theta_1 - T_2\cos\theta_2 = 0 \tag{5.8.3}$$

Equating the Vertical forces

$$T_1\sin\theta_1 + T_2\sin\theta_2 = W \tag{5.8.4}$$

Putting in a matrix form

$$\begin{pmatrix}\cos\theta_1 & -\cos\theta_2\\ \sin\theta_1 & \sin\theta_2\end{pmatrix}\begin{pmatrix}T_1\\T_2\end{pmatrix} = \begin{pmatrix}W\\0\end{pmatrix} \tag{5.8.5}$$

$$\begin{pmatrix}\frac{27}{45} & -\frac{36}{45}\\ \frac{36}{45} & \frac{27}{45}\end{pmatrix}\begin{pmatrix}T_1\\T_2\end{pmatrix} = \begin{pmatrix}70\\0\end{pmatrix} \tag{5.8.6}$$

Solving the system of linear equations in (5.8.6) we get,

$$\begin{pmatrix}T_1\\T_2\end{pmatrix} = \begin{pmatrix}57.71\\-44.21\end{pmatrix} \tag{5.8.7}$$

6 Optimization

6.1. Find the point on the curve $x^2 = 8y$ which is nearest to the point (2,4).
Solution:
Given point,

$$\mathbf{p} = \begin{pmatrix}2\\4\end{pmatrix} \tag{6.1.1}$$

Forming an optimisation problem,
Variable of optimisation

$$\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix} \tag{6.1.2}$$

Objective function is defined as,

$$Z = \min_{\mathbf{x}} \|\mathbf{x} - \mathbf{p}\| \tag{6.1.3}$$

Constraints

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c = 0 \tag{6.1.4}$$

$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \tag{6.1.5}$$

$$\mathbf{b} = \begin{pmatrix}-1/2\\0\end{pmatrix} \tag{6.1.6}$$

$$c = 0 \tag{6.1.7}$$

The problem can be reframed as

$$\begin{aligned} Z = &\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{p}\| \\ \text{s.t.} \quad &\mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c = 0 \end{aligned} \tag{6.1.8}$$

6.2. Show that a right circular cylinder of least curved surface and given volume 125 cm^3 has an altitude equal to $\sqrt{2}$ times the radius of the base.
Solution:
Given,

Symbol	Description
r	radius of cylinder
h	height of cylinder
s	surface area
v	volume

TABLE 6.2

$$s = \pi r h \tag{6.2.1}$$

$$v = \pi r^2 h \tag{6.2.2}$$

Framing as an optimization problem

$$\begin{aligned} &\min_{r,h} \quad s \\ &\text{s.t.} \quad 125 = \pi r^2 h \\ &\qquad h = \sqrt{2} r \end{aligned} \tag{6.2.3}$$

By solving in cvxpy we get the minimum surface area as $41cm^2$

6.3. If a young man rides his motorcycle at 25 km/hour, he had to spend Rs. 2 per km on petrol. If he rides at a faster speed of 40 km/hour, the petrol cost increases at Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to find what is the maximum distance he can travel within one hour. Express this as an LPP.

Solution:

Let us assume that the man travels x km when the speed is 25 km/hour and y km when the speed is 40 km/hour.

Thus, the total distance travelled is given by

$$z = (x + y)km \tag{6.3.1}$$

Now, it is given that the man has Rs 100 to spend on petrol.

$$\text{total cost} \;= 2x + 5y \leq 100 \tag{6.3.2}$$

Now, time taken to travel x km = $\frac{x}{25}hrs$

Time taken to travel y km = $\frac{y}{40}hrs$

Now, it is given that the maximum time is 1 hour. So, we have

$$\frac{x}{25} + \frac{y}{40} \leq 1 \tag{6.3.3}$$

$$8x + 5y \leq 200 \tag{6.3.4}$$

Also, it is clear that $x \geq 0 \text{and} y \geq 0$.

Variable of optimization,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{6.3.5}$$

Objective,

$$z = \mathbf{c}^T\mathbf{x} \tag{6.3.6}$$

$$z = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{6.3.7}$$

Constraints,

$$\mathbf{Px} \leq \mathbf{q} \tag{6.3.8}$$

$$\begin{pmatrix} 2 & 5 \\ 8 & 5 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 100 \\ 200 \\ 0 \\ 0 \end{pmatrix} \tag{6.3.9}$$

The LPP is formed as

$$\max_{\mathbf{x}} \quad z \qquad \text{s.t.} \quad \mathbf{Px} \leq \mathbf{q} \tag{6.3.10}$$

By solving in cvxpy we get the maximum distance as 30km.