

CBSE MATHEMATICS 2020

G V V Sharma*

1 MATRICES

- 1.1. Show that the plane $x - 5y - 2z = 1$ contains the line $\frac{x-5}{3} = y = 2 - z$.

Solution: The plane and line can be expressed in vector form as

$$(1 \ -5 \ -2) \mathbf{x} = 1 \quad (1.1.1)$$

$$\mathbf{x} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \quad (1.1.2)$$

The plane contains the line if

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (1.1.3)$$

The input parameters are

$$\mathbf{m} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{n} = (1 \ -5 \ -2). \quad (1.1.4)$$

\therefore

$$(1 \ -5 \ -2) \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 0, \quad (1.1.5)$$

the given plane contains the given line.

- 1.2. Find a vector \vec{r} equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units.

Solution: Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis vectors (direction vectors of the coordinate axes) such that

$$(\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3) = \mathbf{I} \quad (1.2.1)$$

Then,

$$\frac{\mathbf{e}_1^\top \mathbf{r}}{\|\mathbf{e}_1\| \|\mathbf{r}\|} = \frac{\mathbf{e}_2^\top \mathbf{r}}{\|\mathbf{e}_2\| \|\mathbf{r}\|} = \frac{\mathbf{e}_3^\top \mathbf{r}}{\|\mathbf{e}_3\| \|\mathbf{r}\|} = \cos \theta \quad (1.2.2)$$

which can be expressed as the system of equations

$$\mathbf{e}_1^\top \mathbf{r} = \|\mathbf{r}\| \cos \theta \quad (1.2.3)$$

$$\mathbf{e}_2^\top \mathbf{r} = \|\mathbf{r}\| \cos \theta \quad (1.2.4)$$

$$\mathbf{e}_3^\top \mathbf{r} = \|\mathbf{r}\| \cos \theta \quad (1.2.5)$$

which can be combined to obtain the matrix equation

$$(\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3)^\top \mathbf{r} = \|\mathbf{r}\| \cos \theta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.2.6)$$

$$\Rightarrow \frac{\mathbf{r}}{\|\mathbf{r}\|} = \cos \theta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.2.7)$$

$$\therefore (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3) = \mathbf{I} \quad (1.2.8)$$

From (1.2.7)

$$\left\| \cos \theta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 1 \quad (1.2.9)$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{3}} \quad (1.2.10)$$

From (1.2.10) and (1.2.7)

$$\mathbf{r} = 3\sqrt{3} \times \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.2.11)$$

$$= 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (1.2.12)$$

- 1.3. Find the angle between unit vectors \vec{a} and \vec{b} so that $\sqrt{3} \vec{a} - \vec{b}$ is also a unit vector.

Solution: From the given information,

$$\|\sqrt{3}\mathbf{a} - \mathbf{b}\| = 1 \quad (1.3.1)$$

$$\Rightarrow \|\sqrt{3}\mathbf{a} - \mathbf{b}\|^2 = 1 \quad (1.3.2)$$

which can be expressed as

$$(\sqrt{3}\mathbf{a} - \mathbf{b})^\top (\sqrt{3}\mathbf{a} - \mathbf{b}) = 1 \quad (1.3.3)$$

$$\Rightarrow 3\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\sqrt{3}\mathbf{a}^\top \mathbf{b} = 1 \quad (1.3.4)$$

and simplified to obtain

$$\mathbf{a}^\top \mathbf{b} = \frac{\sqrt{3}}{2} \quad (1.3.5)$$

Thus the desired angle is

$$\cos^{-1} \frac{\sqrt{3}}{2} = 30^\circ \quad (1.3.6)$$

1.4. If $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, Find scalar k so that $\mathbf{A}^2 + \mathbf{I} = k\mathbf{A}$.

Solution: Using the Cayley-Hamilton theorem,

$$\lambda^2 - k\lambda + 1 = 0 \quad (1.4.1)$$

Since the trace of the matrix is equal to the sum of its eigenvalues,

$$k = \lambda_1 + \lambda_2 = \text{tr}(\mathbf{A}) = -3 - 1 = -4 \quad (1.4.2)$$

1.5. Find the coordinates of the point where the line

$$\frac{x-1}{3} = \frac{y+4}{7} = \frac{z+4}{2}$$

cuts the xy -plane.

Solution: The given line can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \quad (1.5.1)$$

and the xy - plane is

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (1.5.2)$$

The desired point can be obtained as

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \quad (1.5.3)$$

Substituting the input parameters

$$\mathbf{m} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, c = 0, \quad (1.5.4)$$

in (1.5.3),

$$\begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + \frac{0 - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}}{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}} \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \quad (1.5.5)$$

$$= \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \quad (1.5.6)$$

$$= \begin{pmatrix} 7 \\ 10 \\ 0 \end{pmatrix} \quad (1.5.7)$$

1.6. The angle between the vectors $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$ is

- a) $-\frac{\pi}{3}$
- b) 0
- c) $\frac{\pi}{3}$
- d) $\frac{2\pi}{3}$

Solution: The angle between the vectors $\mathbf{m}_1, \mathbf{m}_2$ is given by

$$\cos \theta = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \quad (1.6.1)$$

Substituting

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (1.6.2)$$

in (1.6.1),

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1, \quad (1.6.3)$$

$$\left\| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{2} \quad (1.6.4)$$

$$\Rightarrow \cos \theta = \frac{-1}{2} \quad (1.6.5)$$

$$\text{or, } \theta = \frac{2\pi}{3} \quad (1.6.6)$$

1.7. If \mathbf{A} is a non-singular square matrix of order 3 such that $\mathbf{A}^2 = 3\mathbf{A}$, then value of $|\mathbf{A}|$ is

- a) -3
- b) 3
- c) 9

d) 27

$$|\mathbf{A}^2| = |3\mathbf{A}| \quad (1.7.1)$$

$$\Rightarrow |\mathbf{A}|^2 = 3^3 |\mathbf{A}| \quad (1.7.2)$$

yielding

$$|\mathbf{A}| = 27 \quad (1.7.3)$$

after simplification.

1.8. If $|\vec{a}| = 4$ and $-3 \leq \lambda \leq 2$ then $|\lambda \vec{a}|$ lies in

- a) $[0, 12]$
- b) $[2, 3]$
- c) $[8, 12]$
- d) $[-12, 8]$

Solution:

$$\|\lambda \mathbf{a}\| = |\lambda| \|\mathbf{a}\| \quad (1.8.1)$$

$$= 4|\lambda| \quad (1.8.2)$$

 \therefore

$$0 \leq |\lambda| \leq 3, \quad (1.8.3)$$

$$0 \leq 4|\lambda| \leq 12 \quad (1.8.4)$$

1.9. The area of a triangle formed by vertices O, A and B, where $\vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$ is

- a) $3\sqrt{5}$ sq.units
- b) $5\sqrt{5}$ sq.units
- c) $6\sqrt{5}$ sq.units
- d) 4 sq.units

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}, \quad (1.9.1)$$

In general, the area of $\triangle OAB$ can be expressed as

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{O}) \times (\mathbf{B} - \mathbf{O})\| \quad (1.9.2)$$

 \therefore

$$\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = 8, \quad (1.9.3)$$

$$\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10, \quad (1.9.4)$$

$$\begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix} = 4, \quad (1.9.5)$$

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 8 \\ -10 \\ 4 \end{pmatrix}, \quad (1.9.6)$$

and the desired area can be obtained as

$$\frac{1}{2} \left\| \begin{pmatrix} 8 \\ -10 \\ 4 \end{pmatrix} \right\| = 3\sqrt{5} \quad (1.9.7)$$

1.10. The coordinates of the foot of the perpendicular drawn from the point $(2, -3, 4)$ on the y -axis is

- a) $(2, 3, 4)$
- b) $(-2, -3, -4)$
- c) $(0, -3, 0)$
- d) $(2, 0, 4)$

Solution: In general, let \mathbf{P} be any point and the line be

$$L: \mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (1.10.1)$$

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \quad (1.10.2)$$

The equation of the y -axis can be written as

$$\mathbf{x} = \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (1.10.3)$$

Substituting the input parameters

$$\mathbf{P} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (1.10.4)$$

in (1.10.2), the desired point is given by

$$\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (1.10.5)$$

$$= \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \quad (1.10.6)$$

1.11. The distance between parallel planes $2x+y-2z-6=0$ and $4x+2y-4z=0$ is _____ units.**Solution:** The above planes have parameters

$$\mathbf{n} = (2 \ 1 \ -2), c_1 = 6, c_2 = 0 \quad (1.11.1)$$

The distance is then obtained as

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \quad (1.11.2)$$

$$= \frac{6}{3} = 2 \quad (1.11.3)$$

1.12. If $P(1,0,-3)$ is the foot of the perpendicular from the origin to the plane, then the Cartesian equation of the plane is

Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = c \quad (1.12.1)$$

Since \mathbf{P} is a point on the plane, it satisfies the above equation and

$$\mathbf{n}^\top \mathbf{P} = c \quad (1.12.2)$$

The normal vector to the plane is OP . Hence,

$$\mathbf{n} = \mathbf{P} \quad (1.12.3)$$

Substituting the above in (1.12.2),

$$\mathbf{P}^\top \mathbf{P} = c \quad (1.12.4)$$

and the desired equation of the plane is

$$\mathbf{P}^\top \mathbf{x} = \mathbf{P}^\top \mathbf{P} \quad (1.12.5)$$

$$(1 \ 0 \ -3) \mathbf{x} = 10 \quad (1.12.6)$$

after substituting numerical values.

1.13. Find the equation of the plane passing through the points $(1, 0, -2)$, $(3, -1, 0)$ and perpendicular to the plane $2x - y + z = 8$. Also find the distance of the plane thus obtained from the origin.

Solution: Let the equation of the desired plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \quad (1.13.1)$$

From the given information,

$$\mathbf{n}^\top \mathbf{x} = 1 \quad (1.13.2)$$

$$\begin{aligned} (1 \ 0 \ -2) \mathbf{n} &= 1 \\ \Rightarrow (3 \ -1 \ 0) \mathbf{n} &= 1 \\ (2 \ -1 \ 1) \mathbf{n} &= 0 \end{aligned} \quad (1.13.3)$$

From (1.13.3), we obtain the matrix equation

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (1.13.4)$$

Forming the augmented matrix, and choosing the pivot,

$$\left(\begin{array}{ccc|c} \textcircled{1} & 0 & -2 & 1 \\ 3 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array} \right) \quad (1.13.5)$$

$$\leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & \textcircled{1} & -6 & 2 \\ 0 & -1 & 5 & -2 \end{array} \right) \quad (1.13.6)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & \textcircled{1} & 0 \end{array} \right) \quad (1.13.7)$$

yielding

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad (1.13.8)$$

Thus, the equation of the desired plane is

$$(1 \ 2 \ 0) \mathbf{n} = 1 \quad (1.13.9)$$

1.14. If

$$\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix} = \begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$$

then value of x is _____

Solution: Expanding the above determinants,

$$2x^2 - 18 = 0 \quad (1.14.1)$$

$$\Rightarrow x = \pm 3 \quad (1.14.2)$$

1.15. Using integration, Find the area lying above x -axis and included between the circle $x^2 + y^2 = 8x$ and inside the parabola $y^2 = 4x$.

1.16. Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(2,0)$, $B(4,5)$ and $C(6,3)$.

Solution: Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}, \quad (1.16.1)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.16.2)$$

the desired area is the magnitude of

$$\begin{vmatrix} 2 & 4 \\ 5 & 3 \end{vmatrix} \quad (1.16.3)$$

Thus the desired area is 14 units.

1.17. If $A = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix}$, Find A^{-1} and use it to solve the following system of the equations:

$$5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Solution: Forming the augmented matrix and pivoting,

$$\left(\begin{array}{ccc|ccc} \textcircled{5} & -1 & 4 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 5 & -2 & 6 & 0 & 0 & 1 \end{array} \right) \quad (1.17.1)$$

$$\leftrightarrow \left(\begin{array}{ccc|ccc} 5 & -1 & 4 & 1 & 0 & 0 \\ 0 & \textcircled{17} & 17 & -2 & 5 & 0 \\ 0 & 1 & -2 & 1 & 0 & -1 \end{array} \right) \quad (1.17.2)$$

$$\leftrightarrow \left(\begin{array}{ccc|ccc} 17 & 0 & 17 & 3 & 1 & 0 \\ 0 & 17 & 17 & -2 & 5 & 0 \\ 0 & 0 & \textcircled{51} & -19 & 5 & 17 \end{array} \right) \quad (1.17.3)$$

$$\xleftrightarrow[R_2 \leftarrow 3R_2 - R_3]{R_1 \leftarrow 3R_1 - R_3} \quad (1.17.4)$$

$$\left(\begin{array}{ccc|ccc} 51 & 0 & 0 & 28 & -2 & -17 \\ 0 & 51 & 0 & 13 & 10 & -17 \\ 0 & 0 & 51 & -19 & 5 & 17 \end{array} \right) \quad (1.17.5)$$

resulting in

$$A^{-1} = \frac{1}{51} \begin{pmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{pmatrix} \quad (1.17.6)$$

1.18. If x, y, z are different and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0 \quad (1.18.1)$$

then using properties of determinants show that

$$1 + xyz = 0$$

Solution: The given determinant can be expressed as

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \quad (1.18.2)$$

Since

$$\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (1.18.3)$$

and

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \quad (1.18.4)$$

(1.18.2) can be expressed as

$$(1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \quad (1.18.5)$$

The above determinant can be simplified as

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \quad (1.18.6)$$

$$\xleftrightarrow[R_2 \leftarrow R_1 - R_2]{R_3 \leftarrow R_1 - R_3} \begin{vmatrix} 1 & x & x^2 \\ 0 & x - y & x^2 - y^2 \\ 0 & x - z & x^2 - z^2 \end{vmatrix}, \quad (1.18.7)$$

$$= (x - y)(x - z) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x + y \\ 0 & 1 & x + z \end{vmatrix}, \quad (1.18.8)$$

$$= (x - y)(y - z)(z - x) \quad (1.18.9)$$

and (1.18.1) can be obtained from (1.18.5) as

$$(1 + xyz)(x - y)(y - z)(z - x) = 0 \quad (1.18.10)$$

Since

$$x \neq y \neq z, (1 + xyz) = 0 \quad (1.18.11)$$

2 CONTINUOUS MATH

2.1. Evaluate :

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Solution: Let

$$\tan \theta = \sin x \quad (2.1.1)$$

Then

$$\sec^2 \theta d\theta = \cos x dx \quad (2.1.2)$$

From (2.1.1) and (2.1.2)

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx \\ = 2 \int_0^{\frac{\pi}{4}} \theta \tan \theta \sec^2 \theta d\theta \end{aligned} \quad (2.1.3)$$

Letting

$$\begin{aligned} u = \theta, dv = \tan \theta \sec^2 \theta d\theta, \\ v = \int \tan \theta \sec^2 \theta d\theta \\ = \int t dt \quad (t = \tan \theta) \\ = \frac{t^2}{2} = \frac{\tan^2 \theta}{2} \end{aligned} \quad (2.1.4)$$

Thus,

$$\begin{aligned} v du = \frac{\tan^2 \theta}{2} d\theta \\ \Rightarrow \int v du = \int \frac{\tan^2 \theta}{2} d\theta \\ = \int \frac{\sec^2 \theta - 1}{2} d\theta \\ = \frac{\tan \theta - 1}{2} \end{aligned} \quad (2.1.5)$$

and

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx \\ = \left[\theta \frac{\tan^2 \theta}{2} - \frac{\tan \theta - 1}{2} \right]_0^{\frac{\pi}{4}} \\ = \frac{\pi}{8} - \frac{1}{2} \end{aligned} \quad (2.1.6)$$

2.2. Prove that

$$\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \frac{1}{2} \sin^{-1} \left(\frac{4}{5} \right)$$

Solution:

$$\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \tan^{-1} \frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4} \times \frac{2}{9}} \quad (2.2.1)$$

$$= \tan^{-1} \frac{1}{2} \quad (2.2.2)$$

$$= \frac{1}{2} \tan^{-1} \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} \quad (2.2.3)$$

$$= \frac{1}{2} \tan^{-1} \frac{4}{3} \quad (2.2.4)$$

$$= R.H.S \quad (2.2.5)$$

2.3. Differentiate $\sec^2(x^2)$ with respect to x^2 .

Solution:

$$\frac{\sec^2(x^2)}{d(x^2)} = 2 \sec(x^2) \tan(x^2) \quad (2.3.1)$$

2.4. If $y = f(x^2)$ and $f'(x) = e^{(\sqrt{x})}$, then find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = 2x f'(x^2) = 2xe^x \quad (2.4.1)$$

2.5. If

$$\tan^{-1} \left(\frac{y}{x} \right) = \log \sqrt{x^2 + y^2}$$

prove that

$$\frac{dy}{dx} = \frac{x+y}{x-y} \quad (2.5.1)$$

Solution: Let

$$y = x \tan \theta \quad (2.5.2)$$

$$\Rightarrow \frac{dy}{dx} = \tan \theta + x \sec^2 \theta \frac{d\theta}{dx} \quad (2.5.3)$$

Then, (2.5.1) can be expressed as

$$\theta = \log(x \sec \theta) \quad (2.5.4)$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{1}{x \sec \theta} \left(\sec \theta + x \sec \theta \tan \theta \frac{d\theta}{dx} \right) \quad (2.5.5)$$

$$= \frac{1}{x} \left(1 + x \tan \theta \frac{d\theta}{dx} \right) \quad (2.5.6)$$

$$\text{or, } \frac{d\theta}{dx} = \frac{1}{x(1 - \tan \theta)} \quad (2.5.7)$$

From (2.5.3) and (2.5.7)

$$\frac{dy}{dx} = \tan \theta + \frac{\sec^2 \theta}{(1 - \tan \theta)} \quad (2.5.8)$$

$$= \frac{1 + \tan \theta}{1 - \tan \theta} \quad (2.5.9)$$

$$= \frac{x + y}{x - y} \quad (2.5.10)$$

upon substituting from (2.5.1) and simplifying.

2.6. If

$$y = e^{(a \cos^{-1} x)}, -1 < x < 1$$

then show that

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

Solution: From the given information,

$$y_1 = -\frac{ae^{(a \cos^{-1} x)}}{\sqrt{1 - x^2}} \quad (2.6.1)$$

$$= -\frac{ay}{\sqrt{1 - x^2}} \implies \sqrt{1 - x^2} y_1 + ay = 0 \quad (2.6.2)$$

Hence, differentiating the above equation,

$$\sqrt{1 - x^2} y_2 - \frac{2xy_1}{\sqrt{1 - x^2}} + ay_1 = 0 \quad (2.6.3)$$

$$\implies (1 - x^2) y_2 - 2xy_1 - a^2 y = 0 \quad (2.6.4)$$

2.7. If $\cos(\sin^{-1} \frac{2}{\sqrt{5}} + \cos^{-1} x) = 0$, then x is

a) $\frac{1}{\sqrt{5}}$

b) $\frac{-2}{\sqrt{5}}$

c) $\frac{2}{\sqrt{5}}$

d) 1

Solution: Using a simplistic approach, $\therefore \cos \frac{\pi}{2} = 0$,

$$\sin^{-1} \frac{2}{\sqrt{5}} + \cos^{-1} x = \frac{\pi}{2} \quad (2.7.1)$$

$$\implies \frac{\pi}{2} - \cos^{-1} x = -\sin^{-1} \frac{2}{\sqrt{5}} \quad (2.7.2)$$

$$\text{or, } \sin^{-1} x = \sin^{-1} \left(-\frac{2}{\sqrt{5}} \right) \quad (2.7.3)$$

$$\implies x = -\frac{2}{\sqrt{5}} \quad (2.7.4)$$

2.8. The interval in which the function f given by $f(x) = x^2 e^{-x}$ is strictly increasing, is

a) $(\infty, -\infty)$

b) $(\infty, 0)$

c) $(2, \infty)$

d) $(0, 2)$

Solution: Taking the derivative

$$f'(x) = 2xe^{-x} - x^2 e^{-x} \quad (2.8.1)$$

$$= (2x - x^2) e^{-x} \quad (2.8.2)$$

and

$$f'(x) > 0 \quad (2.8.3)$$

$$\implies x(2 - x) > 0 \quad (2.8.4)$$

$$\text{or, } x \in (0, 2) \quad (2.8.5)$$

2.9. The function $f(x) = \frac{x-1}{x(x^2-1)}$ is discontinuous at

a) Exactly one point

b) Exactly two points

c) Exactly three points

d) No point

Solution: The given function can be expressed as

$$f(x) = \frac{x-1}{x(x-1)(x+1)} \quad (2.9.1)$$

Hence, the denominator of the given function vanishes at $x = 0, 1, -1$, which are the points of discontinuity. However, there is a removable discontinuity at $x = 1$.

2.10. The function $f : R \rightarrow [-1, 1]$ defined by $f(x) = \cos x$ is

a) Both one-one and onto

b) Not one-one, but onto

c) one-one, but Not onto

d) Neither one-one, nor onto

Solution:

$$\cos(-2\pi) = \cos(2\pi) = 1 \quad (2.10.1)$$

Hence, $f(x)$ is not one to one. However, it is onto.

2.11. If the radius of the circle is increasing at the rate of 0.5cm/s, then the rate of increase of its circumference is _____

Solution: Let the p be the circumference and r the radius. Then

$$p = 2\pi r \quad (2.11.1)$$

$$\implies \frac{dp}{dt} = 2\pi \frac{dr}{dt} = \pi \quad (2.11.2)$$

2.12. a) The range of the principle value branch of the function $y = \sec^{-1} x$ is _____

Solution: The range is $[0, \pi] - \{\frac{\pi}{2}\}$.

b) The principal value of $\cos^{-1}(\frac{-1}{2})$ is _____

Solution: Since

$$\cos \frac{\pi}{3} = \frac{1}{2}, \quad (2.12.1)$$

$$\cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2} \quad (2.12.2)$$

Thus the desired principal value is $\frac{2\pi}{3}$

2.13. Evaluate :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos^2 x dx$$

Solution: \because the integrand is odd, the given integral is 0.

2.14. Find the value of k, so that the function

$$f(x) = \begin{cases} kx^2 + 5, & x \leq 1 \\ 2, & x > 1 \end{cases}$$

is continuous at $x=1$.

Solution: From the given equation,

$$f(1+) = f(1) \quad (2.14.1)$$

$$\Rightarrow k + 5 = 2 \quad (2.14.2)$$

$$\text{or, } k = -3 \quad (2.14.3)$$

2.15. Find the integrating factor of the differential equation

$$x \frac{dy}{dx} = 2x^2 + y \quad (2.15.1)$$

Solution: The given differential equation can be expressed as

$$\frac{dy}{dx} - \frac{y}{x} = 2x \quad (2.15.2)$$

Letting

$$P(x) = -\frac{1}{x}, Q(x) = 2x, \quad (2.15.3)$$

$$M(x) = C e^{\int P(x) dx} = e^{-\ln x} \quad (2.15.4)$$

$$= \frac{1}{x} \quad (2.15.5)$$

which is the desired integrating factor.

2.16. If

$$f(x) = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$$

Find

$$f'\left(\frac{\pi}{3}\right)$$

Solution: The given equation can be expressed as

$$f(x) = \sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad (2.16.1)$$

$$= \sqrt{2 \sin^2 \frac{x}{2} 2 \cos^2 \frac{x}{2}} \quad (2.16.2)$$

$$= \tan \frac{x}{2} \quad (2.16.3)$$

$$\Rightarrow f'(x) = f'\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad (2.16.4)$$

2.17. Find $f'(x)$ if

$$f(x) = (\tan x)^{(\tan x)} \quad (2.17.1)$$

Solution: Differentiating the above,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left\{ (\tan x)^{(\tan x)} \right\} \\ &= (\tan x) (\tan x)^{(\tan x)-1} \sec^2 x \\ &\quad + (\tan x)^{(\tan x)} \ln (\tan x) \sec^2 x \\ &= \sec^2 (\tan x)^{(\tan x)} (1 + \ln (\tan x)) \end{aligned} \quad (2.17.2)$$

2.18. Find :

$$\int \frac{\tan^3 x}{\cos^3 x} dx \quad (2.18.1)$$

Solution: Letting

$$t = \sec x, dt = \sec x \tan x dx \quad (2.18.2)$$

$$\Rightarrow \int \frac{\tan^3 x}{\cos^3 x} dx = \int (\sec x \tan x)^3 dx \quad (2.18.3)$$

$$= \int t^2 (t^2 - 1) dt \quad (2.18.4)$$

$$= \frac{t^5}{5} - \frac{t^3}{3} \quad (2.18.5)$$

$$= \frac{\sec^5}{5} - \frac{\sec^3}{3} \quad (2.18.6)$$

2.19. Solve the following differential equation:

$$(1 + e^{\frac{y}{x}}) dy + e^{\frac{y}{x}} \left(1 - \frac{y}{x}\right) dx = 0; (x \neq 0) \quad (2.19.1)$$

Solution: Letting

$$y = xt, dy = x dt + t dx \quad (2.19.2)$$

(2.19.1) can be expressed as

$$\begin{aligned}
 (1 + e^t)(x dt + t dx) + e^t(1 - t)dx &= 0 \\
 \Rightarrow x(1 + e^t) dt &+ \{t(1 + e^t) + (1 - t)e^t\} dx = 0 \\
 \Rightarrow \frac{dx}{x} &= -\frac{(1 + e^t)}{t + e^t} dt \\
 \Rightarrow \int \frac{dx}{x} &= -\int \frac{(1 + e^t)}{t + e^t} dt \\
 \text{or, } \ln x &= -\ln\left(\frac{y}{x} + e^{\frac{y}{x}}\right) + C_1 \quad (2.19.3)
 \end{aligned}$$

Thus,

$$y + xe^{\frac{y}{x}} = C \quad (2.19.4)$$

3 DISCRETE MATH

3.1. The relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2)(2, 1)(1, 1)\}$ is

- Symmetric and transitive, but not reflexive
- reflexive and symmetric, but not transitive
- Symmetric, but neither reflexive nor transitive
- An equivalence relation

3.2. Check whether the relation R in the set N set of natural numbers given by $R = \{(a, b) : a \text{ is divisor of } b\}$ is reflexive, symmetric or transitive. Also determine whether R is an equivalence relation.

4 PROBABILITY

4.1. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn randomly one-by-one without replacement and are found to be both kings. Find the probability of the lost card being a king.

Solution: See Tables (4.1.1) and (4.1.2) for

| Event | Description |
|-----------|-----------------------------|
| $X_1 = 1$ | Lost card being king |
| $X_2 = 1$ | One card drawn being a king |
| $X_3 = 2$ | Two cards drawn being kings |

TABLE 4.1.1

| Probability | Value |
|---------------------------------|-------------------------------|
| $\Pr(X_1 = 1)$ | $\frac{4}{52} = \frac{1}{13}$ |
| $\Pr(X_1 = 0)$ | $\frac{12}{13}$ |
| $\Pr(X_2 = 1 X_1 = 1)$ | $\frac{3}{51} = \frac{1}{17}$ |
| $\Pr(X_2 = 1 X_1 = 0)$ | $\frac{4}{51}$ |
| $\Pr(X_3 = 1 X_1 = 1, X_2 = 1)$ | $\frac{2}{50} = \frac{1}{25}$ |
| $\Pr(X_3 = 1 X_1 = 0, X_2 = 1)$ | $\frac{3}{50}$ |
| $\Pr(X_1 = 1 X_2 = 1, X_3 = 1)$ | ? |

TABLE 4.1.2

the input probabilities. The desired probability is then obtained from (4.1.2) as

$$\begin{aligned}
 \Pr(X_1 = 1|X_2 = 1, X_3 = 1) \\
 = \frac{\frac{1}{25} \times \frac{1}{17} \times \frac{1}{13}}{\frac{1}{25} \times \frac{1}{17} \times \frac{1}{13} + \frac{3}{50} \times \frac{4}{51} \times \frac{12}{13}} = \frac{1}{25} \quad (4.1.1)
 \end{aligned}$$

4.2. A fair dice is thrown two times. Find the probability distribution of the number of sixes. Also determine the mean of the number of sixes.

Solution: Let $X = \{0, 1, 2\}$ denote the sample space. Table (4.2.1) gives the desired probability distribution. In general, this can be expressed as a Binomial distribution with probability mass function (pmf)

$$p_X(k) = {}^nC_k (1 - p)^k p^{n-k}, 0 \leq k \leq n \quad (4.2.1)$$

The mean value is then obtained as

$$E[X] = np = \frac{1}{3} \quad (4.2.2)$$

for $n = 2$ and $p = \frac{1}{6}$

| Probability | Value |
|--------------|--|
| $\Pr(X = 0)$ | $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$ |
| $\Pr(X = 1)$ | $2 \times \frac{5}{6} \times \frac{1}{6} = \frac{5}{18}$ |
| $\Pr(X = 2)$ | $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ |

TABLE 4.2.1

5 LINEAR PROGRAMMING

5.1. The corner points of the feasible region of an LPP are $(0,0), (0,8), (2,7), (5,4)$ and $(6,0)$. The maximum profit $P = 3x + 2y$ occurs at the point

Solution: The profit can be expressed as

$$P = (3 \ 2) \mathbf{x} \quad (5.1.1)$$

and the respective values at each of the above points are given by

$$(3 \ 2) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0, \quad (5.1.2)$$

$$(3 \ 2) \begin{pmatrix} 0 \\ 8 \end{pmatrix} = 16 \quad (5.1.3)$$

$$(3 \ 2) \begin{pmatrix} 2 \\ 7 \end{pmatrix} = 20 \quad (5.1.4)$$

$$(3 \ 2) \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 23 \quad (5.1.5)$$

$$(3 \ 2) \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 18 \quad (5.1.6)$$

Hence, the maximum profit is $P = 23$ which occurs at $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

5.2. A cottage industry manufactures pedestal lamps and wooden shades. Both the products require machine time as well as craftsman time in the making. The number of hours required for producing 1 unit of each and the corresponding profit is given in the following table : In a day, the factory has availability of not

| Item | Machine Time | Craftsman Time | Profit(in INR) |
|---------------|--------------|----------------|----------------|
| Pedestal Lamp | 1.5 hours | 3 hours | 30 |
| Wooden shades | 3 hours | 1 hour | 20 |

TABLE 5.2.1

more than 42 hours of machine time and 24 hours of craftsman time.

Assuming that all items manufactured are sold, how should the manufacturer schedule his daily production in order to maximise the profit? Formulate it as an LPP and solve it graphically.

Solution: Let x be the number of lamps and y be the number of wooden shades produced. From the given information, the problem can be formulated as

$$P = \max_{x,y} 30x + 20y \quad (5.2.1)$$

$$1.5x + 3y \leq 42 \quad (5.2.2)$$

$$3x + y \leq 24 \quad (5.2.3)$$

which can be expressed in vector form as

$$P = \max_{\mathbf{x}} (30 \ 20) \mathbf{x} \quad (5.2.4)$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 28 \\ 24 \end{pmatrix} \quad (5.2.5)$$

The feasible region is a quadrilateral with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 14 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad (5.2.6)$$

with respective profit

$$(30 \ 20) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \quad (5.2.7)$$

$$(30 \ 20) \begin{pmatrix} 8 \\ 0 \end{pmatrix} = 240 \quad (5.2.8)$$

$$(30 \ 20) \begin{pmatrix} 0 \\ 14 \end{pmatrix} = 280 \quad (5.2.9)$$

$$(30 \ 20) \begin{pmatrix} 4 \\ 12 \end{pmatrix} = 360 \quad (5.2.10)$$

Thus, the manufacturer should produce 4 pedestal lamps and 12 wooden shades daily.

5.3. Amongst all open (from the top) right circular cylindrical boxes of volume $125\pi \text{ cm}^3$, find the dimensions of the box which has the least surface area.

Solution: Let r be the radius of the cylinder and h be the height. Then the surface area is

$$S = \pi r^2 + 2\pi r h \quad (5.3.1)$$

Also, the volume is

$$V = \pi r^2 h \quad (5.3.2)$$

a) The given problem can then be formulated as

$$S = \min_{r,h} \pi r^2 + 2\pi r h \quad (5.3.3)$$

$$\text{s.t } \pi r^2 h = 1 \quad (5.3.4)$$

which is a geometric programming problem that can be solved using *cvxpy*.

b) Alternatively, from (5.3.1) and (5.3.2)

$$S(r) = \pi r^2 + \frac{2V}{r} \quad (5.3.5)$$

$$\implies S'(r) = 2\pi r - \frac{2V}{r^2} \quad (5.3.6)$$

$$\text{and } S''(r) = 2\pi + \frac{4V}{r^3} > 0 \quad (5.3.7)$$

Thus, $S(r)$ has a minimum which can be obtained from (5.3.6) as

$$2\pi r - \frac{2V}{r^2} = 0 \quad (5.3.8)$$

$$\implies r = \left(\frac{V}{\pi}\right)^{\frac{1}{3}} \quad (5.3.9)$$

$$= 5 \quad \text{and} \quad (5.3.10)$$

$$h = \frac{V}{\pi r^2} = 5 \quad (5.3.11)$$

upon substituting numerical values.

- c) Using gradient descent, the update equation can be expressed as

$$r_{n+1} = r_n - \mu S'(r_n) \quad (5.3.12)$$

$$\begin{aligned}
\Pr(X_1 = 1|X_2 = 1, X_3 = 1) &= \frac{\Pr(X_1 = 1, X_2 = 1, X_3 = 1)}{\Pr(X_2 = 1, X_3 = 1)} \\
&= \frac{\Pr(X_3 = 1|X_1 = 1, X_2 = 1) \Pr(X_2 = 1|X_1 = 1) \Pr(X_1 = 1)}{\sum_{i=0}^1 \Pr(X_1 = i, X_2 = 1, X_3 = 1)} \\
&= \frac{\Pr(X_3 = 1|X_1 = 1, X_2 = 1) \Pr(X_2 = 1|X_1 = 1) \Pr(X_1 = 1)}{\sum_{i=0}^1 \Pr(X_3 = 1, |X_1 = i, X_2 = 1) \Pr(X_2 = 1|X_1 = i) \Pr(X_1 = i)} \quad (4.1.2)
\end{aligned}$$