Matrix Analysis

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CONTENTS

***Abstract*—This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE; JEE and JNTU.**

1 SECTION A

1.1 The value(s) of k for which the quadratic equation $2x^2 + kx + 2 = 0$ has equal roots, is
   a) 4
   b) ± 4
   c) -4
   d) 0

   Solution: A quadratic equation

$$ax^2 + bx + c = 0 \tag{1.0.1}$$

has equal roots only if the discriminant

$$b^2 - 4ac = 0 \tag{1.0.2}$$

Substituting

$$a = 2, b = k, c = 2, \tag{1.0.3}$$
$$\implies k^2 - 4 = 0 \tag{1.0.4}$$
$$\text{or, } k = \pm 4 \tag{1.0.5}$$

1.2 Which of the following is not an A.P. ?
   a) $-1.2, 0.8, 2.8...$
   b) $3, 3 + \sqrt{2}, 3 + 2\sqrt{2}, 3 + 3\sqrt{2}...$
   c) $\frac{4}{3}, \frac{7}{3}, \frac{9}{3}, \frac{12}{3}...$
   d) $\frac{-1}{5}, \frac{-2}{5}, \frac{-3}{5}$

   Solution: a_0, a_1, a_2 can be terms of an AP only if

$$a_1 - a_0 = a_2 - 2a_1 \tag{1.0.6}$$

Considering each of the above cases,

a)

$$2.8 - 1.2 = 1.6 = 2(0.8) \tag{1.0.7}$$

Hence, the given terms are in A.P.

b)

$$3 + \sqrt{2} - 3 = \sqrt{2} \tag{1.0.8}$$
$$3 + 2\sqrt{2} - \left(3 + \sqrt{2}\right) = \sqrt{2} \tag{1.0.9}$$
$$3 + 3\sqrt{2} - \left(3 + 2\sqrt{2}\right) = \sqrt{2} \tag{1.0.10}$$

Hence, the given terms are in A.P.

c)

$$\frac{7}{3} - \frac{4}{3} = 1 \tag{1.0.11}$$
$$\frac{9}{3} - \frac{7}{3} = \frac{2}{3} \tag{1.0.12}$$
$$\frac{12}{3} - \frac{9}{3} = 1 \tag{1.0.13}$$

Hence, the given terms are not in A.P.

1.3 The radius of a sphere (in cm), whose volume is $12\pi cm^3$, is
   a) 3
   b) $3\sqrt{2}$
   c) $3^{\frac{2}{3}}$
   d) $3^{\frac{1}{3}}$

   Solution: The volume of a sphere, given the radius r, is given by

$$V = \frac{4}{3}\pi r^3 \tag{1.0.14}$$

1.4 The distance between the points (m , -n) and (-m , n) is
   a) $\sqrt{m^2 + n^2}$
   b) $m + n$
   c) $2\sqrt{m^2 + n^2}$
   d) $\sqrt{2m^2 + 2n^2}$

   Solution: Letting

$$\mathbf{A} = \begin{pmatrix} m \\ -n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -m \\ n \end{pmatrix} \tag{1.0.15}$$

$$\mathbf{A} - \mathbf{B} = 2\begin{pmatrix} m \\ -n \end{pmatrix} \tag{1.0.16}$$

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Using the definition of the norm in (**??**),

$$\|\mathbf{A}-\mathbf{B}\| = 2\left\|\begin{pmatrix} m \\ -n \end{pmatrix}\right\| \tag{1.0.17}$$

$$= 2\sqrt{\begin{pmatrix} m & -n \end{pmatrix}\begin{pmatrix} m \\ -n \end{pmatrix}} \tag{1.0.18}$$

$$= 2\sqrt{m^2+n^2} \tag{1.0.19}$$

1.5 In Fig. 1.5 ,from an external point P, two tangents $\mathbf{PQ}$ and $\mathbf{PR}$ are drawn to a circle of radius 4cm with center O. If $\angle QPR = 90°$, then length of $\mathbf{PQ}$ is

a) 3
b) 4
c) 2
d) $2\sqrt{2}$

Solution: In general, for a circle with radus r and $\angle QPR = \theta$,

$$PQ = r\cot\frac{\theta}{2} \tag{1.0.20}$$

$$= 4\cot 45° = 4 \tag{1.0.21}$$

upon substituting numerical values.

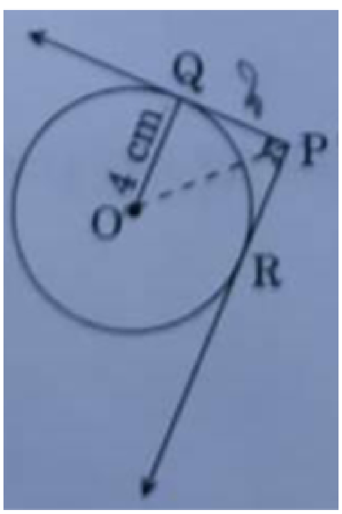

Fig. 1.5.

1.6 On dividing a polynomial p$\left(x\right)$ by x^2-4, the quotient and remainder are found to be x and 3 respectively. The polynomial p$\left(x\right)$

a) 3
b) 4
c) 2
d) $2\sqrt{2}$

Solution: In general, the polynomial

$$p(x) = d(x)q(x) + r(x) \tag{1.0.22}$$

$$= \left(x^2-4\right)x + 3 \tag{1.0.23}$$

$$= x^3 - 4x + 3 \tag{1.0.24}$$

1.7 In Fig 1.7 ,$DE \parallel BC$. If $\frac{AD}{DB} = \frac{3}{2}$ and $AE = 2.7cm$, then EC is equal to

a) $2.0cm$
b) $1.8cm$
c) $4.0cm$
d) $2.7cm$

Solution: Since

$$\frac{AD}{DB} = \frac{AE}{EC} \tag{1.0.25}$$

$$\implies \frac{3}{2} = \frac{2.7}{EC} \tag{1.0.26}$$

$$\text{or, } EC = 1.8 \tag{1.0.27}$$

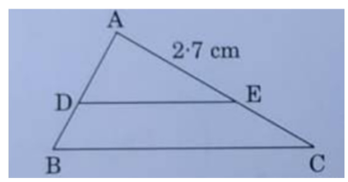

Fig. 1.7.

1.8 a) The point on the x axis which is equidistant from $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

i) $\left(7,0\right)$
ii) $\left(5,0\right)$
iii) $\left(0,0\right)$
iv) $\left(3,0\right)$

Solution: If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{1.0.28}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A}-\mathbf{B}\right)^\top \mathbf{e}_1} = 3 \tag{1.0.29}$$

upon substituting numerical values. Hence, the desired point is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

b) The center of a circle whose end points of a diameter are $\left(-6,3\right)$ and $\left(6,4\right)$ is

i) $\left(8, -1\right)$
ii) $\left(4, 7\right)$
iii) $\left(0, \frac{7}{2}\right)$
iv) $\left(4, \frac{7}{2}\right)$

Solution: Using section formula, the desired point is given by

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{A}}{2} \tag{1.0.30}$$
$$= \frac{1}{2}\left[\begin{pmatrix} -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix}\right] \tag{1.0.31}$$
$$= \frac{1}{2}\begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{1.0.32}$$

1.9 The pair of linear equations,

$$\frac{3x}{2} + \frac{5y}{3} = 7 \text{ and} \tag{1.0.33}$$
$$9x + 10y = 14 \tag{1.0.34}$$

is

a) consistent
b) inconsistent
c) consistent with one solution
d) consistent with many solutions

Solution: The given system can be expressed as the matrix equation

$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 9 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 14 \end{pmatrix} \tag{1.0.35}$$

The augmented matrix can be expressed as

$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} & 7 \\ 9 & 10 & 14 \end{pmatrix} \tag{1.0.36}$$
$$\xleftrightarrow{R_1 \leftarrow 6R_1} \begin{pmatrix} 9 & 10 & 42 \\ 9 & 10 & 14 \end{pmatrix} \tag{1.0.37}$$
$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 9 & 10 & 42 \\ 0 & 0 & 28 \end{pmatrix} \tag{1.0.38}$$

From the above, it is obvious that the rank of the coefficient matrix is not equal to the rank of the augmented matrix. Hence, the system is inconsistent.

1.10 In Fig 1.10, $\left(PQ\right)$ is tangent to the circle with center at O, at the point B. If $\angle AOB = 100°$, then $\angle ABP$ is equal to

a) $50°$
b) $40°$
c) $60°$
d) $80°$

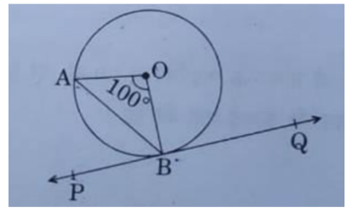

Fig. 1.10.

Solution: In general,

$$\angle ABP = \frac{1}{2}\angle AOB \tag{1.0.39}$$
$$= 50° \tag{1.0.40}$$

2.1 What is the simplest form of $\frac{1+\tan^2 A}{1+\cot^2 A}$
Solution:

$$\frac{1+\tan^2 A}{1+\cot^2 A} = \frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}} \tag{1.0.41}$$
$$= \tan^2 A \tag{1.0.42}$$

2.2 If the probablity of an event E happening is 0.023, then $P\left(\bar{E}\right) =$

2.3 All concentric circles are ________ to each other

2.4 The probability of an event that is sure to happen is, ________

2.5 AOBC is a rectangle whose 3 vertices are $A = \left(0, -3\right)$, $O = \left(0, 0\right)$ and $B = \left(4, 0\right)$. The length of the diagonal is ________
Solution: The length of the diagonal is

$$\left\|\mathbf{A} - \mathbf{B}\right\| = \sqrt{3^2 + 4^2} = 5 \tag{1.0.43}$$

2.6 Write the value of $\sin^2 30° + \cos^2 60°$
Solution: Since

$$\cos 60° = \sin 30°, \tag{1.0.44}$$
$$\sin^2 30° + \cos^2 60° = 2\sin^2 30° \tag{1.0.45}$$
$$= 1 \tag{1.0.46}$$

2.7 a) Form a quadratic polynomial, the sum and product of whose zeros are $\left(-3\right)$ and 2 respectively.
Solution: The desired quadratic polynomial is

$$\left(x + 3\right)\left(x - 2\right) = x^2 - x - 6 \tag{1.0.47}$$

b) Can $\left(x^2-1\right)$ be a remainder while dividing x^4-3x^2+5x-9 by $\left(x^2+3\right)$? ustify the reasons.
Solution: If the given statement be true,

$$\frac{x^4-3x^2+5x-9}{x^2+3} = q(x) + \frac{x^2-1}{x^2+3} \tag{1.0.48}$$

$$= q(x) + \frac{x^2+3-4}{x^2+3} \tag{1.0.49}$$

$$= q(x) + 1 - \frac{4}{x^2+3} \tag{1.0.50}$$

which implies that the remainder is -4 resulting in a contradiction. Hence, the given statement is not true.

2.8 Find the sum of the first 100 natural numbers.
Solution: The sum of the first n natural number is

$$\frac{n\left(n+1\right)}{2} \tag{1.0.51}$$

Sustituting $n = 100$ in the above, the desired sum is

$$50 \times 51 = 2550 \tag{1.0.52}$$

2.9 The LCM of 2 numbers is 182 and their HCF is 13. If one of the numbers is 26, find the other.
Solution: The desired number is obtained as

$$\frac{182 \times 13}{26} = 91 \tag{1.0.53}$$

2.10 In Fig 2.10 the angle of elevation from the top of a tower from a point C on the ground, which is 30m away from the foot of the tower is 30°. Find the height of the tower.
Solution: In general, the height is given by

$$h = d\tan\theta \tag{1.0.54}$$

$$= 30\tan 30^\circ = 30\frac{1}{\sqrt{3}} \tag{1.0.55}$$

$$= 10\sqrt{3} \tag{1.0.56}$$

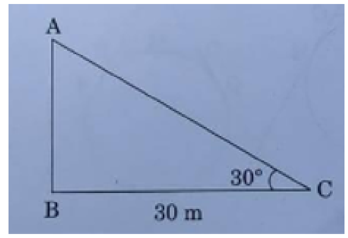

Fig. 2.10.

2 Section B

3.1 A cone and a cylinder have the same radii, but the height of the cone is 3 times that of the cylinder. Find the ratio of their volumes
Solution: Let $h_i, V_i, i = 1, 2$ be the respective heights and volumes of the cone and the cylinder. Then

$$V_1 = \frac{1}{3}\pi r^2 h_1 \tag{2.0.1}$$

$$V_2 = \pi r^2 h_2 \tag{2.0.2}$$

$$\implies \frac{V_1}{V_2} = \frac{h_1}{3h_2} \tag{2.0.3}$$

$$= 1 \tag{2.0.4}$$

$\because h_1 = 3h_2$.

3.2 a) In Fig 3.2, a quadrilateral ABCD is drawn to circumscribe a circle. Prove that $\mathbf{AB} + \mathbf{CD} = \mathbf{BC} + \mathbf{AD}$.
Solution:

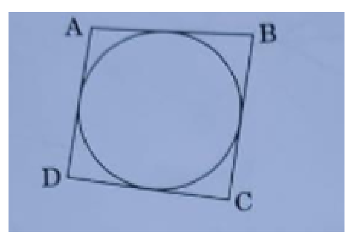

Fig. 3.2.

b) In Fig **??**, find the perimeter of $\triangle ABC$ if $\mathbf{AP} = 12cm$

3.3 Find the mode of the following distribution:

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	4	6	7	12	5	6

3.4 In Fig **??** if $\mathbf{PQ} \parallel \mathbf{BC}$ and $\mathbf{PR} \parallel \mathbf{CD}$, prove that $\frac{QB}{AQ} = \frac{DR}{AR}$.

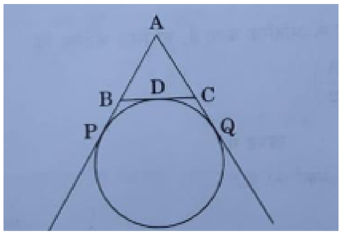

Fig. 3.2.

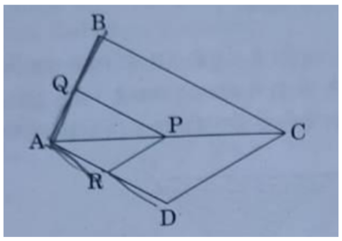

Fig. 3.4.

3.5 a) Show that $5+2\sqrt{7}$ is an irrational number, where $\sqrt{7}$ is given to be an irrational number.

b) Check whether 12^n can end with the digit 0 for any natural number n.
Solution: If 12^n ends with the digit 0,

$$12^n \equiv 0 \pmod{10} \tag{2.0.5}$$

However,

$$12^n \not\equiv 0 \pmod{10} \tag{2.0.6}$$

it should be divisible by 10. 12 is not divisible by 10, neither is 12^2. Let us assume that 12^{n-1} is not divisible by 12. Then,

$$12^n = 12.12^{n-1} \tag{2.0.7}$$

Since neither 12 nor 12^{n-1} are divisible by 10, 12^n is also not divisible by n.

3.6 If A, B and C are interior angles of $\triangle ABC$, then show that

$$\cos\left(\frac{B+C}{2}\right) = \sin\left(\frac{A}{2}\right) \tag{2.0.8}$$

Solution: In a triangle,

$$A+B+C = 180° \tag{2.0.9}$$

$$\implies \left(\frac{B+C}{2}\right) = 90° - \frac{A}{2} \tag{2.0.10}$$

$$\text{or,}\ \cos\left(\frac{B+C}{2}\right) = \sin\left(90° - \frac{A}{2}\right) \tag{2.0.11}$$

yielding (**??**).

3 Section C

3.7 Prove that:
$\left(\sin^4 - \cos^4 + 1\right)\csc^2\theta = 2$

3.8 Find the sum:
$\left(-5\right) + \left(-8\right) + \left(-11\right) + ... + \left(-230\right)$

3.9 a) Construct a $\triangle ABC$ with sides $\mathbf{BC} = 6cm$, $\mathbf{AB} = 5cm$ and $\angle ABC = 60°$. Then construct a triangle whose sides are $\frac{3}{4}$ of the corresponding sides of $\triangle ABC$.

b) Draw a circle of radius $3.5cm$. Take a point P outside the circle at a distance of $7cm$ from the centre of the circle and construct a pair of tangents to the circle from that point.

3.10 In Fig **??**, ABCD is a parallelogram. A semicircle with centre O and the diameter AB has been drawn and it passes through D. If $\mathbf{AB} = 12$ and $\mathbf{OD} \perp \mathbf{AB}$, then find the area of the shaded region. Use $\left(\pi = 3.14\right)$

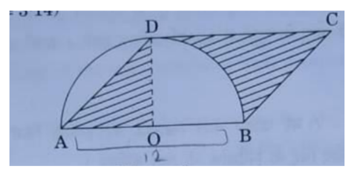

Fig. 3.10.

4.1 A game in a booth at a Diwali Fair involves using a spinner first. Then, if the spinner stops on an even number, the player is allowed to pick a marble from a bag. The spinner and the marbles in the bag are represented in Fig **??**. Prizes are given, when a black marble is picked. Shweta plays the game once.

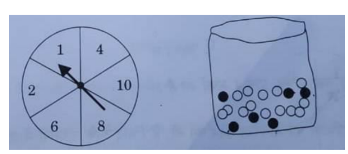

Fig. 4.1.

a) What is the probability that she will be allowed to pick a marble from the bag?

b) Suppose she is allowed to pick a marble from the bag, what is the probability of getting a prize, when it is given that the bag contains 20 balls out of which 6 are black?

4.2 a) A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 added to its denominator. Find the fraction.

b) The present age of a father is three years more than three times the age of his son. Three years hence the father's age will be 10 years more than twice the age of the son. Determine their present ages.

4.3 a) Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix}6, -4\end{pmatrix}$ and $\begin{pmatrix}-2, -7\end{pmatrix}$. Also find the point of intersection
Solution: In general, letting the given points be $\mathbf{A}, \mathbf{B}$,

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{3.0.1}$$

Since the point lies on the y-axis,

$$\mathbf{e}_1^\top \mathbf{P} = 0 \tag{3.0.2}$$

$$\implies k\mathbf{e}_1^\top \mathbf{B} + \mathbf{e}_1^\top \mathbf{A} = 0 \tag{3.0.3}$$

$$\text{or, } k = -\frac{\mathbf{e}_1^\top \mathbf{A}}{\mathbf{e}_1^\top \mathbf{B}} \tag{3.0.4}$$

Substituting in (**??**) and simplifying,

$$\mathbf{P} = \frac{\left(\mathbf{e}_1^\top \mathbf{B}\right)\mathbf{A} - \left(\mathbf{e}_1^\top \mathbf{A}\right)\mathbf{B}}{\left(\mathbf{e}_1^\top \mathbf{B}\right) - \left(\mathbf{e}_1^\top \mathbf{A}\right)} \tag{3.0.5}$$

b) Show that the points $\begin{pmatrix}7, 10\end{pmatrix}$, $\begin{pmatrix}-2, 5\end{pmatrix}$ and $\begin{pmatrix}3, -4\end{pmatrix}$ and vertices of an isosceles right triangle.
Solution: Let the given points be $\mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively. Then, the direction vectors of AB, BC and CA are

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix}7\\10\end{pmatrix} - \begin{pmatrix}-2\\5\end{pmatrix} = \begin{pmatrix}9\\5\end{pmatrix} \tag{3.0.6}$$

$$\mathbf{B} - \mathbf{C} = -\begin{pmatrix}-2\\5\end{pmatrix} - \begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}-5\\9\end{pmatrix} \tag{3.0.7}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix}3\\-4\end{pmatrix} - \begin{pmatrix}7\\10\end{pmatrix} = \begin{pmatrix}-4\\-14\end{pmatrix} \tag{3.0.8}$$

From the above, we find that

$$\left(\mathbf{A} - \mathbf{B}\right)^\top \left(\mathbf{B} - \mathbf{C}\right) = \begin{pmatrix}9 & 5\end{pmatrix}\begin{pmatrix}-5\\9\end{pmatrix} \tag{3.0.9}$$

$$= 0 \tag{3.0.10}$$

$$\left(\mathbf{B} - \mathbf{C}\right)^\top \left(\mathbf{C} - \mathbf{A}\right) = \begin{pmatrix}-5 & 9\end{pmatrix}\begin{pmatrix}-4\\-14\end{pmatrix} \tag{3.0.11}$$

$$= -106 \tag{3.0.12}$$

$$\left(\mathbf{C} - \mathbf{A}\right)^\top \left(\mathbf{A} - \mathbf{B}\right) = \begin{pmatrix}-4 & -14\end{pmatrix}\begin{pmatrix}9\\5\end{pmatrix} \tag{3.0.13}$$

$$= -106 \tag{3.0.14}$$

From the above equations, (**??**) and (**??**),

$$\left(\mathbf{A} - \mathbf{B}\right) \perp \left(\mathbf{B} - \mathbf{C}\right) \tag{3.0.15}$$

$$\angle BCA = \angle CAB \tag{3.0.16}$$

Thus, the triangle is right angled and isosceles.

4.4 Use Euclid Division Lemma to show that the square of any positive integer is either in the form $3q$ or $3q + 1$ for some integer q.

4 Section D

4.5 a) Sum of the areas of two squares is $544m^2$. If the difference of their perimeters is $32m$, find the sides of the two squares.

b) A motorboat whose speed is 18 kmph in still water takes 1 hour more to go 24 km upstream than to return downstream than to return downstream to the same spot. Find the speed of the stream.

4.6 a) For the following data, draw a 'less than' ogive and hence find the median of the distribution.

Age	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of people	5	15	20	25	15	11	9

b) The distribution given below shows the number of wickets taken by bowlers in one-day cricket matches. Find the mean and the median of the number of wickets. Find the mean and the median of the number of wickets taken.

Number of wickets	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of bowlers	5	15	20	25	15	11	9

4.7 A statue $1.6m$ tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is $60°$ and from the same point the angle of elevation of the top of the pedestal is $45°$. Find the height of the pedestal.Use$\left(\sqrt{3} = 1.73\right)$

4.8 a) Obtain other zeros of the polynomial
$p\left(x\right) = 2x^4 - x^3 - 11x^2 + 5x + 5$
if the two of its zeros are $\sqrt{5}$ and $-\sqrt{5}$

b) What minimum must be added to $2x^3 - 3x^2 + 6x + 7$ so that the resulting polynomial will be divisible by $x^2 - 4x + 8$?

4.9 In a cylindrical vessel of radius 10cm, containing some water, 9000 small spherical balls are dropped which are completely immersed in water which raises the water level. If each spherical ball is of radius 0.5 cm then find the rise in the level of water in the vessel.

4.10 If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points, prove that the other two sides are divided in the same ratio.