Matrix Analysis

G V V Sharma*

CONTENTS

***Abstract*—This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE; JEE and JNTU.**

1 SECTION A

1.1 The value(s) of k for which the quadratic equation $2x^2 + kx + 2 = 0$ has equal roots, is

a) 4
b) ± 4
c) -4
d) 0

Solution: A quadratic equation

$$ax^2 + bx + c = 0 \tag{1.0.1}$$

has equal roots only if the discriminant

$$b^2 - 4ac = 0 \tag{1.0.2}$$

Substituting

$$a = 2, b = k, c = 2, \tag{1.0.3}$$
$$\implies k^2 - 4 = 0 \tag{1.0.4}$$
$$\text{or, } k = \pm 4 \tag{1.0.5}$$

1.2 Which of the following is not an A.P. ?

a) $-1.2, 0.8, 2.8...$
b) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}...$
c) $\frac{4}{3}, \frac{7}{3}, \frac{9}{3}, \frac{12}{3}...$
d) $\frac{-1}{5}, \frac{-2}{5}, \frac{-3}{5}$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

Solution: a_0, a_1, a_2 can be terms of an AP only if

$$a_1 - a_0 = a_2 - 2a_1 \tag{1.0.6}$$

Considering each of the above cases,

a)

$$2.8 - 1.2 = 1.6 = 2(0.8) \tag{1.0.7}$$

Hence, the given terms are in A.P.

b)

$$3 + \sqrt{2} - 3 = \sqrt{2} \tag{1.0.8}$$
$$3 + 2\sqrt{2} - \left(3 + \sqrt{2}\right) = \sqrt{2} \tag{1.0.9}$$
$$3 + 3\sqrt{2} - \left(3 + 2\sqrt{2}\right) = \sqrt{2} \tag{1.0.10}$$

Hence, the given terms are in A.P.

c)

$$\frac{7}{3} - \frac{4}{3} = 1 \tag{1.0.11}$$
$$\frac{9}{3} - \frac{7}{3} = \frac{2}{3} \tag{1.0.12}$$
$$\frac{12}{3} - \frac{9}{3} = 1 \tag{1.0.13}$$

Hence, the given terms are not in A.P.

1.3 The radius of a sphere (in cm), whose volume is $12\pi cm^3$, is

a) 3
b) $3\sqrt{2}$
c) $3^{\frac{2}{3}}$
d) $3^{\frac{1}{3}}$

Solution: The volume of a sphere, given the radius r, is given by

$$V = \frac{4}{3}\pi r^3 \tag{1.0.14}$$

1.4 The distance between the points (m , -n) and (-m , n) is

a) $\sqrt{m^2 + n^2}$
b) $m + n$
c) $2\sqrt{m^2 + n^2}$
d) $\sqrt{2m^2 + 2n^2}$

Solution: Letting

$$\mathbf{A} = \begin{pmatrix} m \\ -n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -m \\ n \end{pmatrix} \tag{1.0.15}$$

$$\mathbf{A} - \mathbf{B} = 2\begin{pmatrix} m \\ -n \end{pmatrix} \tag{1.0.16}$$

Using the definition of the norm in (**??**),

$$\|\mathbf{A} - \mathbf{B}\| = 2\left\|\begin{pmatrix} m \\ -n \end{pmatrix}\right\| \tag{1.0.17}$$

$$= 2\sqrt{\begin{pmatrix} m & -n \end{pmatrix}\begin{pmatrix} m \\ -n \end{pmatrix}} \tag{1.0.18}$$

$$= 2\sqrt{m^2 + n^2} \tag{1.0.19}$$

1.5 In Fig. 1.5 ,from an external point P, two tangents $\mathbf{PQ}$ and $\mathbf{PR}$ are drawn to a circle of radius 4cm with center O. If $\angle QPR = 90°$, then length of $\mathbf{PQ}$ is

a) 3
b) 4
c) 2
d) $2\sqrt{2}$

Solution: In general, for a circle with radus r and $\angle QPR = \theta$,

$$PQ = r\cot\frac{\theta}{2} \tag{1.0.20}$$

$$= 4\cot 45° = 4 \tag{1.0.21}$$

upon substituting numerical values.

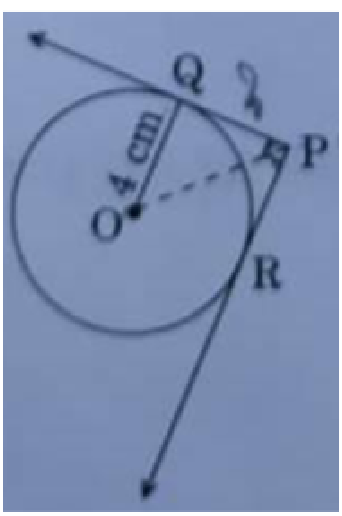

Fig. 1.5.

1.6 On dividing a polynomial $\mathrm{p}\left(x\right)$ by $x^2 - 4$, the quotient and remainder are found to be x and 3 respectively. The polynomial $\mathrm{p}\left(x\right)$

a) 3
b) 4
c) 2
d) $2\sqrt{2}$

Solution: In general, the polynomial

$$p(x) = d(x)q(x) + r(x) \tag{1.0.22}$$

$$= \left(x^2 - 4\right)x + 3 \tag{1.0.23}$$

$$= x^3 - 4x + 3 \tag{1.0.24}$$

1.7 In Fig 1.7 ,$DE \parallel BC$. If $\frac{AD}{DB} = \frac{3}{2}$ and $AE = 2.7cm$, then EC is equal to

a) $2.0cm$
b) $1.8cm$
c) $4.0cm$
d) $2.7cm$

Solution: Since

$$\frac{AD}{DB} = \frac{AE}{EC} \tag{1.0.25}$$

$$\implies \frac{3}{2} = \frac{2.7}{EC} \tag{1.0.26}$$

$$\text{or, } EC = 1.8 \tag{1.0.27}$$

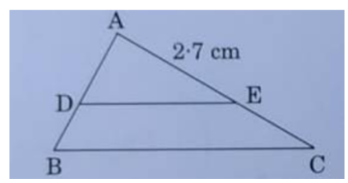

Fig. 1.7.

1.8 a) The point on the x axis which is equidistant from $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

i) $\begin{pmatrix} 7, 0 \end{pmatrix}$
ii) $\begin{pmatrix} 5, 0 \end{pmatrix}$
iii) $\begin{pmatrix} 0, 0 \end{pmatrix}$
iv) $\begin{pmatrix} 3, 0 \end{pmatrix}$

Solution: If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{1.0.28}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1} = 3 \tag{1.0.29}$$

upon substituting numerical values. Hence, the desired point is $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

b) The center of a circle whose end points of a diameter are $\left(-6, 3\right)$ and $\left(6, 4\right)$ is
   i) $\left(8, -1\right)$
   ii) $\left(4, 7\right)$
   iii) $\left(0, \frac{7}{2}\right)$
   iv) $\left(4, \frac{7}{2}\right)$

   Solution: Using section formula, the desired point is given by

$$\mathbf{O} = \frac{\mathbf{B}+\mathbf{A}}{2} \tag{1.0.30}$$
$$= \frac{1}{2}\left[\begin{pmatrix}-6\\3\end{pmatrix} + \begin{pmatrix}6\\4\end{pmatrix}\right] \tag{1.0.31}$$
$$= \frac{1}{2}\begin{pmatrix}0\\7\end{pmatrix} \tag{1.0.32}$$

1.9 The pair of linear equations,

$$\frac{3x}{2} + \frac{5y}{3} = 7 \text{ and} \tag{1.0.33}$$
$$9x + 10y = 14 \tag{1.0.34}$$

is
a) consistent
b) inconsistent
c) consistent with one solution
d) consistent with many solutions

Solution: The given system can be expressed as the matrix equation

$$\begin{pmatrix}\frac{3}{2} & \frac{5}{3}\\ 9 & 10\end{pmatrix}\mathbf{x} = \begin{pmatrix}7\\14\end{pmatrix} \tag{1.0.35}$$

The augmented matrix can be expressed as

$$\begin{pmatrix}\frac{3}{2} & \frac{5}{3} & 7\\ 9 & 10 & 14\end{pmatrix} \tag{1.0.36}$$
$$\xleftrightarrow{R_1 \leftarrow 6R_1} \begin{pmatrix}9 & 10 & 42\\ 9 & 10 & 14\end{pmatrix} \tag{1.0.37}$$
$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix}9 & 10 & 42\\ 0 & 0 & 28\end{pmatrix} \tag{1.0.38}$$

From the above, it is obvious that the rank of the coefficient matrix is not equal to the rank of the augmented matrix. Hence, the system is inconsistent.

1.10 In Fig 1.10, $\left(PQ\right)$ is tangent to the circle with center at O, at the point B. If $\angle AOB = 100°$, then $\angle ABP$ is equal to
a) $50°$
b) $40°$
c) $60°$
d) $80°$

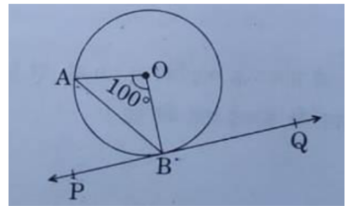

Fig. 1.10.

Solution: In general,

$$\angle ABP = \frac{1}{2}\angle AOB \tag{1.0.39}$$
$$= 50° \tag{1.0.40}$$

2.1 What is the simplest form of $\frac{1+\tan^2 A}{1+\cot^2 A}$
Solution:

$$\frac{1+\tan^2 A}{1+\cot^2 A} = \frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}} \tag{1.0.41}$$
$$= \tan^2 A \tag{1.0.42}$$

2.2 If the probablity of an event E happening is 0.023, then $P\left(\bar{E}\right) =$

2.3 All concentric circles are ________ to each other

2.4 The probability of an event that is sure to happen is, ________

2.5 AOBC is a rectangle whose 3 vertices are $A = \left(0, -3\right)$, $O = \left(0, 0\right)$ and $B = \left(4, 0\right)$. The length of the diagonal is ________
Solution: The length of the diagonal is

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{3^2 + 4^2} = 5 \tag{1.0.43}$$

2.6 Write the value of $\sin^2 30° + \cos^2 60°$
Solution: Since

$$\cos 60° = \sin 30°, \tag{1.0.44}$$
$$\sin^2 30° + \cos^2 60° = 2\sin^2 30° \tag{1.0.45}$$
$$= 1 \tag{1.0.46}$$

2.7 a) Form a quadratic polynomial, the sum and product of whose zeros are $\left(-3\right)$ and 2 respectively.

Solution: The desired quadratic polynomial is

$$(x+3)(x-2) = x^2 - x - 6 \tag{1.0.47}$$

b) Can $\left(x^2-1\right)$ be a remainder while dividing $x^4 - 3x^2 + 5x - 9$ by $\left(x^2+3\right)$? ustify the reasons.
Solution: If the given statement be true,

$$\frac{x^4 - 3x^2 + 5x - 9}{x^2+3} = q(x) + \frac{x^2-1}{x^2+3} \tag{1.0.48}$$
$$= q(x) + \frac{x^2+3-4}{x^2+3} \tag{1.0.49}$$
$$= q(x) + 1 - \frac{4}{x^2+3} \tag{1.0.50}$$

which implies that the remainder is -4 resulting in a contradiction. Hence, the given statement is not true.

2.8 Find the sum of the first 100 natural numbers.
Solution: The sum of the first n natural number is

$$\frac{n(n+1)}{2} \tag{1.0.51}$$

Sustituting $n = 100$ in the above, the desired sum is

$$50 \times 51 = 2550 \tag{1.0.52}$$

2.9 The LCM of 2 numbers is 182 and their HCF is 13. If one of the numbers is 26, find the other.
Solution: The desired number is obtained as

$$\frac{182 \times 13}{26} = 91 \tag{1.0.53}$$

2.10 In Fig 2.10 the angle of elevation from the top of a tower from a point C on the ground, which is 30m away from the foot of the tower is 30°. Find the height of the tower.
Solution: In general, the height is given by

$$h = d\tan\theta \tag{1.0.54}$$
$$= 30\tan 30^\circ = 30\frac{1}{\sqrt{3}} \tag{1.0.55}$$
$$= 10\sqrt{3} \tag{1.0.56}$$

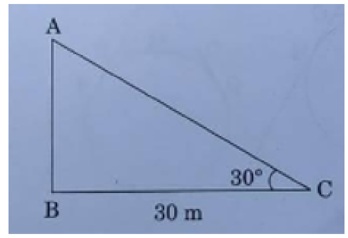

Fig. 2.10.

2 Section B

3.1 A cone and a cylinder have the same radii, but the height of the cone is 3 times that of the cylinder. Find the ratio of their volumes
Solution: Let $h_i, V_i, i = 1, 2$ be the respective heights and volumes of the cone and the cylinder. Then

$$V_1 = \frac{1}{3}\pi r^2 h_1 \tag{2.0.1}$$
$$V_2 = \pi r^2 h_2 \tag{2.0.2}$$
$$\implies \frac{V_1}{V_2} = \frac{h_1}{3h_2} \tag{2.0.3}$$
$$= 1 \tag{2.0.4}$$

$\because h_1 = 3h_2$.

3.2 a) In Fig 3.2, a quadrilateral ABCD is drawn to circumscribe a circle. Prove that $\mathbf{AB} + \mathbf{CD} = \mathbf{BC} + \mathbf{AD}$.
Solution:

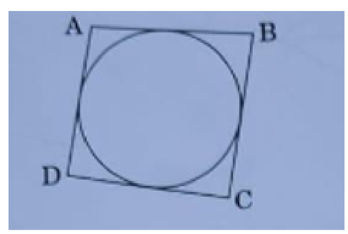

Fig. 3.2.

b) In Fig 3.2, find the perimeter of $\triangle ABC$ if $\mathbf{AP} = 12cm$

3.3 Find the mode of the following distribution:

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	4	6	7	12	5	6

3.4 In Fig 3.4 if $\mathbf{PQ} \parallel \mathbf{BC}$ and $\mathbf{PR} \parallel \mathbf{CD}$, prove that $\frac{QB}{AQ} = \frac{DR}{AR}$.

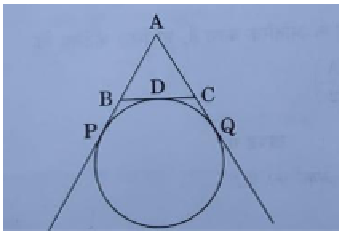

Fig. 3.2.

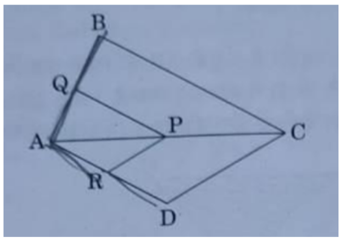

Fig. 3.4.

3.5 a) Show that $5 + 2\sqrt{7}$ is an irrational number, where $\sqrt{7}$ is given to be an irrational number.

b) Check whether 12^n can end with the digit 0 for any natural number n.
Solution: If 12^n ends with the digit 0,

$$12^n \equiv 0 \pmod{10} \tag{2.0.5}$$
$$\implies 12^n \equiv 0 \pmod{2} \tag{2.0.6}$$
$$12^n \equiv 0 \pmod{5} \tag{2.0.7}$$

However,

$$12 \not\equiv 0 \pmod{5} \tag{2.0.8}$$
$$12^2 \not\equiv 0 \pmod{5} \tag{2.0.9}$$

Let

$$12^{n-1} \not\equiv 0 \pmod{5} \tag{2.0.10}$$

Then, it is obvious from (2.0.8) and (2.0.10) that

$$12.12^{n-1} \not\equiv 0 \pmod{5} \tag{2.0.11}$$
$$\implies 12^n \not\equiv 0 \pmod{5} \tag{2.0.12}$$

3.6 If A, B and C are interior angles of $\triangle ABC$, then show that

$$\cos\left(\frac{B+C}{2}\right) = \sin\left(\frac{A}{2}\right) \tag{2.0.13}$$

Solution: In a triangle,

$$A + B + C = 180° \tag{2.0.14}$$
$$\implies \left(\frac{B+C}{2}\right) = 90° - \frac{A}{2} \tag{2.0.15}$$
$$\text{or, } \cos\left(\frac{B+C}{2}\right) = \sin\left(90° - \frac{A}{2}\right) \tag{2.0.16}$$

yielding (2.0.13).

3 Section C

3.7 Prove that

$$\left(\sin^4\theta - \cos^4\theta + 1\right)\csc^2\theta = 2 \tag{3.0.1}$$

Solution: Since

$$\left(\sin^2\theta + \cos^2\theta\right)\left(\sin^2\theta - \cos^2\theta\right) = -\cos 2\theta \tag{3.0.2}$$

the L.H.S in (3.0.1) can be expressed as

$$\left(1 - \cos 2\theta\right)\csc^2\theta = 2\sin^2\theta\csc^2\theta \tag{3.0.3}$$
$$= 2 \tag{3.0.4}$$

3.8 Find the sum

$$\left(-5\right) + \left(-8\right) + \left(-11\right) + ... + \left(-230\right) \tag{3.0.5}$$

Solution: The above series is an A.P. with

$$a_0 = -5, a_n = -230, d = -3 \tag{3.0.6}$$

Since

$$a_n = a_0 + (n-1)\, d, \tag{3.0.7}$$
$$n = \frac{a_n - a_0}{d} + 1 \tag{3.0.8}$$
$$= 46 \tag{3.0.9}$$

Thus the desired sum is given by

$$S_n = \frac{n\left(a_n + a_0\right)}{2} \tag{3.0.10}$$
$$= -235 \times 23 = -5405 \tag{3.0.11}$$

3.9 a) Construct a $\triangle ABC$ with sides $\mathbf{BC} = 6cm$, $\mathbf{AB} = 5cm$ and $\angle ABC = 60°$. Then construct a triangle whose sides are $\frac{3}{4}$ of the corresponding sides of $\triangle ABC$.

b) Draw a circle of radius $3.5cm$. Take a point P outside the circle at a distance of $7cm$

from the centre of the circle and construct a pair of tangents to the circle from that point.

3.10 In Fig 3.10, ABCD is a parallelogram. A semicircle with centre O and the diameter AB has been drawn and it passes through D. If $\mathbf{AB} = 12$ and $\mathbf{OD} \perp \mathbf{AB}$, then find the area of the shaded region. Use $\left(\pi = 3.14\right)$
Solution: From the figure, the radius of the circle is

$$r = \frac{AB}{2} = OD = 6 \tag{3.0.12}$$

The area of the parallelogram is

$$AB \times OD = 2r^2 \tag{3.0.13}$$

The area of the sector DOB is

$$\frac{1}{4}\pi r^2 \tag{3.0.14}$$

Thus, the desired area is

$$\left(2 - \frac{\pi}{4}\right) r^2 = 9\left(8 - \pi\right) \tag{3.0.15}$$

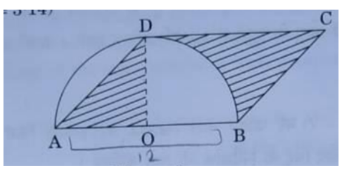

Fig. 3.10.

4.1 A game in a booth at a Diwali Fair involves using a spinner first. Then, if the spinner stops on an even number, the player is allowed to pick a marble from a bag. The spinner and the marbles in the bag are represented in Fig 4.1. Prizes are given, when a black marble is picked. Shweta plays the game once.

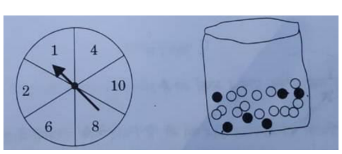

Fig. 4.1.

a) What is the probability that she will be allowed to pick a marble from the bag?
Solution: Let $X \in \{0, 1\}$ represent the odd and even outcomes of the spinner and $Y \in \{0, 1\}$ represent the black and white marbles respectively. Shweta can pick a marble only if she gets an even number in the spinner. Hence, the desired probability is

$$\Pr\left(X = 1\right) = \frac{5}{6} \tag{3.0.16}$$

b) Suppose she is allowed to pick a marble from the bag, what is the probability of getting a prize, when it is given that the bag contains 20 balls out of which 6 are black?
Solution: Shweta will get a prize only if she picks a black marble. The desired probability can be expressed as

$$\Pr\left(Y = 0 | X = 1\right) = \frac{6}{20} = \frac{3}{10} \tag{3.0.17}$$

4.2 a) A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 added to its denominator. Find the fraction.
Solution: Let the desired fraction be $\frac{x}{y}$. From the given information,

$$\frac{x-1}{y} = \frac{1}{3} \tag{3.0.18}$$

$$\frac{x}{y+8} = \frac{1}{4} \tag{3.0.19}$$

The above equations result in the system

$$3x - y = 3 \tag{3.0.20}$$

$$4x - y = 8 \tag{3.0.21}$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \tag{3.0.22}$$

The augmented matrix is obtained as

$$\begin{pmatrix} 3 & -1 & 3 \\ 4 & -1 & 8 \end{pmatrix} \tag{3.0.23}$$

Through pivoting, we obtain

$$\begin{pmatrix} 3 & -1 & 3 \\ 0 & 1 & 12 \end{pmatrix} \tag{3.0.24}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1+R_2}{3}} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 12 \end{pmatrix} \tag{3.0.25}$$

$$\implies \mathbf{x} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{3.0.26}$$

b) The present age of a father is three years more than three times the age of his son.

Three years hence the father's age will be 10 years more than twice the age of the son. Determine their present ages.

Solution: Let the ages of the father and son be x, y respectively. From the given information,

$$x = 3y + 3 \tag{3.0.27}$$

$$x + 3 = 2(y + 3) + 10 \tag{3.0.28}$$

which can be expressed as

$$x - 3y = 3 \tag{3.0.29}$$

$$x - 2y = 13 \tag{3.0.30}$$

$$\implies \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 13 \end{pmatrix} \tag{3.0.31}$$

The augmented matrix for the above matrix equation is

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 1 & -2 & 13 \end{array}\right) \tag{3.0.32}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & 10 \end{array}\right) \tag{3.0.33}$$

$$\xleftrightarrow{R_1 \leftarrow 3R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 33 \\ 0 & 1 & 10 \end{array}\right) \implies \mathbf{x} = \begin{pmatrix} 33 \\ 10 \end{pmatrix} \tag{3.0.34}$$

4.3 a) Find the ratio in which the y-axis divides the line segment joining the points $\begin{pmatrix}6, -4\end{pmatrix}$ and $\begin{pmatrix}-2, -7\end{pmatrix}$. Also find the point of intersection

Solution: In general, letting the given points be $\mathbf{A}, \mathbf{B}$,

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \tag{3.0.35}$$

Since the point lies on the y-axis,

$$\mathbf{e}_1^\top \mathbf{P} = 0 \tag{3.0.36}$$

$$\implies k\mathbf{e}_1^\top \mathbf{B} + \mathbf{e}_1^\top \mathbf{A} = 0 \tag{3.0.37}$$

$$\text{or, } k = -\frac{\mathbf{e}_1^\top \mathbf{A}}{\mathbf{e}_1^\top \mathbf{B}} \tag{3.0.38}$$

Substituting in (3.0.35) and simplifying,

$$\mathbf{P} = \frac{\left(\mathbf{e}_1^\top \mathbf{B}\right)\mathbf{A} - \left(\mathbf{e}_1^\top \mathbf{A}\right)\mathbf{B}}{\left(\mathbf{e}_1^\top \mathbf{B}\right) - \left(\mathbf{e}_1^\top \mathbf{A}\right)} \tag{3.0.39}$$

b) Show that the points $\begin{pmatrix}7, 10\end{pmatrix}$, $\begin{pmatrix}-2, 5\end{pmatrix}$ and $\begin{pmatrix}3, -4\end{pmatrix}$ and vertices of an isosceles right triangle.

Solution: Let the given points be $\mathbf{A}, \mathbf{B}, \mathbf{C}$ respectively. Then, the direction vectors of AB, BC and CA are

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 7 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix} \tag{3.0.40}$$

$$\mathbf{B} - \mathbf{C} = -\begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix} \tag{3.0.41}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 7 \\ 10 \end{pmatrix} = \begin{pmatrix} -4 \\ -14 \end{pmatrix} \tag{3.0.42}$$

From the above, we find that

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 9 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ 9 \end{pmatrix} \tag{3.0.43}$$

$$= 0 \tag{3.0.44}$$

$$(\mathbf{B} - \mathbf{C})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 9 \end{pmatrix} \begin{pmatrix} -4 \\ -14 \end{pmatrix} \tag{3.0.45}$$

$$= -106 \tag{3.0.46}$$

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} -4 & -14 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \end{pmatrix} \tag{3.0.47}$$

$$= -106 \tag{3.0.48}$$

From the above equations, (**??**) and (**??**),

$$(\mathbf{A} - \mathbf{B}) \perp (\mathbf{B} - \mathbf{C}) \tag{3.0.49}$$

$$\angle BCA = \angle CAB \tag{3.0.50}$$

Thus, the triangle is right angled and isosceles.

4.4 Use Euclid Division Lemma to show that the square of any positive integer is either in the form $3q$ or $3q + 1$ for some integer q.

Solution: Let p be any positive integer. Then p can be either $3k, 3k + 1$ or $3k - 1$, for some postive integer k. If

$$p = 3k, \tag{3.0.51}$$

$$p^2 = 9k^2 = 3(q) \tag{3.0.52}$$

$$\text{where } q = 3k^2 \tag{3.0.53}$$

If

$$p = 3k + 1, \tag{3.0.54}$$

$$p^2 = 9k^2 + 6k + 1 = 3(q) + 1, \tag{3.0.55}$$

$$\text{where } q = 3k^2 + 2k \tag{3.0.56}$$

Similarly, if

$$p = 3k - 1, \quad (3.0.57)$$

$$p^2 = 9k^2 - 6k + 1 = 3(q) + 1, \quad (3.0.58)$$

$$\text{where } q = 3k^2 - 2k \quad (3.0.59)$$

4 Section D

4.5 a) Sum of the areas of two squares is $544m^2$. If the difference of their perimeters is $32m$, find the sides of the two squares.
Solution: Let the sides be x, y. From the given information,

$$x^2 + y^2 = 544 \quad (4.0.1)$$

$$4x - 4y = 32 \quad (4.0.2)$$

From the given information, the above equations can be expressed in vector form as

$$\mathbf{x}^\top \mathbf{x} = r^2 \quad (4.0.3)$$

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.0.4)$$

Thus, the desired solution is the point of intersection of the line with the circle in the first quadrant. Using the parametric equation of the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4.0.5)$$

Substituting the above in (4.0.3),

$$(\mathbf{A} + \lambda\mathbf{m})^\top (\mathbf{A} + \lambda\mathbf{m}) = r^2$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top\mathbf{A} + \|\mathbf{A}\|^2 - r^2 = 0 \quad (4.0.6)$$

yielding

$$\lambda = \frac{-\mathbf{m}^\top\mathbf{A} \pm \sqrt{(\mathbf{m}^\top\mathbf{A})^2 - \|\mathbf{m}\|^2\left(\|\mathbf{A}\|^2 - r^2\right)}}{\|\mathbf{m}\|^2} \quad (4.0.7)$$

For this problem, the numerical values are

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, c = 8, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (4.0.8)$$

$$\mathbf{A} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, r^2 = 544 \quad (4.0.9)$$

Substituting the above in (4.0.7),

$$\lambda = 12 \quad (4.0.10)$$

Thus, substituting from (4.0.10) and (4.0.8) in (4.0.5) the desired point of intersection is

$$\mathbf{x} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} + 12\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4.0.11)$$

$$= \begin{pmatrix} 20 \\ 12 \end{pmatrix} \quad (4.0.12)$$

Thus, the sides are $20m$ and $12m$.

b) A motorboat whose speed is 18 kmph in still water takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.
Solution: Let the speed of the stream be x and the speed of the boat be v. From the given information,

$$\frac{d}{v - x} = \frac{d}{x + v} + 1 \quad (4.0.13)$$

where d is the distance traveled upstream. From the above equation,

$$\frac{2xd}{v^2 - x^2} = 1 \quad (4.0.14)$$

$$\implies x^2 + 2xd - v^2 = 0 \quad (4.0.15)$$

$$\text{or, } x = -d \pm \sqrt{d^2 + v^2} \quad (4.0.16)$$

$$= 6 \quad (4.0.17)$$

4.6 a) For the following data, draw a 'less than' ogive and hence find the median of the distribution.

Age	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of people	5	15	20	25	15	11	9

b) The distribution given below shows the number of wickets taken by bowlers in one-day cricket matches. Find the mean and the median of the number of wickets. Find the mean and the median of the number of wickets taken.

Number of wickets	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of bowlers	5	15	20	25	15	11	9

4.7 A statue $1.6m$ tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is $60°$ and from the same point the angle of elevation of the top of the pedestal is $45°$. Find the height of the pedestal.Use$\left(\sqrt{3} = 1.73\right)$

Solution: Let the height of the pedestal be x and the height of the statue be h. Ifthe given angles be $\theta_1 = 60°$ and $\theta_2 = 45°$, from the given information,

$$(h + x)\cot\theta_1 = x\cot\theta_2 \tag{4.0.18}$$

$$\implies x = \frac{h\cot\theta_1}{\cot\theta_2 - \cot\theta_1} \tag{4.0.19}$$

4.8 a) Obtain other zeros of the polynomial $p\left(x\right) = 2x^4 - x^3 - 11x^2 + 5x + 5$ if the two of its zeros are $\sqrt{5}$ and $-\sqrt{5}$

b) What minimum must be added to $2x^3 - 3x^2 + 6x + 7$ so that the resulting polynomial will be divisible by $x^2 - 4x + 8$?

4.9 In a cylindrical vessel of radius 10cm, containing some water, 9000 small spherical balls are dropped which are completely immersed in water which raises the water level. If each spherical ball is of radius 0.5 cm then find the rise in the level of water in the vessel.

4.10 If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points, prove that the other two sides are divided in the same ratio.