CBSE MATHEMATICS 2020

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CONTENTS

1 MATRICES

1.1. Show that the plane $x-5y-2z=1$ contains the line $\frac{x-5}{3}=y=2-z$.
Solution: The plane and line can be expressed in vector form as

$$\begin{pmatrix}1 & -5 & -2\end{pmatrix}\mathbf{x}=1 \tag{1.1.1}$$

$$\mathbf{x}=\begin{pmatrix}5\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-1\end{pmatrix} \tag{1.1.2}$$

The plane contains the line if

$$\mathbf{m}^\top\mathbf{n}=0 \tag{1.1.3}$$

The input parameters are

$$\mathbf{m}=\begin{pmatrix}3\\1\\-1\end{pmatrix},\mathbf{n}=\begin{pmatrix}1 & -5 & -2\end{pmatrix}. \tag{1.1.4}$$

$\because$

$$\begin{pmatrix}1 & -5 & -2\end{pmatrix}\begin{pmatrix}3\\1\\-1\end{pmatrix}=0, \tag{1.1.5}$$

the given plane contains the given line.

1.2. Find a vector $\overrightarrow{r}$ equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units.
Solution: Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be the standard basis vectors (direction vectors of the coordinate axes) such that

$$\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}=\mathbf{I} \tag{1.2.1}$$

Then,

$$\frac{\mathbf{e}_1^\top\mathbf{r}}{\left\|\mathbf{e}_1\right\|\left\|\mathbf{r}\right\|}=\frac{\mathbf{e}_2^\top\mathbf{r}}{\left\|\mathbf{e}_2\right\|\left\|\mathbf{r}\right\|}=\frac{\mathbf{e}_3^\top\mathbf{r}}{\left\|\mathbf{e}_3\right\|\left\|\mathbf{r}\right\|}=\cos\theta \tag{1.2.2}$$

which can be expressed as the system of equations

$$\mathbf{e}_1^\top\mathbf{r}=\left\|\mathbf{r}\right\|\cos\theta \tag{1.2.3}$$
$$\mathbf{e}_2^\top\mathbf{r}=\left\|\mathbf{r}\right\|\cos\theta \tag{1.2.4}$$
$$\mathbf{e}_3^\top\mathbf{r}=\left\|\mathbf{r}\right\|\cos\theta \tag{1.2.5}$$

which can be combined to obtain the matrix equation

$$\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}^\top\mathbf{r}=\left\|\mathbf{r}\right\|\cos\theta\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.2.6}$$

$$\implies \frac{\mathbf{r}}{\left\|\mathbf{r}\right\|}=\cos\theta\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.2.7}$$

$$\because \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}=\mathbf{I} \tag{1.2.8}$$

From (1.2.7)

$$\left\|\cos\theta\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|=1 \tag{1.2.9}$$

$$\implies \cos\theta=\frac{1}{\sqrt{3}} \tag{1.2.10}$$

From (1.2.10) and (1.2.7)

$$\mathbf{r}=3\sqrt{3}\times\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.2.11}$$

$$=3\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.2.12}$$

1.3. Find the angle between unit vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ so that $\sqrt{3}\ \overrightarrow{a}$ - $\overrightarrow{b}$ is also a unit vector.
Solution: From the given information, the coordinate vector is

$$\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \tag{1.3.1}$$

The angle between the vectors is then given by

$$\cos\theta = \frac{1 - \|\mathbf{x}\|^2}{\mathbf{x}^\top \mathbf{R}\mathbf{x}} \tag{1.3.2}$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.3.3}$$

By substituting from (1.3.1) in (1.3.2)

$$\cos\theta = \frac{\sqrt{3}}{2} \implies \theta = 30° \tag{1.3.4}$$

1.4. If $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, Find scalar k so that $\mathbf{A}^2 + \mathbf{I} = k\mathbf{A}$.
Solution: Using the Cayley-Hamilton theorem,

$$\lambda^2 - k\lambda + 1 = 0 \tag{1.4.1}$$

Since the trace of the matrix is equal to the sum of its eigenvalues,

$$k = \lambda_1 + \lambda_2 = \text{tr}\left(\mathbf{A}\right) = -3 - 1 = -4 \tag{1.4.2}$$

1.5. Find the coordinates of the point where the line

$$\frac{x-1}{3} = \frac{y+4}{7} = \frac{z+4}{2}$$

cuts the xy-plane.
Solution: The given line can be expressed as

$$\mathbf{x} = \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \tag{1.5.1}$$

and the xy- plane is

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.5.2}$$

The desired point can be obtained as

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{1.5.3}$$

Substiuting the input parameters

$$\mathbf{m} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \mathbf{c} = 0, \tag{1.5.4}$$

in (1.5.3),

$$\begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + \frac{0 - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix}}{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}} \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \tag{1.5.5}$$

$$= \begin{pmatrix} 1 \\ -4 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} \tag{1.5.6}$$

$$= \begin{pmatrix} 7 \\ 10 \\ 0 \end{pmatrix} \tag{1.5.7}$$

1.6. The angle between the vectors $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$ is
a) $\frac{-\pi}{3}$
b) 0
c) $\frac{\pi}{3}$
d) $\frac{2\pi}{3}$

Solution: The angle between the vectors $\mathbf{m}_1, \mathbf{m}_2$ is given by

$$\cos\theta = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \tag{1.6.1}$$

Substituting

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{1.6.2}$$

in (1.6.1),

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1, \tag{1.6.3}$$

$$\left\| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\| = \sqrt{2} \tag{1.6.4}$$

$$\implies \cos\theta = \frac{-1}{2} \tag{1.6.5}$$

$$\text{or, } \theta = \frac{2\pi}{3} \tag{1.6.6}$$

1.7. If $\mathbf{A}$ is a non-singular square matrix of order 3 such that $\mathbf{A}^2 = 3\mathbf{A}$, then value of $\left|\mathbf{A}\right|$ is
a) -3
b) 3
c) 9
d) 27

$$\left|\mathbf{A}^2\right| = \left|3\mathbf{A}\right| \tag{1.7.1}$$
$$\implies \left|\mathbf{A}\right|^2 = 3^3\left|\mathbf{A}\right| \tag{1.7.2}$$

yielding

$$\left|\mathbf{A}\right| = 27 \tag{1.7.3}$$

after simplification.

1.8. If $\left|\overrightarrow{a}\right| = 4$ and $-3 \le \lambda \le 2$ then $\left|\lambda\overrightarrow{a}\right|$ lies in
   a) $[0,12]$
   b) $[2,3]$
   c) $[8,12]$
   d) $[-12,8]$

   Solution:

$$\left\|\lambda\mathbf{a}\right\| = \left|\lambda\right|\left\|\mathbf{a}\right\| \tag{1.8.1}$$
$$= 4\left|\lambda\right| \tag{1.8.2}$$

$\because$

$$0 \le \left|\lambda\right| \le 3, \tag{1.8.3}$$
$$0 \le 4\left|\lambda\right| \le 12 \tag{1.8.4}$$

1.9. The area of a triangle formed by vertices O, A and B, where $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$ is

   a) $3\sqrt{5}$ sq.units

   b) $5\sqrt{5}$ sq.units

   c) $6\sqrt{5}$ sq.units

   d) 4 sq.units

   Solution: Let

$$\mathbf{A} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-3\\-2\\1\end{pmatrix}, \tag{1.9.1}$$

In general, the area of $\triangle OAB$ can be expressed as

$$\frac{1}{2}\left\|\left(\mathbf{A}-\mathbf{O}\right)\times\left(\mathbf{B}-\mathbf{O}\right)\right\| \tag{1.9.2}$$

$\because$

$$\begin{pmatrix}2 & -2\\3 & 1\end{pmatrix} = 8, \tag{1.9.3}$$
$$\begin{pmatrix}3 & 1\\1 & -3\end{pmatrix} = -10, \tag{1.9.4}$$
$$\begin{pmatrix}1 & -3\\2 & -2\end{pmatrix} = 4, \tag{1.9.5}$$
$$\mathbf{A}\times\mathbf{B} = \begin{pmatrix}8\\-10\\4\end{pmatrix}, \tag{1.9.6}$$

and the desired area can be obtained as

$$\frac{1}{2}\left\|\begin{pmatrix}8\\-10\\4\end{pmatrix}\right\| = 3\sqrt{5} \tag{1.9.7}$$

1.10. The coordinates of the foot of the perpendicular drawn from the point $(2,-3,4)$ on the y-axis is
   a) $(2,3,4)$
   b) $(-2,-3,-4)$
   c) $(0,-3,0)$
   d) $(2,0,4)$

   Solution: In general, let $\mathbf{P}$ be any point and the line be

$$L: \quad \mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{1.10.1}$$

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top\left(\mathbf{P}-\mathbf{A}\right)}{\left\|\mathbf{m}\right\|^2}\mathbf{m} \tag{1.10.2}$$

The equation of the y-axis can be written as

$$\mathbf{x} = \lambda\begin{pmatrix}0\\1\\0\end{pmatrix} \tag{1.10.3}$$

Substituting the input parameters

$$\mathbf{P} = \begin{pmatrix}2\\-3\\4\end{pmatrix}, \mathbf{A} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \mathbf{m} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \tag{1.10.4}$$

in (1.10.2), the desired point is given by

$$\mathbf{x} = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}2\\-3\\4\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} \tag{1.10.5}$$
$$= \begin{pmatrix}0\\-3\\0\end{pmatrix} \tag{1.10.6}$$

1.11. The distance between parallel planes 2x+y-2z-6=0 and 4x+2y-4z=0 is ________ units.

   Solution: The above planes have parameters

$$\mathbf{n} = \begin{pmatrix}2 & 1 & -2\end{pmatrix}, c_1 = 6, c_2 = 0 \tag{1.11.1}$$

The distance is then obtained as

$$d = \frac{\left|c_1 - c_2\right|}{\left\|\mathbf{n}\right\|} \tag{1.11.2}$$
$$= \frac{6}{3} = 2 \tag{1.11.3}$$

1.12. If P(1,0,-3) is the foot of the perpendicular from the origin to the plane, then the Cartesian equation of the plane is ....
Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = c \tag{1.12.1}$$

Since $\mathbf{P}$ is a point on the plane, it satisfies the above equation and

$$\mathbf{n}^\top \mathbf{P} = c \tag{1.12.2}$$

The normal vector to the plane is OP. Hence,

$$\mathbf{n} = \mathbf{P} \tag{1.12.3}$$

Substituting the above in (1.12.2),

$$\mathbf{P}^\top \mathbf{P} = c \tag{1.12.4}$$

and the desired equation of the plane is

$$\mathbf{P}^\top \mathbf{x} = \mathbf{P}^\top \mathbf{P} \tag{1.12.5}$$

$$\begin{pmatrix} 1 & 0 & -3 \end{pmatrix} \mathbf{x} = 10 \tag{1.12.6}$$

after substituting numerical values.

1.13. Find the equation of the plane passing through the points $(1, 0, -2)$, $(3, -1, 0)$ and perpendicular to the plane $2x - y + z = 8$. Also find the distance of the plane thus obtained from the origin.
Solution: Let the equation of the desired plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{1.13.1}$$

From the given information,

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{1.13.2}$$

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & -2 \end{pmatrix} \mathbf{n} = 1 \\ \implies &\begin{pmatrix} 3 & -1 & 0 \end{pmatrix} \mathbf{n} = 1 \\ &\begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \mathbf{n} = 0 \end{aligned} \tag{1.13.3}$$

From (1.13.3), we obtain the matrix equation

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{1.13.4}$$

Forming the augmented matrix, and choosing the pivot,

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 3 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right) \tag{1.13.5}$$

$$\leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -6 & 2 \\ 0 & -1 & 5 & -2 \end{array}\right) \tag{1.13.6}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -6 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right) \tag{1.13.7}$$

yielding

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \tag{1.13.8}$$

Thus, the equation of the desired plane is

$$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \mathbf{n} = 1 \tag{1.13.9}$$

1.14. If

$$\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix} = \begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$$

then value of x is ————-
Solution: Expanding the above determinants,

$$2x^2 - 18 = 0 \tag{1.14.1}$$

$$\implies x = \pm 3 \tag{1.14.2}$$

1.15. Using integration, find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside the parabola $y^2 = 4x$.
Solution: The given circle and parabola can be expressed as conics with parameters

$$\mathbf{V}_1 = \mathbf{I}, \mathbf{u}_1 = -\begin{pmatrix} 4 \\ 0 \end{pmatrix}, f_1 = 0 \tag{1.15.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_2 = -\begin{pmatrix} 2 \\ 0 \end{pmatrix}, f_2 = 0 \tag{1.15.2}$$

The intersection of the given conics is obtained as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu \mathbf{V}_2\right) \mathbf{x} + 2 \left(\mathbf{u}_1 + \mu \mathbf{u}_2\right)^\top \mathbf{x} \tag{1.15.3}$$

$$+ \left(f_1 + \mu f_2\right) = 0 \tag{1.15.4}$$

This conic is a pair of straight lines if and only if

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{1.15.5}$$

$$\left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| < 0 \tag{1.15.6}$$

$$\implies \begin{vmatrix} 1 & 0 & -4-2\mu \\ 0 & 1+\mu & 0 \\ -4-2\mu & 0 & 0 \end{vmatrix} = 0 \tag{1.15.7}$$

upon substituting numerical values, which can be expanded to obtain

$$(\mu+1)(\mu+2)^2 = 0 \tag{1.15.8}$$

yielding

$$\mu = -2 \tag{1.15.9}$$

upon checking for (1.15.6). Thus, the parameters for the pair of straight lines can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.15.10}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \mathbf{0}, \tag{1.15.11}$$

$$f = 0, \tag{1.15.12}$$

$$\implies \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \tag{1.15.13}$$

Thus, the desired pair of straight lines are

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{1.15.14}$$

$$\implies \begin{pmatrix} 1 & \pm 1 \end{pmatrix} \mathbf{x} = 0 \tag{1.15.15}$$

$$\text{or, } \mathbf{x} = \kappa \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix} \tag{1.15.16}$$

upon substituting from (1.15.13). The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \kappa\mathbf{m} \quad \kappa \in \mathbb{R} \tag{1.15.17}$$

with the conic section

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{1.15.18}$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \kappa_i\mathbf{m} \tag{1.15.19}$$

where

$$\kappa_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u})\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)} \Big) \tag{1.15.20}$$

Substituting

$$\mathbf{q} = \mathbf{0}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.15.21}$$

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\begin{pmatrix} 4 \\ 0 \end{pmatrix} f = 0 \tag{1.15.22}$$

in (1.15.20), the intersection parameters κ_i of the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.15.23}$$

with the given circle are

$$\kappa = 0, 4 \tag{1.15.24}$$

and the points of intersection are

$$\mathbf{0}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.15.25}$$

Similarly, the points of intersection of the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{1.15.26}$$

with the given circle are

$$\mathbf{0}, \begin{pmatrix} 4 \\ -4 \end{pmatrix} \tag{1.15.27}$$

Thus the intersection of the parabola with the circle above the x-axis are the points

$$\mathbf{0}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.15.28}$$

From Fig. 1.15.1, the area covered by the parabola is given by

$$\int_0^4 2\sqrt{x} = \frac{4}{3}\left[x^{\frac{3}{2}}\right]_0^4 \tag{1.15.29}$$

$$= \frac{32}{3} \tag{1.15.30}$$

The area covered by the circle is

$$\frac{\pi r^2}{4} = 4\pi \tag{1.15.31}$$

Thus, the desired area is the shaded region in Fig. 1.15.1, and given by

$$4\pi - \frac{32}{3} \tag{1.15.32}$$

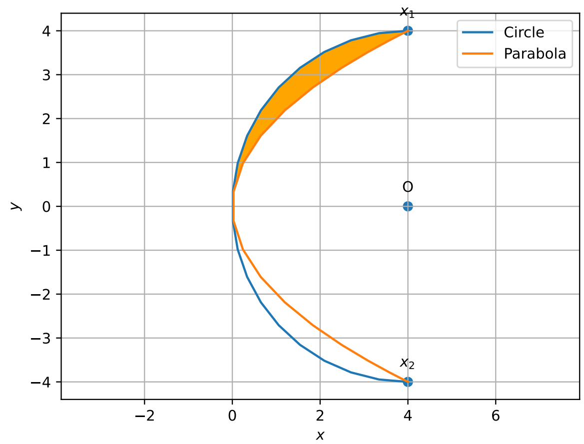

Fig. 1.15.1.

1.16. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A(2,0), B(4,5) and C(6,3).
Solution: Since

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-2\\-5\end{pmatrix}, \tag{1.16.1}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-4\\-3\end{pmatrix}, \tag{1.16.2}$$

the desired area is the magnitude of

$$\begin{vmatrix}2 & 4\\5 & 3\end{vmatrix} \tag{1.16.3}$$

Thus the desired area is 14 units.

1.17. If $A=\begin{pmatrix}5 & -1 & 4\\2 & 3 & 5\\5 & -2 & 6\end{pmatrix}$, Find A^{-1} and use it to solve the following system of the equations:

$$5x-y+4z=5$$
$$2x+3y+5z=2$$
$$5x-2y+6z=-1$$

Solution: Forming the augmented matrix and pivoting,

$$\left(\begin{array}{ccc|ccc}5 & -1 & 4 & 1 & 0 & 0\\2 & 3 & 5 & 0 & 1 & 0\\5 & -2 & 6 & 0 & 0 & 1\end{array}\right) \tag{1.17.1}$$

$$\leftrightarrow\left(\begin{array}{ccc|ccc}5 & -1 & 4 & 1 & 0 & 0\\0 & 17 & 17 & -2 & 5 & 0\\0 & 1 & -2 & 1 & 0 & -1\end{array}\right) \tag{1.17.2}$$

$$\leftrightarrow\left(\begin{array}{ccc|ccc}17 & 0 & 17 & 3 & 1 & 0\\0 & 17 & 17 & -2 & 5 & 0\\0 & 0 & 51 & -19 & 5 & 17\end{array}\right) \tag{1.17.3}$$

$$\xleftrightarrow[R_2\leftarrow 3R_2-R_3]{R_1\leftarrow 3R_1-R_3} \tag{1.17.4}$$

$$\left(\begin{array}{ccc|ccc}51 & 0 & 0 & 28 & -2 & -17\\0 & 51 & 0 & 13 & 10 & -17\\0 & 0 & 51 & -19 & 5 & 17\end{array}\right) \tag{1.17.5}$$

resulting in

$$\mathbf{A}^{-1}=\frac{1}{51}\begin{pmatrix}28 & -2 & -17\\13 & 10 & -17\\-19 & 5 & 17\end{pmatrix} \tag{1.17.6}$$

Thus, letting

$$\mathbf{b}=\begin{pmatrix}5\\2\\-1\end{pmatrix}, \tag{1.17.7}$$

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=\frac{1}{51}\begin{pmatrix}28 & -2 & -17\\13 & 10 & -17\\-19 & 5 & 17\end{pmatrix}\begin{pmatrix}5\\2\\-1\end{pmatrix} \tag{1.17.8}$$

$$=\begin{pmatrix}3\\2\\-2\end{pmatrix} \tag{1.17.9}$$

is the desired solution.

1.18. If x,y,z are different and

$$\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}=0 \tag{1.18.1}$$

then using properties of determinants show that

$$1+xyz=0$$

Solution: The given determinant can be expressed as

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \tag{1.18.2}$$

Since

$$\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \tag{1.18.3}$$

and

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \tag{1.18.4}$$

(1.18.2) can be expressed as

$$(1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \tag{1.18.5}$$

The above determinant can be simplified as

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \tag{1.18.6}$$

$$\xleftrightarrow[R_2 \leftarrow R_1 - R_2]{R_3 \leftarrow R_1 - R_3} \begin{vmatrix} 1 & x & x^2 \\ 0 & x-y & x^2-y^2 \\ 0 & x-z & x^2-z^2 \end{vmatrix}, \tag{1.18.7}$$

$$= (x-y)(x-z) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 1 & x+z \end{vmatrix}, \tag{1.18.8}$$

$$= (x-y)(y-z)(z-x) \tag{1.18.9}$$

and (1.18.1) can be obtained from (1.18.5) as

$$(1+xyz)(x-y)(y-z)(z-x) = 0 \tag{1.18.10}$$

Since

$$x \neq y \neq z, (1+xyz) = 0 \tag{1.18.11}$$

2 Continuous Math

2.1. Evaluate :

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$$

Solution: Let

$$\tan\theta = \sin x \tag{2.1.1}$$

Then

$$\sec^2\theta\, d\theta = \cos x\, dx \tag{2.1.2}$$

From (2.1.1) and (2.1.2)

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\; dx = 2\int_0^{\frac{\pi}{4}} \theta \tan\theta \sec^2\theta\, d\theta \tag{2.1.3}$$

Letting

$$u = \theta, dv = \tan\theta\sec^2\theta\, d\theta,$$

$$v = \int \tan\theta \sec^2\theta\, d\theta = \int t\, dt \quad (t = \tan\theta) = \frac{t^2}{2} = \frac{\tan^2\theta}{2} \tag{2.1.4}$$

Thus,

$$v\, du = \frac{\tan^2\theta}{2}\, d\theta \implies \int v\, du = \int \frac{\tan^2\theta}{2}\, d\theta = \int \frac{\sec^2\theta - 1}{2}\, d\theta = \frac{\tan\theta - \theta}{2} \tag{2.1.5}$$

and

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\; dx = 22\left[\theta\frac{\tan^2\theta}{2} - \frac{\tan\theta - \theta}{2}\right]_0^{\frac{\pi}{4}} = 2\left[\frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)\right] = \frac{\pi}{2} - 1 \tag{2.1.6}$$

2.2. Prove that

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

Solution:

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\tan^{-1}\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}} \tag{2.2.1}$$

$$=\tan^{-1}\frac{1}{2} \tag{2.2.2}$$

$$=\frac{1}{2}\tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} \tag{2.2.3}$$

$$=\frac{1}{2}\tan^{-1}\frac{4}{3} \tag{2.2.4}$$

$$=R.H.S \tag{2.2.5}$$

2.3. Differentiate $\sec^2(x^2)$ with respect to x^2.
Solution:

$$\frac{\sec^2(x^2)}{d\,(x^2)}=22\sec(x^2)\tan(x^2) \tag{2.3.1}$$

2.4. If $y=f(x^2)$ and $f'(x)=e^{\left(\sqrt{x}\right)}$, then find $\frac{dy}{dx}$.
Solution:

$$\frac{dy}{dx}=2xf'(x^2)=2xe^x \tag{2.4.1}$$

2.5. If

$$\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$$

prove that

$$\frac{dy}{dx}=\frac{x+y}{x-y} \tag{2.5.1}$$

. **Solution:** Let

$$y=x\tan\theta \tag{2.5.2}$$

$$\implies\frac{dy}{dx}=\tan\theta+x\sec^2\theta\frac{d\theta}{dx} \tag{2.5.3}$$

Then, (2.5.1) can be expressed as

$$\theta=\log\left(x\sec\theta\right) \tag{2.5.4}$$

$$\implies\frac{d\theta}{dx}=\frac{1}{x\sec\theta}\left(\sec\theta+x\sec\theta\tan\theta\frac{d\theta}{dx}\right) \tag{2.5.5}$$

$$=\frac{1}{x}\left(1+x\tan\theta\frac{d\theta}{dx}\right) \tag{2.5.6}$$

$$\text{or, }\frac{d\theta}{dx}=\frac{1}{x\left(1-\tan\theta\right)} \tag{2.5.7}$$

From (2.5.3) and (2.5.7)

$$\frac{dy}{dx}=\tan\theta+\frac{\sec^2\theta}{(1-\tan\theta)} \tag{2.5.8}$$

$$=\frac{1+\tan\theta}{1-\tan\theta} \tag{2.5.9}$$

$$=\frac{x+y}{x-y} \tag{2.5.10}$$

upon substituting from (2.5.1) and simplifying.

2.6. If

$$y=e^{(a\cos^{-1}x)},-1<x<1$$

then show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Solution: From the given information,

$$y_1=-\frac{ae^{(a\cos^{-1}x)}}{\sqrt{1-x^2}} \tag{2.6.1}$$

$$=-\frac{ay}{\sqrt{1-x^2}}\implies\sqrt{1-x^2}y_1+ay\quad=0 \tag{2.6.2}$$

Hence, differentiating the above equation,

$$\sqrt{1-x^2}y_2-\frac{2xy_1}{\sqrt{1-x^2}}+ay_1=0 \tag{2.6.3}$$

$$\implies\left(1-x^2\right)y_2-2xy_1-a^2y=0 \tag{2.6.4}$$

2.7. If $\cos(\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x)=0$, then x is

a) $\frac{1}{\sqrt{5}}$

b) $\frac{-2}{\sqrt{5}}$

c) $\frac{2}{\sqrt{5}}$

d) 1

Solution: Using a simplistic approach, $\because \cos\frac{\pi}{2}=0$,

$$\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x=\frac{\pi}{2} \tag{2.7.1}$$

$$\implies\frac{\pi}{2}-\cos^{-1}x=\sin^{-1}\frac{2}{\sqrt{5}} \tag{2.7.2}$$

$$\text{or, }\sin^{-1}x=\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \tag{2.7.3}$$

$$\implies x=\frac{2}{\sqrt{5}} \tag{2.7.4}$$

2.8. The interval in which the function f given by $f(x)=x^2e^{-x}$ is strictly increasing, is

a) $(-\infty, \infty)$
b) $(-\infty, 0)$
c) $(2, \infty)$
d) $(0, 2)$

Solution: Taking the derivative

$$f'(x) = 2xe^{-x} - x^2 e^{-x} \tag{2.8.1}$$
$$= \left(2x - x^2\right) e^{-x} \tag{2.8.2}$$

and

$$f'(x) > 0 \tag{2.8.3}$$
$$\implies x(2 - x) > 0 \tag{2.8.4}$$

The above expression results in two possibilities

$$x > 0, (2 - x) > 0 \tag{2.8.5}$$
$$\implies 0 < x < 2 \tag{2.8.6}$$

and

$$x < 0, (2 - x) < 0 \tag{2.8.7}$$
$$\implies x > 0, x < 2 \tag{2.8.8}$$

which is impossible. This can be seen in Fig. 2.8.1,

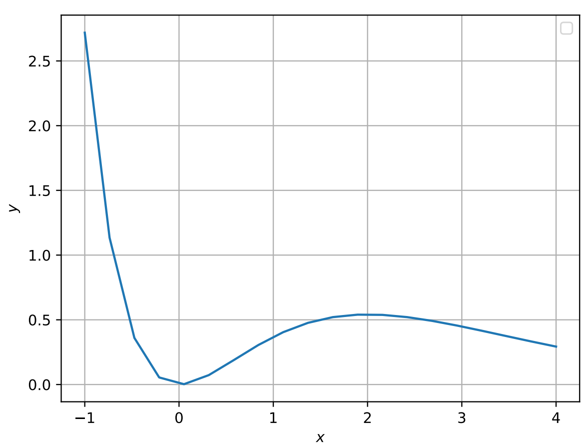

Fig. 2.8.1.

2.9. The function $f(x) = \frac{x-1}{x(x^2-1)}$ is discontinuous at

a) Exactly one point
b) Exactly two points
c) Exactly three points
d) No point

Solution: The given function can be expressed as

$$f(x) = \frac{x - 1}{x(x - 1)(x + 1)} \tag{2.9.1}$$

Hence, the denominator of the given function vanishes at $x = 0, 1, -1$, which are the points of discontinuity. However, there is a removable discontinuity at $x = 1$. This can be seen in Fig. 2.9.1,

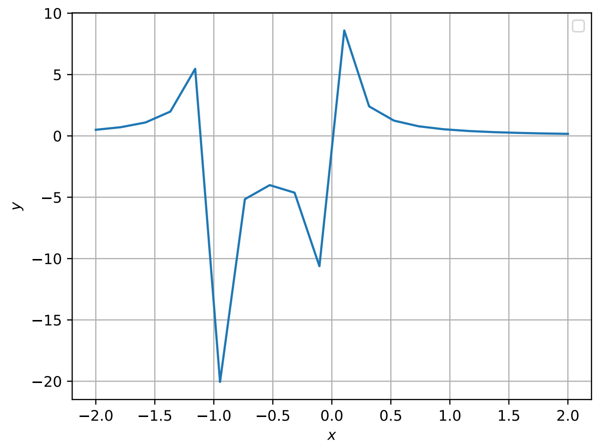

Fig. 2.9.1.

2.10. The function $f : R \to [-1, 1]$ defined by $f(x) = \cos x$ is

a) Both one-one and onto
b) Not one-one, but onto
c) one-one, but Not onto
d) Neither one-one, nor onto

Solution:

$$\cos(-2\pi) = \cos(2\pi) = 1 \tag{2.10.1}$$

Hence, $f(x)$ is not one to one. However, it is onto. This can be seen in Fig. 2.10.1,

2.11. If the radius of the circle is increasing at the rate of 0.5cm/s, then the rate of increase of its circumference is ————

Solution: Let the p be the circumference and r the radius. Then

$$p = 2\pi r \tag{2.11.1}$$
$$\implies \frac{dp}{dt} = 2\pi \frac{dr}{dt} = \pi \tag{2.11.2}$$

2.12. a) The range of the principle value branch of the function $y = \sec^{-1} x$ is ————-

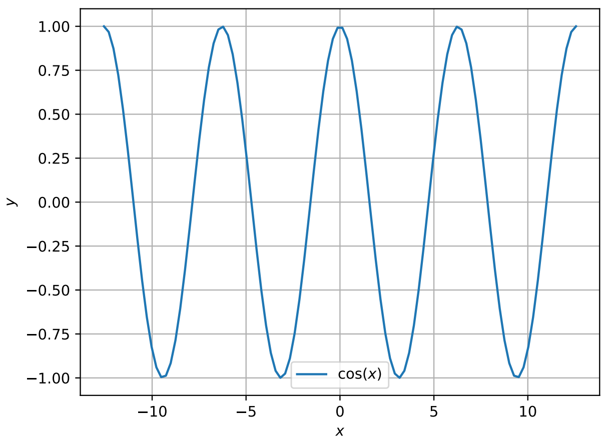

Fig. 2.10.1.

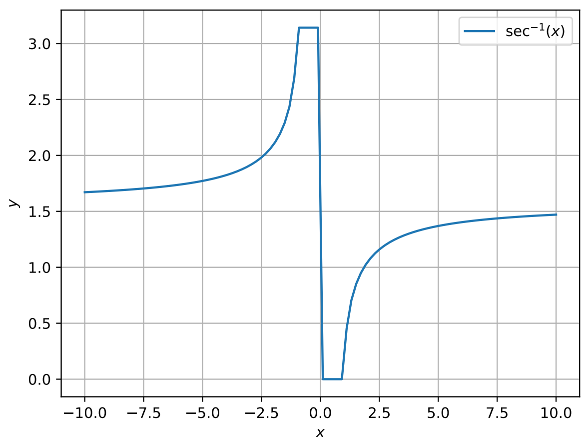

Fig. 2.12.1.

Solution: The range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$. This can be seen in Fig. 2.12.1,

b) The principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is ————

Solution: Since

$$\cos\frac{\pi}{3} = \frac{1}{2}, \tag{2.12.1}$$

$$\cos\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{2} \tag{2.12.2}$$

Thus the desired principal value is $\frac{2\pi}{3}$

2.13. Evaluate :

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x dx$$

Solution: $\because$ the integrand is odd, the given integral is 0. This can be seen in Fig. 2.13.1,

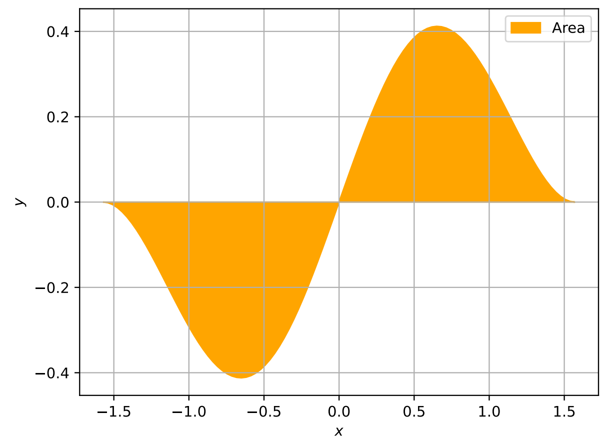

Fig. 2.13.1.

2.14. Find the value of k, so that the function

$$f(x) = \begin{cases} kx^2 + 5, & x \leq 1 \\ 2, & x > 1 \end{cases}$$

is continuous at x=1.

Solution: From the given equation,

$$f(1+) = f(1) \tag{2.14.1}$$

$$\implies k + 5 = 2 \tag{2.14.2}$$

$$\text{or, } k = -3 \tag{2.14.3}$$

This can be seen in Fig. 2.14.1,

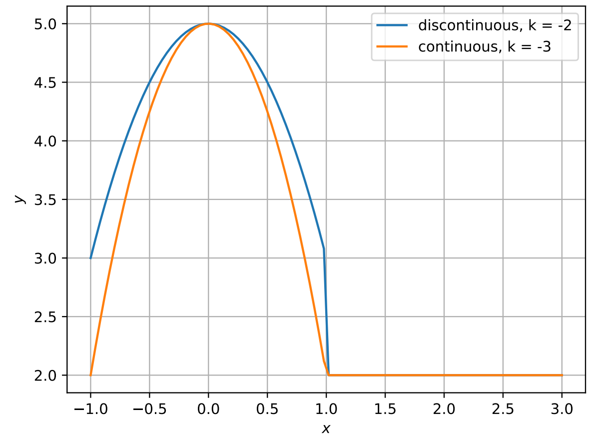

Fig. 2.14.1.

2.15. Find the integrating factor of the differential equation

$$x\frac{dy}{dx} = 2x^2 + y \tag{2.15.1}$$

Solution: The given differential equation can be expressed as

$$\frac{dy}{dx} - \frac{y}{x} = 2x \tag{2.15.2}$$

Letting

$$P(x) = -\frac{1}{x}, Q(x) = 2x, \tag{2.15.3}$$
$$M(x) = Ce^{\int P(x)\,dx} = e^{-\ln x} \tag{2.15.4}$$
$$= \frac{1}{x} \tag{2.15.5}$$

which is the desired integrating factor.

2.16. If

$$f(x) = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$$

Find

$$f'\left(\frac{\pi}{3}\right)$$

Solution: The given equation can be expressed as

$$f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}} \tag{2.16.1}$$
$$= \sqrt{2\sin^2\frac{x}{2} 2\cos^2\frac{x}{2}} \tag{2.16.2}$$
$$= \tan\frac{x}{2} \tag{2.16.3}$$
$$\implies f'(x) = f'\left(\frac{\pi}{3}\right) = \frac{1}{2} \tag{2.16.4}$$

2.17. Find $f'(x)$ if

$$f(x) = (\tan x)^{(\tan x)} \tag{2.17.1}$$

Solution: Differentiating the above,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left\{(\tan x)^{(\tan x)}\right\} \\ &= (\tan x)(\tan x)^{(\tan x)-1}\sec^2 x \\ &+ (\tan x)^{(\tan x)}\ln(\tan x)\sec^2 x \\ &= \sec^2(\tan x)^{(\tan x)}(1+\ln(\tan x)) \end{aligned} \tag{2.17.2}$$

2.18. Find :

$$\int \frac{\tan^3 x}{\cos^3 x}dx \tag{2.18.1}$$

Solution: Letting

$$t = \sec x, dt = \sec x \tan x\,dx \tag{2.18.2}$$
$$\implies \int \frac{\tan^3 x}{\cos^3 x}dx = \int (\sec x \tan x)^3\,dx \tag{2.18.3}$$
$$= \int t^2\left(t^2-1\right)\,dt \tag{2.18.4}$$
$$= \frac{t^5}{5} - \frac{t^3}{3} \tag{2.18.5}$$
$$= \frac{\sec^5}{5} - \frac{\sec^3}{3} \tag{2.18.6}$$

2.19. Solve the following differential equation:

$$(1+e^{\frac{y}{x}})dy + e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx = 0; (x \neq 0) \tag{2.19.1}$$

Solution: Letting

$$y = xt, dy = x\,dt + t\,dx \tag{2.19.2}$$

(2.19.1) can be expressed as

$$\begin{aligned} &\left(1+e^t\right)(x\,dt + t\,dx) + e^t(1-t)dx = 0 \\ &\implies x\left(1+e^t\right)\,dt \\ &+ \left\{t\left(1+e^t\right) + (1-t)\,e^t\right\}\,dx = 0 \\ &\implies \frac{dx}{x} = -\frac{(1+e^t)}{t+e^t}dt \\ &\implies \int\frac{dx}{x} = -\int\frac{(1+e^t)}{t+e^t}dt \\ &\text{or, } \ln x = -\ln\left(\frac{y}{x}+e^{\frac{y}{x}}\right) + C_1 \end{aligned} \tag{2.19.3}$$

Thus,

$$y + xe^{\frac{y}{x}} = C \tag{2.19.4}$$

3 Discrete Math

3.1. The relation R in the set $\{1,2,3\}$ given by $R = \{(1,2)(2,1)(1,1)\}$ is

a) Symmetric and transitive, but not reflexive

b) reflexive and symmetric, but not transitive

c) Symmetric, but neither reflexive nor transitive

d) An equivalence relation

3.2. Check whether the relation R in the set N set of natural numbers given by R= {(a,b):a is divisor of b} is reflexive, symmetric or transitive. Also determine whether R is an equivalence relation.

4 Probability

4.1. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn randomly one-by-one without replacement and are found to to be both kings. Find the probability of the lost card being a king.
Solution: See Tables (4.1.1) and (4.1.2) for

Event	Description
$X_1 = 1$	Lost card being king
$X_2 = 1$	One card drawn being a king
$X_3 = 2$	Two cards drawn being kings

TABLE 4.1.1

Probability	Value
$\Pr(X_1 = 1)$	$\frac{4}{52} = \frac{1}{13}$
$\Pr(X_1 = 0)$	$\frac{12}{13}$
$\Pr(X_2 = 1 \vert X_1 = 1)$	$\frac{3}{51} = \frac{1}{17}$
$\Pr(X_2 = 1 \vert X_1 = 0)$	$\frac{4}{51}$
$\Pr(X_3 = 1 \vert X_1 = 1, X_2 = 1)$	$\frac{2}{50} = \frac{1}{25}$
$\Pr(X_3 = 1 \vert X_1 = 0, X_2 = 1)$	$\frac{3}{50}$
$\Pr(X_1 = 1 \vert X_2 = 1, X_3 = 1)$	?

TABLE 4.1.2

the input probabilities. The desired probability is then obtained from (4.1.2) as

$$\Pr(X_1 = 1|X_2 = 1, X_3 = 1) = \frac{\frac{1}{25} \times \frac{1}{17} \times \frac{1}{13}}{\frac{1}{25} \times \frac{1}{17} \times \frac{1}{13} + \frac{3}{50} \times \frac{4}{51} \times \frac{12}{13}} = \frac{1}{25} \tag{4.1.1}$$

4.2. A fair dice is thrown two times. Find the probability distribution of the number of sixes. Also determine the mean of the number of sixes.
Solution: Let $X = \{0, 1, 2\}$ denote the sample space. Table (4.2.1) gives the desired probability distribution. In general, this can be expressed as a Binomial distribution with probability mass function (pmf)

$$p_X(k) = {}^nC_k (1-p)^k p^{n-k}, 0 \le k \le n \tag{4.2.1}$$

The mean value is then obtained as

$$E[X] = np = \frac{1}{3} \tag{4.2.2}$$

for $n = 2$ and $p = \frac{1}{6}$

Probability	Value
$\Pr(X = 0)$	$\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$
$\Pr(X = 1)$	$2 \times \frac{5}{6} \times \frac{1}{6} = \frac{5}{18}$
$\Pr(X = 2)$	$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

TABLE 4.2.1

5 Linear Programming

5.1. The corner points of the feasible region of an LPP are (0,0),(0,8),(2,7),(5,4) and (6,0). The maximum profit P=3x+2y occurs at the point ....
Solution: The profit can be expressed as

$$P = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \tag{5.1.1}$$

and the respective values at each of the above points are given by

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0, \tag{5.1.2}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix} = 16 \tag{5.1.3}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = 20 \tag{5.1.4}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 23 \tag{5.1.5}$$

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 18 \tag{5.1.6}$$

Hence, the maximum profit is $P = 23$ which occurs at $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

5.2. A cottage industry manufactures pedestal lamps and wooden shades. Both the products require machine time as well as craftsman time in the making. The number of hours required for producing 1 unit of each and the corresponding profit is given in the following table : In a day, the factory has availability of not more than 42 hours of machine time and 24 hours of craftsman time.
Assuming that all items manufactured are sold, how should the manufacturer schedule his daily

Item	Machine Time	Craftsman Time	Profit(in INR)
Pedestal Lamp	1.5 hours	3 hours	30
Wooden shades	3 hours	1 hour	20

TABLE 5.2.1

production in order to maximise the profit? Formulate it as an LPP and solve it graphically.
Solution: Let x be the number of lamps and y be the number of wooden shades produced. From the given information, the problem can be formulated as

$$P = \max_{x,y} 30x + 20y \tag{5.2.1}$$
$$1.5x + 3y \le 42 \tag{5.2.2}$$
$$3x + y \le 24 \tag{5.2.3}$$

which can be expressed in vector form as

$$P = \max_{\mathbf{x}} \begin{pmatrix} 30 & 20 \end{pmatrix} \mathbf{x} \tag{5.2.4}$$
$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 28 \\ 24 \end{pmatrix} \tag{5.2.5}$$
$$\mathbf{x} \succeq \mathbf{0} \tag{5.2.6}$$

a) *Graphical solution:*
From Fig. 5.2.1, the feasible region is a quadrilateral with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 14 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \end{pmatrix} \tag{5.2.7}$$

with respective profit

$$\begin{pmatrix} 30 & 20 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{5.2.8}$$
$$\begin{pmatrix} 30 & 20 \end{pmatrix} \begin{pmatrix} 8 \\ 0 \end{pmatrix} = 240 \tag{5.2.9}$$
$$\begin{pmatrix} 30 & 20 \end{pmatrix} \begin{pmatrix} 0 \\ 14 \end{pmatrix} = 280 \tag{5.2.10}$$
$$\begin{pmatrix} 30 & 20 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \end{pmatrix} = 360 \tag{5.2.11}$$

Thus, the manufacturer should produce 4 pedestal lamps and 12 wooden shades daily.

b) *Lagrange Multipliers:* The given problem is expressed in the form

$$P = -\min_{\mathbf{x}} \begin{pmatrix} 30 & 20 \end{pmatrix} \mathbf{x} \tag{5.2.12}$$
$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 28 \\ 24 \\ 0 \\ 0 \end{pmatrix} \tag{5.2.13}$$

The Lagrangian, defined as the linear combination of the loss function and the constraints is defined as

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\lambda}) = -\begin{pmatrix} 30 & 20 \end{pmatrix} \mathbf{x} \\ + \lambda_1 \left[\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} - 28 \right] \\ + \lambda_2 \left[\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} - 24 \right] + \lambda_3 \left[\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} \right] \\ + \lambda_4 \left[\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} \right] \end{aligned} \tag{5.2.14}$$

Taking the derivative

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = 0, \tag{5.2.15}$$

we obtain

$$\lambda_1 + 3\lambda_2 - \lambda_3 = 30 \tag{5.2.16}$$
$$2\lambda_1 + \lambda_2 - \lambda_4 = 20 \tag{5.2.17}$$
$$x_1 + 2x_2 = 28 \tag{5.2.18}$$
$$3x_1 + x_2 = 24 \tag{5.2.19}$$
$$x_1 = 0 \tag{5.2.20}$$
$$x_2 = 0 \tag{5.2.21}$$

It is obvious that $x_1 = 0, x_2 = 0$ are infeasible. Hence, considering only λ_1, λ_2 as the active multipliers, the above equations can be expressed as

$$\begin{pmatrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 28 \\ 24 \end{pmatrix} \tag{5.2.22}$$

yielding the optimal solution as

$$\begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 6 \\ 8 \end{pmatrix} \tag{5.2.23}$$

5.3. Amongst all open (from the top) right circular cylindrical boxes of volume $125\pi\ cm^3$, find the dimensions of the box which has the least surface area.
Solution: Let r be the radius of the cylinder and h be the height. Then the surface area is

$$S = \pi r^2 + 2\pi r h \tag{5.3.1}$$

Also, the volume is

$$V = \pi r^2 h \tag{5.3.2}$$

a) The given problem can then be formulated as

$$S = \min_{r,h} \pi r^2 + 2\pi r h \tag{5.3.3}$$
$$\text{s.t} \quad \pi r^2 h = 125 \tag{5.3.4}$$

which is a *disciplined geometric programming* (DGP) problem that can be solved using $cvxpy$. DGP is a subset of *log-log-convex program* (LLCP). An LLCP is defined as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le \tilde{f}_i, \quad i = 1, \dots, m \\ & g_i(x) = \tilde{g}_i, \quad i = 1, \dots, p, \end{array} \tag{5.3.5}$$

where the functions f_i are log-log convex, $\tilde{f}_i$ are log-log concave, and the functions g_i and $\tilde{g}_i$ are log-log affine. An optimization problem with constraints of the above form in which the goal is to maximize a log-log concave function is also an LLCP. A function

$$f : D \subseteq \mathbf{R}^n_{++} \to \mathbf{R} \tag{5.3.6}$$

is said to be log-log convex if the function

$$F(u) = \log f(e^u) \tag{5.3.7}$$

with domain

$$\{u \in \mathbf{R}^n : e^u \in D\} \tag{5.3.8}$$

is convex (where $\mathbf{R}^n_{++}$ denotes the set of positive reals and the logarithm and exponential are meant elementwise); the function F is called the log-log transformation of f. The function f is log-log concave if F is concave, and it is log-log affine if F is affine. LLCPs are problems that become convex after the variables, objective functions, and constraint functions are replaced with their logs, an operation that we refer to as a log-log transformation. LLCPs generalize geometric programming.

b) Alternatively, from (5.3.1) and (5.3.2)

$$S(r) = \pi r^2 + \frac{2V}{r} \tag{5.3.9}$$
$$\implies S'(r) = 2\pi r - \frac{2V}{r^2} \tag{5.3.10}$$
$$\text{and } S''(r) = 2\pi + \frac{4V}{r^3} > 0 \tag{5.3.11}$$

Thus, $S(r)$ has a minimum which can be obtained from (5.3.10) as

$$2\pi r - \frac{2V}{r^2} = 0 \tag{5.3.12}$$
$$\implies r = \left(\frac{V}{\pi}\right)^{\frac{1}{3}} \tag{5.3.13}$$
$$= 5 \quad \text{and} \tag{5.3.14}$$
$$h = \frac{V}{\pi r^2} = 5 \tag{5.3.15}$$

upon substituting numerical values. This is verified in Fig. 5.3.1.

c) Using gradient descent, the update equation can be expressed as

$$r_{n+1} = r_n - \gamma S'(r_n) \tag{5.3.16}$$

where $r_0 = 2$ and $\gamma = 0.001$ are chosen by the user. These values need to be suitably guessed for the algorithm to converge.

$$\Pr\left(X_1=1|X_2=1,X_3=1\right)=\frac{\Pr\left(X_1=1,X_2=1,X_3=1\right)}{\Pr\left(X_2=1,X_3=1\right)}$$
$$=\frac{\Pr\left(X_3=1|X_1=1,X_2=1\right)\Pr\left(X_2=1|X_1=1\right)\Pr\left(X_1=1\right)}{\sum_{i=0}^{1}\Pr\left(X_1=i,X_2=1,X_3=1\right)}$$
$$=\frac{\Pr\left(X_3=1|X_1=1,X_2=1\right)\Pr\left(X_2=1|X_1=1\right)\Pr\left(X_1=1\right)}{\sum_{i=0}^{1}\Pr\left(X_3=1,|X_1=i,X_2=1\right)\Pr\left(X_2=1|X_1=i\right)\Pr\left(X_1=i\right)} \quad (4.1.2)$$

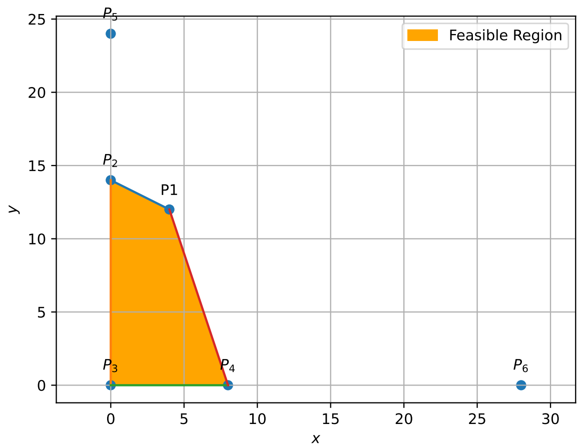

Fig. 5.2.1.

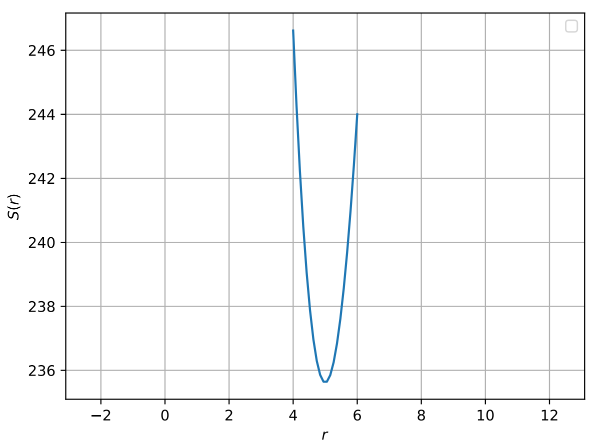

Fig. 5.3.1.