> 1. The matrix, $M_i = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$, can represent Euclidean transformations, where $R_i \in \mathbb{R}^{3\times 3}$ is an orthonormal matrix, $\det(\mathbf{R}_i) = 1$, and $\mathbf{t}_i \in \mathbb{R}^{3\times 1}$ is a vector. Please prove that the set $\{M_i\}$ forms a group.

Answer:

We consider two sets:

$$SE(3) = \{\mathbf{M}|\mathbf{M} = \left[\begin{array}{c:c} \mathbf{R} & \mathbf{t} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right], \mathbf{R} \in R^{3\times 3}, \mathbf{t} \in R^3, \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, det(\mathbf{R}) = 1\}$$
$$G = \{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R}\mathbf{R}^T = I, det(\mathbf{R}) = 1\}.$$

Let $g_1 = R_1 \in G, g_2 = R_2 \in G$, $M_1 = \left[\begin{array}{c:c} g_1 & \mathbf{t_1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \in A$, $M_2 = \left[\begin{array}{c:c} g_2 & \mathbf{t_2} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \in A$.

(1) Verify closure.

According our assumption, we can get $g_1 g_2 = R_1 R_2$, which is true under ordinary matrix multiplication.

$$\begin{aligned} A_1 \times A_2 &= \left[\begin{array}{c:c} g_1 & \mathbf{t_1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c:c} g_2 & \mathbf{t_2} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \\ &= \left[\begin{array}{c:c} g_1 g_2 & g_1\mathbf{t_2} + \mathbf{t_1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \end{aligned}$$

where the "×" refers to the standard multiplication operation between matrices.

$$\because g_1 g_2 = R_1 R_2$$
$$\therefore (g_1 g_2)(g_1 g_2)^T = (R_1 R_2)(R_1 R_2)^T = R1R2R_2^T R_1^T = I$$
$$\because det(g_1 g_2) = det(R_1 R_2) = det(R_1) det(R_2) = 1$$
$$\therefore g_1 g_2 \in G,\ g_1\mathbf{t_2} + \mathbf{t_1} \in R^3$$
$$\therefore A_1 \times A_2 \in SE(3)$$

The set is closed under the multiplication operation. In other words, if A and B are any two matrices in SE(3), AB ∈ SE(3).

(2) Verify associativity

The multiplication operation between matrices is associative. In other words, if A, B, and C are any three matrices ∈ SE(3), then (AB) C = A (BC).

(3) Verify identity element

For every element A ∈ SE(3), there is an identity element given by the 4×4 identity matrix,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SE(3),$$

such that $\mathbf{AI} = \mathbf{A}$.

(4) Verify inverse element for each group element.

We assume that $M = \left[\begin{array}{c:c} g & \mathbf{t} \\ \hdashline \mathbf{0}_{1\times3} & 1 \end{array}\right] \in SE(3)$, we first need to prove the inverse element of g_1 belongs to G. Let $g = R \in G$

$$\because (R^TR)^{-1} = R^{-1}(R^{-1})^T = I,\ det(R^{-1}) = \frac{1}{det(R)} = 1$$

$$\therefore R^{-1} \in G$$

$$\because g = R, gR^{-1} = RR^{-1} = I, R^{-1}g = R^{-1}R = I.$$

Therefore, the inverse element of g is $g^{-1}, g^{-1} \in G$.

For $M \in SE(3)$, we can easily find out its inverse element:

$$A^{-1} = \left[\begin{array}{c:c} R^{-1} & -R^{-1}\mathbf{t} \\ \hdashline \mathbf{0}_{1\times3} & 1 \end{array}\right] = \left[\begin{array}{c:c} R^{T} & -R^{T}\mathbf{t} \\ \hdashline \mathbf{0}_{1\times3} & 1 \end{array}\right],$$

and $M^{-1} \in SE(3)$.

Therefore, the set $\{M_i\}$ forms a group.

> 2. When deriving the Harris corner detector, we get the following matrix *M* composed of first-order partial derivatives in a local image patch *w*,
>
> $$M = \begin{bmatrix} \sum_{(x_i,y_i)\in w}(I_x)^2 & \sum_{(x_i,y_i)\in w}(I_xI_y) \\ \sum_{(x_i,y_i)\in w}(I_xI_y) & \sum_{(x_i,y_i)\in w}(I_y)^2 \end{bmatrix},$$
>
> a) Please prove that *M* is positive semi-definite.
>
> b) In practice, *M* is usually positive definite. If *M* is positive definite, prove that in the Cartesian coordinate system, $[x, y]M\begin{bmatrix} x \\ y \end{bmatrix} = 1$, represents an ellipse.
>
> c) Suppose that *M* is positive definite and its two eigen-values are λ_1 and λ_2 and $\lambda_1 > \lambda_2 > 0$. For the ellipse defined by $[x, y]M\begin{bmatrix} x \\ y \end{bmatrix} = 1$, prove that the length of its semi-major axis is $\frac{1}{\sqrt{\lambda_2}}$ while the length of its semi-minor axis is $\frac{1}{\sqrt{\lambda_1}}$.

Answer:

a)

Let $x = [u\ v]^T$

$$x^TMx = [u\ v]\begin{bmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = u^2I_x^2 + 2uvI_xI_y + v^2I_y^2 = (uI_x + vI_y)^2 \geq 0.$$

For any $x \in R^2$, $x^TMx \geq 0$. Therefore, *M* is positive semi-definite and hence eigenvalues are always non-negative.

b)

From a), we can see that if *M* is positive definite, then $x^T Mx > 0$. Any equation of ellipse centered at the origin can be written as a quadratic form $k = a_1u^2 + a_2uv + a_3v^2, k > 0$. Therefore, $[x, y]M \begin{bmatrix} x \\ y \end{bmatrix} = x^2I_x^2 + 2xyI_xI_y + y^2I_y^2 = 1$ can represent an ellipse, where $k = 1, a_1 = x^2, a_2 = 2xy, a_3 = y^2$.

c)

According to Principal Axis Theorem, we can we can always diagonalize symmetric matrices – that is we can find a matrix P for a symmetric matrix M such what $P^\top MP = D$ where D is a diagonal matrix. The columns of P that produce this result are found to be the eigenvectors of M.

Let $x = Py$, where x is in standard coordinates and y is in a different coordinate system where the columns of P are the basis vectors of that coordinate system. Then we can rewite the ellipse equation:

$$K = x^\top Mx\top = y^\top P^\top MPy = y^\top Dy,$$

where D is a diagonal matrix. Now we can get:

$$K = y^\top Dy = [y_1 \quad y_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2.$$

We've just eliminated our cross term. We can use the standard formula for an ellipse to match coefficients. We find the following:

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = k = \lambda_1 y_1^2 + \lambda_2 y_2^2$$
$$\therefore r_1 = \lambda_1^{-\frac{1}{2}} \text{ and } r_2 = \lambda_2^{-\frac{1}{2}}$$

> 3. In the lecture, we talked about the least square method to solve an over-determined linear system $A\mathbf{x} = b, A \in \mathbb{R}^{m\times n}, \mathbf{x} \in \mathbb{R}^{n\times 1}, m > n, \operatorname{rank}(A) = n$. The closed form solution is $\mathbf{x} = \left(A^TA\right)^{-1}A^Tb$. Try to prove that A^TA is non-singular (or in other words, it is invertible).

Answer:

We can prove $col(A^TA) = col(A)$ to demonstrate $rank(A^TA) = rank(A)$, according to Rank Theorem, to prove that A^TA is non-singular (invertible).

Let $\mathcal{A}_1 = \operatorname{col}(A) \text{ and } \mathcal{A}_2 = \operatorname{col}\left(A^tA\right)$, then we have:

$$\mathcal{A}_2 = \left\{A^tA\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\right\} \subseteq \left\{A^t\boldsymbol{y} : \boldsymbol{y} \in \mathbb{R}^m\right\} = \mathcal{A}_1.$$

Thus $\mathcal{A}_1^\perp \subseteq \mathcal{A}_2^\perp$. In fact, by the defifinition of orthogonal complement. Indeed:

$$\boldsymbol{u} \in \mathcal{A}_1^\perp \quad \Rightarrow \quad \boldsymbol{u}\cdot\boldsymbol{v} = 0, \forall \boldsymbol{v} \in \mathcal{A}_1 \quad \Rightarrow \quad \boldsymbol{u}\cdot\boldsymbol{v} = 0, \forall \boldsymbol{v} \in \mathcal{A}_2 \quad \Rightarrow \quad \boldsymbol{u} \in \mathcal{A}_2^\perp.$$

However, we can also show $\mathcal{A}_2^\perp \subseteq \mathcal{A}_1^\perp$ as follow:

$$\boldsymbol{u} \in \mathcal{A}_2^\perp \quad \Rightarrow \quad \boldsymbol{u}^tA^tA = 0 \quad \Rightarrow \quad \|A\boldsymbol{u}\|^2 = \boldsymbol{u}^tA^tA\boldsymbol{u} = 0 \quad \Rightarrow \quad \boldsymbol{u}^tA^t = 0 \quad \Rightarrow \quad \boldsymbol{u} \in \mathcal{A}_1^\perp$$

Thus $\mathcal{A}_1^{\perp} = \mathcal{A}_2^{\perp}$ and $\mathcal{A}_1 = \left(\mathcal{A}_1^{\perp}\right)^{\perp} = \left(\mathcal{A}_2^{\perp}\right)^{\perp} = \mathcal{A}_2$, that is, $col(A^T A) = col(A)$.

Therefore, is $A \in \mathbb{R}^{m \times n}$ has full column rank (rank(A) = *n*) , then rank($A^T A$)=n. So $A^T A$ is non-singular.