Due: Monday, 21st November, 12:00 (AEST)

There are two options for submission:

- Through WebCMS/give as a single pdf file, maximum size 2Mb. Prose should be typed, not handwritten. Use of LaTeX is encouraged, but not required.
- Through the Inspera online assessment system (details TBA).

You may submit through either or both systems – if you submit through both systems please indicate which version should be marked.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (12 marks)

Consider the following two algorithms that naïvely compute the sum and product of two $n \times n$ matrices.

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sum(A,B):
  for i ∈ [0,n):
    for j ∈ [0,n):
      C[i,j] = A[i,j] + B[i,j]
    end for
  end for
  return C
```

```
product(A,B):
  for i ∈ [0,n):
    for j ∈ [0,n):
      C[i,j] = add{A[i,k] * B[k,j] : k ∈ [0,n)}
    end for
  end for
  return C
```

Assuming that adding and multiplying matrix elements can be carried out in $O(1)$ time, and add will add the elements of a set S in $O(|S|)$ time:

(a) Give an asymptotic upper bound, in terms of n, for the running time of sum. (3 marks)

(b) Give an asymptotic upper bound, in terms of n, for the running time of product. (3 marks)

When n is even, we can define a recursive procedure for multiplying two $n \times n$ matrices as follows. First, break the matrices into smaller submatrices:

$$A = \begin{pmatrix} S & T \\ U & V \end{pmatrix} \qquad B = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where S, T, U, V, W, X, Y, Z are $\frac{n}{2} \times \frac{n}{2}$ matrices. Then it is possible to show:

$$AB = \begin{pmatrix} SW + TY & SX + TZ \\ UW + VY & UX + VZ \end{pmatrix}$$

where $SW + TY$, $SX + TZ$, etc. are sums of products of the smaller matrices. If n is a power of 2, each smaller product (SW, TY, etc) can be computed recursively, until the product of 1×1 matrices needs to be computed – which is nothing more than a simple multiplication, taking $O(1)$ time.

Assume n is a power of 2, and let $T(n)$ be the worst-case running time for computing the product of two $n \times n$ matrices using this method.

(c) With justification, give a recurrence equation for $T(n)$. (4 marks)

(d) Find an asymptotic upper bound for $T(n)$. (2 marks)

Problem 2 (18 marks)

Recall from Assignment 2 the neighbourhood of eight houses:

As before, each house wants to set up its own wi-fi network, but the wireless networks of neighbouring houses – that is, houses that are either next to each other (ignoring trees) or over the road from one another (directly opposite) – can interfere, and must therefore be on different channels. Houses that are sufficiently far away may use the same wi-fi channel. Again we would like to solve the problem of finding the minimum number of channels needed, but this time we will solve it using techniques from logic and from probability. Rather than directly asking for the minimum number of channels required, we ask if it is possible to solve it with just 2 channels. So suppose each wi-fi network can either be on channel hi or on channel lo. Is it possible to assign channels to networks so that there is no interference?

(a) Formulate this problem as a problem in propositional logic. Specifically:

- (i) Define your propositional variables (4 marks)
- (ii) Define any propositional formulas that are appropriate and indicate what propositions they represent. (4 marks)
- (iii) Indicate how you would solve the problem (or show that it cannot be done) using propositional logic. It is sufficient to explain the method, you do not need to provide a solution. (2 marks)
- (iv)* Explain how to modify your answer(s) to (i) and (ii) if the goal was to see if it is possible to solve with 3 channels rather than 2. (4 marks)

(b) Suppose each house chooses, uniformly at random, one of the two network channels. What is the probability that there will be no interference? (4 marks)

Problem 3 (12 marks)

Prove the following results hold in all Boolean Algebras:

(a) For all x: $(x \wedge 1') \vee (x' \wedge 1) = x'$ (4 marks)

(b) For all x, y: $(x \wedge y) \vee x = x$ (4 marks)

(c) For all x, y: $y' \wedge ((x \vee y) \wedge x') = 0$ (4 marks)

Proof assistant

https://cgi.cse.unsw.edu.au/~cs9020/cgi-bin/logic/22T3/boolean_algebra/assignment3a

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Problem 4* (4 marks)

Show that there are no three element Boolean Algebras.

Problem 5 (12 marks)

Prove or disprove the following logical equivalences:

(a) $\neg(p \to q) \equiv (\neg p \to \neg q)$ (4 marks)

(b) $((p \wedge q) \to r) \equiv (p \to (q \to r))$ (4 marks)

(c) $((p \vee (q \vee r)) \wedge (r \vee p)) \equiv ((p \wedge q) \vee (r \vee p))$ (4 marks)

Proof assistant

https://cgi.cse.unsw.edu.au/~cs9020/cgi-bin/logic/22T3/prop_logic/assignment3b

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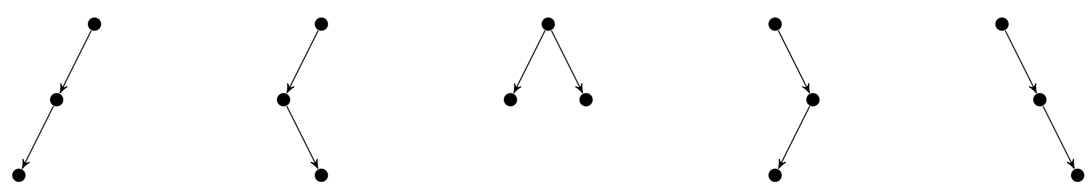
Problem 6 (16 marks)

Recall from Assignment 2 the definition of a binary tree data structure: either an empty tree, or a node with two children that are trees.

Let $T(n)$ denote the number of binary trees with n nodes. For example $T(3) = 5$ because there are five binary trees with three nodes:



(a) Using the recursive definition of a binary tree structure, or otherwise, derive a recurrence equation for $T(n)$. (6 marks)

A **full binary tree** is a non-empty binary tree where every node has either two non-empty children (i.e. is a fully-internal node) or two empty children (i.e. is a leaf).

(b) Using observations from Assignment 2, or otherwise, explain why a full binary tree must have an odd number of nodes. (2 marks)

(c) Let $B(n)$ denote the number of full binary trees with n nodes. Derive an expression for $B(n)$, involving $T(n')$ where $n' \leq n$. *Hint: Relate the internal nodes of a full binary tree to $T(n)$.* (4 marks)

A well-formed formula is in **Negated normal form** if it consists of just $\wedge$, $\vee$, and literals (i.e. propositional variables or negations of propositional variables). For example, $(p \vee (\neg q \wedge \neg r))$ is in negated normal form; but $(p \vee \neg(q \vee r))$ is not.

Let $F(n)$ denote the number of well-formed, negated normal form formulas[1] there are that use precisely n propositional variables exactly one time each. So $F(1) = 2$, $F(2) = 16$, and $F(4) = 15360$.

(d) Using your answer for part (c), give an expression for $F(n)$. (4 marks)

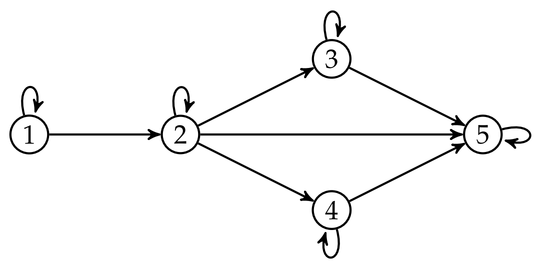
[1]Note: we do not assume $\wedge$ and $\vee$ are associative

Remark

The $T(n)$ are known as the Catalan numbers. As this question demonstrates they are very useful for counting various tree-like structures.

Problem 7 (16 marks)

Consider the following directed graph:



and consider the following process:

- Initially, start at 1.
- At each time step, choose one of the outgoing edges from your current location uniformly at random, and follow it to the next location. For example, if your current location was 2, then with probability $\frac{1}{4}$ you would move to 3; with probability $\frac{1}{4}$ you would move to 4; with probability $\frac{1}{4}$ you would move to 5; and with probability $\frac{1}{4}$ you would stay at 2.

Let $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, $p_5(n)$ be the probability your location after n time steps is 1, 2, 3, 4, or 5 respectively. So $p_1(0) = 1$ and $p_2(0) = p_3(0) = p_4(0) = p_5(0) = 0$.

(a) Express $p_1(n+1)$, $p_2(n+1)$, $p_3(n+1)$, $p_4(n+1)$, and $p_5(n+1)$ in terms of $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, and $p_5(n)$. (5 marks)

(b) Prove ONE of the following:

(i) For all $n \in \mathbb{N}$: $p_1(n) = \frac{1}{2^n}$ (5 marks)

(ii) For all $n \in \mathbb{N}$: $p_2(n) = 2\left(\frac{1}{2^n} - \frac{1}{4^n}\right)$ (6 marks)

(iii) For all $n \in \mathbb{N}$: $p_3(n) = p_4(n) = (n-2)\frac{1}{2^n} + \frac{2}{4^n}$ (7 marks)

(iv) For all $n \in \mathbb{N}$: $p_5(n) = 1 - (2n-1)\frac{1}{2^n} - \frac{2}{4^n}$ (8 marks)

Note

Clearly state which identity you are proving. A maximum of 8 marks is available for this question and marks will be awarded based on level of technical ability demonstrated. You may assume the identities which you are not proving.

(c) For each $n \in \mathbb{N}$ let X_n be the random variable that has value:

- 0 if your location at time n is 1;

- 1 if your location at time n is 2;
- 2 if your location at time n is 3 or 4; and
- 3 if your location at time n is 5

(i. e. X_n is the length of the longest path from 1 to your location at time n).

What is the expected value of X_3? (3 marks)

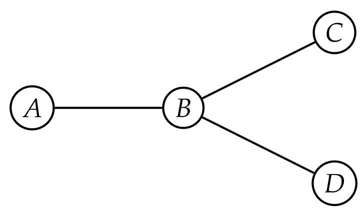
Remark

This is an example of a Markov chain – a very useful model for stochastic processes.

Problem 8 (10 marks)

In this question we are going to look at modelling the spread of a virus in a network (or news in a social network).

Consider the following graph:



and consider the following process:

- Initially, at time $n = 0$, A is infected and no other vertices are infected.
- At each time step, each infected vertex does the following:
  - for each uninfected neighbour, spread the infection to that vertex with probability $\frac{1}{2}$.

  So if A and B were infected and C and D were not, then in one time step, the virus would spread to both C and D with probability $\frac{1}{4}$; spread to C only with probability $\frac{1}{4}$; spread to D only with probability $\frac{1}{4}$; and not spread any further with probability $\frac{1}{4}$.

Let $p_A(n)$, $p_B(n)$, $p_C(n)$, $p_D(n)$ be the probability that A, B, C, D (respectively) are infected after n time steps. So $p_A(0) = 1$ and $p_B(0) = p_C(0) = p_D(0) = 0$.

(a) Express $p_D(n+1)$ in terms of $p_A(n)$, $p_B(n)$, $p_C(n)$ and $p_D(n)$. (4 marks)

(b) Find an expression for $p_D(n)$ in terms of n only. You do not need to prove the result, but you should briefly justify your answer. *Hint: Try to relate this system with Question 7* (4 marks)

(c)* What is the expected number of infected vertices after $n = 3$ time steps? (2 marks)

Advice on how to do the assignment

Collaboration is encouraged, but all submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf file.
- When giving answers to questions, we always would like you to prove/explain/motivate your answers. You are being assessed on your understanding and ability.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of external resources). You may make use of external material provided it is properly referenced[2] – however, answers that depend too heavily on external resources may not receive full marks if you have not adequately demonstrated ability/understanding.

[2] Proper referencing means sufficient information for a marker to access the material. Results from the lectures or textbook can be used without proof, but should still be referenced.