Assignment 3

z5395765

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Problem1

(a) Give an asymptotic upper bound, in terms of n, for the running time of sum.

We execute the operation $'+'$ in the loop for $n \times n$ times. So the upper bound for the running time of sum is $O(n^2)$.

(b) Give an asymptotic upper bound, in terms of n, for the running time of product.

We execute the operation *add* in the loop for $n \times n$ times and every time, *add* will add the elements of a set S in $O(|S|)$ time. In this case, $S = n$, so the upper bound for the running time of product is $O(n^3)$.

(c) With justification, give a recurrence equation for $T(n)$.

As stated in the question, the product of 1×1 matrices takes $O(1)$ time, so the $T(1) = O(1)$. As for the submatrices of $n \times n$ matrices A and B, it takes $O(n/2) \times 8$ time to compute the product and $O(1) \times 4$ to compute the addition.
Therefore, $T(n) = 8O(n/2) + 4O(1)$, $T(1) = O(1)$.

(d) Find an asymptotic upper bound for $T(n)$.

According to the Master Theorum, in this case, $a = 8, b = 2, c = 0, d = 3, k = 0$, $c < d$, so the upper bound for $T(n)$ is $O(n^3)$.

Problem2

(a) Formulate this problem as a problem in propositional logic. Specifically:

(i) Define your propositional variables

Assume that houses in the upper row is House1-House4, the houses in the lower row is House5-House8.

H1: House1 is on channel hi.

H2: House2 is on channel hi.

H3: House3 is on channel hi.

H4: House4 is on channel hi.

H5: House5 is on channel hi.

H6: House6 is on channel hi.

H7: House7 is on channel hi.

H8: House8 is on channel hi.

(ii) Define any propositional formulas that are appropriate and indicate what propositions they represent.

The propositional formulas and the propositions are defined as below:

$I_{1,2} : H_1 \leftrightarrow \neg H_2$ indicates that If House1 is on channel hi, then House2 is on channel lo, and vice versa.
$I_{2,3} : H_2 \leftrightarrow \neg H_3$ indicates that If House2 is on channel hi, then House3 is on channel lo, and vice versa.
$I_{3,4} : H_3 \leftrightarrow \neg H_4$ indicates that If House3 is on channel hi, then House4 is on channel lo, and vice versa.
$I_{5,6} : H_5 \leftrightarrow \neg H_6$ indicates that If House5 is on channel hi, then House6 is on channel lo, and vice versa.
$I_{6,7} : H_6 \leftrightarrow \neg H_7$ indicates that If House6 is on channel hi, then House7 is on channel lo, and vice versa.
$I_{7,8} : H_7 \leftrightarrow \neg H_8$ indicates that If House7 is on channel hi, then House8 is on channel lo, and vice versa.
$I_{1,5} : H_1 \leftrightarrow \neg H_5$ indicates that If House1 is on channel hi, then House5 is on channel lo, and vice versa.
$I_{2,6} : H_2 \leftrightarrow \neg H_6$ indicates that If House2 is on channel hi, then House6 is on channel lo, and vice versa.
$I_{3,7} : H_3 \leftrightarrow \neg H_7$ indicates that If House3 is on channel hi, then House7 is on channel lo, and vice versa.
$I_{4,8} : H_4 \leftrightarrow \neg H_8$ indicates that If House4 is on channel hi, then House8 is on channel lo, and vice versa.

(iii) Indicate how you would solve the problem (or show that it cannot be done) using propositional logic. It is sufficient to explain the method, you do not need to provide a solution.

In order to assign channels to networks so that there is no interference, we need to satisfy all the proposition I defined in the question(ii), so the propsitional logic would be:

$$I_{1,2} \wedge I_{2,3} \wedge I_{3,4} \wedge I_{5,6} \wedge I_{6,7} \wedge I_{7,8} \wedge I_{1,5} \wedge I_{2,6} \wedge I_{3,7} \wedge I_{4,8} = \psi$$

To show that ψ is satisfiable, we create a truth table involving $H_1 - H_8, I_{1,2} - I_{4,8}, \psi$ and see if there is any valuation which makes all of the $I_{i,j}$ true.

(iv) Explain how to modify your answer(s) to (i) and (ii) if the goal was to see if it is possible to solve with 3 channels rather than 2.

We can define 3 variables for each house(H_i, M_i, L_i). In order to make the neighbour houses not interfered by each other, we set the formula as $(H_i \leftrightarrow \neg H_j) \wedge (M_i \leftrightarrow \neg M_j) \wedge (L_i \leftrightarrow \neg L_j)$ if ith house and the jth house are neighbour.

(b) Suppose each house chooses, uniformly at random, one of the two network channels. What is the probability that there will be no interference?

Suppose ith house and the jth house are neighbours, $P(I_{i,j}$ is true$) = \frac{1}{2}$. From the answer of (ii), $P = \frac{1}{2^{10}} = \frac{1}{1024}$.

Problem3

Prove the following results hold in all Boolean Algebras:

(a) Show that $(x \wedge 1') \vee (x' \wedge 1) = x'$

$$
\begin{aligned}
(x \wedge 1') \vee (x' \wedge 1) &= ((x \wedge 1') \vee x') \wedge ((x \wedge 1') \vee 1) && \text{(Distributivity of } \vee \text{ over } \wedge) \\
&= (x' \vee (x \wedge 1')) \wedge ((x \wedge 1') \vee 1) && \text{(Commutatitivity of } \vee) \\
&= ((x' \vee x) \wedge (x' \vee 1')) \wedge ((x \wedge 1') \vee 1) && \text{(Distributivity of } \vee \text{ over } \wedge) \\
&= ((x \vee x') \wedge (x' \vee 1')) \wedge ((x \wedge 1') \vee 1) && \text{(Commutatitivity of } \vee) \\
&= (1 \wedge (x' \vee 1')) \wedge ((x \wedge 1') \vee 1) && \text{(Complement with } \vee) \\
&= ((1 \wedge x') \vee (1 \wedge 1')) \wedge ((x \wedge 1') \vee 1) && \text{(Distributivity of } \wedge \text{ over } \vee) \\
&= ((1 \wedge x') \vee 0) \wedge ((x \wedge 1') \vee 1) && \text{(Complement with } \wedge) \\
&= ((x' \wedge 1) \vee 0) \wedge ((x \wedge 1') \vee 1) && \text{(Commutatitivity of } \wedge) \\
&= (x' \vee 0) \wedge ((x \wedge 1') \vee 1) && \text{(Identity of } \wedge) \\
&= x' \wedge ((x \wedge 1') \vee 1) && \text{(Identity of } \vee) \\
&= x' \wedge (1 \vee (x \wedge 1')) && \text{(Commutatitivity of } \vee) \\
&= x' \wedge ((1 \vee x) \wedge (1 \vee 1')) && \text{(Distributivity of } \vee \text{ over } \wedge) \\
&= x' \wedge ((1 \vee x) \wedge 1) && \text{(Complement with } \vee) \\
&= x' \wedge (1 \wedge (1 \vee x)) && \text{(Commutatitivity of } \wedge) \\
&= x' \wedge ((1 \wedge 1) \vee (1 \wedge x)) && \text{(Distributivity of } \wedge \text{ over } \vee) \\
&= x' \wedge (1 \vee (1 \wedge x)) && \text{(Idempotence of } \wedge) \\
&= x' \wedge (1 \vee (x \wedge 1)) && \text{(Commutatitivity of } \wedge) \\
&= x' \wedge (1 \vee x) && \text{(Identity of } \wedge) \\
&= x' \wedge (x \vee 1) && \text{(Commutatitivity of } \vee) \\
&= x' \wedge 1 && \text{(Annihilation of } \vee) \\
&= x' && \text{(Identity of } \wedge)
\end{aligned}
$$

(b) Show that $(x \wedge y) \vee x = x$

$$\begin{aligned}
(x \wedge y) \vee x &= x \vee (x \wedge y) && \text{(Commutatitivity of } \vee) \\
&= (x \vee x) \wedge (x \vee y) && \text{(Distributivity of } \vee \text{ over } \wedge) \\
&= x \wedge (x \vee y) && \text{(Idempotence of } \vee) \\
&= (x \vee 0) \wedge (x \vee y) && \text{(Identity of } \vee) \\
&= x \vee (0 \wedge y) && \text{(Distributivity of } \vee \text{ over } \wedge) \\
&= x \vee (y \wedge 0) && \text{(Commutatitivity of } \wedge) \\
&= x \vee 0 && \text{(Annihilation of } \wedge) \\
&= x && \text{(Identity of } \vee)
\end{aligned}$$

(c) Show that $y' \wedge ((x \vee y) \wedge x') = 0$

$$\begin{aligned}
y' \wedge ((x \vee y) \wedge x') &= y' \wedge (x' \wedge (x \vee y)) && \text{(Commutatitivity of } \wedge) \\
&= y' \wedge ((x' \wedge x) \vee (x' \wedge y)) && \text{(Distributivity of } \wedge \text{ over } \vee) \\
&= y' \wedge ((x \wedge x') \vee (x' \wedge y)) && \text{(Commutatitivity of } \wedge) \\
&= y' \wedge (0 \vee (x' \wedge y)) && \text{(Complement with } \wedge) \\
&= y' \wedge ((x' \wedge y) \vee 0) && \text{(Commutatitivity of } \vee) \\
&= y' \wedge (x' \wedge y) && \text{(Identity of } \vee) \\
&= y' \wedge (y \wedge x') && \text{(Commutatitivity of } \wedge) \\
&= (y' \wedge y) \wedge x' && \text{(Associativity of } \wedge) \\
&= (y \wedge y') \wedge x' && \text{(Commutatitivity of } \wedge) \\
&= 0 \wedge x' && \text{(Complement with } \wedge) \\
&= x' \wedge 0 && \text{(Commutatitivity of } \wedge) \\
&= 0 && \text{(Annihilation of } \wedge)
\end{aligned}$$

Problem4

Show that there are no three element Boolean Algebras.

Assume $(T, \vee, \wedge, ', 0, 1)$ is a three element Boolean Algebra. Assume $T = 0, 1, \alpha$, $\alpha \neq 0, 1$. Consider $' : T \to T$:

$0' = 1, 1' = 0$, α' could either be 0, 1 or α.

If $\alpha' = 0$, then $\alpha = 1$, contradition.

If $\alpha' = 1$, then $\alpha = 0$, contradition.

If $\alpha' = \alpha$, $0 = \alpha \wedge \alpha' = \alpha \wedge \alpha = \alpha$, contradition.

In conclusion, all the cases we consider cannot exist, so there are no three element Boolean Algebras.

Problem5

Prove or disprove the following logical equivalences:

(a) Show that $\neg(p \to q) \equiv (\neg p \to \neg q)$

Counter example: Suppose p is True, q is True, $\neg(p \to q)$ is False, while $(\neg p \to \neg q)$ is True. Therefore, $\neg(p \to q) \not\equiv (\neg p \to \neg q)$.

(b) Show that $((p \wedge q) \to r) = (p \to (q \to r))$

$$\begin{aligned}
((p \wedge q) \to r) &\equiv \neg(p \wedge q) \vee r && \text{(Implication)}\\
&\equiv (\neg p \vee \neg q) \vee r && \text{(De Morgan's, } \neg \text{ over } \wedge)\\
&\equiv \neg p \vee (\neg q \vee r) && \text{(Associativity of } \vee)\\
&\equiv p \to (\neg q \vee r) && \text{(Implication)}\\
&\equiv (p \to (q \to r)) && \text{(Implication)}
\end{aligned}$$

(c) Show that $((p \vee (q \vee r)) \wedge (r \vee p)) = ((p \wedge q) \vee (r \vee p))$

$$\begin{aligned}
((p \vee (q \vee r)) \wedge (r \vee p)) &\equiv (p \vee (r \vee q)) \wedge (r \vee p) && \text{(Commutatitivity of } \vee)\\
&\equiv (r \vee p) \wedge (p \vee (r \vee q)) && \text{(Commutatitivity of } \wedge)\\
&\equiv ((r \vee p) \wedge p) \vee ((r \vee p) \wedge (r \vee q)) && \text{(Distributivity of } \wedge \text{ over } \vee)\\
&\equiv (r \vee p) \wedge (p \vee (r \vee q)) && \text{(Distributivity of } \wedge \text{ over } \vee)\\
&\equiv (r \vee p) \wedge ((p \vee r) \vee q) && \text{(Associativity of } \vee)\\
&\equiv (r \vee p) \wedge ((r \vee p) \vee q) && \text{(Commutatitivity of } \vee)\\
&\equiv (r \vee (p \vee p)) \wedge ((r \vee p) \vee q) && \text{(Idempotence of } \vee)\\
&\equiv ((r \vee p) \vee p) \wedge ((r \vee p) \vee q) && \text{(Associativity of } \vee)\\
&\equiv (r \vee p) \vee (p \wedge q) && \text{(Distributivity of } \vee \text{ over } \wedge)\\
&\equiv ((p \wedge q) \vee (r \vee p)) && \text{(Commutatitivity of } \vee)
\end{aligned}$$

Problem6

(a) Using the recursive definition of a binary tree structure, or otherwise, derive a recurrence equation for $T(n)$.

Let $T(k, n)$ represents the number of trees with n nodes in total, k nodes in T_l. The number of nodes of T_l can be any integer between 0 and n-1, and all the cases are disjoint. Therefore:

$$\begin{aligned}
T(n) &= T(0, n) + T(1, n) + ... + T(n-2, n) + T(n-1, n)\\
&= T(0) \times T(n-1) + T(1) \times T(n-2) + ... + T(n-2) \times T(1) + T(n-1) \times T(0)\\
&= \Sigma_{k=0}^{n-1} T(k) \times T(n-1-k)
\end{aligned}$$

Therefore, a recurrence equation for $T(n)$: $T(n) = \Sigma_{k=0}^{n-1} T(k) \times T(n-1-k)$.

(b) Using observations from Assignment 2, or otherwise, explain why a full binary tree must have an odd number of nodes.

Using the observation from Assignment2, leaves = fullly internals + 1 and nodes = fully internals + leaves. Therefore, nodes = 2fully internals +1. Suppose the number of fully internals is k, the number of nodes is 2k+1. Therefore, a full binary tree must have an odd number of nodes.

(c) Let $B(n)$ denote the number of full binary trees with n nodes. Derive an expression for B(n), involving $T(n')$ where $n' \leq n$

By observation, we know that the number of internals of a full binary tree with n nodes equals to the number of binary trees with n nodes($T(n)$). Therefore, suppose the number of internals of a full bianary tree is k, then $B(2k+1) = T(k)$. Therefore,

$$B(n) = \begin{cases} 0, & n \text{ is even}, \\ T(\frac{n-1}{2}), & n \text{ is odd}. \end{cases}$$

(d) Using your answer for part (c), give an expression for $F(n)$.

We can set a parse tree for any fomulas in NNF. If we use n propositional variables exactly one time each, we know that the nodes in total of the parse tree is $2n-1$. There are $B(2n-1)$ ways to shape the tree, 2^{n-1} ways to fill $n-1$ internal nodes, $2^n n!$ ways to fill n leaves. Therefore, $F(n) = B(2n-1)2^{2n-1}n!$.

Problem7

(a) Express $p_1(n+1)$, $p2_(n+1)$, $p_3(n+1)$, $p_4(n+1)$, and $p_5(n+1)$ in terms of $p_1(n)$, $p_2(n)$, $p_3(n)$, $p_4(n)$, and $p_5(n)$.

$p_1(n+1) = \frac{1}{2}p_1(n)$

$p_2(n+1) = \frac{1}{2}p_1(n) + \frac{1}{4}p_2(n)$

$p_3(n+1) = \frac{1}{4}p_2(n) + \frac{1}{2}p_3(n)$

$p_4(n+1) = \frac{1}{4}p_2(n) + \frac{1}{2}p_4(n)$

$p_5(n+1) = \frac{1}{4}p_2(n) + \frac{1}{2}p_3(n) + \frac{1}{2}p_4(n) + p_5(n)$

(b) Prove ONE of the following:

(i) Prove that for all $n \in N$: $p_1(n) = \frac{1}{2^n}$

From (a), we know that $p_1(n+1) = \frac{1}{2}p_1(n)$. Therefore through unwinding:

$$\begin{aligned} p_1(n) &= \tfrac{1}{2}p_1(n-1) \\ &= \tfrac{1}{2} \times \tfrac{1}{2}p_1(n-2) \\ &= ... \\ &= \tfrac{1}{2^{n-1}}p_1(1) \\ &= \tfrac{1}{2^n} \end{aligned}$$

(ii) For all $n \in N : p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$ Let P(n) be the proposition that $p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$.

Base case, n = 0, $P_2(0) = 0 = 2(\frac{1}{2^0} - \frac{1}{4^0})$.

Induction case: $p_2(n) \implies p_2(n+1)$.

Assume $p_2(n)$ holds, which means $p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$.

$$\begin{aligned} p_2(n+1) &= \tfrac{1}{2}p_1(n) + \tfrac{1}{4}p_2(n) \\ &= \tfrac{1}{2}\tfrac{1}{2^n} + \tfrac{1}{4}2(\tfrac{1}{2^n} - \tfrac{1}{4^n}) \\ &= \tfrac{1}{2^n} - \tfrac{2}{4^{n+1}} \\ &= 2(\tfrac{1}{2^{n+1}} - \tfrac{1}{4^{n+1}}) \end{aligned}$$

Therefore, $P_2(n+1)$ holds. Therefore, For all $n \in N : p_2(n) = 2(\frac{1}{2^n} - \frac{1}{4^n})$

(c) What is the expected value of $X3$?

Using the result of part(b), we know that when $n = 3$:

$$\begin{aligned} p_1(3) &= \tfrac{1}{8} \\ p_2(3) &= \tfrac{7}{32} \\ p_3(3) &= \tfrac{5}{32} \\ p_4(3) &= \tfrac{5}{32} \\ p_5(3) &= \tfrac{11}{32} \end{aligned}$$

Therefore,

$$\begin{aligned} p(X_3 = 0) &= \tfrac{1}{8} \\ p(X_3 = 1) &= \tfrac{7}{32} \\ p(X_3 = 2) &= \tfrac{10}{32} \\ p(X_3 = 3) &= \tfrac{11}{32} \end{aligned}$$

Therefore, the expected value of $X3$: $E = 0 \times \frac{1}{8} + 1 \times \frac{7}{32} + 2 \times \frac{10}{32} + 3 \times \frac{11}{32} = \frac{15}{8}$.

Problem8

(a) Express $p_D(n+1)$ in terms of $p_A(n)$, $p_B(n)$, $p_C(n)$ and $p_D(n)$.

After n+1 time steps, D is infected indicates that A and B are infected after n time steps. There are 4 possible cases for the next action of the state that A,B are infected, which is no more infection, only C is infected, only D is infected and both C and D are infected. Therefore, the possibility of D infected in the next step is $\frac{1}{2}$. Therefore, $p_D(n+1) = p_A(n) \times p_B(n) \times \frac{1}{2}$.

(b) Find an expression for $p_D(n)$ in terms of n only. You do not need to prove the result, but you should briefly justify your answer.

Relate this system to Q7, we can link infected vertices set $\{A\}$ with state 1, infected vertices set $\{A, B\}$ with state2, infected vertices set $\{A, B, C\}$ with state3, infected vertices set $\{A, B, D\}$ with state4, infected vertices set $\{A, B, C, D\}$ with state5. In this way:

$$p_D(n) = 1 - \frac{n+1}{2^n}$$

(c) What is the expected number of infected vertices after n = 3 time steps?

Suppose X_3 is the number of infected vertices after 3 time steps .

$$p(X_3 = 1) = \frac{1}{8}$$
$$p(X_3 = 2) = \frac{7}{32}$$
$$p(X_3 = 3) = \frac{10}{32}$$
$$p(X_3 = 4) = \frac{11}{32}$$

$E = 1 \times \frac{1}{8} + 2 \times \frac{7}{32} + 3 \times \frac{10}{32} + 4 \times \frac{11}{32} = \frac{23}{8}$.
So the expected number of infected vertices is $\frac{23}{8}$.