Lecture Notes of Computer Architecture

Zhou Fan
ACM Class, SJTU

Contents

1

1.1

1.1.1

|CPU| | |
| |
|I/O | |I/O | | |
| | | | | |

1.1.2

1 ( ) [1]

$$F(2, n, L, U) = \{\pm 0.a_1 a_2 \dots a_n \times 10^m\} \cup \{0\}$$

$a_1 = 1$ $a_i \in \{0, 1\}$. m $L \le m \le U$. n 2

2 (IEEE)

1. : $t = 24, L = -126, U = 127$
2. : $t = 53, L = -1022, U = 1023$
3. Underflow Limit: $UFL = 0.1 \times 2^L$. $0 < x < UFL$ $fl(x) = 0$.
4. Overflow Limit: $OFL = 0.11 \dots 1 * 2^U$. $x > OFL$ $fl(x) = \infty$.
5. : $UFL \le x \le OFL$ $fl(x)$ $x = 0.a_1 a_2 \dots a_n \dots \times 2^m$. $a_{n+1} = 1$ $d_t + 1$

3 ( )

$$\begin{aligned} x - fl(x) &= 2^m \times 0.0 \dots 0 a_{n+2} \dots \\ &= 2^m \times [2^{-(t+2)} + 2^{-(t+3)} + \cdots] \\ &= 2^m \times 2^{-(t+1)} \end{aligned}$$

$$\frac{x - fl(x)}{x} < \frac{x - fl(x)}{0.5 \times 2^m} = 2^{-t}$$

ε

4

$$fl(x) = x(1 + \delta), \quad |\delta| \le \varepsilon$$

[1] floating Number System

1.1.3

$+ \quad \Rightarrow$

5 $\boldsymbol{(x+y)}$ $x,\, y$ $x+y$

1. m
2.
3.
4. ……
5.

$$fl(x)+fl(y)=x(1+\delta_x)+y(1+\delta_y)$$

1.2

1.2.1

1.

2.

3. $y = f(x)$ $f(x^*)$ Taylor

4. rounding-off

1.2.2

6 ( ) x x^*

$$e(x^*) = x^* - x.$$

ε^*

$$|e(x^*)| \leq \varepsilon^*$$

7 ( ) x x^*

$$e_r(x^*) = \frac{x^* - x}{x}$$

ε_r^*

$$|e_r(x^*)| \leq \varepsilon_r^*$$

x

$$\bar{e}_r(x^*) = \frac{x^* - x}{x^*}$$

$$|\bar{e}_r(x^*)| \leq \frac{\varepsilon^*}{|x^*|}.$$

$\varepsilon^* \ll 1$

$$|e_r - \bar{e}_r| = O((\varepsilon_r^*)^2)$$

8 ( ) $x \in R$ x^* x^* x n n n

$$|x^* - x| \leq \frac{1}{2} \times 10^{m-n}$$

x^* n n

9 ( ) $x = 0.a_1a_2\dots a_n \times 10^m$ n

$$\left|\frac{x^* - x}{x}\right| \leq \frac{1}{2a_1} \times 10^{1-n}.$$

$$\left|\frac{x^* - x}{x}\right| \leq \frac{1}{2(1 + a_1)} \times 10^{1-n},$$

x^* n

$x \geq 0.a_1$ $a_1 \neq 0$

1.2.3

10 ( )

1. / : $\varepsilon(x^* \pm y^*) \le \varepsilon_x^* + \varepsilon_y^*$
2. : $\varepsilon(x^* y^*) \le |x^*|\varepsilon_y^* + |y^*|\varepsilon_x^*$
3. : $\varepsilon(\frac{x^*}{y^*}) = \frac{|x^*|\varepsilon_y^* + |y^*|\varepsilon_x^*}{|y^*|^2}$

x y x^* y^* $x^* \pm y^*$ $x \pm y$

$$\begin{aligned} & |x^* \pm y^* - (x \pm y)| \le |x^* - x| + |y^* - y| \le \varepsilon_x^* + \varepsilon_y^* \\ \Rightarrow \quad & \varepsilon(x^* \pm y^*) \le \varepsilon_x^* + \varepsilon_y^* \end{aligned}$$

$+1 - 1$

11 ( ) $A = f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ $\mathbf{x}^*$ $\mathbf{x}$ Peano Taylor A

$$\begin{aligned} e(A^*) &= f(\mathbf{x}^*) - f(\mathbf{x}) \\ &= \sum_{p=1}^{q} \mathrm{d}^k f(\mathbf{x}^*) + o(||x^* - x||^q) \\ & \text{q=1} \\ &= \sum_{k=1}^{n} \partial_k f(\mathbf{x}^*)(x^* - x) + + o(||x^* - x||^q) \end{aligned}$$

$$\begin{aligned} \varepsilon(A^*) &\approx \sum_{k=1}^{n} \partial_k f(\mathbf{x}^*) \varepsilon(x^*) \\ \varepsilon_r(A^*) &= \frac{\varepsilon(A^*)}{|A^*|} \end{aligned}$$

1.2.4

12 n

$S_n = \sum_i^n a_i$

$$\begin{aligned} & S_2^* = fl(a_1 + a_2) = (a_1 + a_2)(1 + \varepsilon_2), \quad |\varepsilon_2| \le \varepsilon = 2^{-t} \\ & \cdots\cdots \\ & S_n^* = fl(S_{n-1}^* + a_n)(1 + \varepsilon_n), \quad |\varepsilon_n| \le \varepsilon \end{aligned}$$

S_n^* $\varepsilon_1 = 0$

$$S_n^* = \sum_{k=1}^{n} a_k \prod_{p=k}^{n} (1 + \varepsilon_p)$$

$$\prod_{i=k}^{n}(1+\varepsilon_k) \approx 1+\sum_{i=k}^{n}\varepsilon_k$$

$$\begin{aligned} S_n^* &\approx \sum_{k=1}^{n} a_k(1+\sum_{p=k}^{n}\varepsilon_p) \\ &= S_n + \sum_{k=1}^{n} a_k \sum_{p=k}^{n}\varepsilon_p \end{aligned}$$

$|\varepsilon_i| \leq \varepsilon$

$$|S_n^* - S_n| \leq \sum_{k=1}^{n}|a_k|\sum_{p=k}^{n}|\varepsilon_p| \leq \varepsilon\sum_{k=1}^{n}|a_k|(n-k+1)$$

$n-k+1$ k [18] ■

1.3

13 ( )

1.

2.

3.

14 ($\boldsymbol{e^A}$) e^A $A \in \mathbf{R}^{n\times n}$

$$e^A = e^{(A/2^n)2^n} = B^{2^n}$$

B B^{2^n} B $x \to 0$ e^x Taylor

$$e^x = 1 + x + \cdots + \frac{x^n}{n!} + \cdots$$

n $A/2^n \approx 0$

$$B \approx I + C + \frac{1}{2}C^2, \quad C = A/2^n$$

B^2

$$B^2 \approx I + 2(C + \frac{1}{2}C^2) + (C + \frac{1}{2}C^2)^2$$

15 ( ) (1) $p(z), z \in \mathbf{R}$

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1}$$

b_n

$$b_0 = a_0, \quad b_k = a_k + b_{k-1}z$$

b_n

$$p'(z) = \sum_{k=0}^{n-1} b_k z^{n-1-k}$$

$x - z$ $p(x)$ $b_n(x)$

$$p(x) = (x - z)q(x) + b_n(x),$$

$x = z$ 0 $p(z) = b_n(z)$ b_n

16 ( ) x_0 x_1 x x_1 x_0 x ω

$$\bar{x} = x_1 + \omega(x_1 - x_0)$$

x

17 ($\boldsymbol{\pi}$) π π_n $2n$

$$\widetilde{\pi}_n = \frac{1}{3}(4\pi_{2n} - \pi_n)$$

π_n $\widetilde{\pi}_n$ π

$$|\pi_n - \pi| = O(\frac{1}{n^2}), \quad |\widetilde{\pi}_n - \pi| = O(\frac{1}{n^4})$$

π_n.

$$\pi_n = n\sin\frac{\pi}{n} = \pi - \frac{\pi^3}{3!}\frac{1}{n^2} + \frac{\pi^5}{5!}\frac{1}{n^4} - \cdots \Rightarrow |\pi_n - \pi| = O(\frac{1}{n^2})$$

$\widetilde{\pi}_n$.

$$\begin{aligned}\widetilde{\pi}_n &= \pi_{2n} + k(\pi_{2n} - \pi_n) = (1+k)\pi_{2n} - k\pi_n \\ &= (1+k)(\pi - \frac{\pi^3}{3!}\frac{1}{4n^2} + \cdots) - k(\pi - \frac{\pi^3}{3!}\frac{1}{n^2} + \cdots) \\ &= \pi - (\frac{k+1}{4} - k)\frac{\pi^3}{3!}\frac{1}{n^2} + O(\frac{1}{n^4})\end{aligned}$$

$k = \frac{1}{3}$

$$|\widetilde{\pi}_n - \pi| = O(\frac{1}{n^4}) \quad \blacksquare$$

π_n $n = 3$ 6 $\pi_3 = 3\sqrt{3}/2$

$$\pi_{2n} = \sqrt{2n(n - \sqrt{n^2 - \pi_n^2})},$$

$\pi_2 n$ π_n

2

2.1

18 ( ) $\{a_n\}$ $\{b_n\}$

$$0 \le a_1 \le a_2 \le \cdots \le a_n$$
$$0 \le b_1 \le b_2 \le \cdots \le b_n$$

$$\sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i b_{k_i} \ge \sum_{i=1}^{n} a_i b_{n-i+1}$$

19 (-)

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$a_1 = a_2 = \cdots = a_n$

$\prod a_i = 1$

$$a_1 = \frac{\alpha_1}{\alpha_2}, \quad \ldots, \quad a_{n-1} = \frac{\alpha_{n-1}}{\alpha_n}, \quad a_n = \frac{\alpha_n}{\alpha_1}$$

$$\frac{\alpha_1}{\alpha_2} + \cdots + \frac{\alpha_n}{\alpha_1} \ge n$$

$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$

$$\text{L.H.S} \ge \alpha_1 \frac{1}{\alpha_1} + \cdots + \alpha_n \frac{1}{\alpha_n} = n \quad \blacksquare$$