科学计算 Exercise 4

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8. 解：

$$\begin{aligned}
\varphi_0(x) &= 1 \\
\varphi_1(x) &= x - \frac{(x, \varphi_0(x))}{(\varphi_0(x), \varphi_0(x))}\varphi_0(x) = x - \frac{\int_{-1}^{1}(1+x^2)x\mathrm{d}x}{\int_{-1}^{1}(1+x^2)\mathrm{d}x} = x \\
\varphi_2(x) &= x^2 - \frac{(x^2, \varphi_0(x))}{(\varphi_0(x), \varphi_0(x))}\varphi_0(x) - \frac{(x^2, \varphi_1(x))}{(\varphi_1(x), \varphi_1(x))}\varphi_1(x) \\
&= x^2 - \frac{\int_{-1}^{1}(1+x^2)x^2\mathrm{d}x}{\int_{-1}^{1}(1+x^2)\mathrm{d}x} - \frac{\int_{-1}^{1}(1+x^2)x^3\mathrm{d}x}{\int_{-1}^{1}(1+x^2)x^2\mathrm{d}x}x \\
&= x^2 - \frac{2}{5} \\
\varphi_3(x) &= x^3 - \frac{(x^3, \varphi_0(x))}{(\varphi_0(x), \varphi_0(x))}\varphi_0(x) - \frac{(x^3, \varphi_1(x))}{(\varphi_1(x), \varphi_1(x))}\varphi_1(x) - \frac{(x^3, \varphi_2(x))}{(\varphi_2(x), \varphi_2(x))}\varphi_2(x) \\
&= x^3 - \frac{\int_{-1}^{1}(1+x^2)x^3\mathrm{d}x}{\int_{-1}^{1}(1+x^2)\mathrm{d}x} - \frac{\int_{-1}^{1}(1+x^2)x^4\mathrm{d}x}{\int_{-1}^{1}(1+x^2)x^2\mathrm{d}x}x - \frac{\int_{-1}^{1}(1+x^2)(x^2-\frac{2}{5})x^3\mathrm{d}x}{\int_{-1}^{1}(1+x^2)(x^2-\frac{2}{5})^2\mathrm{d}x}(x^2 - \frac{2}{5}) \\
&= x^3 - \frac{9}{14}x
\end{aligned}$$

10. 证明：令 $x = cos\theta$，则有

$$\begin{aligned}
\int_{-1}^{1}\frac{[T_n(x)]^2}{\sqrt{1-x^2}}\mathrm{d}x &= \int_{\pi}^{0}\frac{\cos^2 n\theta}{\sin\theta}\mathrm{d}(\cos\theta) = \int_{\pi}^{0}\frac{\cos^2 n\theta}{\sin\theta}(-\sin\theta)\mathrm{d}\theta \\
&= \int_{0}^{\pi}\cos^2 n\theta\mathrm{d}\theta = \int_{0}^{n\pi}\cos^2 t\mathrm{d}\frac{t}{n} \\
&= \frac{1}{n}\int_{0}^{n\pi}\cos^2 t\mathrm{d}t = \int_{0}^{\pi}\cos^2 t\mathrm{d}t \\
&= \int_{0}^{\pi}\frac{1-\cos 2t}{2}\mathrm{d}t = \int_{0}^{2\pi}\frac{1-\cos u}{4}\mathrm{d}u = \frac{\pi}{2}
\end{aligned}$$

11. 解：$T_3(x)$ 在 $[-1,1]$ 上的零点为

$$x_1=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2},\ x_2=\cos\frac{\pi}{2}=0,\ x_3=\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$$

以 x_1,x_2,x_3 做插值点，$f(x)=e^x$ 的二次插值多项式为

$$\begin{aligned}P_2(x)&=f(x_3)+f[x_3,x_2](x-x_3)+f[x_3,x_2,x_1](x-x_3)(x-x_2)\\&=e^{-\frac{\sqrt{3}}{2}}+\frac{1-e^{-\frac{\sqrt{3}}{2}}}{\frac{\sqrt{3}}{2}}(x+\frac{\sqrt{3}}{2})+\frac{2(e^{\frac{\sqrt{3}}{2}}+e^{-\frac{\sqrt{3}}{2}}-2)}{3}(x+\frac{\sqrt{3}}{2})x\\&=\frac{2}{3}(e^{-\frac{\sqrt{3}}{2}}+e^{\frac{\sqrt{3}}{2}}-2)x^2+\frac{e^{\frac{\sqrt{3}}{2}}-e^{-\frac{\sqrt{3}}{2}}}{\sqrt{3}}x+1\end{aligned}$$

$$R_2(x)=f(x)-P_2(x)=\frac{f^{(3)}(\xi)}{3!}\omega_3(x)$$

记

$$M_3=\|f^{(3)}(x)\|_\infty=\max_{-1\le x\le 1}|f^{(3)}(x)|=e$$

则有

$$\begin{aligned}\max_{-1\le x\le 1}|R_2(x)|&\le\frac{M_3}{3!}\max_{-1\le x\le 1}|(x-x_1)(x-x_2)(x-x_3)|\\&=\frac{e}{3!}\frac{1}{2^2}=\frac{e}{24}\end{aligned}$$

即误差界为 $\frac{e}{24}$.

12. 解：若取 $\Phi=\mathrm{span}\{1,x\}$，令 $t=T(x)=ax+b$，使得 $T(0)=-1,T(1)=1$. 得 $t=2x-1$，则 $t\in[-1,1]$，且

$$x=\frac{t+1}{2},\ f(x)=g(t)=x^2+3x+2=\frac{1}{4}t^2+2t+\frac{15}{4}$$

设最佳平方逼近多项式为 $Q(t)$，则

$$f(t)-Q(t)=\frac{1}{4}\widetilde{P}_2$$

其中 $\widetilde{P}_2$ 为首项系数为 1 的二次 Legendre 多项式，

$$\widetilde{P}_2=\frac{2!}{4!}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(t^2-1)^2=t^2-\frac{1}{3}$$

则

$$Q(t)=f(t)-\frac{1}{4}\widetilde{P}_2=\frac{1}{4}t^2+2t+\frac{15}{4}-\frac{1}{4}(t^2-\frac{1}{3})=2t+\frac{23}{6}$$

最佳平方逼近多项式为

$$P(x)=Q(t)=4x+\frac{11}{6}$$

若取 $\Phi=\mathrm{span}\{1,x,x^2\}$，最佳平方逼近多项式为 $f(x)$ 本身，即

$$P(x)=x^2+3x+2$$