Proof of Product of Even and Odd Numbers

Dillon Roller

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1 Proof that the product of two even numbers are even

Definition: An even number is any number that is divisible by 2. This means that any even number can be written in the form $x = 2n$ where $n \in \mathbb{Z}$ (The set of integers is denoted by $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

Let $a, b \in \mathbb{Z}$, where a and b are even numbers. This means they can be written as follows:

$$\begin{aligned} a &= 2m \\ b &= 2n \end{aligned}$$

where $m, n \in \mathbb{Z}$. Then:

$$\begin{aligned} ab &= (2m)(2n) \\ &= 4mn \\ &= 2(2mn) \end{aligned}$$

Let $l = 2mn$. Since $\mathbb{Z}$ is closed under multiplication, then $2mn \in \mathbb{Z}$, and therefore, $l \in \mathbb{Z}$. Then ab is, by definition, an even number, since $ab = 2l$, where $l \in \mathbb{Z}$. Therefore, any even number times an even number is even itself. $\blacksquare$

2 Proof that the product of two odd numbers are odd

Definition: An odd number is any number that is not divisible by 2. This means that any odd number can be written in the form $x = 2n + 1$ where $n \in \mathbb{Z}$ (The set of integers is denoted by $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

Let $a, b \in \mathbb{Z}$, where a and b are odd numbers. This means they can be written as follows:

$$
\begin{aligned}
a &= 2m + 1 \\
b &= 2n + 1
\end{aligned}
$$

where $m, n \in \mathbb{Z}$. Then:

$$
\begin{aligned}
ab &= (2m + 1)(2n + 1) \\
&= 4mn + 2m + 2n + 1 \\
&= 2(2mn + m + n) + 1
\end{aligned}
$$

Let $l = 2mn + m + n$. Since $\mathbb{Z}$ is closed under addition and multiplication, then $2mn + m + n \in \mathbb{Z}$, and therefore, $l \in \mathbb{Z}$. Then ab is, by definition, an odd number, since $ab = 2l + 1$, where $l \in \mathbb{Z}$. Therefore, any odd number times an odd number is odd itself. $\blacksquare$

3 Proof that the product of an even number and an odd number is even

Let $a, b \in \mathbb{Z}$, where a is even and b is odd. This means they can be written as follows:

$$
\begin{aligned}
a &= 2m \\
b &= 2n + 1
\end{aligned}
$$

where $m, n \in \mathbb{Z}$. Then:

$$
\begin{aligned}
ab &= (2m)(2n + 1) \\
&= 4mn + 2m \\
&= 2(2mn + m)
\end{aligned}
$$

Let $l = 2mn + m$. Since $\mathbb{Z}$ is closed under addition and multiplication, then $2mn + m \in \mathbb{Z}$, and therefore, $l \in \mathbb{Z}$. Then ab is, by definition, an even number, since $ab = 2l$, where $l \in \mathbb{Z}$. Therefore, any even number times an odd number is even itself. $\blacksquare$