EKN(812)-PS 1 memo

Question 1

We want to find the demands for the following utilities:
a)

$$\bullet u(x,y) = y + \frac{x^{1-\epsilon^{-1}}}{1-\epsilon^{-1}}$$

- The marshallian demands and own-price elasticity of x are given by:

$$x^m = (\frac{p_x}{p_y})^{-\epsilon}$$
$$y^m = \frac{m}{p_y} - (\frac{p_x}{p_y})^{1-\epsilon}$$
$$\eta_x = -\epsilon$$

- The hicksian demands are as follows:

$$x^h = (\frac{p_x}{p_y})^{-\epsilon}$$
$$y^h = \frac{u*}{1-\epsilon^{-1}}(\frac{p_x}{p_y})^{-\epsilon}$$

b)

$$u(x,y) = (\alpha x^\rho + (1-\alpha)y^\rho)^{\frac{1}{\rho}}$$

• The marshallian demands and own price elasticity are:

$$
\begin{aligned}
x^m &= \frac{mp_x^{r-1}}{p_x^r + p_y^r(\frac{\alpha}{1-\alpha})^{r-1}}, \\
y^m &= \frac{mp_y^{r-1}}{p_y^r + p_x^r(\frac{\alpha}{1-\alpha})}, \\
with\ r &= \frac{\rho}{\rho - 1}; \\
\eta_x &= -\sigma + \frac{\alpha^\sigma(\theta - a)}{p_x^{\sigma-1}(\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma})}, \\
\text{with } \sigma &= \frac{1}{1-\rho}
\end{aligned}
$$

•The hicksian demands are given by:

$$
\begin{aligned}
x^h &= u(\frac{\alpha}{p_x})^\sigma \Big(\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}\Big)^{-\frac{\sigma}{\sigma-1}} \\
y^h &= u(\frac{\alpha}{p_y})^\sigma \Big(\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}\Big)^{-\frac{\sigma}{\sigma-1}} \\
\text{with } \sigma &= \frac{1}{1-\rho}
\end{aligned}
$$

c)

$$
u(x, y) = -\frac{1}{2}(a(x \bar{-} x)^2 + 2b(\bar{x} - x)(\bar{y} - y) + (\bar{y} - y)^2)
$$

•The marshallian demands are given by:

$$
\begin{aligned}
x^m &= \frac{m - p_y(\bar{y} - \frac{\bar{x}}{k})}{p_x + \frac{p_y}{k}} \\
y^m &= \frac{m - p_x(\bar{x} - k\bar{y})}{p_x k + p_y} \\
with\ k &= \frac{p_y b - p_x}{p_x b - p_y a}
\end{aligned}
$$

•The hessian of the utility function is:

$$
H_u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = \begin{pmatrix} -a & -b \\ -b & -1 \end{pmatrix}
$$

u_{xx} is negative by definition of a, and in order to ensure concavity all is left is to ensure that the determinant of the matrix is positive:

$$\begin{aligned} D_H &> 0 \\ u_{xx}u_{yy} - u_{xy}^2 &> 0 \\ a - b^2 &> 0 \\ -\sqrt{a} < b &< \sqrt{a} \end{aligned}$$

•The first order conditions for the maximization of this utility subject to the budget constraint gives the following identity:

$$\frac{p_x}{p_y} = \frac{u_x}{u_y} = \frac{a(\bar{x} - x) + b(\bar{y} - y)}{b(\bar{x} - x) + (\bar{y} - y)}$$

To ensure that the solution is interior the parameters m, p_x, p_y must be chosen so that the optimal values (x^*, y^*) respect the identity. In particular, the bliss point cannot be reached at optimum. One way of achieving this is to pose that the bliss point is unaffordable and $p_x\bar{x} + p_y\bar{y} > m$.

Question 2

We want to maximise the following utility, subject to a budget constraint:

$$u(x, y) = \frac{x^{1-\frac{1}{\epsilon}}}{1 - \frac{1}{\epsilon}} + \frac{y^{1-\frac{1}{2\epsilon}}}{1 - \frac{1}{2\epsilon}}$$

The corresponding FOCs are:

$$\begin{aligned} x^{-\frac{1}{\epsilon}} &= \lambda p_x \\ y^{-\frac{1}{2\epsilon}} &= \lambda p_y \end{aligned}$$

From which we derive the following identity:

$$y = \left(\frac{p_x}{p_y}\right)^{2\epsilon} x^2$$

Replacing in the budget constraint yields:

$$p_x x + p_y \left(\frac{p_x}{p_y}\right)^{2\epsilon} x^2 - m = 0$$

This quadratic fonctions has a single positive solution given by:

$$x^m = \frac{-p_x + \left(p_x^2 + 4p_y\left(\frac{p_x}{p_y}\right)^{2\epsilon} m\right)^{\frac{1}{2}}}{2p_y\left(\frac{p_x}{p_y}\right)^{2\epsilon}}$$

And the demand for the other good is:

$$\begin{aligned} y^m &= \left(\frac{-p_x + \left(p_x^2 + 4p_y\left(\frac{p_x}{p_y}\right)^{2\epsilon} m\right)^{\frac{1}{2}}}{2p_y\left(\frac{p_x}{p_y}\right)^{2\epsilon}}\right)^2 \left(\frac{p_x}{p_y}\right)^{2\epsilon} \\ &= \left(-p_x + \left(p_x^2 + 4p_y\left(\frac{p_x}{p_y}\right)^{2\epsilon} m\right)^{\frac{1}{2}}\right)^2 \frac{\left(\frac{p_x}{p_y}\right)^{2\epsilon}}{4p_y^2\left(\frac{p_x}{p_y}\right)^{4\epsilon}} \\ y^m &= \frac{\left(-p_x + \left(p_x^2 + 4p_y\left(\frac{p_x}{p_y}\right)^{2\epsilon} m\right)^{\frac{1}{2}}\right)^2}{4p_y^2\left(\frac{p_x}{p_y}\right)^{2\epsilon}} \end{aligned}$$

Pose

$$x^m(m) = \frac{f(m)}{k}$$

$$y^m(m) = \frac{f(m)^2}{2k}$$

Income elasticities are given by:

$$\eta_x = \frac{mkf'(m)}{kf(m)} = \frac{mf'(m)}{f(m)}$$

and

$$\eta_y = \frac{2kmf'(m)f(m)}{kf(m)^2} = \frac{2mf'(m)}{f(m)}$$

The y good is the relative "luxury" and the x good the relative "necessity".

Question 3

Consider $x^h(p_x, p_y, U_0)$ the compensated demand for good x in a two goods system with x and y. Hicksian demands are homogenous of degree 0 in prices only. So by Euler's homogeneous function theorem:

$$\frac{\partial x^h}{\partial p_x} p_x + \frac{\partial x^h}{\partial p_y} p_y = 0$$

Dividing by x, we get:

$$\epsilon^h_{11} + \epsilon^h_{12} = 0$$

Since $\epsilon^h_{11} < 0$ we have $\epsilon^h_{12} > 0$, hence the two goods are substitutes.

Question 4

a) An agent of type k buys an amount x of good x for $p_x x = km$ and an amount y of good y for $p_y y = (1-k)m$. Then the standard bundle for type k is the pair $\{(x, y) = (\frac{km}{p_x}, \frac{(1-k)m}{p_y})\}$.

b) The average demand for x is given by:

$$\bar{x} = (1-0)\int_0^1 \frac{km}{p_x} dk = \frac{m}{2p_x}$$

Similarly, the demand for y is given by:

$$\bar{y} = (1-0)\int_0^1 \frac{(1-k)m}{p_y} dk = \frac{m}{2p_y}$$

c) We want to determine how much to compensate the agents to ensure that the average bundle $(\bar{x}, \bar{y})$ is still affordable after a price increase from p_x to p'_x. In other words the new income m' must be:

$$\begin{aligned} m' &= \bar{x}p'_x + \bar{y}p_y \\ &= \frac{m}{2p_x}p'_x + \frac{m}{2p_y}p_y \\ &= \frac{m}{2}(\frac{p'_x}{p_x} + 1) \end{aligned}$$

The new demand x' then becomes:

$$\begin{aligned} x' &= \frac{m'}{2p'_x} \\ &= \frac{m}{2}(\frac{p'_x}{p_x} + 1)\frac{1}{2p'_x} \\ &= \frac{m}{4}(\frac{1}{p_x} + \frac{1}{p'_x}) \\ x' &= \frac{x}{2} + \frac{m}{4p'_x} \end{aligned}$$

The demands are downward sloping despite the fact that the agents are "irrational" (non-utility maximizers).

d)Since we have $k \in [0,1]$, the average demand coincides with the market demand for goods x and y. The corresponding (own) price elasticity of demand is:

$$\begin{aligned}\eta_x &= \frac{p_x}{x}\frac{\partial}{\partial x}(\frac{m}{2p_x}) \\ &= -\frac{2p_x^2}{m}\frac{m}{4p_x^2} \\ \eta_x &= -\frac{1}{2}\end{aligned}$$

Question 5

Done in class.

Question 6

Done in class.

Question 7

We want to maximize consumption of a good vector x under rationing. The quantity of one good x_0 is fixed, and the consumer optimizes on the remaining goods. Under no rationing marshallian demands are given by:

$$\max_x U(x) \text{ st. } px = m$$

Such a program solves for $x^* = \{x_i^m(p,m)\}$ a function of income and the price vector. Under rationing, the problem becomes

$$\begin{aligned}\max_x U(x) \text{ st. } px &= m \\ \text{and } x_0 &= \bar{x_0}\end{aligned}$$

This new program now solves for $x^r = \{x_i^m(p,m,\bar{x_0})\}$ Note that

$$U(x^*(p,m)) \geq U(x^*(p,m,\bar{x_0}))$$

because at best $\bar{x_0}$ is at the marshallian optimum.

Under what conditions do we have :

$$x_i^m(p,m) = x_i^r(p,m,\bar{x_0})$$

This would be the case when $\bar{x_0} = x_0^m(p,m)$ and as a result we have the following idenity:

$$x_i^m(p,m) = x_i^r(p,m,x_0^m(p,m))$$

This identity states that for the demands under rationing equal those of a non-rationed regime only if the fixed value of x_0 corresponds to its optimal value under the non-rationed regime $x_1^*(p,m)$. Differentiating with respect to p_i gives:

$$\frac{\partial x_i^m(p_i)}{\partial p_i} = \frac{\partial x_i^r(x_0(p_i),p_i)}{\partial p_i}$$

$$\frac{\partial x_i^m}{\partial p_i} = \frac{\partial x_i^r}{\partial p_i} + (\frac{\partial x_i^r}{\partial x_0})(\frac{\partial x_0^m}{\partial p_i}) \tag{1}$$

Similarly, differentiating with respect to income gives:

$$\frac{\partial x_i^m(p_i)}{\partial m} = \frac{\partial x_i^r(x_0(m),m)}{\partial m}$$

$$\frac{\partial x_i^m}{\partial m} = \frac{\partial x_i^r}{\partial m} + (\frac{\partial x_i^r}{\partial x_0})(\frac{\partial x_0^m}{\partial m}) \tag{2}$$

Multiplying (2) by x_i and adding with (1) gives:

$$\Big(\frac{\partial x_i^m}{p_i} + \frac{\partial x_i^m}{\partial m}x_i^m\Big) = \Big(\frac{\partial x_i^r}{p_i} + \frac{\partial x_i^r}{\partial m}x_i^r\Big) + \frac{\partial x_i^r}{\partial x_0}\Big(\frac{\partial x_0^m}{p_i} + \frac{\partial x_0^m}{\partial m}x_i^m\Big) \tag{3}$$

We use Slutsky's identity to transform the expressions in brackets, which gives:

$$\frac{\partial h_i}{p_i} = \frac{\partial h_i^s}{\partial p_i} + \frac{\partial x_i^r}{\partial x_0}\Big(\frac{\partial h_0}{\partial p_i}\Big) \tag{4}$$

where the h_i are hicksian demands and h_i^s are the rationed hickisan demands (see lecture 3 slides), which can be rearranged as follows:

$$\frac{\partial h_i}{p_i} - \frac{\partial h_i^s}{\partial p_i} = \frac{\partial x_i^r}{\partial x_0}\Big(\frac{\partial h_0}{\partial p_i}\Big) \tag{5}$$

From the case of rationed hicksian demands (see slides lecture 3), we know that:

$$\frac{\partial h_i}{p_i} - \frac{\partial h_i^s}{\partial p_i} = \frac{\left(\frac{\partial h_i}{\partial p_0}\right)^2}{\frac{\partial h_0}{\partial p_0}} \tag{6}$$

From (5) and (6), we obtain:

$$\frac{\partial x_i^r}{\partial x_0}\left(\frac{\partial h_0}{\partial p_i}\right) = \frac{\left(\frac{\partial h_i}{\partial p_0}\right)^2}{\frac{\partial h_0}{\partial p_0}} \tag{7}$$

Note that by the symmetry property of hicksian demands, we have $\frac{\partial h_0}{\partial p_i} = \frac{\partial h_i}{\partial p_0}$, which simplifies (7) to :

$$\frac{\partial x_i^r}{\partial x_0} = \frac{\frac{\partial h_i}{\partial p_0}}{\frac{\partial h_0}{\partial p_0}} \tag{8}$$

we use (8) to substitute in (1):

$$\frac{\partial x_i^m}{\partial p_i} - \frac{\partial x_i^r}{\partial p_i} = \frac{\frac{\partial h_i}{\partial p_0} . \frac{\partial x_0^m}{\partial p_i}}{\frac{\partial h_0}{\partial p_0}} \tag{9}$$

$$= \frac{\sigma_{i0} . \frac{\partial x_0^m}{\partial p_i}}{\sigma_{00}} \tag{10}$$

Note that by slutsky's identity applied to $\frac{\partial x_0^m}{\partial p_i}$, the numerator can be written:

$$\sigma_{i0} . (\sigma_{0i} - \frac{\partial x_0^m}{\partial m} x_i^m)$$

which substituted in (10) gives:

$$\frac{\partial x_i^m}{\partial p_i} - \frac{\partial x_i^r}{\partial p_i} = \frac{\sigma_{i0}^2 - \sigma_{i0} \frac{\partial x_0^m}{\partial m} x_i^m}{\sigma_{00}} \tag{11}$$

Hence the difference between marshallian elasticities of standard and rationed marshallian demands will depend on the sign the the numerator of expression (11), and the relative magnitude of income and substitution effects.