EKN 812- ps2 Memo

March 25, 2019

Question 1

a) The student's problem is written as follows:

$$\max_x \sum_i S_i \text{ st. } \sum_i t_i \leq T$$

Inspecting the production function $f_i(t_i, K) = min(KT_i^\theta, \bar{S}_i)$ we realise that there are N upper bounds: $t_i \leq \bar{t_i} = (\bar{S}_i/K)^{\frac{1}{\theta}}$. The lagrangian of the problem becomes:

$$L = \sum_i S_i - \lambda(\sum_i t_i - T) - \sum_i \mu_i(t_i - \bar{t_i})$$

.

The FOCs to this problem are:

$[t_i] : \theta K t_i^{\theta-1} - \lambda - \mu_i \leq 0$ with equality if $t_i* > 0$

There are 2^{N+1} possibilities for the different multipliers, but we can eliminate some of them by studying the implications for the KKT conditions. The general logic here is to guess a certain pattern for the multipliers, and work out what the implied time allocation would be in each case.

- Let us assume that $\lambda = 0$ and at least one $\mu_i = 0$. Then $\sum_j t_j^* < T$ but also $t_i^* < \bar{t_i}$. So if $t_i > 0$, we have $\theta K t_i^{\theta-1} = \lambda + \mu_i = 0 + 0 = 0$. This cannot happen. We can then infer that $\lambda = 0 \implies \mu_i > 0$ for all i. In that case, the students get full marks and $\sum_i t_i^* = \sum_i \bar{t_i} = \sum_i (\frac{\bar{S}_i}{K})^{\frac{1}{\theta}} < T \iff \frac{1}{T}\sum_i \bar{S}_i^{1/\theta} < K^{1/\theta}$ i.e. K is large enough that that student can finish all of the questions with time to spare..

- Now let's consider the cases where $\lambda > 0$

Since the global time constraint binds, we have the following

$$\theta K t_i^{\theta-1} = \lambda + \mu_i \implies (\theta K)^{\frac{1}{\theta-1}} t_i = (\lambda + \mu_i)^{\frac{1}{\theta-1}} \implies (\theta K)^{\frac{1}{\theta-1}} T = \sum_i (\lambda + \mu_i)^{\frac{1}{\theta-1}}$$

We need to find out which $\mu_i > 0$ and which $\mu_i = 0$. The strategy will consist in determining the best order to chose in answering the questions e.g. the easier ones first or the harder ones first, as captured by associated points, depending on the characteristics of the production function.

For all i, the KKT conditions state that $\theta K t_i^{\theta-1} \leq \lambda + \mu_i$. This condition holds with equality whenever $t_i^* > 0$.

- Case 1: Assume $\theta < 1$ so $t_i^{\theta-1} \to \infty$ as $t_i \to 0$. Then we always have $\theta K t_i^{\theta-1} = \lambda + \mu_i$.Note that the time required to obtain full marks is $\bar{t_i} = (\frac{\bar{S_i}}{K})^{\frac{1}{\theta}}$. For each i, evaluating our first order conditions at $\bar{t_i}$, we have $\theta K \bar{t_i} = \theta K^{1/\theta} \bar{S_i}^{\frac{\theta-1}{\theta}}$.
At $t_i = \bar{t_i}$, we have $\theta K t_i^{\theta-1} = \theta K^{1/\theta} \bar{S_i}^{\frac{\theta-1}{\theta}}$. Besides, $\frac{\theta-1}{\theta} < 0$ by assumption on θ. So, evaluated at the upper bound, the marginal product of time spent per question is decreasing in the maximum score $\bar{S_i}$. As such, if questions are ranked by maximum score $S_1 < S_2 < ... < S_N$, the corresponding marginal products can be ranked: $\bar{S_1}^{\frac{\theta-1}{\theta}} > \bar{S_2}^{\frac{\theta-1}{\theta}} ... > \bar{S_N}^{\frac{\theta-1}{\theta}}$. As a result, given that $\lambda > 0$, the questions that will bind for μ_i are those for which the maximum score is lower i.e. the "easier" questions. (Formally, if $t_i^* = \bar{t_i}$ and $\bar{S_i} > \bar{S_j}$ then $t_j^* = \bar{t_j}$).

- Case 2: Assume $\theta > 1$. If $t_i^* = \bar{t_i}$, we have

$$\theta K t_i^* = \theta K^{1/\theta} \bar{S_i}^{\frac{\theta-1}{\theta}} = \lambda + \mu_i > \lambda$$

Now since $\theta > 0$, so is $\frac{\theta-1}{\theta}$ and by ranking the questions as before, we have

$$S_1 < S_2 < ... < S_N \implies \bar{S_1}^{\frac{\theta-1}{\theta}} < \bar{S_2}^{\frac{\theta-1}{\theta}} ... < \bar{S_N}^{\frac{\theta-1}{\theta}}$$

, so students get maximum points for "harder questions" (formally, if $\bar{S_i} > \bar{S_j}$ and $t_j^* = \bar{t_j}$, then $t_i^* = \bar{t_i}$; if not $\lambda = \theta K^{1/\theta} \bar{S_i}^{\frac{\theta-1}{\theta}} > \theta K^{1/\theta} \bar{S_j}^{\frac{\theta-1}{\theta}} > \lambda$, which is a contradiction).That is, if students get the maximum points for any of the questions, they do so for the ones with high $\bar{S_i}$. Note this is a statement that is true for given K. Of course, they could still get the maximum points for all of the questions (e.g. if K is large) or for none of them (e.g. if K is small).

Question: Could we have $\theta K (t^*)^{\theta-1} = \lambda$ for $t^* \in (0, \bar{t_i})$? We conjecture that we can partition the questions as follows:
We assume that $\underset{i}{\Sigma} \bar{t_i} > T$, and find j^* such that $\sum_{i=j^*}^N \leq T$ but $\sum_{i=j^*-1}^N > T$.

We can show that

$$t_i^* = 0 \text{ if } i = 1, ..., j^* - 2$$

$$t_{j-1}^* = T - \sum_{j^*}^{N} \text{ if } i = j^* - 1$$

$$t_i^* = \bar{t}_i \text{ if } i \geq j^*$$

$$\text{and } \lambda = \theta K (t_{j-1}^*)^{\theta-1}$$

As a matter of fact, it is not possible to do better locally. Consider reallocating ϵ from $t_{j^*-1}^*$. Then we would lose $\theta K (t_{j-1}^*)^{\theta-1}$ and gain nothing if we allocate to $i \leq j^* - 2$, so it is not worth it. Similarly, reallocation some time from $i \geq j^*$ would yield a loss of $\epsilon \theta K \bar{S}_i^{\theta-1}$ and gain $0.\epsilon$, so a net loss. Hence the proposed partition cannot be improved upon.

c)Now witht he new production function $f_i(t_i, K) = \bar{S}_i(1 - exp(-Kt_i))$. Inspecting this function, we note that $\frac{\partial f_i}{\partial t_i} = \bar{S}_i K exp(-Kt_i)$, that $\bar{S}_i$ can only be reached at infinity, and that evaluating $\frac{\partial f_i}{\partial t_i}$ zero gives $\bar{S}_i K$. Since $t_i < \bar{t}_i$, the first order conditions for all i are limited to :

$$\bar{S}_i(1 - exp(-Kt_i)) \leq \lambda$$

with equality if $t_i^* > 0$.

- Case 1: All $t_i^* > 0$, then

$$\sum_{i=1}^{N} log(\bar{S}_i K) - K \sum_i t_i = N log(\lambda)$$

$$\iff log(\lambda) = (\frac{\sum_i log \bar{S}_i}{N}) + log K - \frac{KT}{N}$$

So

$$log(\bar{S}_i K) - Kt_i = log(\lambda)$$

$$\iff log \bar{S}_i + log K - (N^{-1} \sum_j log \bar{S}_j + log K - \frac{KT}{N}) = KT_i^*$$

$$\iff log \bar{S}_i - N^{-1} \sum_j log \bar{S}_j = K(t_i^* - \frac{T}{N})$$

So $\sigma_{\bar{S}}^2 = K cov(t_i^*, \bar{S}_i) \iff cov(t_i^*, \bar{S}_i) = \frac{\sigma_{\bar{S}_i}^2}{K}$. So as K increases, the distribution of t_i^* is more equal. However, the correlation betweem t_i^* and $\bar{S}_i$) is 1 regardless of K.

- we have a corner solution if

$$log(\bar{S}_i) < log(\lambda) \iff log\bar{S}_i - N^{-1}\sum_j log\bar{S}_j < \frac{-KT}{N}$$

As a result, for a given K, students will tend to have more corner solutions if questions have very different $\bar{S}_i$, or for a given distribution of $\bar{S}_i$, students with lower Ks will attempt fewer questions.

d)If students value time outside the exam at rate v, their corresponding utility is:

$$\underset{t_i}{max} U = \underset{i}{\Sigma} S_i + v(T - \underset{i}{\Sigma} t_i)$$

The first order conditions are:

$$\frac{\partial f_i}{\partial t_i}(t_i, K) \leq v$$

with equality if $t_i^* = 0$.
Note we are ignoring the possibility of hitting $\bar{S}_i$ now because, as discussed in (c), with this production function it would take infinitely long to do so.

e)Could we have people leaving early? First note that we have:

$$t_i^* > 0 \iff log(\bar{S}_i K) - log(v) = Kt_i^* \text{ and}$$
$$t_i^* = 0 \iff log(\bar{S}_i K) < log(v)$$

so $\sum_{t_i} t_i = \sum_{t_i^*>0} t_i^* = \sum_{t_i^*>0}(log(\bar{S}_i K) - log(v)) = K \sum_{t_i^*>0} t_i^*$
If $\sum_i t_i^* = T$, then

$$\sum_i^N (log(\bar{S_i}K) - log(v)) = KT$$

This possibility seems quite unlikely. Rather, there should be some i's for which $t_i^* = 0 \iff log(\bar{S_i}K) < log(v)$. It is even possible that $t_i^* = 0$ for all i, so students just never show up. For a given $\bar{S}_i$, students attempt fewer questions if $log(v) - log(K)$ is large i.e. for high values of v of low values of K.

f) Students do not collide because their utility does not depend on the other students.

g) This occurs when relative ranking is more important than the measurement of absolute knowledge. For instance, under the assumption that the level of skill/knowledge of cohorts remain somewhat constant from year to year, it would make sense to grade to the curve.

h) There is an incentive to collude to spend less time on the exam and enjoy time outside while getting the same grade. However there is an incentive to cheat too as such an arrangement makes it easy to get a better grade by not shirking as agreed.

Question 2

a)The profit function of the firm serving the two markets is:

$$\pi = y_1 p_1(y_1) + y_2 p_2(y_2) - c(y)$$

where $y = y_1 + y_2$.

Maximising profit yields:

$$\begin{aligned} p_1(y_1) + p_1'(y_1)y_1 &= c_1'(y) \\ p_2(y_2) + p_2'(y_2)y_2 &= c_2'(y) \end{aligned}$$

Noting that the marginal costs are equal, we have:

$$p_1(y_1) + p_1'(y_1)y_1 = p_2(y_2) + p_2'(y_2)y_2$$

This yields:

$$\frac{p_1(y_1)}{p_2(y_2)} = \frac{1 - \epsilon_2^{-1}}{1 - \epsilon_1^{-1}}$$

b) With taxation, the profit function of the firm becomes:

$$\pi = y_1(p_1(y_1) - t) + y_2 p_2(y_2) - c(y)$$

Maximising this yields the following first order conditions:

$$\begin{aligned} \pi_1(y_1, y_2, t) &= 0 \\ \pi_2 = (y_1, y_2, t) &= 0 \end{aligned}$$

and the sufficient second order conditions (the hessian matrix is negative definite to ensure the existence of a local maximum):

$$\begin{aligned} \pi_{11} < 0, \pi_{22} &< 0 \\ \pi_{11}\pi_{22} - \pi_{12}^2 &> 0 \end{aligned}$$

Solving these equations yield the explicit choice functions

$$y_1 = y_1^*(t)$$
$$y_2 = y_2^*(t)$$

These two equations are simultaneous solutions of the maximization problem, and without any further restriction, a change in t yields a change in both y_1 and y_2. Substituting back into the first order conditions:

$$\pi_1(y_1^*(t), y_2^*(t), t) = 0$$
$$\pi_2(y_1^*(t), y_2^*(t), t) = 0$$

Differentiating with respect to t yields:

$$\pi_{11}\frac{\partial y_1^*}{\partial t} + \pi_{12}\frac{\partial y_2^*}{\partial t} + \pi_{1t} = 0$$
$$\pi_{21}\frac{\partial y_1^*}{\partial t} + \pi_{22}\frac{\partial y_2^*}{\partial t} + \pi_{2t} = 0$$

In matrix form, this system can be written:

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1^*}{\partial t} \\ \frac{\partial y_2^*}{\partial t} \end{pmatrix} = \begin{pmatrix} -\pi_{1t} \\ -\pi_{2t} \end{pmatrix} \tag{1}$$

This system can be solved for $\frac{\partial y_1^*}{\partial t}$ and $\frac{\partial y_2^*}{\partial t}$ using Cramer's rule:

$$\frac{\partial y_1^*}{\partial t} = \frac{-\pi_{1t}\pi_{22} + \pi_{2t}\pi_{12}}{|H|} \tag{2}$$
$$\frac{\partial y_2^*}{\partial t} = \frac{-\pi_{2t}\pi_{11} + \pi_{1t}\pi_{21}}{|H|} \tag{3}$$

where H is the determinant of the hessian matrix of the profit function π:

$$H = \begin{vmatrix} \pi_{11}\pi_{12} \\ \pi_{21}\pi_{22} \end{vmatrix} = \pi_{11}\pi_{22} - \pi_{12}^2 > 0 \tag{4}$$

We have:

$$\pi_1 = y_1\frac{\partial p_1}{\partial y_1} + p_1 - \frac{\partial c}{\partial y_1} - t \tag{5}$$

$$\pi_{11} = y_1\frac{\partial^2 p_1}{\partial y_1^2} + 2\frac{\partial p_1}{\partial y_1} - \frac{\partial^2 c}{\partial y_1^2} \tag{6}$$

$$\pi_{12} = \frac{\partial^2 c}{\partial y_1 \partial y_2} \tag{7}$$

$$\pi_{1t} = 1 \tag{8}$$

$$\pi_2 = y_2\frac{\partial p_2}{\partial y_2} + p_2 - \frac{\partial c}{\partial y_2} \tag{9}$$

$$\pi_{22} = y_2\frac{\partial^2 p_2}{\partial y_1^2} + 2\frac{\partial p_2}{\partial y_2} - \frac{\partial^2 c}{\partial y_2^2} \tag{10}$$

$$\pi_{2t} = 0 \tag{11}$$

Therefore (2) and (3) can be explicited as follows:

$$\frac{\partial y_1^*}{\partial t} = \frac{-y_2\frac{\partial^2 p_2}{\partial y_1^2} - 2\frac{\partial p_2}{\partial y_2} + \frac{\partial^2 c}{\partial y_2^2}}{|H|} \tag{12}$$

$$\frac{\partial y_2^*}{\partial t} = \frac{\frac{\partial^2 c}{\partial y_1 \partial y_2}}{|H|} \tag{13}$$

As a result, the output of good y_2 resulting from the tax introduction will depend on the sign of the numerator of (13) $\frac{\partial^2 c}{\partial y_1 \partial y_2}$.

c)Assuming demand decreases in market 2, total output can increase if the quantity $\frac{\partial y_1^*}{\partial t}$ is outweighed by the increase in quantity $\frac{\partial y_2^*}{\partial t}$ i.e. if :

$$-y_2\frac{\partial^2 p_2}{\partial y_1^2} - 2\frac{\partial p_2}{\partial y_2} + \frac{\partial^2 c}{\partial y_2^2} + \frac{\partial^2 c}{\partial y_1 \partial y_2} > 0 \tag{14}$$

Question 3

a)Agent has income $y_a \sim N(\mu_a.\sigma_a)$ and CARA utility with absolute risk aversion α. His certaintty equivalent is given by:

$$\tilde{c_a^s} = \mu_a - \frac{\alpha\sigma^2}{2}$$

b)After entering into a marriage, a's new income is given by the random variable $c_a^m = \pi_0 + \pi_1 y = \pi_0 + \pi_1(y_a + y_b)$, which is a linear combination of normal distributions. As such we have:

$$c_a \sim N(\pi_0 + \pi_1(\mu_a + \mu_b), \pi_1^2(\sigma_a^2 + \sigma_b^2))$$

The corresponding certainty equivalent is then given by:

$$\tilde{c_a^m} = \pi_0 + \pi_1 - \frac{\alpha\pi_1^2(\sigma_a^2 + \sigma_b^2)}{2}$$

Similarly, the certainty equivalent of agent b is given by:

$$\tilde{c_b^m} = \pi_0 + (1 - \pi_1)(\mu_a + \mu_b) - \frac{\beta(1 - \pi_1)^2(\sigma_a^2 + \sigma_b^2)}{2}$$

c)Simplifying with $\mu_a = \mu_b = \mu$ and $\sigma_a = \sigma_b = \sigma$, we have:

$$\tilde{c_a} + \tilde{c_b} = 2\mu - (\alpha\pi_1^2 + \beta(1 - \pi_1)^2)\sigma^2$$

The surplus from marriage is then:

$$\begin{aligned}
S(\pi_0, \pi_1) &= \tilde{c_a^m} + \tilde{c_b^m} - \tilde{c_a^s} - \tilde{c_b^s} \\
&= 2\mu - (\alpha\pi_1^2 + \beta(1 - \pi_1)^2)\sigma^2 - 2\mu + \alpha\frac{\sigma^2}{2} + \beta\frac{\sigma^2}{2} \\
&= \big(- \alpha\pi_1^2 - \beta(1 - \pi_1)^2 + \frac{\alpha + \beta}{2}\big)\sigma^2
\end{aligned}$$

Maximising S yields:

$$\pi_1^* = \frac{\beta}{\alpha + \beta}$$

We now verify that this value of (π_1^*) maximise the surplus from marriage:

$$\begin{aligned}
S(\pi_1^*) &= \big(- \alpha(\frac{\beta}{\alpha + \beta})^2 - \beta(1 - (\frac{\beta}{\alpha + \beta}))^2 + \frac{\alpha + \beta}{2}\big)\sigma^2 && (15) \\
&= \frac{\alpha^2 + \beta^2}{2(\alpha + \beta)} > 0 && (16)
\end{aligned}$$

So the surplus is positive as required.

d)If $\beta = 0$, then the aggregate suprlus $\frac{S(\pi_1^*)}{\sigma^2} = \frac{\alpha^{2/2}}{2} = \frac{\alpha}{2}$. So the surplus is larger when α is larger i.e. as a is more risk-averse. We should then expect to see more risk averse a's being matched to less risk-averse b's.