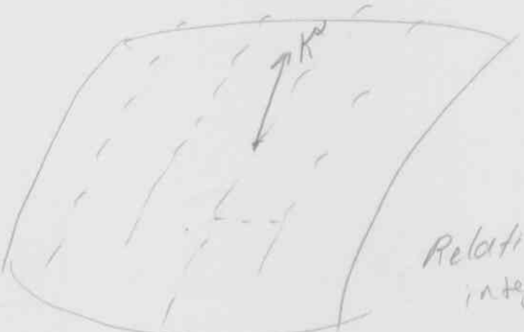
Geodesic Congruences



K_α = null normal

K^α = tangent to null geodesics

$K_\alpha K^\alpha = 0$

Relative behaviour of the integral curves : $\nabla_\beta K_\alpha \sim 16$ (4, 4)

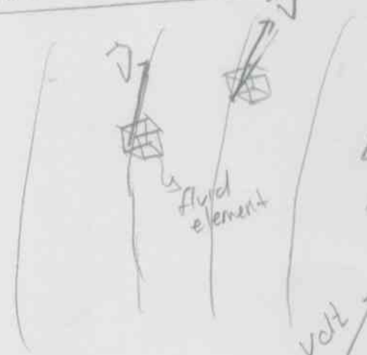
u^α

$\nabla_\beta u_\alpha$ timelike → null

Newtonian

$\partial_j V_k \sim 9$ (3, 3)

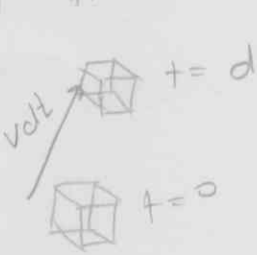
Fluid Mechanics – mass conservation



mass conservation : $\partial_t \rho + \partial_j (\rho V^j) = 0$ — derivative at a fixed point

continuity eqn

During a time interval dt fluid element moves by $\vec{V}dt$



"Material change"

$$D\rho = \rho(t+dt, \vec{x}+\vec{V}dt) - \rho(t, \vec{x})$$

$$= \cancel{\rho(t,x)} + \partial_t \rho\, dt + V^j dt\, \partial_j \rho + \cancel{\rho(t,x)}$$

$$\frac{D\rho}{dt} = \partial_t \rho + V^j \partial_j \rho \sim u^\alpha \nabla_\alpha \rho$$

Newtonian version

$$0 = \underbrace{\partial_t \rho + V^j \partial_j \rho}_{\frac{D\rho}{dt}} + \rho\, \partial_j V^j$$

$\frac{D}{dt}$: Material derivative

$$\frac{D\rho}{dt} + \rho\, \vec{\nabla}\cdot\vec{V} = 0 \quad \Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{V} = -\frac{1}{\rho}\frac{D\rho}{dt}}$$

fractional rate of change of density

Density changes ~~not~~ because of the mass, because of the volume

$\rho = \frac{m}{V}$

$\frac{D\rho}{dt} = -\frac{m}{V^2}\frac{dV}{dt} = -\frac{\rho}{V}\frac{DV}{dt}$

$$\boxed{\frac{1}{V}\frac{DV}{dt} = \vec{\nabla}\cdot\vec{v}}$$

Fractional rate of change of volumes

$\Theta \equiv \vec{\nabla}\cdot\vec{v} = \frac{1}{V}\frac{DV}{dt}$ = "rate of expansion"

Decomposition of $\partial_a v_b \rightarrow 9$ component

$v_{ab} \equiv \partial_a v_b$

$v_{(ab)} \equiv \frac{1}{2}(v_{ab} + v_{ba})$

$\underbrace{v_{ab}}_{9} = \underbrace{v_{(ab)}}_{6} + \underbrace{v_{[ab]}}_{3}$

$v_{[ab]} \equiv \frac{1}{2}(v_{ab} - v_{ba})$

$v_{(ab)} = \frac{1}{3}\underbrace{\Theta}_{1}\delta_{ab} + \underbrace{v_{\langle ab\rangle}}_{5}$; $\delta^{ab} v_{\langle ab\rangle} = 0$

$\Theta = \delta^{ab} v_{(ab)}$; $v_{\langle ab\rangle} = v_{(ab)} - \frac{1}{3}\Theta\delta_{ab}$

$\underbrace{\partial_a v_b}_{9} = \frac{1}{3}\underbrace{\Theta}_{1}\delta_{ab} + \underbrace{v_{\langle ab\rangle}}_{5} + \underbrace{v_{[ab]}}_{3}$

$v_{[xy]} = \frac{1}{2}(\partial_x v_y - \partial_y v_x) = \frac{1}{2}(\vec{\nabla}\times\vec{v})_z$

$v_{[yz]} = \frac{1}{2}(\partial_y v_z - \partial_z v_y) = \frac{1}{2}(\vec{\nabla}\times\vec{v})_x$

$v_{[zx]} = \frac{1}{2}(\vec{\nabla}\times\vec{v})_y$

$\sigma_{ab} = v_{\langle ab\rangle}$ = "rate of sheer"

$\omega_{ab} = v_{[ab]}$ = "rate of rotation"

Focus on 2D flow on xy-plane ; $v_z = 0$, $\partial_z v_x = \partial_z v_y = 0$

$\Theta = \vec{\nabla}\cdot\vec{v} = \partial_x v_x + \partial_y v_y$

Shear

$\Theta = 0$, $\omega_{ab} = 0 \rightarrow \omega_{xy} = \frac{1}{2}(\partial_x v_y - \partial_y v_x) = 0$

$\partial_x v_y = \partial_y v_x$

$\partial_x v_x + \partial_y v_y = 0$

$$\sigma_{ab} = v_{(ab)} - \cancel{\frac{1}{3}\Theta\delta_{ab}}$$

$$\sigma_{xx} = \frac{1}{2}(\partial_x v_x + \partial_x v_x) = \partial_x v_x \equiv \sigma_+$$

$$\sigma_{xy} = \frac{1}{2}(\partial_x v_y + \partial_y v_x) = \sigma_{yx} = \sigma_+$$

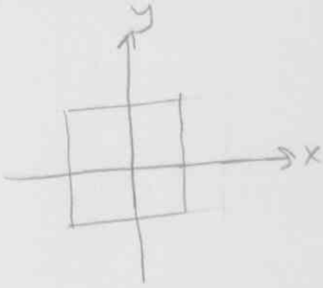
$$\sigma_{yy} = \partial_y v_y = -\partial_x v_x = -\sigma_+$$

$$\sigma_{ab} = \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix}$$

Square fluid element around the origin in frame $\vec{v}(0) = 0$

Velocity around origin: $v_x = \underbrace{v_x(0,0)}_{0} + \underbrace{\partial_x v_x(0,0)}_{\sigma_+}x + \underbrace{\partial_y v_x(0,0)}_{\sigma_x}y \ldots$

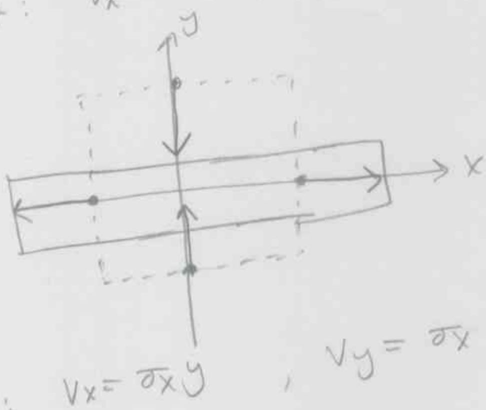
$v_y = \underbrace{v_y(0,0)}_{0} + \underbrace{\partial_y v_y(0,0)}_{-\sigma_+}y + \underbrace{\partial_x v_y(0,0)}_{\sigma_x}x$



$$v_x = \sigma_+ x + \sigma_x y$$

$$v_y = \sigma_x x - \sigma_+ y$$

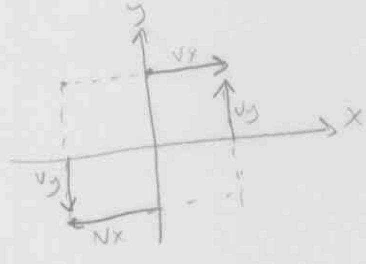
σ_+: $v_x = \sigma_+ x$, $v_y = -\sigma_+ y$



→ stretch in the x-direction
squeeze in the y-direction

→ shear (no change in volume)

σ_x: $v_x = \sigma_x y$, $v_y = \sigma_x x$



→ shear (no change in volume)

Rotation

$\Theta = 0, \ \sigma_{ab} = 0, \ \omega_{ab} \neq 0 \quad \rightarrow \quad \omega_{xy} = \frac{1}{2}(\partial_x v_y - \partial_y v_x) = \omega_z$

$\partial_x v_x + \partial_y v_y = 0$

$\partial_x v_x = 0$

$\partial_y v_y = 0$

$\partial_x v_y + \partial_y v_x = 0 \quad \rightarrow \quad \omega_z = \frac{1}{2}(\partial_x v_y + \partial_x v_y) = \partial_x v_y = -\partial_y v_x$

$\partial_x v_y = \omega_z$

$\partial_y v_x = -\omega_z$

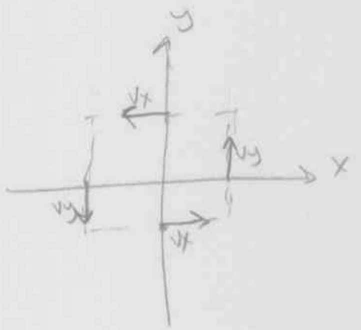
$$v_x = \underbrace{v_x(0,0)}_{0} + \cancel{\partial_x v_x(0,0)}\, x + \underbrace{\partial_y v_x(0,0)}_{-\omega_z}\, y$$

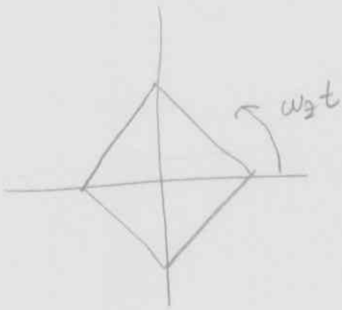
$$v_y = \underbrace{v_y(0,0)}_{0} + \underbrace{\partial_x v_y(0,0)}_{\omega_z}\, x + \cancel{\partial_y v_y(0,0)}\, y$$

$v_x = -\omega_z y \qquad \vec{V} = \vec{\omega} \times \vec{r}$ = rotation ; $\vec{\omega}$ = angular velocity ≡ rate of rotation

due to having only $-z$ component

$v_y = \omega_z x$





no change in volume

$$\partial_a v_b = \frac{1}{3}\Theta \delta_{ab} + \sigma_{ab} + \omega_{ab}$$

$\frac{1}{3}\Theta\delta_{ab}$: rate of expansion ; σ_{ab}: rate of shear ; ω_{ab}: rate of rotation

Space time velocity field

u^α: normalized velocity field, $g_{\alpha\beta} u^\alpha u^\beta = -1$, not necessarily geodesic

Congruence of timelike worldlines (of a fluid element) to which u^α is tangent

$u^\alpha \overset{*}{=} (1,0,0,0), \quad u_\alpha \overset{*}{=} (-1,0,0,0)$

$P^\alpha{}_\beta \overset{*}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \delta_{jk}$

Acceleration vector $a^\alpha \equiv \frac{Du^\alpha}{d\tau} = u^\beta \nabla_\beta u^\alpha$ is not zero because it is not on geodesics

$$u_\alpha a^\alpha = u_\alpha u^\beta \nabla_\beta u^\alpha = \frac{1}{2} u^\beta \underbrace{\nabla_\beta (\underbrace{u_\alpha u^\alpha}_{-1})}_{0} = 0$$

$\underline{\underline{u_\alpha a^\alpha = 0}}$

$\underline{\text{Projection operator}}$: $P^\alpha_\beta = \underbrace{g^\alpha{}_\beta}_{\delta^\alpha_\beta} + u^\alpha u_\beta$

- $P^\alpha{}_\beta u^\beta = (g^\alpha{}_\beta + u^\alpha u_\beta) u^\beta = u^\alpha - u^\alpha = 0$

- $P^\alpha{}_\beta P^\beta{}_\gamma = P^\alpha{}_\gamma$

- $P^\alpha{}_\alpha = \overset{\text{dimension 4}}{d} - 1 = 3$

Decomposition of vector

$\underline{u^\alpha}$ and its integral curves → preffered time → "time"

orthogonal directions → preffered spatial directions → "space"

$\underset{\downarrow \atop 4}{A^\alpha} = \underbrace{\overset{\nearrow 1}{A} u^\alpha}_{time} + \underbrace{A_\perp^\alpha}_{spatial} \; 3 \quad : \quad u_\alpha A^\alpha_\perp = 0$

- $u_\alpha A^\alpha = A (\underbrace{u_\alpha u^\alpha}_{-1}) + \underbrace{u_\alpha A_\perp^\alpha}_{0} \quad \Rightarrow \quad A = - u_\alpha A^\alpha$

- $P^\beta{}_\alpha A^\alpha = A \underbrace{P^\beta{}_\alpha u^\alpha}_{0} + P^\beta{}_\alpha A_\perp^\alpha$
$= (g^\beta{}_\alpha + u^\beta u_\alpha) A^\alpha_\perp = A^\beta_\perp + u^\beta \underbrace{u_\alpha A^\alpha_\perp}_{0} \quad \Rightarrow \quad A^\beta_\perp = P^\beta{}_\alpha A^\alpha$

- $A = -u_\alpha A^\alpha$
- $A^\alpha_\perp = P^\alpha{}_\beta A^\beta$

Decomposition of tensor

$$\underset{\downarrow \atop 16}{A^{\alpha\beta}} = \underset{\downarrow \atop {1 \atop \text{time-time}}}{A u^\alpha u^\beta} + \underset{\downarrow \atop {3 \atop \text{time-space}}}{u^\alpha B^\beta_\perp} + \underset{\downarrow \atop {3 \atop \text{space-time}}}{C^\alpha_\perp u^\beta} + \underset{\downarrow \atop {9 \atop \text{space-space}}}{D^{\alpha\beta}_{\perp\perp}}$$

- $B^\alpha_\perp . u_\alpha = 0$
- $C^\alpha{}_\perp . u_\alpha = 0$
- $D^{\alpha\beta}_{\perp\perp} u_\alpha = D^{\alpha\beta}_{\perp\perp} u_\beta = 0$

$$A^{\alpha\beta} = g^\alpha{}_\mu g^\beta{}_\nu A^{\mu\nu} = (P^\alpha_\mu - u^\alpha u_\mu)(P^\beta_\nu - u^\beta u_\nu) A^{\mu\nu}$$
$$= u^\alpha u^\beta (u_\mu u_\nu A^{\mu\nu}) - u^\alpha (u_\mu P^\beta{}_\nu A^{\mu\nu}) + (-P^\alpha_\mu u_\nu A^{\mu\nu}) u^\beta + P^\alpha_\mu P^\beta{}_\nu A^{\mu\nu}$$

$A = u_\mu u_\nu A^{\mu\nu}$

$B^\beta_\perp = -u_\mu P^\beta_\nu A^{\mu\nu}$

$C^\alpha_\perp = -P^\alpha_\mu u_\nu A^{\mu\nu}$

$D^{\alpha\beta}_{\perp\perp} = P^\alpha_\mu P^\beta{}_\nu A^{\mu\nu}$

Decompose $\nabla_\alpha u_\beta \equiv A_{\alpha\beta}$

side calculation: $u^\alpha \nabla_\beta u_\alpha = \frac{1}{2} u^\alpha \nabla_\beta u_\alpha + \frac{1}{2} u_\alpha \nabla_\beta u^\alpha$

$$= \frac{1}{2} \nabla_\beta (\underbrace{u_\alpha u^\alpha}_{-1}) = 0$$

$u^\alpha \nabla_\beta u_\alpha = 0$ ✓

$$A = u^\mu u^\nu A_{\mu\nu} = u^\mu \underbrace{u^\nu \nabla_\mu u_\nu}_{0} = 0$$

$$B^{\perp}_{\beta} - \underline{u^\mu} P_\beta{}^\nu \underline{\nabla_\mu u_\nu} = -P_\beta{}^\nu a_\nu = -(g_\beta{}^\nu + \underbrace{u_\beta u^\nu}_{0}) \underline{a_\nu} = -a_\beta$$

$$C^{\perp}_{\alpha} = -P_\alpha{}^\mu \underbrace{u^\nu \nabla_\mu u_\nu}_{0} = 0$$

$$D^{\perp\perp}_{\alpha\beta} = P_\alpha{}^\mu P_\beta{}^\nu \nabla_\mu u_\nu = (g_\alpha{}^\mu + u_\alpha u^\mu)(g_\beta{}^\nu + u_\beta \underbrace{u^\nu}_{0}) \underline{\nabla_\mu u_\nu}$$

$$= (g_\alpha{}^\mu + u_\alpha u^\mu) \nabla_\mu u_\beta = \nabla_\alpha u_\beta + u_\alpha a_\beta$$

$$\nabla_\alpha u_\beta = \underbrace{-u_\alpha a_\beta}_{\text{time-space}} + \underbrace{(\nabla_\alpha u_\beta + u_\alpha a_\beta)}_{\text{space-space}}$$

→ Analogus to newtonian $\partial_j v_k$

further decomposition (trace, symmetric trace free, antisymmetric)

$$\nabla_\alpha u_\beta + u_\alpha a_\beta = \underbrace{\frac{1}{3} \Theta}_{\text{Rate of (expansion of worldlines}} P_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{\text{shear}} + \underbrace{\omega_{\alpha\beta}}_{\text{rotation)}}$$

- $\sigma_{\alpha\beta} P^{\alpha\beta} = 0$. Traceless, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$
- $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$
- $\underbrace{\underline{\sigma_{\alpha\beta}} u^\alpha = 0, \ \underline{\omega_{\alpha\beta}} u^\alpha = 0}_{\text{spatial}}$ time

$$\nabla_{(\alpha} u_{\beta)} + u_{(\alpha} a_{\beta)} = \frac{1}{3} \Theta P_{\alpha\beta} + \sigma_{\alpha\beta}$$

$$\underbrace{P^{\alpha\beta} (\nabla_{(\alpha} u_{\beta)} + u_{(\alpha} a_{\beta)})} = \frac{1}{\cancel{3}} \Theta \underbrace{P^{\alpha\beta} P_{\alpha\beta}}_{\cancel{3}} = \Theta$$

$$(g^{\alpha\beta} + u^\alpha u^\beta)(\nabla_\alpha u_\beta + u_\alpha a_\beta) = \nabla_\alpha u^\alpha + \underbrace{u_\alpha a^\alpha}_{0} + u^\alpha \underbrace{u^\beta \nabla_\alpha u_\beta}_{0} + u^\alpha \underbrace{u^\beta u_\alpha a_\beta}_{0} = \nabla_\alpha u^\alpha$$

$$\Theta = \nabla_\alpha u^\alpha$$

$$\sigma_{\alpha\beta} = \nabla_{(\alpha} u_{\beta)} + u_{(\alpha} a_{\beta)} - \frac{1}{3} \Theta P_{\alpha\beta}$$

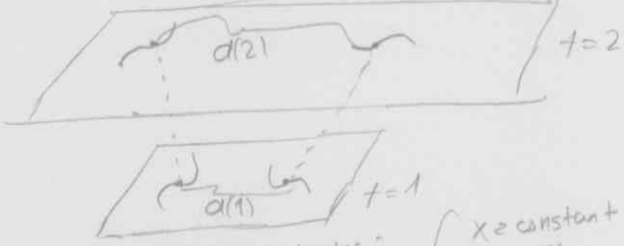
$$\omega_{\alpha\beta} = \nabla_{[\alpha} u_{\beta]} + u_{[\alpha} a_{\beta]}$$

$\Theta = \nabla_\alpha u^\alpha$: Tells us whether worldlines are diverging or converging.

Final decomposition

$$\underbrace{\nabla_\alpha u_\beta}_{12} = \underbrace{-u_\alpha a_\beta}_{3} + \frac{1}{3} \underbrace{\Theta}_{1} P_{\alpha\beta} + \underbrace{\sigma_{\alpha\beta}}_{5} + \underbrace{\omega_{\alpha\beta}}_{3}$$

Example 1 <FRW cosmology



$$ds^2 = -dt^2 + a^2(t)\left(dx^2+dy^2+dz^2\right)$$

World lines of galaxies: $\begin{cases} x = \text{constant} \\ y = \text{''} \\ z = \text{''} \end{cases}$

cosmological flows $u^\alpha = (1,0,0,0)$

$$u_\alpha = (-1,0,0,0)$$

$$\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^\gamma_{\alpha\beta} u_\gamma = -\Gamma^\gamma_{\alpha\beta} u_\gamma = \Gamma^t_{\alpha\beta}$$

$$\Gamma^t_{\alpha\beta} = \frac{1}{2}\underbrace{g^{t\lambda}}_{g^{tt}=-1}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right)$$

$$\Gamma^t_{\alpha\beta} = -\frac{1}{2}\Big(\underbrace{\partial_\alpha g_{t\beta}}_{0} + \underbrace{\partial_\beta g_{t\alpha}}_{0} - \partial_t g_{\alpha\beta}\Big)$$

only g_{ij}, $i=j$ elements has time dependence

$$\Gamma^t_{\alpha\beta} = \frac{1}{2}\partial_t g_{\alpha\beta} = \frac{1}{\cancel{2}}\cdot\cancel{2}\, a\dot{a} = a\dot{a}\begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}$$

$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta = \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix} a^2$$

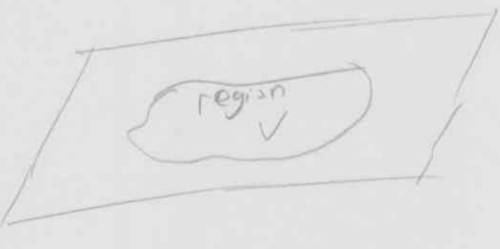
$$\nabla_\alpha u_\beta = \frac{\cancel{a}\dot{a}}{a^{\cancel{2}}} P_{\alpha\beta} \Rightarrow \boxed{\nabla_\alpha u_\beta = \underbrace{\frac{\dot{a}}{a}}_{=\frac{1}{3}\Theta} P_{\alpha\beta}}$$

: Pure trace term $\Theta = 3\frac{\dot{a}}{a}$

$$\Theta = 3\frac{\dot{a}}{a} = \frac{1}{a^3}\frac{d}{dt}\left(a^3\right)$$

$$= \frac{1}{(\text{Volume})}\frac{d}{dt}(\text{Volume})$$

$\Rightarrow$ Fractional rate of volume change



Example 2 timelike curves in Schwarzschild

$$f = \left(1 - \frac{2M}{r}\right)$$

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$u^\alpha = \gamma(\overset{t}{1}, \overset{r}{0}, \overset{\theta}{\underbrace{\Omega}_{\text{constant}}}, \overset{\varphi}{0}) \qquad \gamma = \Big(\underbrace{1 - \frac{2M}{r} - \Omega^2 r^2}_{>0}\Big)^{-1/2}$$

- Vector field is defined everywhere because curves do not have to be geodesics

r = constant

circular orbits

$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + \frac{1}{3}\Theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$a^\alpha = u^\beta \nabla_\beta u^\alpha \qquad a^r = -\frac{\gamma^2(r-2M)(M-\Omega^2 r^3)}{r^3} \qquad \left(\text{geodesic} - \Omega^2 = M/r^3 \ ; \ r > 3M \text{ for } \gamma > 0\right)$$

$$\Theta = \nabla_\alpha u^\alpha = \gamma\Omega \cot\theta$$

$$\sigma_{tt} = -\frac{1}{3}\Omega^3\gamma^3 r(r-2M)\cot\theta \qquad \omega_{tr} = \Omega^2\gamma^3(r-3M)$$

$$\omega_{r\theta} = \Omega\gamma^3(r-3M)$$

$$\sigma_{t\theta} = \frac{1}{3}\Omega^2\gamma^3 r(r-2M)\cot\theta$$

$$\sigma_{rr} = -\frac{1}{3}\Omega\gamma\frac{r}{r-2M}\cot\theta$$

$$\sigma_{\theta\theta} = -\frac{1}{3}\Omega\gamma^2 r(r-2M)\cot\theta$$

$$\sigma_{\theta\varphi} = \frac{2}{3}\Omega\gamma r^2 \sin\theta\cos\theta$$

Hypersurface orthogonal

specialized flow such that worldlines are orthogonal family of hypersurfaces

↳ $\Phi(x^\alpha) = \text{constant}$

$\Rightarrow \omega_{\alpha\beta} = 0$ • If we have hypersurface orthogonality rotation tensor vanishes

Key input $\Rightarrow u^\alpha$ = unit normal to hypersurfaces

Normal vector $n_\alpha \propto \partial_\alpha \Phi$

$u^{\alpha} \to$ future $+1$, $u_\alpha \to$ past (-1)

$$u_\alpha = -e^{\chi}\nabla_\alpha \Phi$$

$$-1 = g^{\alpha\beta}u_\alpha u_\beta = e^{2\chi} g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi$$

$$e^{-2\chi} = -g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi$$

Differentiating above equation

$$e^{-2\chi}(-2\nabla_\mu\chi) = -\nabla_\mu(g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi) = -g^{\alpha\beta}(\nabla_\mu\nabla_\alpha\Phi\underline{\nabla_\beta}\Phi + \underline{\nabla_\alpha}\Phi\nabla_\mu\nabla_\beta\Phi)$$

$$= -(\nabla^\alpha\Phi\nabla_\mu\nabla_\alpha\Phi + \nabla^\alpha\Phi\nabla_\mu\nabla_\alpha\Phi)$$

$$= -2\underline{\nabla^\alpha\Phi}\nabla_{\mu\alpha}\Phi = \cancel{2}e^{-\chi}u^\alpha\nabla_{\mu\alpha}\Phi$$

$$\nabla_{\alpha\mu}\Phi = \nabla_{\mu\alpha}\Phi$$

$$\boxed{\nabla_\mu\chi = -e^{-\chi}u^\alpha\nabla_{\mu\alpha}\Phi}$$

$$\nabla_\beta u_\alpha = -e^{\chi}\left(\nabla_\beta \chi \nabla_\alpha \Phi + \nabla_{\beta\alpha}\Phi\right)$$

$$= -e^{\chi}\left(\underline{-e^{\chi} u^\mu \nabla_{\mu\beta}\Phi \nabla_\alpha \Phi} + \nabla_{\beta\alpha}\Phi\right)$$

$$= -e^{\chi}\left(u^\mu u_\alpha \nabla_{\mu\beta}\Phi + \nabla_{\beta\alpha}\Phi\right)$$

$$= -e^{\chi}\underbrace{\left(u^\mu u_\alpha + g^\mu{}_\alpha\right)}_{P^\mu{}_\alpha}\nabla_{\mu\beta}\Phi = -e^{\chi} P^\mu{}_\alpha \nabla_{\mu\beta}\Phi$$

$$\nabla_\beta u_\alpha = -e^{\chi} P^\mu{}_\alpha \nabla_{\mu\beta}\Phi$$

$$a_\alpha = u^\beta \nabla_\beta u_\alpha = -e^{\chi} u^\beta P^\mu{}_\alpha \nabla_{\mu\beta}\Phi$$

$$\underbrace{\nabla_\beta u_\alpha + u_\beta a_\alpha} = \frac{1}{3}\Theta P_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}$$

$$-e^{\chi}\left(P_\alpha{}^\mu \nabla_{\mu\beta}\Phi + u_\beta u^\nu P_\alpha{}^\mu \nabla_{\mu\nu}\Phi\right) = -e^{\chi}\underbrace{\left(g_\beta{}^\nu + u_\beta u^\nu\right)}_{P_\beta{}^\nu} P_\alpha{}^\mu \nabla_{\mu\nu}\Phi$$

$$\nabla_\beta u_\alpha + u_\beta a_\alpha = -e^{\chi}\underbrace{P_\beta{}^\nu P_\alpha{}^\mu \nabla_{\mu\nu}\Phi}_{\text{symmetric in } \alpha\beta}, \quad \boxed{\omega_{\alpha\beta}=0}$$

Raychaudhuri's equation : Evolution equation for Θ : $\frac{D\Theta}{d\tau} = \cdots$

$$\frac{D}{d\tau}(\nabla_\alpha u_\beta) = u^\mu \nabla_\mu(\nabla_\alpha u_\beta) = u^\mu \nabla_{\mu\alpha} u_\beta = u^\mu \underbrace{\left(\nabla_{\mu\alpha} - \nabla_{\alpha\mu}\right) u_\beta}_{-R^\nu{}_{\beta\mu\alpha} u_\nu} + \frac{u^\mu \nabla_{\alpha\mu} u_\beta}{\nabla_\alpha \underbrace{\left(u^\mu \nabla_\mu u_\beta\right)}_{a_\beta} - \nabla_\alpha u^\mu \nabla_\mu u_\beta}$$

$$g^{\alpha\beta}\left(\frac{D}{d\tau}(\nabla_\alpha u_\beta) = \underbrace{-R_{\nu\beta\mu\alpha} u^\nu u^\mu}_{-R_{\mu\alpha\nu\beta} u^\mu u^\nu = -R_{\alpha\mu\beta\nu} u^\mu u^\nu} + \nabla_\alpha a_\beta - (\nabla_\alpha u^\mu)(\nabla_\mu u_\beta)\right)$$

$$\frac{D}{d\tau}(\Theta) = -R^\beta{}_{\mu\beta\nu} u^\mu u^\nu + \nabla_\alpha a^\alpha - (\nabla_\alpha u_\mu)(\nabla^\mu u^\alpha)$$

$$-(\nabla_\alpha u_\beta)(\nabla^\beta u^\alpha) + \nabla_\alpha a^\alpha$$

$$\frac{D}{d\tau}\Theta = -R_{\alpha\beta} u^\alpha u^\beta - (\nabla_\alpha u_\beta)(\nabla^\beta u^\alpha) + \nabla_\alpha a^\alpha$$

$$\nabla_\alpha u_\beta = -u_\alpha a_\beta + \frac{1}{3}\Theta P_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

$$\nabla^\beta u^\alpha = -u^\beta a^\alpha + \frac{1}{3}\Theta P^{\alpha\beta} + \sigma^{\alpha\beta} - \omega^{\alpha\beta}$$

$$(\nabla_\alpha u_\beta)(\nabla^\beta u^\alpha) = \frac{1}{3}\Theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta}$$

$$\boxed{\frac{D\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta} u^\alpha u^\beta + \nabla_\alpha a^\alpha}$$

R's eqn

$\Theta = \nabla_\alpha u^\alpha = \frac{1}{V}\frac{d}{d\lambda}V$: Fractional rate of volume change

Focusing theorem -

- Congruence is geodesic $\left(\frac{Du^\alpha}{d\tau} = a^\alpha = 0\right)$
- Congruence is hypersurface orthogonal $(\omega_{\alpha\beta} = 0)$

$$\frac{D\Theta}{d\tau} = -\Big(\underbrace{\tfrac{1}{3}\Theta^2}_{\text{spatial}} + \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\text{spatial}} + R_{\alpha\beta}u^\alpha u^\beta\Big) \leq 0$$

$(\text{spatial})^2 \geq 0$?

- Ricci condition

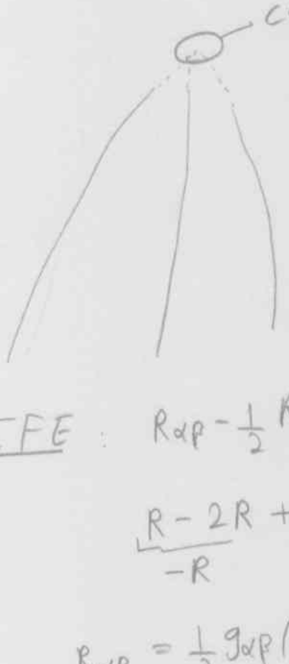
$$R_{\alpha\beta}u^\alpha u^\beta \geq 0$$

$\Theta > 0$

Θ decreases

Gravity pulls the geodesics together $\Rightarrow$ gravity is attractive

caustic $\equiv$ singularity of flow



$\Theta < 0$

$\Theta \to -\infty$

EFE : $R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$

$\underbrace{R - 2R}_{-R} + 4\Lambda = 8\pi T \quad \Rightarrow \quad -R = -4\Lambda + 8\pi T$

$$R_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta}(4\Lambda - 8\pi T) - \Lambda g_{\alpha\beta} + 8\pi T_{\alpha\beta}$$

$$R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) + \Lambda g_{\alpha\beta}$$

$$R_{\alpha\beta}u^\alpha u^\beta = 8\pi\left(T_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}T(-1)\right) + \Lambda(-1)$$

$$\boxed{R_{\alpha\beta}u^\alpha u^\beta = 8\pi\left(T_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2}T\right) - \Lambda}$$

Perfect fluid: $T_{\alpha\beta} = \mu V_\alpha V_\beta + p(g_{\alpha\beta} + V_\alpha V_\beta)$ $\qquad V_\mu V^\mu = -1$

$$T_{\alpha\beta} u^\alpha u^\beta = \mu (V\cdot u)^2 + p\left(-1 + (V\cdot u)^2\right)$$

$$T = \mu(-1) + p(4-1) = -\mu + 3p$$

$$T_{\alpha\beta} u^\alpha u^\beta + \frac{1}{2} T = \mu\left[(V\cdot u)^2 - \frac{1}{2}\right] + p\left[-1 + (V\cdot u)^2 + \frac{3}{2}\right]$$

Choose

$u^\alpha = V^\alpha$ $\qquad T_{\alpha\beta} u^\alpha u^\beta + \frac{1}{2} T = \frac{1}{2}(\mu + 3p)$

$u\cdot V = -1$

$R_{\alpha\beta} u^\alpha u^\beta = 4\pi(\mu + 3p) - \Lambda$ $\quad < 0$ (for our universe) $\quad \Lambda \gg$

$\qquad\qquad > 0$ (for a star)

Example - cosmology

$$\Theta = \frac{1}{V}\frac{dV}{dt}$$

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

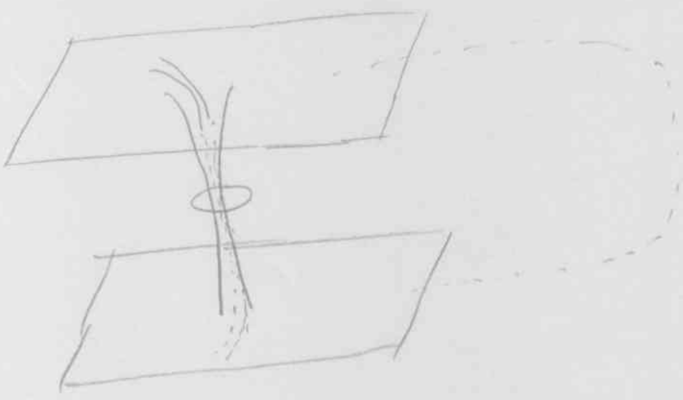
$$\Theta = 3\frac{\dot a}{a} = \frac{1}{a^3}\frac{d}{dt}a^3, \quad \sigma_{\alpha\beta} = 0$$

$$\frac{D\Theta}{d\tau} = 3\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right) = 3\frac{\ddot a}{a} - \cancel{3\frac{\dot a^2}{a^2}} = \cancel{-\frac{1}{3} 9 \frac{\dot a^2}{a^2}} - 4\pi(\mu + 3p) + \Lambda$$

$$3\frac{\ddot a}{a} = \underbrace{-4\pi(\mu + 3p)}_{\text{focus}} \underbrace{+\Lambda}_{\text{defocuses}}$$

$\Lambda \gg$ $\longrightarrow$ R's equation, in cosmology case Friedmann eqn

Example - wormhole

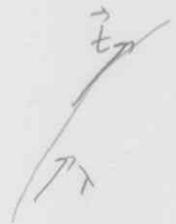


$R_{\alpha\beta} u^\alpha u^\beta < 0$

Congiuences of null geodesics

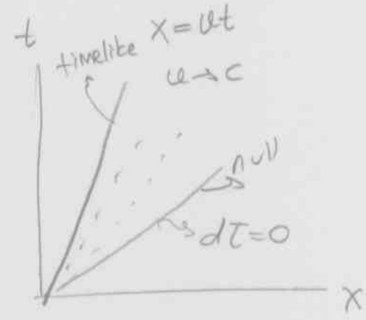
Null geodesic : ~~Path of photon~~
path of light ray

Null geodesic

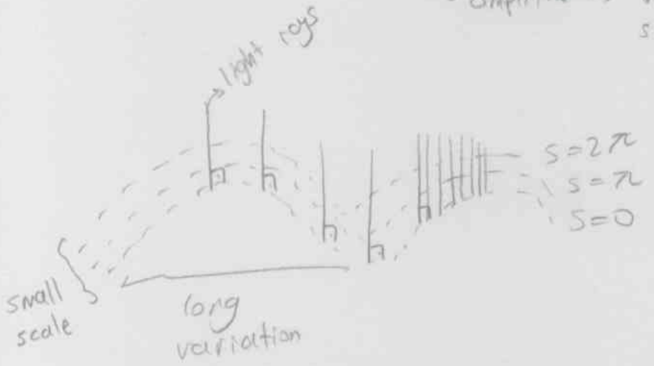


$$\frac{Dt^\alpha}{d\lambda} = k t^\alpha \xrightarrow{\text{null}} \frac{Dt^\alpha}{d\lambda} = k t^\alpha$$

$$t_\alpha t^\alpha < 0 \qquad\qquad t_\alpha t^\alpha = 0$$



light ? $\quad \vec{E}, \vec{B} \sim A e^{-iS}$; S → phase ⇒ various rapidly ; A → amplitude ⇒ various slowly



light beam: collection of light rays

Dipole scalar field

Scalar field Φ , $\Box \Phi = 0$ (flat space time)

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

$$\Phi = \underbrace{\frac{1}{r}\left(1 + \frac{i}{\omega r}\right)}_{\text{Amplitude} \equiv A} \cos\theta \, e^{-i\underbrace{\omega(t-r)}_{\text{phase} \equiv S}}$$

$\omega \gg 1$

$\delta r = \lambda = \frac{2\pi}{\omega}$, $\delta S = 2\pi$ (large variation)

$$\delta A \sim \frac{\delta r}{r^2} \rightarrow \frac{\delta A}{A} \sim r \frac{\delta r}{r} = \frac{\lambda}{r}$$

wave zone $r \gg \lambda$

$$\frac{\delta A}{A} \ll 1$$

$S = \omega(t - r)$ $\quad K_\alpha \equiv -\nabla_\alpha \left(\frac{S}{\omega}\right)$ = normal to phase fronts S = constant

$$= -\nabla_\alpha (t - r)$$

$$= (\underset{t}{-1}, \underset{r}{1}, \underset{\theta}{0}, \underset{\varphi}{0})$$

$$\left.\begin{array}{l} K_\alpha = (-1, 1, 0, 0) \\ K^\alpha = (1, 1, 0, 0) \end{array}\right\} K_\alpha K^\alpha = 0$$

$$k^\beta \nabla_\beta k_\alpha = -k^\beta \nabla_\beta \nabla_\alpha \left(\frac{S}{\omega}\right) = -k^\beta \nabla_{\beta\alpha} \underbrace{\left(\frac{S}{\omega}\right)}_{\text{scalar}}$$

$$= -k^\beta \nabla_{\alpha\beta}\left(\frac{S}{\omega}\right) = k^\beta \nabla_\alpha k_\beta = \frac{1}{2} \nabla_\alpha (\underbrace{k_\beta k^\beta}_{0}) = 0$$

$$\boxed{\frac{Dk^\alpha}{d\lambda} = k^\beta \nabla_\beta k^\alpha = 0}$$

Geodesics

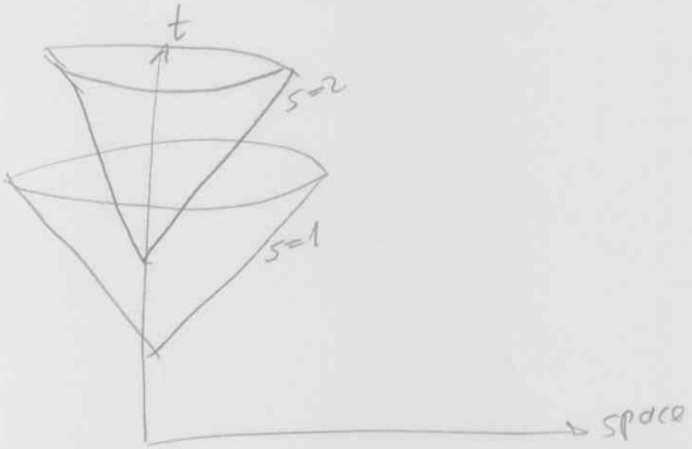
$$k^\alpha = \frac{dx^\alpha}{d\lambda} = (1,1,0,0)$$

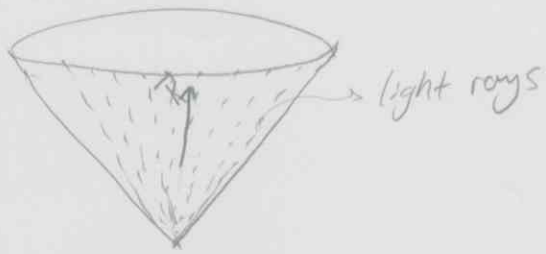
$$\frac{dt}{d\lambda} = 1 \quad , \frac{dr}{d\lambda} = 1, \frac{d\theta}{d\lambda} = 0 \, , \quad \frac{d\varphi}{d\lambda} = 0$$

$t = \lambda + \text{cnst} \quad , \theta = \text{const}$

$r = \lambda + \text{cnst} \quad , \varphi = \text{const}$

$$\frac{s}{\omega} = t - r = \text{const}$$





K_α = normal to phase front s

K^α = tangent to each null geodesics

Each null geodesic is tangent to surface

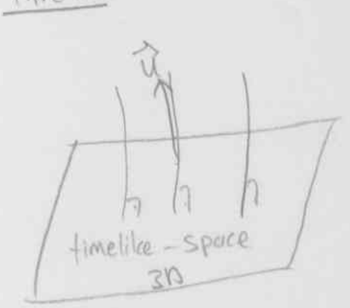
K^α is normal and tangent

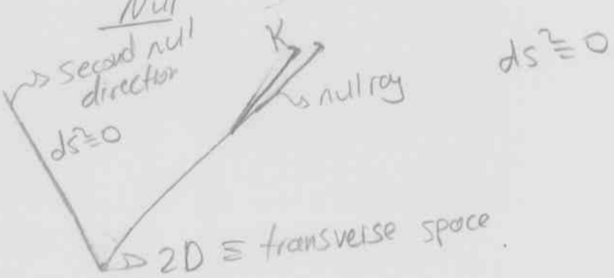
Calculate $\nabla_\alpha k_\beta$: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin^2\theta \end{pmatrix}$

$$\Theta = \nabla_\alpha k^\alpha = g^{\alpha\beta} \nabla_\alpha k_\beta = \frac{1}{r^2} r + \frac{1}{r^2 \sin^2\theta} r \sin^2\theta = \frac{2}{r} = \frac{1}{4\pi r^2} \frac{d}{d\lambda}(4\pi r^2)$$

Fractional change in cross-sectional area

Timelike





Transverse Projection

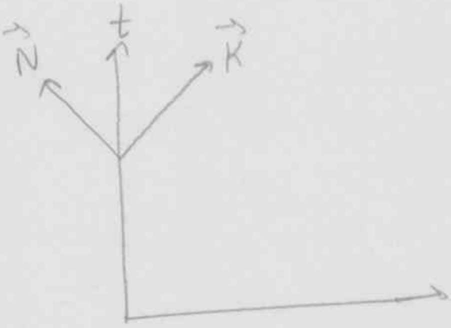
- Given null vector: k^α
- choose a second null direction: N^α

$$k_\alpha k^\alpha = 0$$
$$N_\alpha N^\alpha = 0$$
$$\xrightarrow{\text{impose}} \quad N_\alpha k^\alpha = -1$$

N is not unique

Example: $k^\alpha = (1,1,0,0)$

$k_\alpha = (-1,1,0,0)$



$$N_\alpha = \frac{1}{2}(-1,-1,0,0)$$
$$N^\alpha = \frac{1}{2}(1,-1,0,0)$$

$$N^\mu = \left(\frac{1}{2}(1+\alpha^2+\beta^2),\ -\frac{1}{2}(1-\alpha^2-\beta^2),\ \frac{\alpha}{r},\ \frac{\beta}{r\sin\theta} \right) \rightarrow \begin{array}{l} N^\mu N_\mu = 0 \\ N_\mu k^\mu = -1 \end{array} \quad \text{for any } \alpha, \beta$$

Decomposition of vector

$$\underset{4}{A^\alpha} = \underset{1}{A_1} k^\alpha + \underset{1}{A_2} N^\alpha + \underset{2}{A_\perp^\alpha} ; \qquad \begin{array}{l} A_\perp^\alpha k_\alpha = 0 \\ A_\perp^\alpha N_\alpha = 0 \end{array}$$

$$\left.\begin{array}{l} k_\alpha A^\alpha = -A_2 \\ N_\alpha A^\alpha = -A_1 \end{array}\right\} \begin{array}{l} A_1 = -N_\alpha A^\alpha \\ A_2 = -k_\alpha A^\alpha \end{array}$$

$$A_\perp^\alpha = A^\alpha + N_\beta A^\beta k^\alpha + k_\beta A^\beta N^\alpha$$
$$= \underbrace{(g^\alpha{}_\beta + k^\alpha N_\beta + N^\alpha k_\beta)}_{\equiv \Omega^\alpha{}_\beta \text{ : Transverse projector}} A^\beta$$

$$A_\perp^\alpha = \Omega^\alpha{}_\beta A^\beta$$

- $\Omega^\alpha{}_\gamma \Omega^\gamma{}_\beta = \Omega^\alpha{}_\beta$
- $\Omega^\alpha{}_\alpha = 2$

Decomposition of tensor

$$A^{\alpha\beta} = A_1 k^\alpha k^\beta + A_2 k^\alpha N^\beta + A_3 N^\alpha k^\beta + A_4 N^\alpha N^\beta$$
$$+ k^\alpha B^\beta_{1\perp} + N^\alpha B^\beta_{2\perp} + C^\alpha_{1\perp} k^\beta + C^\alpha_{2\perp} N^\beta + D^{\alpha\beta}_{\perp\perp}$$

2 2 2 2 4

$$A = \begin{array}{c|ccc} & k & N & T \\ k & A_1 & A_2 & B_1 \\ N & A_3 & A_4 & B_2 \\ T & C_1 & & \overleftrightarrow{D} \\ & C_2 & & \end{array}$$

$$k_\beta B^\beta_{1/2\perp} = N_\beta B^\beta_{1/2\perp} = 0$$

$$k_\beta C^\beta_{1/2} = N_\beta C^\beta_{1/2} = 0$$

$$k_\alpha D^{\alpha\beta}_{1/2\perp\perp} = k_\beta D^{\alpha\beta}_{1/2\perp\perp} = N_\alpha D^{\alpha\beta}_{1/2\perp\perp} = N_\beta D^{\alpha\beta}_{1/2\perp\perp} = 0$$

$$A^{\alpha\beta} = g^\alpha_\mu \, g^\beta_\nu \, A^{\mu\nu}$$
$$= (\Omega^\alpha_\mu - k^\alpha N_\mu - N^\alpha k_\mu)(\Omega^\beta_\nu - k^\beta N_\nu - N^\beta k_\nu) A^{\mu\nu}$$

$$A_1 = N_\mu N_\nu A^{\mu\nu} \qquad \checkmark\ B^\beta_{1\perp} = -N_\mu \Omega^\beta_\nu A^{\mu\nu}, \qquad B^\beta_{2\perp} = -k_\mu \Omega^\beta_\nu A^{\mu\nu}$$

$$A_2 = N_\mu k_\nu A^{\mu\nu} \qquad \checkmark\ C^\alpha_{1\perp} = -\Omega^\alpha_\mu N_\nu A^{\mu\nu}, \qquad C^\alpha_{2\perp} = -\Omega^\alpha_\mu k_\nu A^{\mu\nu}$$

$$A_3 = k_\mu N_\nu A^{\mu\nu} \qquad D^{\alpha\beta}_{\perp\perp} = \Omega^\alpha_\mu \Omega^\beta_\nu A^{\mu\nu}$$

$$A_4 = k_\mu k_\nu A^{\mu\nu}$$

Geometric optics

↳ Maxwells eqns (in vacuum)

$$\nabla_\beta F^{\alpha\beta} = 0$$
$$\nabla_{[\alpha} F_{\beta\gamma]} = 0 \ \checkmark$$

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$$
$$\bar{A}_\alpha = A_\alpha + \nabla_\alpha f \Rightarrow \bar{F}_{\alpha\beta} = F_{\alpha\beta}$$

Lorentz gauge: $\nabla_\alpha A^\alpha = 0$

doesnt fix gauge uniquely

$$\bar{A}_\alpha = A_\alpha + \nabla_\alpha f \Rightarrow \nabla_\alpha \bar{A}^\alpha = \nabla_\alpha A^\alpha + \nabla_\alpha \nabla^\alpha f$$
$$\nabla_\alpha \nabla^\alpha f = 0$$
$$\Box f = 0$$

$$0 = \nabla_\beta (\nabla_\alpha A^\beta - \nabla^\beta A_\alpha)$$
$$= \nabla_\beta \nabla_\alpha A^\beta - \Box A_\alpha$$
$$= (\nabla_{\beta\alpha} - \nabla_{\alpha\beta}) A^\beta + \nabla_\alpha \underbrace{(\nabla_\beta A^\beta)}_{0} - \Box A_\alpha$$
$$= R^\beta{}_{\mu\beta\alpha} A^\mu - \Box A_\alpha$$
$$= R_{\mu\alpha} A^\mu - \Box A_\alpha = 0$$

$$\Box A^\alpha - R^\alpha{}_\beta A^\beta = 0$$

phase varies rapidly

amplitude varies slowly

$$A^\alpha = \left(a^\alpha + O(\varepsilon)\right) e^{-i(S/\varepsilon)} \qquad \varepsilon \ll 1$$

$$k_\alpha = -\nabla_\alpha S$$

$$\nabla_\beta A^\alpha = -\frac{i}{\varepsilon} \nabla_\beta S \left(a^\alpha + \ldots\right) e^{-i(S/\varepsilon)} + O(\varepsilon^1)$$

$$= \frac{i}{\varepsilon} k_\beta a^\alpha e^{-iS/\varepsilon}$$

$$\nabla_\alpha A^\alpha = 0 \Rightarrow k_\alpha a^\alpha = 0$$

$$\cdot\ \Box A^\alpha = \nabla^\beta \nabla_\beta A^\alpha = \frac{i}{\varepsilon} \nabla^\beta \left[\left(k_\beta a^\alpha\right) e^{-iS/\varepsilon}\right] = \frac{i}{\varepsilon}\left(-\frac{i}{\varepsilon}\right) \nabla^\beta S\, k_\beta a^\alpha e^{-iS/\varepsilon} + O(1/\varepsilon)$$

$$= -\frac{1}{\varepsilon^2} k^\beta k_\beta a_\alpha e^{-iS/\varepsilon} + O(1/\varepsilon)$$

$$\Box A^\alpha - R^\alpha{}_\beta A^\beta = 0 \Rightarrow \boxed{k^\beta k_\beta = 0} \rightarrow$$ Maxwell eqn's lead k to be a null vector

$$R_{\alpha\beta} A^\beta = O(1)$$

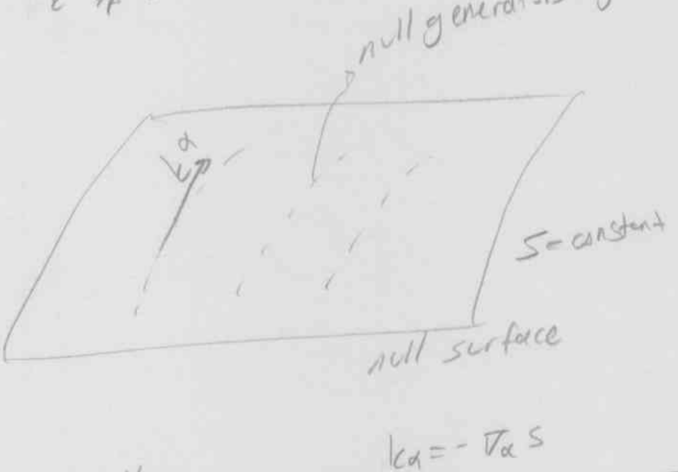
surfaces of constant S $\rightarrow$ light cone (surfaces)

Maxwell: $k_\alpha k^\alpha = 0$

k^α tangent to null geodesics (light rays)

$$k_\alpha = -\nabla_\alpha S$$

$$k^\beta \nabla_\beta k^\alpha = 0$$



$$k_\alpha = -\nabla_\alpha S$$

$$k_\alpha = -\nabla_\alpha S$$

Decompse $\nabla_\alpha K_\beta$

$$\nabla_\beta k_\alpha = -\nabla_{\beta\alpha} S = -\nabla_{\alpha\beta} S = \nabla_\alpha k_\beta$$

$$\nabla_\alpha k_\beta = B k_\alpha k_\beta + k_\alpha B_\beta^\perp + B_\alpha^\perp k_\beta + \underbrace{B^{\perp\perp}_{\alpha\beta}}$$ (transvarse part $\rightarrow$ useful part

$$B = N^\mu N^\nu \nabla_\mu k_\nu$$

$$B^\perp_\beta = -N^\mu \Omega^\nu{}_\beta \nabla_\mu k_\nu$$

$$B^{\perp\perp}_{\alpha\beta} = \Omega^\mu{}_\alpha \Omega^\nu{}_\beta \nabla_\mu k_\nu$$

$$A_2 = N^\mu k^\nu \nabla_\mu k_\nu = \frac{1}{2} N^\mu \nabla_\mu (k_\nu k^\nu) = 0$$

$$A_3 = k^\mu N^\nu \underline{\nabla_\mu k_\nu} = 0 \quad , A_4' = k^\mu k^\nu \nabla_\mu k_\nu = 0 \ , \ B_{2\beta}^\perp = -\Omega_\beta{}^\nu \underline{k^\mu \nabla_\mu k_\nu} = 0$$

$$C_{2\alpha}^\perp = -\Omega_\alpha{}^\mu \underline{k^\nu \nabla_\mu k_\nu} = 0$$

$$\underbrace{B^{\perp\perp}_{\alpha\beta}}_{3} = \frac{1}{2}\Omega_{\alpha\beta}\,\underset{1}{\Theta} + \underset{2}{\sigma_{\alpha\beta}} \quad ; \quad \sigma_{\alpha\beta}\Omega^{\alpha\beta} = 0$$

due to symmetry

$(+\,\omega_{\alpha\beta})$ → if it is not hypersurface orthogonal

$$\Theta = \Omega^{\alpha\beta} B^{\perp\perp}_{\alpha\beta} = \nabla_\alpha k^\alpha$$
$$\sigma_{\alpha\beta} = B^{\perp\perp}_{\alpha\beta} - \frac{1}{2}\Omega_{\alpha\beta}\Theta$$

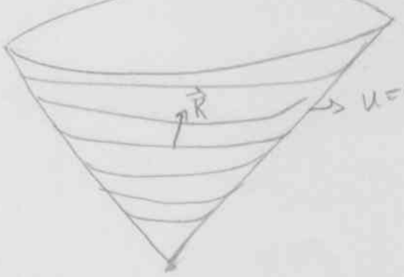
$$\nabla_\alpha k^\alpha = g^{\alpha\beta}\nabla_\alpha k_\beta = (-k^\alpha N^\beta - N^\alpha k^\beta + \Omega^{\alpha\beta})\nabla_\alpha k_\beta$$
$$= \Omega^{\alpha\beta}\nabla_\alpha k_\beta = \Omega^{\alpha\beta}(B\, k_\alpha k_\beta + k_\alpha B^\perp_\beta + k_\beta B^\perp_\alpha + B^{\perp\perp}_{\alpha\beta})$$
$$= \Omega^{\alpha\beta} B^{\perp\perp}_{\alpha\beta} = \Theta$$

Example: null cones in Schwarzschild

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad f = 1 - \frac{2M}{r}$$
$$= -f\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dv} + r^2 d\Omega^2$$

$$\left.\begin{array}{l} u = t - r^* \\ v = t + r^* \end{array}\right\} \quad r^* = \int\frac{dr}{f} = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

$$K_\alpha = -\nabla_\alpha u = -\nabla_\alpha(t - r^*) = (-1, \tfrac{1}{f}, 0, 0)$$
$$K^\alpha = (\tfrac{1}{f}, 1, 0, 0) \rightarrow \frac{dt}{d\lambda} = \frac{1}{f}$$
$$\frac{dr}{d\lambda} = 1$$
$$\frac{d\theta}{d\lambda} = \frac{d\varphi}{d\lambda} = 0$$



→ u = constant ≡ outgoing light cones

$$\frac{dt}{dr} = \frac{1}{f} \Rightarrow dt = \frac{dr}{f}$$
$$t = r^* + \text{const}$$

→ v = constant ≡ ingoing light cones

→ we do not care for now

$$\nabla_\alpha k_\beta = \begin{pmatrix} -M/r^2 , & \frac{M}{r^2 f} , & 0 , & 0 \\ \frac{M}{r^2 f} , & -\frac{M}{r^2 f^2} , & 0 , & 0 \\ 0 , & 0 , & r , & 0 \\ 0 & 0 & 0 , & r\sin^2\theta \end{pmatrix}$$

$$N_\alpha \propto -\nabla_\alpha v = (-1, -\tfrac{1}{f}, 0, 0)$$

$$N^\alpha N_\alpha = 0$$
$$N_\alpha k^\alpha = -1$$

$$N_\alpha = \frac{1}{2}f\left(-1, -\tfrac{1}{f}, 0, 0\right)$$

$B = -\frac{M}{r^2}$

$B^{\perp}_{\alpha} = 0$

$B^{\perp\perp}_{\alpha\beta} = \frac{1}{2}\Theta\,\Omega_{\alpha\beta}\,,\ \sigma_{\alpha\beta} = 0$

$\Theta = \nabla_\alpha k^\alpha = \frac{2}{r} = \frac{1}{4\pi r^2}\frac{d}{d\lambda}\left(4\pi r^2\right)\,,\quad \Omega_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

<u>Example</u> – anisotropic cosmology

$ds^2 = -dt^2 + a^2(t)\,dx^2 + b^2(t)\,dy^2 + c^2(t)\,dz^2$

Kasner-space time – vacuum solution

$\left.\begin{array}{l} a(t) = t^\alpha \\ b(t) = t^\beta \\ c(t) = t^\gamma \end{array}\right\}\quad \begin{array}{l}\alpha+\beta+\gamma = 1 \\ \alpha^2+\beta^2+\gamma^2 = 1\end{array}$

Lets take null directions along z-direction

$= -dt^2 + c^2(t)\,dz^2 + \dots$

$= -c^2\underbrace{\left(\frac{1}{c}dt - dz\right)}_{du}\underbrace{\left(\frac{1}{c}dt + dz\right)}_{dv} + \dots$

$u = \int\frac{dt}{c} - z\,,\quad v = \int\frac{dt}{c} + z$

$\Rightarrow$ right moving light surfaces

$k_\alpha = -\nabla_\alpha u = \left(-\frac{1}{c}, 0, 0, 1\right)$

$k^\alpha = \left(\frac{1}{c}, 0, 0, \frac{1}{c^2}\right)$

$\frac{dt}{d\lambda} = \frac{1}{c}\,,\quad \frac{dz}{d\lambda} = \frac{1}{c^2}$

$\frac{dt}{dz} = c \qquad dz = \frac{dt}{c}$

$z = \int\frac{dt}{c} + \text{constant}$

$N_\alpha\ \alpha - \nabla_\alpha v = \left(-\frac{1}{c}, 0, 0, -1\right)$

$N_\alpha = \frac{c^2}{2}\left(-\frac{1}{c}, 0, 0, -1\right)$

$\nabla_\alpha k_\beta = B k_\alpha k_\beta + k_\alpha B^{\perp}_{\beta} + B^{\perp}_{\alpha} k_\beta + \frac{1}{2}\Theta\,\Omega_{\alpha\beta} + \sigma_{\alpha\beta}$

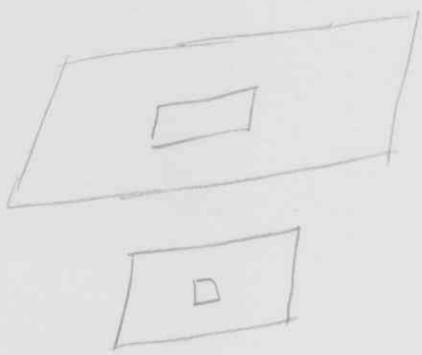
$B = \dot{c}$

$B^{\alpha}_{\perp} = 0$

$\Theta = \nabla_\alpha k^\alpha = \frac{1}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right)$

$\sigma_{xx} = \frac{a^2}{2c}\left(\frac{\dot a}{a} - \frac{\dot b}{b}\right)$

$\sigma_{yy} = \frac{b^2}{2c}\left(-\frac{\dot a}{a} + \frac{\dot b}{b}\right)$



Raychaudhuri's equation - evolution equation for Θ

$$k^\mu \nabla_\mu (\nabla_\alpha k_\beta) = -(\nabla_\alpha k_\mu)(\nabla^\mu k_\beta) - R_{\mu\alpha\nu\beta} k^\mu k^\nu$$

(same as timelike case) $u^\alpha \to k^\alpha$ page 9

$$k^\mu \nabla_\mu \Theta \equiv \frac{D\Theta}{d\lambda} = -(\nabla_\alpha k_\beta)(\nabla^\beta k^\alpha) - R_{\mu\nu} k^\mu k^\nu$$

$\nabla_\alpha k_\beta = \nabla_\beta k_\alpha$

$$\frac{D\Theta}{d\lambda} = -(B k_\alpha k_\beta + k_\alpha B^\perp_\beta + B_\alpha^\perp k_\beta + B^{\perp\perp}_{\alpha\beta})(B k^\alpha k^\beta + k^\alpha B^\beta_\perp + B^\alpha_\perp k^\beta + B^{\alpha\beta}_{\perp\perp}) - R_{\alpha\beta} k^\alpha k^\beta$$

$$= -B^{\perp\perp}_{\alpha\beta} B^{\alpha\beta}_{\perp\perp} - R_{\alpha\beta} k^\alpha k^\beta$$

$$= -\left(\frac{1}{2}\Theta \Omega_{\alpha\beta} + \sigma_{\alpha\beta}\right)\left(\frac{1}{2}\Theta \Omega^{\alpha\beta} + \sigma^{\alpha\beta}\right) - R_{\alpha\beta} k^\alpha k^\beta$$

$$= -\frac{1}{2}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{\alpha\beta} k^\alpha k^\beta$$

$$\frac{D\Theta}{d\lambda} = -\Big(\frac{1}{2}\underbrace{\Theta^2}_{\substack{\text{spatial}\\ \geq 0}} + \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\substack{\text{spatial}\\ \geq 0}} + R_{\alpha\beta} k^\alpha k^\beta\Big)$$

R's eqn

Focusing theorem

$\frac{D\Theta}{d\lambda} \leq 0 \Rightarrow \Theta$ will decrease in future $\to$ gravity is attractive

Ricci condition

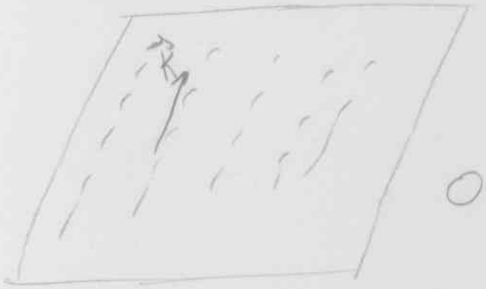
$$R_{\alpha\beta} k^\alpha k^\beta \geq 0$$

$$8\pi\left(T_{\alpha\beta} - \cancel{\frac{1}{2} T g_{\alpha\beta}}\right) k^\alpha k^\beta \geq 0$$

EFE

$T_{\alpha\beta} k^\alpha k^\beta \geq 0 \to$ An observer moving with a speed of light measuring the energy density positive

Stronger than strong energy condition



Stationary Black hole $\rightarrow \Theta = 0$

$$\frac{D\Theta}{d\lambda} = 0$$

Stationary BH Horizon //

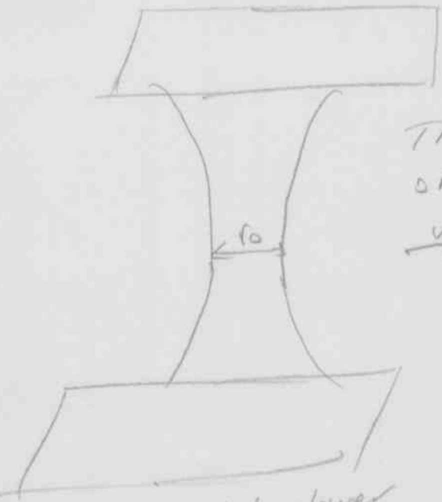
$$0 = -\left(\sigma_{\alpha\beta}\sigma^{\alpha\beta} + R_{\alpha\beta}k^{\alpha}k^{\beta}\right)$$

$$\sigma_{\alpha\beta} = 0 \ , \ R_{\alpha\beta}k^{\alpha}k^{\beta} = 0$$

1-) $ds^2 = -dt^2 + dl^2 + r^2(l)\, d\Omega^2$

$r(0) = r_0$

$r \to \pm\infty$, $|l|$



There is no singularities or such to not to have wormhols

Θ changes sign

$$\frac{D\Theta}{d\tau} = -\Big(\underbrace{\frac{1}{3}\Theta^2}_{+} + \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{+} + R_{\alpha\beta}u^\alpha u^\beta\Big)$$

$\frac{D\Theta}{d\tau} < 0$; $\Theta > 0$, still expands but slower
$\Theta < 0$, converges $\Theta \to -\infty$

$\frac{D\Theta}{d\tau} > 0$; $\Theta > 0$, $\Theta \to \infty$
$\boxed{\Theta < 0}$, distence would decreases at first then increases ($\Theta \to \infty$)
→ That is what we want

so $\frac{D\Theta}{d\tau}$ must be positive, to hold that
$R_{\alpha\beta} u^\alpha u^\beta < 0$: strong energy condition fails

or if we consider null geodesics

$$\frac{D\Theta}{d\lambda} = -\Big(\underbrace{\frac{1}{2}\Theta^2}_{>0} + \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{>0} + \underbrace{R_{\alpha\beta}k^\alpha k^\beta}_{<0}\Big)$$

$R_{\alpha\beta}k^\alpha k^\beta = T_{\alpha\beta}k^\alpha k^\beta < 0$
→ null energy condition fails

2-) $u^\alpha = (1,0,0,0)$ Ch2

$t \to \tau$ for geodesics

a-) $$\frac{Du^\alpha}{d\tau} = u^\beta \nabla_\beta u^\alpha = u^\beta(\cancel{\partial_\beta u^\alpha} + \Gamma^\alpha_{\beta\lambda} u^\lambda)$$
$$= u^\beta(\Gamma^\alpha_{\beta 0} u^0) = u^0 \Gamma^\alpha_{00} u^0$$

$$\Gamma^\alpha_{\beta 0} = \frac{1}{2} g^{\alpha\lambda}(\partial_\beta g_{\lambda 0} + \partial_0 g_{\beta\lambda} - \partial_\lambda g_{\beta 0})$$

$$\Gamma^\alpha_{00} = \frac{1}{2} g^{\alpha\lambda}(\partial_0 2 g_{0\lambda} - \partial_\lambda g_{00})$$

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = 0$$

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(0 - \partial_1 g_{00}) = 0$$

$$\Gamma^2_{00} = 0$$

$$\Gamma^3_{00} = 0$$

$$\frac{Du^\alpha}{d\tau} = (0,0,0,0) = 0 = \underline{u^\beta \nabla_\beta u^\alpha = 0}$$

b-) $$\Theta = \nabla_\alpha u^\alpha = \underbrace{\partial_\alpha u^\alpha}_{0} + \Gamma^\alpha_{\alpha\lambda} u^\lambda = \Gamma^0_{0\lambda} u^\lambda + \Gamma^i_{i\lambda} u^\lambda$$
$$= \Gamma^0_{00} u^0 + \Gamma^i_{i0} u^0 = (\underbrace{\Gamma^0_{00}}_{0} + \Gamma^i_{i0}) = \Gamma^i_{i0} = 3\frac{\dot a}{a}$$

$$\Gamma^i_{i0} = \frac{1}{2} g^{i\mu}(\partial_i g_{\mu 0} + \partial_0 g_{i\mu} - \cancel{\partial_\mu g_{0i}})$$

$$\Gamma^1_{10} = \frac{1}{2} g^{11}(\cancel{\partial_1 g_{10}} + \partial_0 g_{11}) = \cancel{\frac{1}{2}}\left(\frac{\cancel{1-kr^2}}{\cancel{a^2}}\right) \cdot \frac{\cancel{2} a\dot a}{\cancel{1-kr^2}} = \frac{\dot a}{a}$$

$$\Gamma^2_{20} = \frac{1}{2} g^{22}(\partial_2 g_{20} + \partial_0 g_{22}) = \frac{\dot a}{a}$$

$$\Gamma^3_{30} = \frac{\dot a}{a}$$

$$\Theta = 3\frac{\dot a}{a} = \frac{1}{a^3}\frac{d}{d\lambda}(a^3) \iff \Theta = \frac{1}{V}\frac{d}{d\lambda} V$$

$$\nabla_\alpha u_\beta = \cancel{\partial_\alpha u_\beta} - \Gamma^\lambda_{\alpha\beta} u_\lambda = -\Gamma^0_{\alpha\beta} u_0 = \Gamma^0_{\alpha\beta}$$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{00}(\underline{\partial_\alpha g_{0\beta}} + \underline{\partial_\beta g_{0\alpha}} - \partial_0 g_{\alpha\beta})$$

$\alpha = i, \beta = 0$ $\quad \Gamma = 0$

$\alpha = 0, \beta = i$ $\quad \Gamma = 0$

$\alpha = i, \beta = j$ $\quad \Gamma^0_{ij} = \frac{1}{2}(-1)\cdot(-\partial_0 g_{ij}) = \frac{1}{2}\partial_0 a^2 \delta_{ij} = \underline{\underline{3a\dot a}}$

There is only $\underset{\alpha=\beta}{\Gamma^0_{11}, \Gamma^0_{22}, \Gamma^0_{33}}$ elements of $\nabla_\alpha u_\beta$ which means just Θ

2c-)
$$\frac{d\Theta}{d\tau} = u^\beta \nabla_\beta (\nabla_\alpha u^\alpha) = -\frac{1}{3}\Theta^2 - \underbrace{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}_{\substack{=0 \\ \text{for FRW}}} + \underbrace{\omega_{\alpha\beta}\omega^{\alpha\beta}}_{\substack{=0 \text{ for} \\ \text{hypersurface} \\ \text{orthogonality}}} - R_{\alpha\beta}u^\alpha u^\beta + \underbrace{\nabla_\alpha a^\alpha}_{=0 \text{ for geodesics}}$$

also $=0$ for this case

$$\frac{d\Theta}{d\tau} = -\frac{1}{3}\left(3\frac{\dot{a}}{a}\right)^2 - R_{\alpha\beta}u^\alpha u^\beta$$
$$= -3\frac{\dot{a}^2}{a^2} - 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)u^\alpha u^\beta$$
$$= -3\frac{\dot{a}^2}{a^2} - 8\pi\left(T_{\alpha\beta}u^\alpha u^\beta + \frac{1}{2}T\right)$$

$T_{\alpha\beta} = \mu u_\alpha u_\beta + p(g_{\alpha\beta} + u_\alpha u_\beta)$

$T = -\mu + 3p$

$$= -3\frac{\dot{a}^2}{a^2} - 8\pi\left(\mu + 0 - \frac{\mu}{2} + \frac{3p}{2}\right)$$
$$= \underbrace{-\cancel{3\frac{\dot{a}^2}{a^2}} - 4\pi(\mu + 3p)}_{RHS}$$

$$LHS = \frac{d\Theta}{d\tau} = \frac{d}{d\tau}\left(3\frac{\dot{a}}{a}\right) = 3\left(\frac{\ddot{a}}{a} - \cancel{\frac{\dot{a}^2}{a^2}}\right)$$

$$\boxed{3\frac{\ddot{a}}{a} = -4\pi(\mu + 3p)}$$

3-)
$$u^\alpha = \frac{1}{\sqrt{1-\frac{3M}{r}}}\left(1, 0, \sqrt{M/r^3}, 0\right)$$

a-)
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta$$

$$u_\alpha = \frac{1}{\sqrt{1-\frac{3M}{r}}}\left(\frac{2M}{r} - 1, 0, \sqrt{Mr}, 0\right)$$

$$u_\alpha u^\alpha = \frac{1}{1-\frac{3M}{r}}\left(\frac{2M}{r} - 1 + \frac{M}{r}\right) = \left(\cancel{\frac{3M}{r}} - 1\right)\frac{1}{\cancel{1-\frac{3M}{r}}} = \boxed{-1 = u^\alpha u_\alpha} \rightarrow \text{timelike}$$

$$u^\beta \nabla_\beta u^\alpha = u^\beta\left(\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\lambda}u^\lambda\right) + u^2\left(\cancel{\partial_2 u^\alpha} + \Gamma^\alpha_{20}u^0 + \Gamma^\alpha_{22}u^2\right)$$
$$= u^0\left(\cancel{\partial_0 u^\alpha} + \Gamma_0{}^\alpha{}_0 u^0 + \Gamma^\alpha_{02}u^2\right)$$

$$\Gamma^\alpha_{00} = \frac{1}{2}g^{\alpha\lambda}\left(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right)$$
$$\Gamma^0_{00} = \frac{1}{2}g^{00}\left(\partial_0 g_{00}\right) = 0$$
$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left(-\partial_1 g_{00}\right) = +\left(1-\frac{2M}{r}\right)\left(\frac{M}{r^2}\right)$$
$$\Gamma^2_{00} = 0$$
$$\Gamma^3_{00} = 0$$

$$\Gamma^{\alpha}_{02} = \frac{1}{2} g^{\alpha\lambda} (\partial_0 g_{2\lambda} + \partial_2 g_{0\lambda} - \cancel{\partial_\lambda g_{02}})$$

$$\Gamma^{0}_{02} = 0$$

$$\Gamma^{1}_{02} = 0$$

$$\Gamma^{2}_{02} = \frac{1}{2} g^{22} (\partial_0 g_{22}) = 0$$

$$\Gamma^{\alpha}_{22} = \frac{1}{2} g^{\alpha\lambda} (2\partial_2 g_{\lambda 2} - \partial_\lambda g_{22})$$

$$\Gamma^{0}_{22} = 0$$

$$\Gamma^{1}_{22} = \frac{1}{2} g^{11} (-\partial_1 g_{22}) = \frac{1}{\cancel{2}} \left(\frac{2M}{r} - 1\right) \cancel{2} r = 2M - r$$

$$u^\beta \nabla_\beta u^\alpha = u^0 \Gamma^{\alpha}_{00} u^0 + u^0 \Gamma^{\alpha}_{02} u^2 + u^2 \Gamma^{\alpha}_{20} u^0 + u^2 \Gamma^{\alpha}_{22} u^2$$

$$\alpha = 0 \quad = 0 \quad + 0 \quad + 0 = 0$$

$$\alpha = 1 \quad = \left(\frac{1}{1 - \frac{3M}{r}}\right) \left(1 - \frac{2M}{r}\right) \frac{M}{r^2} + \frac{M}{r^3} \cdot \frac{1}{\left(1 - \frac{3M}{r}\right)} (2M - r)$$

$$= \left(\frac{1}{1 - \frac{3M}{r}}\right) \left(\frac{r - 2M}{r^3} \cdot M - \frac{r - 2M}{r^3} \cdot M\right) = 0$$

$$\alpha = 2 \quad = 0$$

$$\alpha = 3, \quad u^3 = 0$$

$$\underline{\underline{u^\beta \nabla_\beta u^\alpha = 0}}$$

b-) $\Theta = \nabla_\alpha u^\alpha = \underbrace{\partial_\alpha u^\alpha}_{0} + \Gamma^{\alpha}_{\alpha\beta} u^\beta = \Gamma^{\alpha}_{\alpha 0} u^0 + \Gamma^{\alpha}_{\alpha 2} u^2$

$$\Gamma^{\alpha}_{\alpha 0} = \frac{1}{2} g^{\alpha\alpha} (\underline{\partial_0} g_{0\alpha} + \cancel{\partial_\alpha g_{0\alpha}} - \cancel{\partial_\alpha g_{0\alpha}}) = 0$$

$$\Gamma^{\alpha}_{\alpha 2} = \frac{1}{2} g^{\alpha\alpha} (\partial_2 g_{\alpha\alpha} + \cancel{\partial_\alpha g_{2\alpha}} - \cancel{\partial_\alpha g_{2\alpha}}) = \frac{1}{2} g^{\alpha\alpha} \partial_2 g_{\alpha\alpha}$$

$$\Gamma^{3}_{32} = \frac{1}{\cancel{2}} \frac{1}{\cancel{r^2 \sin^2\theta}} \cdot \cancel{r^2} \cdot \cancel{2} \cdot \sin\theta \cos\theta = \cot\theta$$

$$\Theta = \cot\theta \, u^\theta = \frac{\cos\theta}{\sin\theta} u^\theta \qquad \begin{matrix} \theta \to 0^+ & \Theta \to +\infty \\ \theta \to \pi^- & \Theta \to -\infty \end{matrix}$$

c-) <u>gok uzun</u>

d-) <u>yerine koyucan</u>

Q3 4