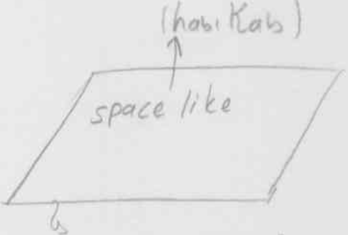
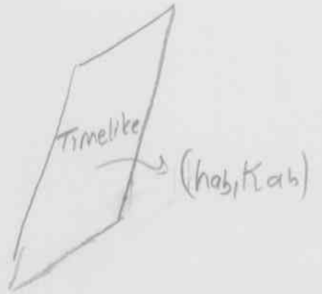
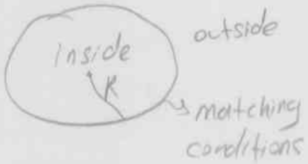
Hypersurfaces (3-D submanifolds)



"moment of time"

Initial value problem of GR





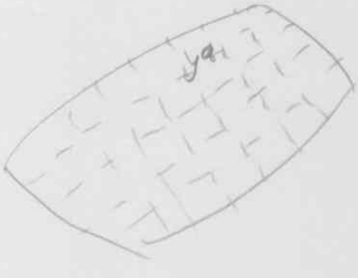
Description

2D sphere in 3D space

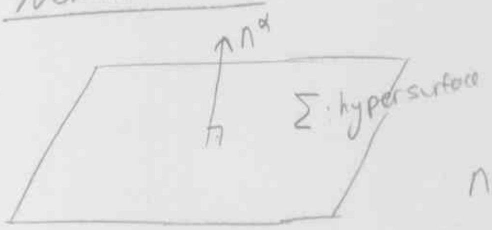
- constraint on coordinates: $x^2+y^2+z^2=R^2$
- embedding relations: $x = R\sin\theta\cos\varphi$
  $y = R\sin\theta\sin\varphi$
  $z = R\cos\theta$

Constraint: $\Phi(x^\alpha) = \text{const}$

embedding relations: $\underset{\text{spacetime coordinates}}{x^\alpha} = x^\alpha(y^a) \to$ intrinsic coordinates on hypersurface



Normal vector



Normal vector - unit vector: $n_\alpha n^\alpha = \varepsilon = \begin{cases} -1 & \Sigma \text{ spacelike} \\ 1 & \Sigma \text{ timelike} \end{cases}$

- point in direction of increasing Φ (convention)

$$n_\alpha \propto \nabla_\alpha \Phi$$

$$n_\alpha = \varepsilon \mu \nabla_\alpha \Phi$$

$$\mu = |g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi|^{-1/2}$$

Ex $t = \text{const}$

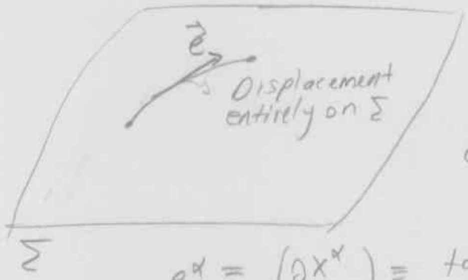
$\Phi = t = \text{const}$

$\nabla_\alpha \Phi = (1,0,0,0)$

$n_\alpha = (-1,0,0,0)$

$n^\alpha = (1,0,0,0)$

Tangent vectors:



Intrinsic displacement

$x^\alpha = x^\alpha(y^a) \rightarrow$ curves on Σ

$$dx^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a}\right) dy^a = e^\alpha_a\, dy^a$$

$$e^\alpha_a \equiv \underbrace{\left(\frac{\partial x^\alpha}{\partial y^a}\right)}_{} \equiv \text{ tangent vectors (basis on } \Sigma)$$

→ space-time vector

→ hypersurface co vector

Sphere

$\Phi = x^2+y^2+z^2$

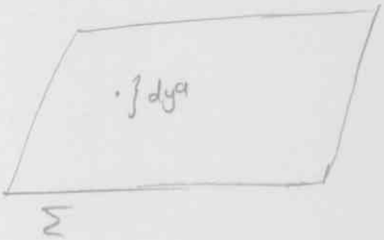
$\nabla_\alpha \Phi = (2x, 2y, 2z)$

$n_\alpha = \left(\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$e^\alpha_\theta = (R\cos\theta\cos\varphi, R\cos\theta\sin\phi, -R\sin\theta)$

$e^\alpha_\varphi = (-R\sin\theta\sin\phi, R\sin\theta\cos\phi, 0)$

Induced metric



Infinitesmall displacement on hypersurface

$$ds^2|_\Sigma = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b} dy^b\right)$$

$$= \underbrace{g_{\alpha\beta}\, e^\alpha_a e^\beta_b}_{h_{ab} \equiv \text{ induced metric}}\, dy^a dy^b = h_{ab}\, dy^a dy^b$$

$$h_{ab} = g_{\alpha\beta}\, e^\alpha_a e^\beta_b$$

$$h_{\theta\theta} = g_{\alpha\beta}\, e^\alpha_\theta e^\beta_\theta = g_{11}\, e^1_\theta e^1_\theta + g_{22}\, e^2_\theta e^2_\theta + g_{33}\, e^3_\theta e^3_\theta = R^2$$ → Due to $g_{\alpha\beta} = \eta_{\alpha\beta}$

$h_{\varphi\varphi} = R^2 \sin^2\theta$

$h_{\theta\varphi} = 0$

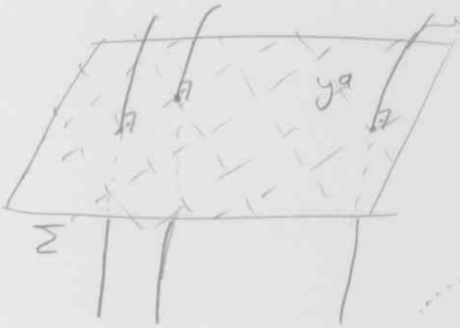
$$ds^2|_{\text{sphere}} = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Intrinsic geometry : $h_{ab} \rightarrow \Gamma^{c}_{ab} \rightarrow R^{c}{}_{abd}$

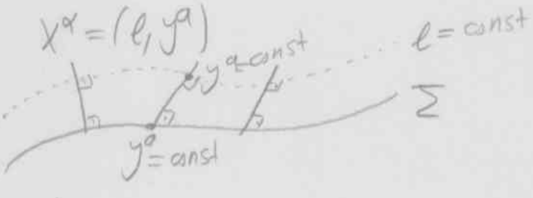
- h^{ab} : inverse metric
- $h = \det(h_{ab})$
- $h^{ab} h_{bc} = \delta^{a}{}_{c}$

Completeness relation: $g^{\alpha\beta} = \varepsilon n^{\alpha} n^{\beta} + \underbrace{h^{ab} e^{\alpha}{}_{a} e^{\beta}{}_{b}}_{\text{"}p^{\alpha\beta}\text{"}}$

Gaussian normal coordinates



$\ell \equiv$ proper time (or distance) measured on each orthogonal geodesic with $\ell = 0$ on Σ



$$ds^2 = \varepsilon d\ell^2 + g_{ab}(\ell, y) dy^a dy^b \qquad X^{\alpha} \overset{*}{=} (\ell, y^a)$$

$$h_{ab} = g_{ab}(\ell = 0, y)$$

Given curves are geodesics, due to focussing theorem curves will cross at some point

So these coordinates are legit around the neighbourhood of the hypersurface

$$\det(g_{\alpha\beta}) \overset{*}{=} \varepsilon \det(\underset{\ell=0}{\underrightarrow{g_{ab}}}) = \varepsilon h$$

$$n^{\alpha} \overset{*}{=} (1, 0, 0, 0) \qquad e^{\alpha}_{1} \overset{*}{=} (0, 1, 0, 0)$$

$$n_{\alpha} \overset{*}{=} (\varepsilon, 0, 0, 0) \qquad e^{\alpha}_{2} \overset{*}{=} (0, 0, 1, 0)$$

$$e^{\alpha}_{3} \overset{*}{=} (0, 0, 0, 1)$$

$$dV = \sqrt{-g}\, d^4x \overset{*}{=} \sqrt{-\varepsilon h}\, d^4x$$

Spacetime volume element in Gaussian coordinates at Σ

$$dV \overset{*}{=} \sqrt{-\varepsilon h}\, d\ell\, d^3y$$

$$-\varepsilon h : \begin{cases} h, & \text{space like} \\ -h, & \text{timelike} \end{cases}$$

4D Integration

$$\int_V f\, dV$$

$$dV = \sqrt{-g}\, d^4x$$

Levi-Civita tensor (volume form)

$$\underbrace{\varepsilon_{\alpha\beta\gamma\delta}}_{\text{tensor}} \equiv \sqrt{-g}\, \underbrace{[\alpha\beta\gamma\delta]}_{\text{permutation symbol}} = \begin{cases} +1 & \text{even permutation of } 0123 \\ -1 & \text{odd} \quad '' \quad '' \quad '' \\ 0 & \text{if any two indices agree} \end{cases}$$

$[0123] = 1$

$[0132] = -1$

$[0021] = 0$

Matrix $M^\alpha{}_\beta$

$$\det(M^\alpha{}_\beta) = [\alpha\beta\gamma\delta]\, M^\alpha{}_0 M^\beta{}_1 M^\gamma{}_2 M^\delta{}_3 = \begin{vmatrix} M^0{}_0 & M^0{}_1 & M^0{}_2 & M^0{}_3 \\ M^1{}_0 & M^1{}_1 & \dots & M^1{}_3 \\ M^2{}_0 & \dots & & M^2{}_3 \\ M^2{}_0 & \dots & & M^3{}_3 \end{vmatrix}$$

Coordinate transformation: $x^\alpha \to z^\mu$

$$g_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial z^\mu} \frac{\partial x^\beta}{\partial z^\nu} \longrightarrow \det(g_{\mu\nu}) = \det(g_{\alpha\beta})\, \det\left(\left(\frac{\partial x^\alpha}{\partial z^\mu}\right)\right)^2$$

$$\sqrt{-\det(g_{\mu\nu})} = \sqrt{-\det(g_{\alpha\beta})}\, \det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right)$$

$$dV(z) = \sqrt{-\det(g_{\mu\nu})}\, d^4z$$

$$= \sqrt{-\det(g_{\alpha\beta})}\, \underbrace{\det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right) d^4z}_{d^4x} = dV(x)$$

Integrating using z-coordinates

$$dV[z] = \sqrt{-\det(g_{\mu\nu})}\, d^4z$$

Consider $\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial x^\alpha}{\partial z^0} \frac{\partial x^\beta}{\partial z^1} \frac{\partial x^\gamma}{\partial z^2} \frac{\partial x^\delta}{\partial z^3}\, dz^0 dz^1 dz^2 dz^3$

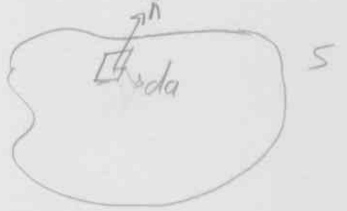
$$= \sqrt{-\det(g_{\alpha\beta})}\, \underbrace{[\alpha\beta\gamma\delta] \frac{\partial x^\alpha}{\partial z^0} \frac{\partial x^\beta}{\partial z^1} \frac{\partial x^\gamma}{\partial z^2} \frac{\partial x^\delta}{\partial z^3}}_{\det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right) = \text{Jacobian}}\, dz^0 dz^1 dz^2 dz^3$$

$$= \underbrace{\sqrt{-\det(g_{\alpha\beta})}\, \det\left(\frac{\partial x^\alpha}{\partial z^\mu}\right)}_{\sqrt{-\det(g_{\mu\nu})}} dz^0 dz^1 dz^2 dz^3 = dV(z)$$

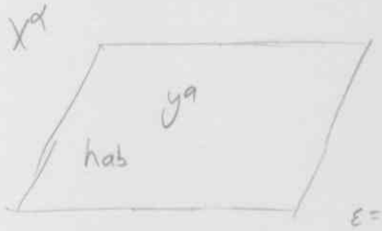
$z^\mu \equiv X^\alpha$: $\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial X^\alpha}{\partial z^0} \frac{\partial X^\beta}{\partial z^1} \frac{\partial X^\gamma}{\partial z^2} \frac{\partial X^\delta}{\partial z^3} d^4 z$

$$= \varepsilon_{0123}\, d^4 x = \sqrt{-g}\, d^4 x = dV[x]$$

3D- hypersurface



$d\vec{S} = da\, \hat{n}$



consider:

$\varepsilon_{\alpha\beta\gamma\delta} \frac{\partial X^\beta}{\partial y^1} \frac{\partial X^\gamma}{\partial y^2} \frac{\partial X^\delta}{\partial y^3} d^3 y = d\Sigma_\alpha$ ⤷ vector - valued element on Σ

$\varepsilon = \pm 1$

$d\Sigma = \sqrt{-\varepsilon \det(h_{ab})}\, dy^1 dy^2 dy^3$

⤷ undirected surface element

$n_\alpha d\Sigma$: directed surface element

$$\boxed{d\Sigma_\alpha = \varepsilon n_\alpha d\Sigma}$$

⤷

Gaussian coordinates

$X^\alpha \overset{*}{=} (l, y^a)$

$n^\alpha \overset{*}{=} (1,0,0,0)$

$n_\alpha \overset{*}{=} (\varepsilon,0,0,0)$

$e^\alpha_1 \overset{*}{=} (0,1,0,0), \dots$

$\sqrt{-g} \overset{*}{=} \sqrt{-\varepsilon h}$

$$d\Sigma_\alpha = \varepsilon_{\alpha\beta\gamma\delta}\, e^\beta_1 e^\gamma_2 e^\delta_3 d^3 y = \sqrt{-g}\, [\alpha\beta\gamma\delta]\, e^\beta_1 e^\gamma_2 e^\delta_3\, d^3 y$$

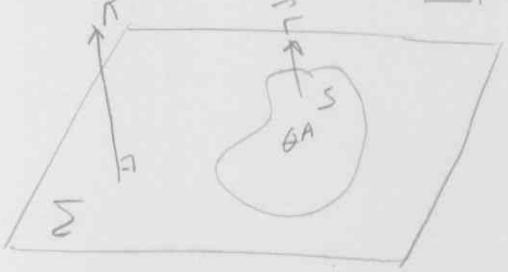
$$\overset{*}{=} \sqrt{-\varepsilon h}\, [\alpha 1 2 3]\, d^3 y \overset{*}{=} \sqrt{-\varepsilon h}\, d^3 y\, \delta^l_\alpha$$

$$n_\alpha d\Sigma \overset{*}{=} \varepsilon\, \delta^l_\alpha \sqrt{-\varepsilon h}\, d^3 y$$

$$d\Sigma_\alpha \overset{*}{=} \varepsilon n_\alpha d\Sigma$$

$$\boxed{d\Sigma_\alpha = \varepsilon n_\alpha d\Sigma}$$

2D surface



two normals: n^α, r^α → normal to S, in Σ

↓ normal to Σ

$$n_\alpha n^\alpha = \varepsilon$$

$$r_\alpha r^\alpha = -\varepsilon$$

surface element: $dS_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma\delta} \dfrac{\partial x^\gamma}{\partial \theta^2} \dfrac{\partial x^\delta}{\partial \theta^3} d\theta^2$ anti-symmetric in $\alpha\beta$

$$= (r_\alpha n_\beta - n_\alpha r_\beta)\, dS$$

$dS = \sqrt{\det(g_{AB})}\ d^2\theta$

↳ induced metric in θ^A coordinates

Gauss's Theorem

V: 4D region in spacetime

Σ: ∂V ≡ boundary of V ≡ hypersurface

A^α - arbitrary vector field

$$\int_V \nabla_\alpha A^\alpha \, dV = \oint_\Sigma A^\alpha \, d\Sigma_\alpha$$ (check the book for the prove)

Stokes Theorem

Σ: 3D hypersurface

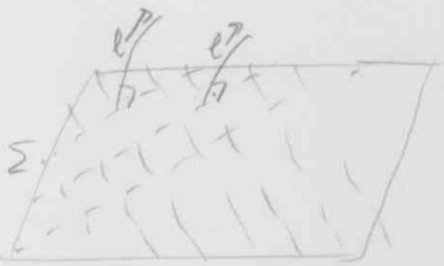
S: boundary of Σ = ∂Σ (2D)

$B^{\alpha\beta}$: antisymmetric tensor field

$$\int_\Sigma \nabla_\beta B^{\alpha\beta} \, d\Sigma_\alpha = \frac{1}{2} \oint B^{\alpha\beta} \, dS_{\alpha\beta}$$ (check the book for the prove)

Extrinsic curvature

Gaussian coordinates



$ds^2 = \varepsilon \, d\ell^2 + g_{ab}(\ell, y) \, dy^a dy^b$; $\ell \equiv$ proper distance (or time)

$\varepsilon = \begin{cases} -1 & \Sigma \text{ spacelike} \\ 1 & \Sigma \text{ timelike} \end{cases}$

$h_{ab} = g_{ab}(\ell=0, y)$
↳ induced metric

$x^\alpha \overset{*}{=} (\ell, y^a)$

$n^\alpha \overset{*}{=} (1,0,0,0)$, $n_\alpha \overset{*}{=} (\varepsilon,0,0,0)$

$e^\alpha_a \overset{*}{=} \delta^\alpha_a$

Intrinsic geometry of Σ

- $h_{ab} \rightarrow D_a \rightarrow R^a{}_{bcd}$

Extrinsic geometry (how the surface bends)

- K_{ab} (extrinsic curvature)

Distances have gone down : $h_{ab} \searrow$

Distances have gone up . $h_{ab} \nearrow$

So, "bending" ~ $\partial_\ell h_{ab} \sim K_{ab}$

Definition : $K_{ab} = \frac{1}{2} \partial_\ell g_{ab} \Big|_{\ell=0}$ (Defined in terms of gaussian coordinates)

$\Gamma^a_{bc} = \Gamma^a_{bc}[h]$ (3D) vs $\Gamma^\alpha_{\beta\gamma}[g]$ (4D)

$R^a{}_{bcd} = R^a{}_{bcd}[h]$ (3D) vs $R^\alpha{}_{\beta\gamma\delta}[g]$ (4D)

$R_{ab} = R_{ab}[h]$ ${}^4R_{\alpha\beta}[g]$

$D_a[h]$ vs $\nabla_\alpha[g]$

Do the calculations ✓

non-vanishing christoffel symbols

$$ {}^4\Gamma^{\ell}_{ab} = -\frac{1}{2}\,\varepsilon\,\partial_\ell g_{ab} \xrightarrow{\Sigma} -\varepsilon K_{ab} $$

check these

$$ {}^4\Gamma^{a}_{\ell b} = K^a{}_b \equiv h^{am} K_{mb} $$

$$ {}^4\Gamma^{a}_{bc} = \Gamma^a_{bc}[h] $$

non-vanishing Riemann component

$$ {}^4R_{\ell a \ell b} = -\frac{1}{2}\,\partial_\ell^2 g_{ab} + K_a{}^m K_{mb} $$

$$ {}^4R_{\ell abc} = D_c K_{ab} - D_b K_{ac} $$

$$ {}^4R_{abcd} = R_{abcd}[h] - \varepsilon\left(K_{ac}K_{bd} - K_{ad}K_{bc}\right) $$

non vanishing Einstein components

$$ {}^4G_{\ell\ell} = -\frac{1}{2}\left(\varepsilon R[h] + K_{ab}K^{ab} - K^2\right) \quad ; \quad K = h^{ab} K_{ab} $$

$$ {}^4G_{\ell a} = D_b K^b{}_a - D_a K $$

${}^4G_{ab}$ = complicated, involves $\partial_\ell^2 g_{ab}$

Covariant definition of K_{ab}

Claim: $K_{ab} = e^\alpha_a\, e^\beta_b\, \nabla_\alpha n_\beta$

4D: scalars (same in all 4D coordinates X^α)
3D: tensor, 2 indexed (symmetric)

Gaussian coordinates: $K_{ab} \overset{*}{=} \delta^\alpha_a\, \delta^\beta_b\, \nabla_\alpha n_\beta \overset{*}{=} \nabla_a n_b$

$$ \overset{*}{=} \left(\cancel{\partial_\alpha} n_\beta - {}^4\Gamma^\alpha_{ab}\, n_\alpha\right) \overset{*}{=} -\varepsilon\, {}^4\Gamma^{\ell}_{\underline{ab}} \overset{*}{=} +\varepsilon\left(-\varepsilon\, \underline{K_{ab}}\right) $$

$$ \overset{*}{=} \underbrace{\varepsilon\cdot\varepsilon}_{1} K_{ab} \overset{*}{=} K_{ab} $$

(equality in Gaussian coordinates)
equality in all coordinates
scalars → equality

$K_{ab} = K_{ba}$

$$ K_{ab} = e^\alpha_a\, e^\beta_b\, \nabla_\alpha n_\beta = e^\alpha_a\, e^\beta_b\, \nabla_b n_\alpha = \frac{1}{2}\, e^\alpha_a\, e^\beta_b \left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha + n^\gamma \underbrace{\nabla_\gamma g_{\alpha\beta}}_{0}\right) $$

$$ = \frac{1}{2}\, e^\alpha_a\, e^\beta_b\, \mathcal{L}_n g_{\alpha\beta} $$

Gauss-Codazzi equations

$${}^4R_{\mu\alpha\beta\gamma}\, n^\mu e^\alpha_a\, e^\beta_b\, e^\gamma_c = D_c K_{ab} - D_b K_{ac}$$

$${}^4R_{\alpha\beta\gamma\delta}\, e^\alpha_a\, e^\beta_b\, e^\gamma_c\, e^\delta_d = R_{abcd} - \varepsilon(K_{ac}K_{bd} - K_{ad}K_{bc})$$

$${}^4G_{\mu\nu}\, n^\mu n^\nu = -\frac{1}{2}(\varepsilon R + K_{ab}K^{ab} - K^2)$$

$${}^4G_{\mu\alpha}\, n^\mu e^\alpha_a = D_b K^b{}_a - D_a K$$

Covariant at X^α

Ex1: 2D sphere

3D: $X^\alpha = (x, y, z)$

2D: $y^a = (\theta, \varphi)$

embedding relations

$x = R\sin\theta\cos\varphi$

$y = R\sin\theta\sin\varphi$

$z = R\cos\theta$

$\varepsilon = 1$, timelike

$n_\alpha = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$$

$$h_{ab} = g_{\alpha\beta}\, e^\alpha_a\, e^\beta_b = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$K_{ab} = e^\alpha_a\, e^\beta_b\, \nabla_\alpha n_\beta = \begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix} = \frac{1}{R} h_{ab}$$

check this

$$K = \frac{2}{R}$$

Ex2 cyclinder

3D: $X^\alpha = (x, y, z)$

2D: $y^a = (z, \varphi)$

embedding relations: $\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = z \end{cases}$

$\Phi = x^2 + y^2$

$n_\alpha = (\cos\varphi, \sin\varphi, 0)$

$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \quad h_{ab}\,dy^a dy^b = \underbrace{dz^2 + R^2 d\varphi^2}_{\text{flat if}}$$

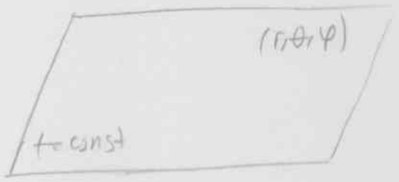
$d\varphi \to R\,d\varphi$

$$K_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}, \quad K = \frac{1}{R}$$

$\varepsilon = 1$, timelike

Ex3: t = constant slice of Schwarzschild

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2$$



$$h_{ab} = \begin{pmatrix} f^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$x^\alpha = (t, r, \theta, \varphi)$

$y^a = (r, \theta, \varphi)$

$$\left.\begin{aligned} t &= t \\ r &= r \\ \theta &= \theta \\ \varphi &= \varphi \end{aligned}\right\} \text{Embedding relation}$$

$K_{ab} = 0$: there is no time dependence (Nothing changes as we go up the hypersurfaces)

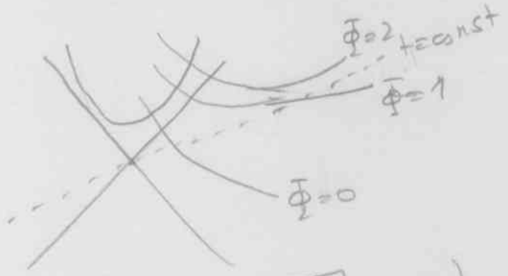
→ static space time

Ex4: Painlevé - Gullstrand slice of Schwarzschild

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

$$f = 1 - \frac{2M}{r}$$

spacelike surface: $\Phi = t + \int \frac{\sqrt{2M/r}}{f} dr = t + 4M\left\{\sqrt{r/2M} + \frac{1}{2} \ln\left(\frac{\sqrt{r/2m} - 1}{\sqrt{r/2m} + 1}\right)\right\}$



$n_\alpha = -\nabla_\alpha \Phi = -\left(1, \frac{\sqrt{2M/r}}{f}, 0, 0\right)$

$n_\alpha n^\alpha = -1$, spacelike

intrinsic coordinates: $y^a = (r, \theta, \varphi)$

Embedding relations: $t = \text{const} - \int \frac{\sqrt{2M/r}}{f} dr$

$r = r, \theta = \theta, \varphi = \varphi$

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$$

$$e^\alpha_r = \left(-\frac{\sqrt{2M/r}}{f}, 1, 0, 0\right)$$

$$e^\alpha_\theta = (0, 0, 1, 0)$$

$$e^\alpha_\varphi = (0, 0, 0, 1)$$

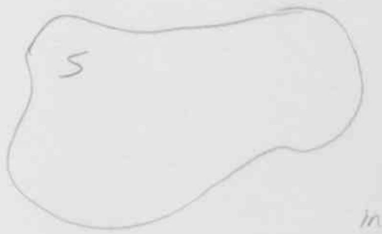
$$h_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$h_{ab} dy^a dy^b = dr^2 + r^2 d\Omega^2 = \text{flat}!$ → It is possible for any asymptoticaly flat space

$$K_{ab} = \begin{pmatrix} \frac{1}{2}\sqrt{2M/r^3} & & \\ & -\sqrt{2Mr} & \\ & & -\sqrt{2Mr}\sin^2\theta \end{pmatrix}$$

$$K = -\frac{3}{2}\sqrt{2M/r^3}$$

Gauss : 2D surfaces in 3D space (flat)



Page 10, 2nd eqn

LHS would be zero in flatspace and $(\varepsilon=1)$: timelike

$$\underbrace{R_{abcd}}_{\text{intrinsic}} = \underbrace{K_{ac}K_{bd} - K_{ad}K_{bc}}_{\text{Extrinsic}}$$

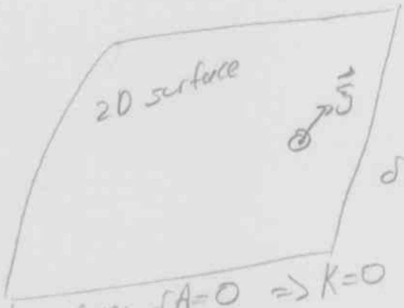
intrinsic ↓ Ricci scalar

Extrinsic ↓ Eigenvalues (K_{ab}) : K_1, K_2

$\frac{1}{2} R = K_1 K_2$

$R = \frac{1}{r_1 r_2}$

$K_1 = \frac{1}{r_1}$, $K_2 = \frac{1}{r_2}$



$A = \int \sqrt{h}\, d^2y$

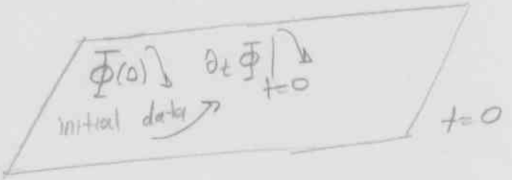
$\delta A = \int K \vec{\xi} \cdot d\vec{S}$

minimal surface: $\delta A = 0 \Rightarrow K = 0$

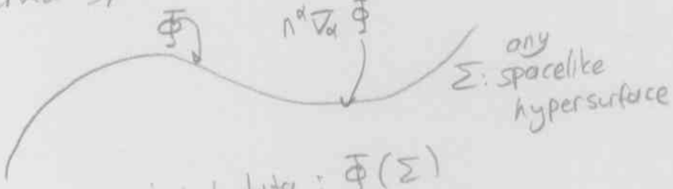
Initial-value problem

flat 4D space time

$\Box \Phi = 0$



Curved space time



initial data : $\Phi(\Sigma)$

$n^\alpha \nabla_\alpha \Phi(\Sigma)$

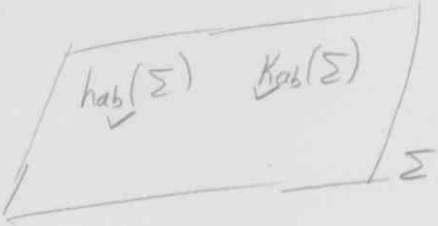
GR

initial moment ≡ any space like hypersurface

$\cancel{g_{\alpha\beta}}(\Sigma)$, $n^\mu \nabla_\mu \cancel{g_{\alpha\beta}}(\Sigma)$ (?)

$g_{\alpha\beta}$ would give more than we need

we need h_{ab}, K_{ab}



EFE: $\partial_t h_{ab} = \ldots$

$\partial_t K_{ab} = \ldots$

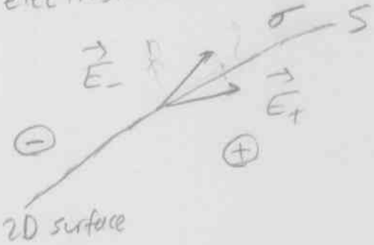
Constraints which we get from Gauss-Codazzi eqn's

$$R - K_{ab}K^{ab} + K^2 = 16\pi\, T_{\mu\nu} n^\mu n^\nu \equiv \text{Energy density} \equiv 16\pi\rho$$

$$D_b K^b{}_a - D_a K = 8\pi\, T_{\mu\nu} n^\mu e^\nu{}_a \equiv \text{matter current} = 8\pi\, j_a$$

Junction conditions

In electrostatics



$\vec{E}.\hat{n} = \sigma/\varepsilon_0$ $\hat{n} \rightarrow$ normal

$\vec{E}.\vec{l} = 0$ $\vec{l} \rightarrow$ tangent

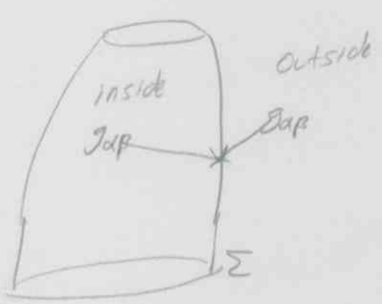
$\vec{E} = -\vec{\nabla} V$

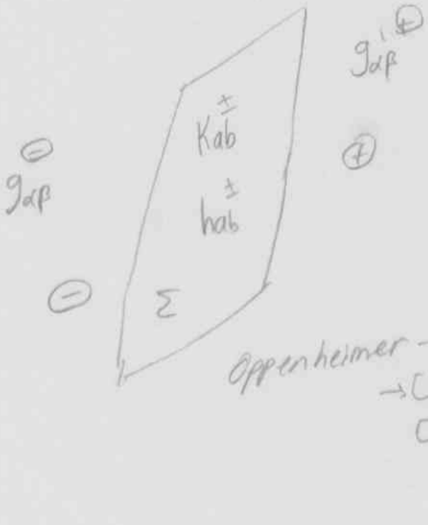
$[\ldots] = (\ldots)_+ - (\ldots)_-$

$[V] = 0$

$[\partial_n V] = -\sigma/\varepsilon_0 \longrightarrow \sigma = \int \rho\, dl$

gravitational collapse





$[h_{ab}] = 0$

$[K_{ab}] - K h_{ab} = 16\pi S_{ab}$

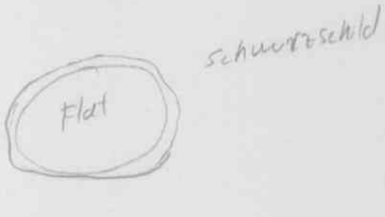
$S_{ab} = \int T_{\alpha\beta}\, e^\alpha_a e^\beta_b\, dl$

↳ surface stress tensor

Oppenheimer-Snyder

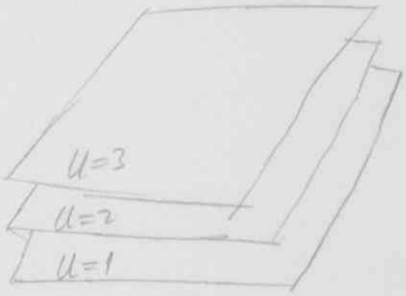
$\rightarrow [h_{ab}] = 0$

$[K_{ab}] = 0$



Null Hypersurfaces

Description :



continious stack (foliation) of null hypersurfaces

$u(x^\alpha) = \text{const}$

$K_\alpha \propto -\nabla_\alpha u$

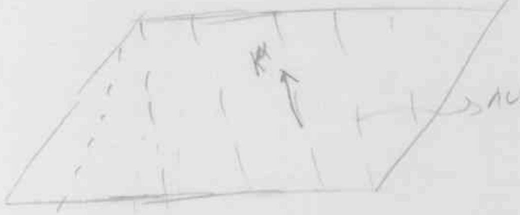
$K_\alpha = -e^{\chi} \nabla_\alpha u \quad ; \quad K_\alpha K^\alpha = 0$

$$\nabla_\beta K_\alpha = -e^{\chi}(\nabla_\beta \chi \, \nabla_\alpha u + \nabla_{\beta\alpha} u)$$
$$= \nabla_\beta \chi \, K_\alpha - e^{\chi} \nabla_{\beta\alpha} u$$

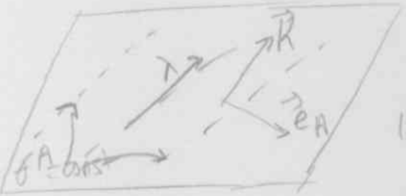
$$K^\beta \nabla_\beta K_\alpha = (\underbrace{K^\beta \nabla_\beta \chi}_{K}) K_\alpha - e^{2\chi} \underbrace{\nabla^\beta u \, \nabla_{\beta\alpha} u}$$

$$\nabla^\beta u \, \nabla_{\alpha\beta} u = \frac{1}{2} \nabla_\alpha \underbrace{(\underbrace{\nabla^\beta u}_{null} \, \underbrace{\nabla_\beta u}_{null})}_{0}$$

$K^\beta \nabla_\beta K_\alpha = K K_\alpha$ - non affine parametrization



null generators of hypersurface



$\theta^A = (\theta^2; \theta^3)$
↳ constant on each null generator

Intrinsic coordinates: $y^a = (\lambda, \theta^A)$
↳ non-affine parameter

Embedding relations: $x^\alpha = x^\alpha(\lambda, \theta^A)$

Tangent vectors: $e^\alpha_\lambda = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} \equiv K^\alpha$

$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda \equiv$ transverse vectors

$K_\alpha e^\alpha_A = 0 \; , \; K_\alpha K^\alpha = 0$

Second null vector:

$$N_\alpha N^\alpha = 0$$

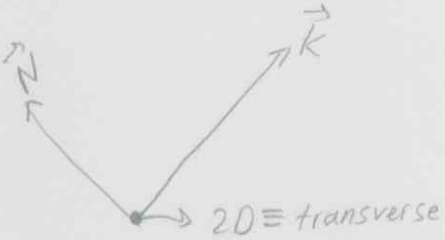
$$N_\alpha K^\alpha = -1$$

$$N_\alpha e^\alpha{}_A = 0$$

Induced metric: $h_{\lambda\lambda} = g_{\alpha\beta}\, e^\alpha_\lambda\, e^\beta_\lambda = g_{\alpha\beta} K^\alpha K^\beta = 0$

$$h_{\lambda A} = g_{\alpha\beta}\, e^\alpha_\lambda\, e^\beta_A = K_\alpha e^\alpha_A = 0$$

$$h_{AB} = g_{\alpha\beta}\, e^\alpha_A\, e^\beta_B = \Omega_{AB}$$



$$ds^2\big|_\Sigma = \Omega_{AB}\, d\theta^A d\theta^B$$

↳ transverse 2D metric

Example - Light cone in Schwarzschild

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

light cones: $u = t - r^* = t - \int \frac{dr}{f} = \text{const}$

$$K_\alpha = -\nabla_\alpha u = (-1, \tfrac{1}{f}, 0, 0)$$

$K^\alpha = (\tfrac{1}{f}, 1, 0, 0)$ $\qquad \frac{dt}{d\lambda} = \frac{1}{f} \qquad \frac{d\theta}{d\lambda} = 0 \qquad \lambda \equiv r$

$\qquad \frac{dr}{d\lambda} = 1 \qquad \frac{d\varphi}{d\lambda} = 0$

$$y^a = (\lambda \equiv r, \theta, \varphi)$$

$x^\alpha(y^a)$: $t = u + \int \frac{dr}{f}$

$$r = r$$
$$\theta = \theta$$
$$\varphi = \varphi$$

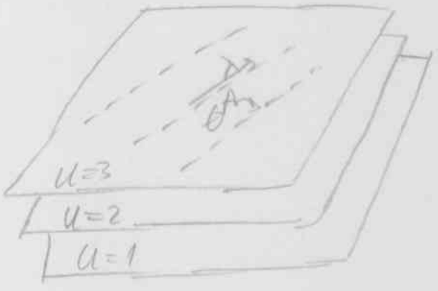
$$e^\alpha_r = \frac{\partial x^\alpha}{\partial r} = (\tfrac{1}{f}, 1, 0, 0) = K^\alpha$$

$$e^\alpha_\theta = (0, 0, 1, 0) \qquad \Omega_{AB} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

$$e^\alpha{}_\varphi = (0, 0, 0, 1)$$

Spacetime metric

- Neighborhood of the foliation
- Construct x^α



$x^\alpha \overset{*}{=} (u, \lambda, \theta^A)$

u = constant on each null hypersurface

λ = parameter on each generator

θ^A = constant on each generator

$K_\alpha = -e^\chi \nabla_\alpha u$

In these coordinates: $K_\alpha \overset{*}{=} (-e^\chi, 0, 0, 0)$

$K^\alpha \overset{*}{=} (0, 1, 0, 0) = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} = e^\alpha_\lambda = K^\alpha$

$e^\alpha_\theta = (0, 0, 1, 0)$

$e^\alpha_\varphi = (0, 0, 0, 1)$

Consequences: $K^\alpha = g^{\alpha\beta} K_\alpha$

$0 = K^u = g^{u\beta} K_\beta = g^{uu} K_u = -e^\chi \underset{0}{\underline{g^{uu}}} \Rightarrow g^{uu} = 0$

$1 = K^\lambda = g^{\lambda\beta} K_\beta = g^{\lambda u} K_u = -e^\chi g^{u\lambda} \Rightarrow g^{u\lambda} = -e^{-\chi}$

$0 = K^A = -e^\chi g^{uA} \Rightarrow g^{uA} = 0$

Inverse metric

$g^{uu} = 0$

$g^{u\lambda} = -e^{-\chi}$

$g^{uA} = 0$

$g^{\lambda\lambda} = \overbrace{e^{-\chi}}^{\text{To make it simple}} V$

$g^{\lambda A} = e^{-\chi} W^A$

$g^{AB} = \Omega^{AB}$ = inverse matrix of Ω_{AB}

Invert the 4x4 matrix $g^{\alpha\beta} \longrightarrow g_{\alpha\beta}$

$g_{uu} = -e^\chi V + \Omega_{AB} W^A W^B$

$g_{u\lambda} = -e^\chi$

$g_{uA} = \Omega_{AB} W^B \equiv W_A$

$g_{\lambda\lambda} = 0$

$g_{\lambda A} = 0$

$g_{AB} = \Omega_{AB}$

$ds^2 = -e^\chi V du^2 - 2e^\chi du d\lambda + \Omega_{AB}(d\theta^A + W^A du)(d\theta^B + W^B du)$

$ds^2\big|_\Sigma \overset{du=0}{=} \Omega_{AB}\, \theta^A \theta^B$

$\det(g_{\alpha\beta}) \overset{*}{=} -e^{2\chi} \det(\Omega_{AB})$

$\sqrt{-g} \overset{*}{=} e^\chi \sqrt{\Omega}$

Integration on null hypersurface

$$\int_\Sigma A^\alpha d\Sigma_\alpha \overset{?}{=}$$

Timelike, spacelike $\rightarrow d\Sigma_\alpha = \underbrace{\varepsilon n_\alpha d\Sigma}_{\text{not defined in null case}}$

More primitive : $d\Sigma_\alpha = \varepsilon_{\alpha\beta\gamma\delta} e^\beta_1 e^\gamma_2 e^\delta_3 dy^1 dy^2 dy^3$

$$y^1 = \lambda, \quad e^\beta_1 = k^\beta$$

$$y^2, y^3 = \theta^2, \theta^3$$

$$d\Sigma_\alpha = \varepsilon_{\alpha\beta\gamma\delta} k^\beta e^\gamma_2 e^\delta_3 d\lambda d^2\theta$$

$= , dS_{\alpha\beta} k^\beta d\lambda$; $\underbrace{dS_{\alpha\beta}}$ $= \varepsilon_{\alpha\beta\gamma\delta} e^\gamma_2 e^\delta_3 d^2\theta$: Definition is the same as page ⑥

$\hookrightarrow$ surface element on cross sectional surfaces

specilize to (u, λ, θ^A) coordinates

$$d\Sigma_\alpha \overset{*}{=} e^\chi \sqrt{\Omega}\, [\alpha\beta\gamma\delta]\, k^\beta e^\gamma_2 e^\delta_3 d\lambda d^2\theta$$

$$\overset{*}{=} e^\chi \sqrt{\Omega}\, [\alpha \lambda 2 3]\, \underbrace{k^\lambda}_{1} \underbrace{e^2_2}_{1} \underbrace{e^3_3}_{1} d\lambda d^2\theta$$

$$\overset{*}{=} e^\chi \sqrt{\Omega}\, \delta_\alpha^{\ u} d\lambda d^2\theta$$

Only non-vanishing component

$$d\Sigma_u = (e^\chi d\lambda) \sqrt{\Omega}\, d^2\theta$$

$$k_u = (-e^\chi)$$

$$d\Sigma_\alpha \overset{*}{=} (-k_\alpha d\lambda) \underbrace{\sqrt{\Omega}\, d^2\theta}_{dS}$$

equality holds in any coordinate system

$$d\Sigma_\alpha \doteq (-k_\alpha d\lambda) dS$$

$$dS_{\alpha\beta} \overset{*}{=} e^\chi \sqrt{\Omega}\, [\alpha\beta 2 3]\, d^2\theta$$

$\alpha, \beta = u, \lambda$

$\alpha, \beta = \lambda, u$

$dS_{u\lambda} \overset{*}{=} e^\chi \sqrt{\Omega}\, d^2\theta$, $k_u = -e^{-\chi}$, $N_\lambda = -1$

$\overset{*}{=} k_u N_\lambda dS$

$$dS_{\lambda u} \overset{*}{=} -dS_{u\lambda} = -k_u N_\lambda dS$$

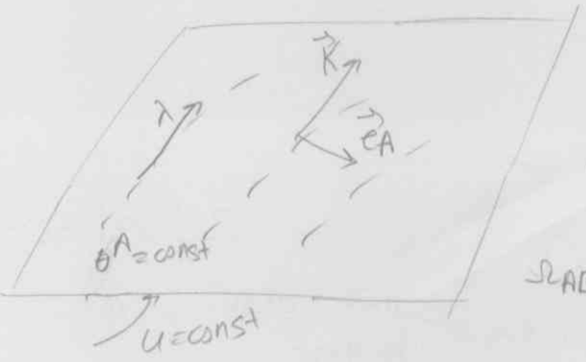
$$dS_{\alpha\beta} \overset{*}{=} 2 k_{[\alpha} N_{\beta]} dS$$

$K_\alpha \overset{*}{=} (-e^{\chi}, 0, 0, 0)$ $\quad N_\alpha N^\alpha = 0, \; N_\alpha K^\alpha = -1, \; N_\alpha e^\alpha{}_A = 0$

$K^\alpha \overset{*}{=} (0, 1, 0, 0)$ $\quad N_\alpha \overset{*}{=} (-\frac{1}{2} V, -1, 0, 0)$

$e^\alpha_\theta \overset{*}{=} (0, 0, 1, 0)$

$e^\alpha_\varphi \overset{*}{=} (0, 0, 0, 1)$



$$K^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta_A}$$

$$e^\alpha{}_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_x$$

$$\Omega_{AB} = g_{\alpha\beta} \, e^\alpha{}_A \, e^\beta{}_B = \text{transverse metric}$$

$$ds^2 = -e^{\chi} du \, (V du + 2 d\lambda) + \Omega_{AB} (d\theta^A + w^A du)(d\theta^B + w^B du)$$

Lie identity: $\mathcal{L}_K \, e^\alpha{}_A = 0$

$$K^\beta \nabla_\beta e^\alpha{}_A = e^\beta{}_A \nabla_\beta K^\alpha$$

<u>Transport equations</u>

$$K^\beta \nabla_\beta K^\alpha = k \, K^\alpha$$

$$K^\beta \nabla_\beta e^\alpha{}_A = \omega_A K^\alpha + B_A{}^B e^\alpha_B$$

$\rightarrow$ No component along N^α

$$K_\alpha K^\beta \nabla_\beta e^\alpha_A = K_\alpha (e^\beta_A \nabla_\beta K^\alpha) = \frac{1}{2} e^\beta_A \nabla_\beta (\underbrace{K_\alpha K^\alpha}_{0}) = 0$$

$$k = \partial_\lambda \chi$$

$$\omega_A = \frac{1}{2} (\partial_A \chi - e^{-\chi} \Omega_{AB} \partial_\lambda w^B)$$

$$B_{AB} = \frac{1}{2} \partial_\lambda \Omega_{AB}$$

$B_{AB} = e^\alpha_A e^\beta_B \nabla_\alpha K_\beta$ = purely transverse part of $\nabla_\alpha K_\beta$

$$\begin{aligned} e^\alpha_A e^\beta_B \nabla_\alpha K_\beta &= e^\beta_B (e^\alpha_A \nabla_\alpha K_\beta) \\ &= e^\beta_B (K^\alpha \nabla_\alpha e_{A\beta}) = e^\beta_B (\omega_A K_\beta + B_A{}^C e_{C\beta}) \\ &= B_A{}^C \underbrace{e^\beta_B e_{C\beta}}_{\Omega_{BC}} = B_{AB} \end{aligned}$$

$$\frac{dA}{d\lambda} = \int_D \partial_\lambda \sqrt{\Omega}\, d^2\theta = \int_D \Theta \sqrt{\Omega}\, d^2\theta = \int_S \Theta\, dS$$

$$\frac{dA}{d\lambda} = \int_S \Theta\, dS$$

↘ Fractional rate of change of cross-sectional area

$$\Theta = \frac{1}{dS} \frac{d}{d\lambda} \delta S$$

Because λ is not affine, $\Theta \neq \nabla_\alpha k^\alpha$

$$\nabla_\alpha k^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, k^\alpha \right) \overset{*}{=} \frac{1}{e^{\chi}\sqrt{\Omega}} \partial_\lambda \left(e^{\chi} \sqrt{\Omega} \right) \overset{*}{=} \underbrace{\partial_\lambda \chi}_{K} + \underbrace{\frac{1}{\sqrt{\Omega}} \partial_\lambda \sqrt{\Omega}}_{\Theta}$$

$$\overset{*}{=} K + \Theta$$

$$\Theta = \nabla_\alpha k^\alpha - K$$

Gauss - Codazzi equations

${}^4R_{\alpha\beta\gamma\delta} \longrightarrow$ geometrical quantities on null hypersurfaces

$$R_{\mu\nu\lambda\alpha} k^\mu N^\nu k^\lambda e^\alpha_A = \partial_\lambda w_A - \partial_A K + B_A{}^B w_B$$

$\vdots$

$${}^4R_{\mu\nu} k^\mu k^\nu = -\partial_\lambda \Theta + \underset{\hookrightarrow \text{not affine}}{\underline{K\Theta}} - \frac{1}{2} \Theta^2 - \sigma_{AB} \sigma^{AB} \quad ; \text{ Raychaudri's eqn}$$

$\vdots$

$${}^4R_{\mu\alpha} k^\mu e^\alpha_A = \partial_\lambda w_A - \partial_A K - \frac{1}{2} \partial_A \Theta + D_B \sigma_A{}^B + \Theta w_A$$