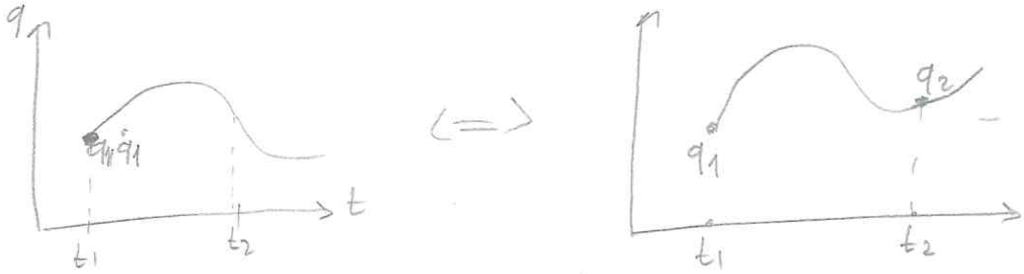
Lagrangian Mechanics

a-) Hamilton's Principle of least action

Motion : Generalized coordinates $q(t)$

"Initial value problem" "Boundary value problem"



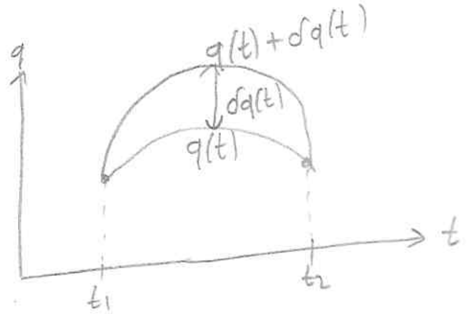
Principle of least action

Motions of mechanical system in $t \in (t_1, t_2)$ are given by extremals of the action functional

$$S = S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$$

L: Lagrangian : determines dynamical system

To derive eom, consider "fixed points" $\delta q(t_1) = 0 = \delta q(t_2)$



We seek extremum

$$\delta S = S[q(t) + \delta q(t)] - S[q(t)] = 0$$

- We have identity

$$\delta \frac{d}{dt} = \frac{d}{dt} \delta$$

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L\, dt = \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$

$$= \int_{t_1}^{t_2} dt \underbrace{\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right]}_{\text{Eom}} \delta q + \left[\frac{\partial L}{\partial \dot{q}} \underbrace{\delta q}_{= 0} \right]_{t_1}^{t_2} = 0$$

at the boundaries

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \right) = 0 \quad , i = 1, \dots N$$

N → # of degree of freedom

$$L = \underset{\text{kinetic}}{\underset{\downarrow}{T}} - V \to \text{potential} \quad , \; L = L(q, \dot{q}; t)$$

- L is not unique

$$L'(q, \dot{q}; t) = L(q, \dot{q}; t) + \frac{d\lambda(q; t)}{dt} \quad \text{gives same eom}$$

Proof by inserting the L' to the eom

or

$$S' = \int L' dt = \int \left(L + \frac{d\lambda}{dt}\right) dt = S + \int \frac{d\lambda}{dt} dt = S + \lambda \Big|_{t_1}^{t_2}$$

$$\delta S' = \delta S + \left(\frac{\partial \lambda}{\partial q} \underset{0}{\underset{\parallel}{\delta q}}\right)\Bigg|_{t_1}^{t_2} \qquad \delta S' = \delta S$$

6-) Integrals of motion

Solving (E-L) — Numerically / Perturbatively / Analytically - Integrable systems

Definition: Integral (Constant) of motion is $I = I(q, \dot{q}; t)$ such that

$$\frac{dI}{dt} = 0 \quad \text{for any } q(t)$$

Noether's Theorem

Connections between symmetries and integrals of motion

Terminology

- Statement is valid on-shell: is valid when provided eom are satisfied

  Off-shell: No conditions imposed

- Types of symmetries

consider transformations

$$t \to t' = t + \bar{\delta} t$$

$$q \to q'(t') = q(t) + \bar{\delta} q(t)$$

$\bar{\delta} t = \varepsilon \, \Delta t$, $\bar{\delta} q = \varepsilon \, A q$

ε: arameter "How big"

Δt: generator "what transformation"

Symmetries

- Discrete  90° ... ε discrete (No conserved quantitres)
- Continious ... ε arbitrary small

Symmetries

- Global ... ε = constant   1st Noether's theorem : conserved quantities
- Local (Gauge) ... $\varepsilon = \varepsilon(t)$ 2nd Noether's theorem : Bianchi identities

<u>Noether's Theorem (V1)</u>

For every <u>global continious</u> symmetry of the system, there is a corresponding on-shell integral of motion

<u>Ex1</u>: $L = L(q, \dot{q}, \cancel{t})$, $L \neq L(t)$

$$E = \frac{\partial L}{\partial \dot{q}^I} \dot{q}^I - L \quad : \text{conserved energy}$$

<u>Proof</u>: $$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} = \dot{q}(E - L) \overset{\text{on-shell}}{=} 0$$

<u>Ex2</u>: $L = L(\dot{q}; q, t)$ cyclic coordinate , $L \neq L(q)$

$$p = \frac{\partial L}{\partial \dot{q}} \quad : \text{generalized momentum}$$

<u>Noether's Theorem (V2 - Explicit formula)</u>

Let $\bar{\delta}$ be a global continious symmetry, that is off shell we find $\bar{\delta} t, \bar{\delta} q$, s.t

$$\bar{\delta} S = 0 \Rightarrow I = \frac{\partial L}{\partial \dot{q}} \bar{\delta} q + \left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)\bar{\delta} t \quad \text{is an on-shell integral of motion}$$

<u>EX</u>: $(\bar{\delta} t, \bar{\delta} q) = \varepsilon(1, 0)$ .. Time translation

$\Rightarrow I$ = Energy

- $(\bar{\delta} t, \bar{\delta} q) = \varepsilon(0, 1)$ .. Space translation

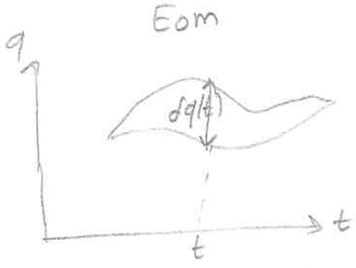
$\Rightarrow I$ = Momentum

Homogeneity of space time

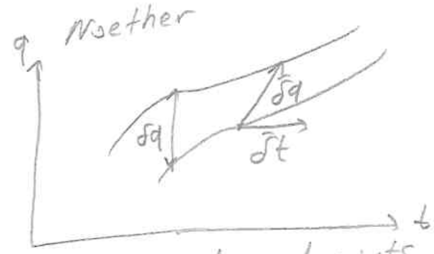
$\Rightarrow I =$ Angular momentum

Proof:

- Distinguish δ and $\bar{\delta}$



Fixed End Points



Not fixed end points

- $\bar{\delta} t = t' - t$

$\bar{\delta} dt = dt' - dt = d\bar{\delta} t = \frac{d\bar{\delta} t}{dt} dt$

- $\bar{\delta} q(t) = q'(t') - q(t) = q'(t) + \bar{\delta} t \frac{dq'(t)}{dt} \dots - q(t)$

$= \delta q(t) + \bar{\delta} t \frac{dq'(t)}{dt}$

$\bar{\delta} = \delta + \bar{\delta} t \frac{d}{dt}$ for any $f(q, \dot{q}; t)$

$$\bar{\delta} S = 0 = \int \bar{\delta}(L\,dt) = \int [(\bar{\delta} L) dt + L(\bar{\delta} dt)] = 0$$

$$= \int \left(\delta L + \bar{\delta} t \frac{dL}{dt} + L \frac{d\bar{\delta} t}{dt}\right) dt = \int \left[\delta L + \frac{d}{dt}(L \bar{\delta} t)\right] dt$$

$$= \int \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q + \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q\right) dt + \int \frac{d}{dt}(L \bar{\delta} t) dt$$

$$= \int_{t_1}^{t_2} \left\{ \frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{q}} \delta q + L \bar{\delta} t}_{I}\Big) + \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q \right\} dt = 0$$

The second term vanishes on-shell, while the first yields that

$I = \frac{\partial L}{\partial \dot{q}} \delta q + L \bar{\delta} t$ is an integral of motion, that is for a fixed trajectory we have $I\big|_{t_1} = I\big|_{t_2}$

Expressing finally I in terms of $\bar{\delta} q$ variations we recover the statement of the theorem.

Noether's Theo...

i-) Observe that the action S is invariant under a global transforma... (constant transformation parameter ϵ), that is, we find a transformation $\bar{\delta} q = \epsilon \bar{\Delta} q$, $\bar{\delta} t = \epsilon \bar{\Delta} t$, for which we <u>off-shell</u> have $\bar{\delta} S = 0$

ii-) Promote ϵ to $\epsilon(t)$ with fixed end points, then we must have

$\epsilon(t_1) = \epsilon(t_2)$

$$\bar{\delta} S = \int_{t_1}^{t_2} dt\, \dot{\epsilon}\, I$$

as the variation vanishes for constant ϵ

iii-) Integrating by parts the last expression

$$\bar{\delta} S = -\int_{t_1}^{t_2} dt\, \epsilon\, \dot{I} = 0$$

R.h.s must be zero on-shell, when the eom are valid $\epsilon(t)$ represents arbitrary variations which is the role of δq. This implies $\dot{I} = 0$ and I is integrals of motion

a-) $L \neq L(q)$

This implies that we have the following global symmetry $q \to q + \epsilon$, with ϵ constant. That is $\bar{\delta} q = \epsilon$, $\bar{\delta} t = 0$ is a symmetry transformation under $\bar{\delta} q = \delta q$ ... so we get $\bar{\delta} S = 0$. We now promote $\epsilon \to \epsilon(t)$

$$\bar{\delta} S = \int \bar{\delta}(L\,dt) = \int dt\, \bar{\delta} L + \int L\, \cancel{d(\underbrace{\bar{\delta} t}_{0})}$$

$$= \int dt\, \bar{\delta} L = \int dt \left(\delta L + \cancel{\underbrace{\bar{\delta} t}_{0}} \frac{dL}{dt} \right) = \int dt\, \delta L$$

$$= \int dt \left(\underbrace{\frac{\partial L}{\partial q}}_{0} \overset{\epsilon}{\delta q} + \frac{\partial L}{\partial \dot{q}} \overset{\dot{\epsilon}}{\delta \dot{q}} \right) = \int \frac{\partial L}{\partial \dot{q}} \dot{\epsilon}\, dt$$

which is the form we have in (i). $I = \frac{\partial L}{\partial \dot{q}}$: conserved momentum

b-) $L \neq L(t)$

$t \to t + \epsilon$, $\bar{\delta} t = \epsilon$ $\quad \bar{\delta} q = 0 = \delta q + \epsilon \dot{q}$, $\quad \epsilon \to \epsilon(t)$

$$\bar{\delta} S = \int \bar{\delta}(L\,dt) = \int \left(\bar{\delta} L + L \frac{d \bar{\delta} t}{dt} \right) dt$$

$$\bar{\delta} L = \frac{\partial L}{\partial q} \underbrace{\bar{\delta} q}_{0} + \frac{\partial L}{\partial \dot{q}} \bar{\delta} \dot{q} = \frac{\partial L}{\partial \dot{q}} (\delta \dot{q} + \epsilon \ddot{q}) = \frac{\partial L}{\partial \dot{q}} \left(\frac{d}{dt} (\bar{\delta} q - \epsilon \dot{q}) + \epsilon \ddot{q} \right) = -\dot{\epsilon} \dot{q} \frac{\partial L}{\partial \dot{q}}$$

$$\bar{\delta} S = \int \dot{\epsilon} \left(L - \dot{q} \frac{\partial L}{\partial \dot{q}} \right) dt \Rightarrow I = L - \frac{\partial L}{\partial \dot{q}} \dot{q}$$