- set
- Topological Space → Hausdorff
- Topological Manifold
- Differantiable Manifold
- Riemann Geometry
- $\boxed{\text{Semi-Riemann Geometry}}$

- Einstein Equation
- Einstein-Hilbert action
- Solutions

Hartong
Obers
"An action Principle for Neutonian Gravity"

$$\nabla^2 \phi = 4\pi G \rho$$

Some set-Theoretical Concepts → Chris Pope, Gravitational Physics

A set is a collection of objects elements or for us, points on manifold

1) A set is a subset of V, if all element of U is also an element of V.

2) If there exist elements in V that are not the elements of U, then U is a proper subset.

3) If U is a subset of V, then the complement of U in V is denoted by $V-U$. That is the set off elements of V that are not in U.

4) Two sets are equal ($U=V$) if every element of U is an element of V and vice versa. This can also be stated as U is a subset of V, and V is a subset of U

5) U and V can form a union

$$U \cup V$$

which is the set of elements in U or in V

6) The intersection

$$U \cap V$$

is the set of elements in U and V.

7) Two sets are disjoint if $U \cap V = \emptyset$

Topological Space → John Mc Greevy, General Relativity

A topological space X is a set with topology $\mathcal{O}$, a family of open sets $\{U_\alpha \subset X\}_{\alpha \in A}$ which satisfies.

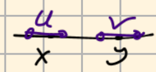
1) $\mathcal{O} \in X, \emptyset$ (The whole space X is open, and so is the empty set)

2) The intersection of opensets is also an open set), i.e $U_1, U_2 \in \mathcal{O}$ then $U_1 \cap U_2 \in \mathcal{O}$

3) The union of opensets is an open set, i.e $U_\alpha \in D$ the $\bigcup_\alpha U_\alpha \in D$

Example 1

$X = \mathbb{R}$, $D = \{(a,b),\ a<b \in \mathbb{R}\}$, a collection of open intervals
↳ Random iki noktayı ayırt edebiliriz



Example 2

X, $D = \{X, \phi\}$
↳ Burada edemem

The topological space X is said to obey the Hausdorff axiom, and hence to be a Hausdorff space. If, for any pair of distinct points $x_1, x_2 \in X$, there exist disjoint open sets O_1 and O_2, each containing just one of the two points.

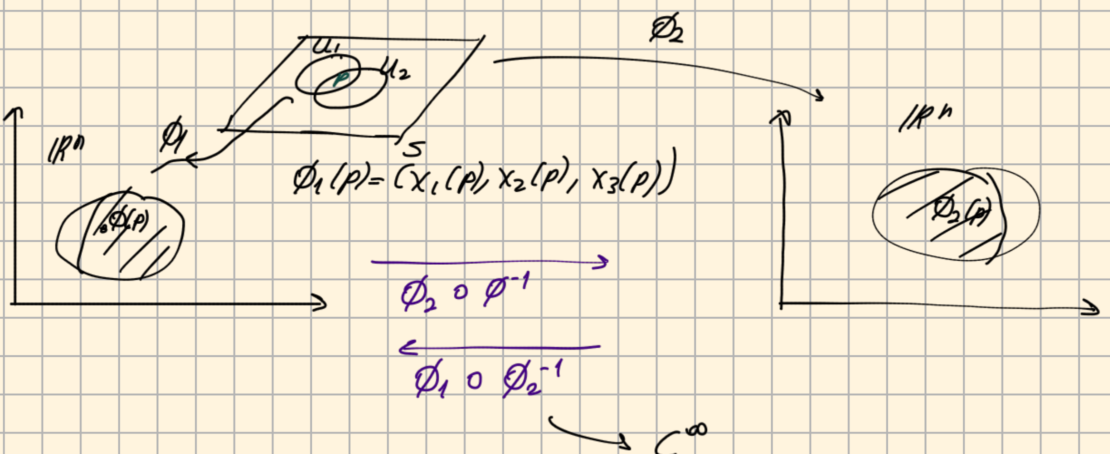
Manifolds

1) A topological space S

2) A open cover $\{U_i\}$, which one known as patches

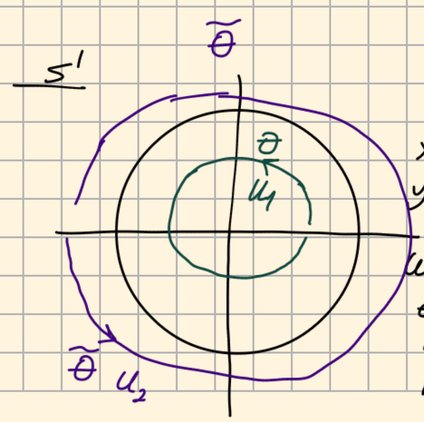
3) A set (called an atlas) of maps $\phi : U_i \rightarrow R^n$ called charts which define a 1-1 relation between the points in U_i and points in an open ball in $\mathbb{R}^n$

Such that if U_1 and U_2 intersects then both $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ are smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$



$\Rightarrow$ Differantiable Manifold

S^1



$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

We can think of a circle as a real line where the left end of the is identified with the right end. Thus for a unit circle, we take the coordinate interval $0 \leq \theta \leq 2\pi$ with the rule that $\theta = 2\pi$ is identified with the point $\theta = 0$

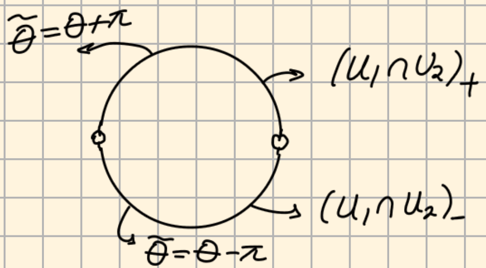
1) Use θ to describe points on the circle corresponding to $0 < \theta < 2\pi$

2) Introduce a new singular coordinate $\tilde{\theta}$, which starts from $\tilde{\theta}=0$ over the left handside at $\theta=\pi$. We shall use $\tilde{\theta}$ only in $0<\tilde{\theta}<2\pi$

3) U_1 covers all points on S^1 except $(1,0)$

U_2 covers all points on S^1 except $(-1,0)$

4) The intersection of U_1 and U_2 compromises all points except $(x,y)=(1,0)$ and $(-1,0)$. Thus it compromises two disconnected open intervals, consists of points on S^1 above the x-axis and the other consists on S^1 that lie below the x-axis

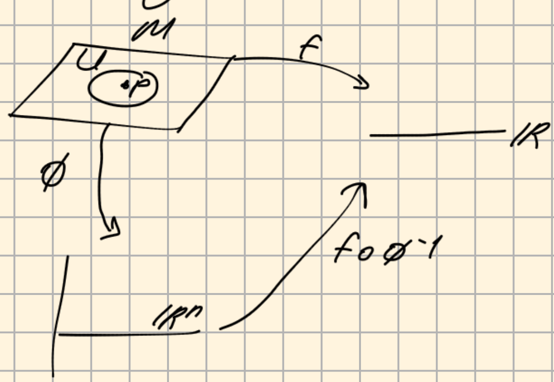


Functions and Manifolds

$$f: M \longrightarrow \mathbb{R}$$

that gives a real number. For each point p in M. If, for some open set U in M we have a coordinate chart ϕ such that U is mapped by ϕ into $\mathbb{R}$; then we have the mapping.

$$f \circ \phi^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}$$



Mukhanov Cosmology

Tangent Vectors

Suppose we consider some patch U in manifold M for which we introduce local coordinates x^i in the usual way. Now, consider a path passing through U, which may therefore be described by specifying the values of the coordinates along the path. We can do this by introducing a parameter λ that increases monotonically along the path, and so points in M along the path are specified by

$$x^i = x^i(\lambda)$$

Then

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^i}\frac{dx^i}{d\lambda} = \Big(\underbrace{\frac{dx^i}{d\lambda}}_{\text{components of the vector field}} \overbrace{\partial_i}^{\text{basis of the vector field}}\Big) f$$

$$= \underbrace{Vf}_{V = V^i \partial_i \text{ where } V^i = \frac{dx^i}{d\lambda}}$$

Then we define the directional derivative

$$f \longrightarrow Vf = \frac{df}{d\lambda}$$

- This obeys the linearity property

$$V(f+g) = Vf + Vg$$

- Liebniz property

$$V(fg) = gVf + fVg$$

- If we have two different tangent vectors at a point "p", let us call them $\frac{\partial}{\partial x}$ and $\tilde{V} = \frac{\partial}{\partial \tilde{x}}$

$$(V + \tilde{V})f = Vf + \tilde{V}f$$

The space of tangent vectors at a point $p \in M$ is called the "Tangent space at p" denoted by $T_p(M)$. Its dimension n is the dimension of the the manifold

$$x^i = x_p^i + \xi^i$$

$$f(x) = f(x_p) + \xi^i \partial_i f \Big|_{x = x_p}$$

$$x^i \longrightarrow x'^i(x^i) \quad : \text{ General coordinate transformation}$$

$$V = V^i \partial_i = V'^i \partial'_i$$

on the other hand

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j}\,;$$

$$\boxed{V'^i = \frac{\partial x'^i}{\partial x^j} V^j}$$

Having defined $T_p(M)$, the tangent space at the point $p \in M$. We can define the so called "tangent bundle" as the space of all possible tangent vectors at all possible points.

$$T(M) = \underbrace{\bigcup}_{p \in M} T_p(M)$$

Non-Coordinate basis for the Tangent space

x^i (coordinates on M)

$\left.\frac{\partial}{\partial x^i}\right|_p$ (coordinates on $T_p(M)$)

$\hookrightarrow$ coordinate basis

We introduce quantities $E_a{}^i$ $\qquad 1 < a < n$

and take our basis to be

$$\underbrace{E_a}_{\text{tangent space index}} = \underbrace{E_a{}^i(x)}_{\det E \neq 0,\ \text{inverse vielbein}} \underbrace{\partial_i}_{\text{world index}}$$

(It is like one hand is on Minkowski and other hand on manifold)

In addition to the general coordinate transformations $x^i \to x'^i(x^i)$, we can also make transformation on the tangent space index.

In other words, we can make transformations from one choice of non-coordinate basis $E_a{}^i$ to another, say $E_a'^i$. This transformation itself can be different at different points in M

$$E_a \to E'_a = \Lambda_a{}^b(x)\, E_b$$

Then

$$V = V^i \partial_i = V^a E_a = \underbrace{V^a E_a{}^i}_{V^i = V^a E_a{}^i} \partial_i$$

Let $V^a \to V'^a = \Lambda^a{}_b(x)\, V^b$

$$V = V^a E_a \to V'^a E'_a = \overbrace{\Lambda^a{}_b \Lambda_a{}^c}^{\delta_b{}^c} V^b E_c$$

<u>Covectors</u>

For every vector space X, there exists the notion of its dual space X^*, which is the space of linear maps

$$X^*: X \to R$$

What this means is that if V is any vector in X, and ω is any covector in X^* in X^* then there exists a rule of making a real number from V an ω

$$\langle \omega | V \rangle \in \mathbb{R}$$

The operation is linear

$$\langle \omega | U+V \rangle = \langle \omega | U \rangle + \langle \omega | V \rangle$$

$$\langle \omega | \lambda V \rangle = \lambda \langle \omega | V \rangle$$

$V = V^a E_a$ and $\omega = \omega_a e^a$

The basis and dual satisfies

$$\langle e^a | E_b \rangle = \delta^a_b$$

Then

$$\langle \omega | V \rangle = \omega_a V^b \langle e^a | E_b \rangle$$
$$= \omega_a V^b \delta_b{}^a = \omega_a V^a$$

Under the change of basis E_a, $E_a = \Lambda_a{}^b(x) E_b$

follows that the dual basis must transform as

$$e^a \to e'^a = \Lambda^a{}_b \, e^b$$

$\langle e_a | E^b \rangle \longrightarrow \langle e'_a | E'^b \rangle$

$$= \Lambda_a{}^c \, \Lambda^b{}_d \langle e_c | E^d \rangle$$
$$= \Lambda_a{}^c \, \Lambda^b{}_d \, \delta_c{}^d$$
$$= \underbrace{\Lambda_a{}^c \, \Lambda^b{}_c}_{\delta_a{}^b}$$

Thus

$\omega_a \to \omega_a' = \Lambda^b{}_a \, \omega_b$ so that

$\omega = \omega_a e^a = \omega'_a \, e'^a$

At every point, $p \in M$ we define the cotangent space $T_p^*(M)$ that is the dual of $T_p(M)$

$$T^*_M = \underbrace{\bigcup}_{p \in M} T^*_p(M) \Rightarrow \text{cotangent bundle}$$

An example of a covector is the differantial of a function. Suppose $f(x)$ is a function on M

Its differantial is called a differantial one-form

$$df = \partial_i f \, dx^i$$

or

$\langle df | V \rangle = Vf$ for any vector V.

- $\langle dx^i | \partial_j \rangle = \delta^i_j$

then

$$\langle df | V \rangle = \partial_i f \, V^j \overbrace{\langle dx^i | \partial_j \rangle}^{\delta^i_j} = V^i \partial_i f$$

- $\omega = \omega_i \, dx^i$

then (under $x'^i = x'^i(x^i)$)

$$\omega' = \omega'_i \, dx'^i = \underbrace{\omega_i' \frac{\partial x'^i}{\partial x^j}}_{\omega_j} dx^j = \omega_j dx^j$$

$$\omega_j = \frac{\partial x'^i}{\partial x^j} \omega'_i$$

Thus

$$V^i \to V'^i = \frac{\partial x'^i}{\partial x^j} V^j$$

$$\omega_i \to \omega'_i = \frac{\partial x'^j}{\partial x^i} \omega_j$$

Thus

$$V^i \omega_i \to V'^i \omega'_i = \underbrace{\frac{\partial x'^i}{\partial x^j}} V^j \underbrace{\frac{\partial x^k}{\partial x'^i}} \omega_k$$

$$= \delta^i_j V^j \, \delta^k_i \, \omega_k = \delta_j{}^k V^j \omega_k = V^i \omega_i$$

Tensor Product

The tensor product of two vector spaces X and Y is denoted by $X \otimes Y$. It obeys a distribution law in the sense that if X, Y and Z are vector spaces then

$$X \otimes (Y+Z) = (X \otimes Y) + (X \otimes Z)$$

if the elements of vector spaces X an Y are denoted by x and y, respectively, then the tensor product $X \otimes Y$, is spanned by the elements of the form $X \otimes Y$.

The following rules are satisfied

- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$
- $\lambda (x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$
- $0 \otimes x = x \otimes 0 = 0$

Canonical QFT
Group Theory
GR
Supersymmetry
PI QFT
Cosmology
Black hole
AdS/CFT
String theory
Particle Physics
dif geo

If α_i is a basis of vectors for X and β_j for Y, then $\alpha_i \otimes \beta_j$ for all i, j gives a basis for $X \otimes Y$

$$x = \sum_i x_i \alpha_i \ , \quad y = \sum_j y_j \beta_j$$

Then

$$z = \sum_{i,j} z_{ij}\, \alpha_i \otimes \beta_j$$

Tensors

Tensors are the objects that live in the tensor product space, involving p-factors of the tangent space $T_p(M)$ and q-factors of the cotangent space $T_q^*(M)$

$$T = T^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q}\, \partial_{i_1} \otimes \dots \partial_{i_p}\, dx^{j_1} \dots \otimes dx^{j_q} \quad (p,q) \text{ tensor}$$

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^{i'}} \underbrace{\frac{\partial}{\partial x^j}}_{\partial_j}$$

$$T = T^{i'_1 i'_2 \dots i'_p}{}_{j'_1 \dots j'_q} \frac{\partial x^{k_1}}{\partial x'^{i_1}} \dots \frac{\partial x^{k_p}}{\partial x'^{i_p}} \frac{\partial x'^{j_1}}{\partial x^{m_1}} \dots \frac{\partial x'^{j_q}}{\partial x^{m_q}}\, \partial_{k_1} \otimes \dots \partial_{k_p} \otimes dx^{m_1} \dots dx^{m_q}$$

then

$$T'^{i_1 \dots i_p}{}_{j_1 \dots j_q} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \dots \frac{\partial x'^{i_p}}{\partial x^{k_p}} \frac{\partial x^{m_1}}{\partial x^{j_1}} \dots \frac{\partial x^{m_q}}{\partial x^{j_q}}\, T^{k_1 \dots k_p}{}_{m_1 \dots m_q}$$

- V is a $(1,0)$ tensor (vector)

  ω is a $(0,1)$ tensor (co-vector)

  $W = W_i{}^j\, \partial_i \otimes dx^j$
  ↳ $(1,1)$ tensor

- $\delta^i{}_j$ is a $(1,1)$-tensor

The metric tensor

If U and V are any vectors then a metric g is a symmetric bilinear map that maps U and V into reals

$$g(U,V) \in \mathbb{R}$$

with the following properties

- $g(U,V) = g(V,U)$
- $g(\lambda U, \mu V) = \lambda \mu\, g(U,V)$

We also demand that the metric is non-degenerate

$$g(U,V) = 0 \text{ for all } V \text{ if } V = 0$$

$g(U,V) = g_{ij} U^i V^j$

$g = g_{ij}\, dx^i \otimes dx^j$

$\hookrightarrow$ (0,2) tensor

- symmetric $\Rightarrow$ $g_{ij} = g_{ji}$
- non-degenerate $\Rightarrow$ $\det g \neq 0$

In a coordinate patch, g_{ij} has s-positive and t negative eigen values

$s-t$ = signature of the metric; will be the same in all coordinates

- $t=0$, $s=n = \dim M$
  Riemann geometry

- $t=1$, $s=n-1$
  Pseudo-Riemann Geometry

If U and V are vectors then $g_{ij} U^i V^j$ is a scalar

$$U_i = g_{ij} U^j$$

$$V^i = g^{ij} V_j$$

$$\partial_i V^j \longrightarrow \partial'_i V'^j = \frac{\partial x^k}{\partial x'^i} \frac{\partial}{\partial x^k}\left(\frac{\partial x'^j}{\partial x^l} V^l \right)$$

Derivative operatörü genel koordinat dönüşümü altında co-vector gibi dönüşmüyor.

is not a (1,1) tensor under General Coordinate Transformation

$$= \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} \partial_k V^l + \frac{\partial x^k}{\partial x'^i} \frac{\partial^2 x'^j}{\partial x^k \partial x^l} V^l$$

$$\nabla_j V^i = \partial_j V^i + \Gamma^i{}_{jk} V^k$$

Consider

$$W_\mu{}^\nu = \partial_\mu V^\nu \xrightarrow{?} W'_\mu{}^\nu = \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} W_\rho{}^\sigma$$

$$\partial_\mu V^\nu \longrightarrow \partial'_\mu V'^\nu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho}\left(\frac{\partial x'^\nu}{\partial x^\sigma} V^\sigma \right)$$

$$= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \underbrace{\partial_\rho V^\sigma}_{W_\rho{}^\sigma} + \underbrace{\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\sigma \partial x^\rho}}_{\text{extra}} V^\sigma$$

$$x^\mu \longrightarrow x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{global}} x^\nu$$

$$x^\mu \xrightarrow[\text{Local}]{} x'^\mu = x'^\mu(x^\nu)$$

$\phi(x) \longrightarrow$ complex scalar

i-) $\phi(x) \longrightarrow e^{-i\alpha}\phi(x)$

$$\partial_\mu\phi \to \partial_\mu(e^{-i\alpha}\phi(x)) = e^{-i\alpha}\partial_\mu\phi$$

ii-) $\phi(x) \longrightarrow e^{-i\alpha(x)}\phi(x)$

$$\partial_\mu\phi \to \partial_\mu(e^{-i\alpha(x)}\phi(x)) = e^{-i\alpha(x)}\partial_\mu\phi - i\,\partial_\mu\alpha(x)\,e^{-i\alpha(x)}\phi$$

iii-) $A_\mu \to A_\mu + \partial_\mu\alpha = A'_\mu$

$$D_\mu\phi = \partial_\mu\phi + iA_\mu\phi$$

$$D_\mu\phi \longrightarrow \partial_\mu\phi' + iA'_\mu\phi'$$

$$e^{-i\alpha}\partial_\mu\phi - i\phi e^{-i\alpha}\partial_\mu\alpha + iA_\mu e^{-i\alpha}\phi + ie^{-i\alpha}\phi\,\partial_\mu\alpha$$

$$= \partial_\mu(e^{-i\alpha}\phi) + i(A_\mu + \partial_\mu\alpha)e^{-i\alpha}\phi$$

$$= e^{-i\alpha}(\partial_\mu\phi + iA_\mu\phi) - \cancel{i\phi\,\partial_\mu\alpha\, e^{-i\alpha}} + \cancel{i\,\partial_\mu\alpha\, e^{-i\alpha}\phi}$$

$$= e^{-i\alpha}D_\mu\phi$$

$\Gamma^\rho_{\mu\nu} \equiv$ Christoffel connection

$$\underbrace{\nabla_\mu}_{\text{covariant derivative}} V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho$$

- $\nabla'_\mu V'^\nu = \underbrace{\partial'_\mu V'^\nu}_{1} + \underbrace{\Gamma'^\nu_{\mu\rho}V'^\rho}_{2} = \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\nabla_\rho V^\sigma}_{3}$

$$= \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\cancel{\partial_\rho V^\sigma} + \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho\partial x^\sigma}V^\sigma}_{1} + \underbrace{\frac{\partial x'^\rho}{\partial x^\sigma}\Gamma'^\nu_{\mu\rho}V^\sigma}_{2}$$

$$= \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\left(\cancel{\partial_\rho V^\sigma} + \Gamma^\sigma_{\rho\lambda}V^\lambda\right)}_{3}$$

Then

$$-\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\rho\partial x^\sigma}V^\sigma + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\tau}\Gamma^\tau_{\alpha\sigma}V^\sigma = \frac{\partial x'^\rho}{\partial x^\sigma}\Gamma'^\nu_{\mu\rho}V^\sigma$$

leading to

$$\Gamma'^{\nu}_{\mu\rho} = \frac{\partial x^{\sigma}}{\partial x'^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\tau}} \Gamma^{\tau}_{\alpha\sigma} - \underbrace{\frac{\partial x^{\sigma}}{\partial x'^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\nu}}{\partial x^{\alpha} \partial x^{\sigma}}}_{\text{non-tensor part}}$$

i-) (M, g, Γ)

ii-) $\longrightarrow$ a-) $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\rho} V^\rho$

b-) Lets consider a scalar ϕ

$$\partial_\mu \phi \longrightarrow \partial_\mu' \phi' = \frac{\partial}{\partial x'^{\mu}} \phi'(x')$$

$$= \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi$$

So, for a scalar field $\nabla_\mu \phi = \partial_\mu \phi$

iii-) $\nabla_\mu (W_\nu V^\rho) = (\nabla_\mu W_\nu) V^\rho + W_\nu (\nabla_\mu V^\rho)$

iV-) $W_\mu V^\mu$ is a scalar i.e. $\partial_\mu (W_\nu V^\nu) = \nabla_\mu (W_\nu V^\nu)$

$$\Rightarrow (\partial_\mu W_\nu) V^\nu + \cancel{W_\nu (\partial_\mu V^\nu)} = (\nabla_\mu W_\nu) V^\nu + W_\nu \underbrace{\nabla_\mu V^\nu}_{(\cancel{\partial_\mu V^\nu} + \Gamma^{\nu}_{\mu\rho} V^\rho)}$$

$$\Rightarrow (\nabla_\mu W_\nu = \partial_\mu W_\nu - \Gamma^{\rho}_{\mu\nu} W_\rho) V^\nu$$

$$\nabla_\mu T^{\nu_1 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} = \partial_\mu T^{\nu_1 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} + \Gamma^{\nu_1}_{\mu\lambda} T^{\lambda \nu_2 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} + \Gamma^{\nu_2}_{\mu\lambda} T^{\nu_1 \lambda \nu_3 \cdots \nu_p}{}_{\rho_1 \cdots \rho_q} \cdots + \Gamma^{\nu_p}_{\mu\lambda} T^{\nu_1 \cdots \nu_{p-1} \lambda}{}_{\rho_1 \cdots \rho_q}$$
$$- \Gamma^{\lambda}_{\mu\rho_1} T^{\nu_1 \cdots \nu_p}{}_{\lambda \rho_2 \cdots \rho_q} \cdots - \Gamma^{\lambda}_{\mu\rho_q} T^{\nu_1 \cdots \nu_p}{}_{\rho_1 \cdots \rho_{q-1} \lambda}$$

Metric

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \cancel{\Gamma^{\lambda}_{\mu\rho}} g_{\nu\lambda} = 0$$

$$-/ \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \cancel{\Gamma^{\lambda}_{\rho\mu}} g_{\lambda\nu} - \cancel{\Gamma^{\lambda}_{\rho\nu}} g_{\mu\lambda} = 0$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \cancel{\Gamma^{\lambda}_{\nu\rho}} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0$$

- $\nabla g = 0$
- $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$

(1) − (2) + (3) = 0

$$g^{\rho\sigma} / \; \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$: Affine connection

(M, g)

$$g_{\mu\nu} \nabla_\rho V^\nu = \nabla_\rho (g_{\mu\nu} V^\nu) = \nabla_\rho V_\mu$$

Some properties of Covariant Derivative

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu :$$

Curved space de nasil olur.

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad \text{diyelim}$$

$$= (\partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho) - (\partial_\nu A_\mu - \Gamma^\rho_{\nu\mu} A_\rho)$$

$$= \partial_\mu A_\nu - \cancel{\Gamma^\rho_{\mu\nu}} A_\rho - \partial_\nu A_\mu + \cancel{\Gamma^\rho_{\nu\mu}} A_\rho$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

So $\quad F_{\mu\nu} \to F'_{\mu\nu} = \dfrac{\partial x^\rho}{\partial x'^\mu} \dfrac{\partial x^\sigma}{\partial x'^\nu} F_{\rho\sigma}$ $\qquad$ For anti-symmetric tensor: $\quad \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

Divergence

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$$

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

$$= \underbrace{\frac{1}{2} g^{\mu\sigma} (\partial_\mu g_{\nu\sigma} - \cancel{\partial_\sigma g_{\nu\mu}})}_{0} + \frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\mu\sigma}$$

$\mu \to \sigma$ $\qquad \cancel{\partial_\sigma g_{\nu\mu}}$

$\sigma \to \mu$

$$= \frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\mu\sigma} = \frac{1}{2} tr(g^{-1} \partial_\nu g)$$

$$g_{\mu\nu} g^{\nu\mu} = \delta_\mu^\mu$$

Let M be invertible square matrix

then

$$\log \det M = tr \log M$$

$$\det M = e^{tr \log M}$$

$$\Downarrow \qquad \Downarrow$$

$$\prod_i \lambda_i \qquad e^{\sum \log \lambda_i}$$

$$\partial_\nu (tr \log g) = tr\left(\frac{1}{g} \partial_\nu g\right)$$

$$\Rightarrow \sqrt{\det g} = e^{tr \log \sqrt{g}} \qquad \frac{\frac{1}{2\sqrt{g}} \partial_\nu g}{\sqrt{g}} = \left(\frac{1}{2} \overbrace{g^{-1} \partial_\nu g}^{tr(\;)}\right) \sqrt{g}$$

$$e^{tr \log g}$$

$$\partial_\nu \left(e^{tr \log \sqrt{g}}\right) = \sqrt{g}$$

$$\partial_\nu \det g = tr(g^{-1} \partial_\nu g) \det g$$

$$\det \sqrt{g}$$

$$\partial_\nu \sqrt{\det g} = \partial_\nu \sqrt{g} = \frac{1}{2\sqrt{g}} \partial_\nu g = \frac{1}{2} \sqrt{g}\, tr(g^{-1} \partial_\nu g)$$

$$\Rightarrow \quad \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} = \frac{1}{2} tr(g^{-1} \partial_\nu g)$$

✓ $\Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{g}}\, \partial_\nu \sqrt{g}$ det $g = g < 0$ ise $\sqrt{g} \to \sqrt{-g}$ olarak yazarım.

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{g}}\, \partial_\mu \sqrt{g}\, V^\mu = \frac{1}{\sqrt{g}}\, \partial_\mu (\sqrt{g}\, V^\mu) \checkmark$$

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2} g^{\mu\mu}\left(\cancel{\partial_\mu g_{\mu\nu}} + \partial_\nu g_{\mu\mu} - \cancel{\partial_\mu g_{\mu\nu}}\right)$$

$$= \frac{1}{2} g^{\mu\mu}\, \partial_\nu g_{\mu\mu}$$

$$= \frac{1}{2} \mathrm{Tr}(g^{-1} \partial_\nu g) \,//$$

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\, \partial_\mu (\sqrt{g}\, V^\mu)$$

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \underbrace{\Gamma^{\mu}_{\mu\nu}}_{\frac{1}{\sqrt{g}} \partial_\nu \sqrt{g}} V^\nu$$

$$= \frac{1}{\sqrt{g}}\, \partial_\nu \sqrt{g}\, V^\nu$$

$$= \frac{1}{\sqrt{g}}\, \partial_\mu \sqrt{g}\, V^\mu$$

Example

$$V^\mu = g^{\mu\nu} \partial_\nu f$$

$$\nabla_\mu \partial^\mu f = \frac{1}{\sqrt{g}}\, \partial_\mu (\sqrt{g}\, g^{\mu\nu} \partial_\nu f)$$

(r, θ)

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \;;\; g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \;;\; \det g = r^2$$

$$\nabla^2 f = \nabla_i \partial^i f = \frac{1}{r}\, \partial_r (r \partial_r f) + \frac{1}{r}\, \partial_\theta \left(\frac{1}{r} \partial_\theta f\right)$$

$$= \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

küreselde

$$g = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/(r^2 \sin^2\theta) \end{pmatrix} \qquad \det g = r^4 \sin^2\theta$$

$$\nabla^2 \phi = \frac{1}{r^2 \sin\theta}\left[\partial_r (r^2 \sin\theta\, \partial_r \Phi) + \partial_\theta (\sin\theta\, \partial_\theta \Phi) + \partial_\phi \left(\frac{1}{\sin\theta}\, \partial_\phi \Phi \right]\right.$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

$$D_\mu \Phi = \partial_\mu \Phi + i A_\mu \Phi$$

$$D_\mu D_\nu \Phi = \partial_\mu D_\nu \Phi + i A_\nu D_\nu \Phi = \partial_\mu (\partial_\nu \Phi + i A_\mu \Phi) + i A_\nu (\partial_\mu \Phi + i A_\mu \Phi)$$

$$= \partial_\mu \partial_\nu \Phi + i \partial_\mu A_\nu \Phi + i A_\mu \partial_\nu \Phi + i A_\nu \partial_\mu \Phi - A_\mu A_\nu \Phi$$

$$- \quad D_\nu D_\mu \Phi = \partial_\nu \partial_\mu \Phi + i \partial_\nu A_\mu \Phi + i A_\nu \partial_\mu \Phi + i A_\mu \partial_\nu \Phi - A_\nu A_\mu \Phi$$

$$[D_\mu, D_\nu] \Phi = i(\partial_\mu A_\nu - \partial_\nu A_\mu) \Phi = i F_{\mu\nu} \Phi$$

For gravity

$$\nabla_\mu \nabla_\nu V^\rho = \partial_\mu (\nabla_\nu V^\rho) - \Gamma^{\lambda}_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^{\rho}_{\mu\lambda} \nabla_\nu V^\lambda$$

$$= \partial_\mu (\partial_\nu V^\rho + \Gamma^{\rho}_{\nu\lambda} V^\lambda) - \Gamma^{\lambda}_{\mu\nu} (\partial_\lambda V^\rho + \Gamma^{\rho}_{\lambda\sigma}) + \Gamma^{\rho}_{\mu\lambda} (\partial_\nu V^\lambda + \Gamma^{\lambda}_{\nu\sigma} V^\sigma)$$

$$= \cancel{\partial_\mu \partial_\nu V^\rho} + \cancel{(\partial_\mu \Gamma^\rho_{\nu\lambda})} V^\lambda + \Gamma^\rho_{\nu\lambda}\, \partial_\mu V^\lambda - \cancel{\Gamma^\lambda_{\mu\nu}(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma)} + \cancel{\Gamma^\rho_{\mu\lambda}\, \partial_\nu V^\lambda} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} V^\sigma$$

$$- \nabla_\nu \nabla_\mu V^\rho = \cancel{\partial_\nu \partial_\mu V^\rho} + \cancel{(\partial_\nu \Gamma^\rho_{\mu\lambda})} V^\lambda + \Gamma^\rho_{\mu\lambda}\, \partial_\nu V^\lambda - \cancel{\Gamma^\lambda_{\nu\mu}(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma)} + \cancel{\Gamma^\rho_{\nu\lambda}\, \partial_\mu V^\lambda} + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} V^\sigma$$

$$[\nabla_\mu, \nabla_\nu] V^\rho = \left(\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}\right) V^\lambda = \underline{R^\rho{}_{\lambda\mu\nu}}\, V^\lambda$$

→ Riemann Tensor

<u>Symmetries of the Riemann tensor</u> Chris Pope Section 4

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$$

$$\Gamma_{\mu\rho\sigma} = g_{\mu\lambda} \Gamma^\lambda{}_{\rho\sigma} = g_{\mu\lambda}\left[\frac{1}{2} g^{\lambda\tau}(\partial_\rho g_{\tau\sigma} + \partial_\sigma g_{\rho\tau} - \partial_\tau g_{\rho\sigma})\right]$$

$$= \frac{1}{2}(\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\rho\mu} - \partial_\mu g_{\rho\sigma})$$

$$R_{\alpha\sigma\mu\nu} = g_{\rho\alpha} R^\rho{}_{\sigma\mu\nu}$$

$$= g_{\rho\alpha}\, \partial_\mu (g^{\rho\lambda} \Gamma_{\lambda\nu\sigma}) - g_{\rho\alpha}\, \partial_\nu (g^{\rho\lambda} \Gamma_{\alpha\mu\lambda}) + \Gamma_{\alpha\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma_{\alpha\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$= g_{\rho\alpha}(\partial_\mu g^{\rho\lambda})\Gamma_{\lambda\nu\sigma} + g_{\rho\alpha} g^{\rho\lambda}\, \partial_\mu \Gamma_{\lambda\nu\sigma} - g_{\rho\alpha}(\partial_\nu g^{\rho\lambda})\Gamma_{\alpha\mu\lambda} - g_{\rho\alpha} g^{\rho\lambda} \partial_\nu \Gamma_{\alpha\mu\lambda} + \Gamma_{\alpha\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma_{\alpha\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$\partial_\lambda (g_{\mu\nu} g^{\nu\rho}) = \partial_\lambda \delta^\rho_\mu = 0 \quad \Rightarrow \quad g_{\mu\nu}\, \partial_\lambda g^{\nu\rho} = -g^{\nu\rho}\, \partial_\lambda g_{\mu\nu}$$

$$0 = \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma}$$

$$g_{\mu\nu}\, \partial_\lambda g^{\nu\rho} = -g^{\nu\rho}(\Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma})$$

$$= -\Gamma^\rho_{\lambda\mu} - \Gamma_{\mu\lambda\nu} g^{\nu\rho}$$

- $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
- $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$
- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
- $R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$

$$R_{\alpha\sigma\mu\nu} = g_{\alpha\rho} R^\rho{}_{\sigma\mu\nu}$$

$$= g_{\alpha\rho}(\partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})$$

$$= g_{\alpha\rho}\, \partial_\mu (g^{\rho\lambda}\Gamma_{\lambda\sigma\nu}) - g_{\alpha\rho}\, \partial_\nu (g^{\rho\lambda}\Gamma_{\lambda\sigma\mu}) + \Gamma_{\alpha\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$= (g_{\alpha\rho}\, \partial_\mu g^{\rho\lambda})\Gamma_{\lambda\sigma\nu} - (g_{\alpha\rho}\, \partial_\nu g^{\rho\lambda})\Gamma_{\lambda\sigma\mu} + \partial_\mu \Gamma_{\alpha\sigma\nu} - \partial_\nu \Gamma_{\alpha\sigma\mu} + \Gamma_{\alpha\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma_{\alpha\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$= (-g^{\rho\lambda}\, \partial_\mu g_{\alpha\rho})\Gamma_{\lambda\sigma\nu} + (g^{\rho\lambda}\, \partial_\nu g_{\alpha\rho})\Gamma_{\lambda\sigma\mu} + \ldots$$

$$= -g^{\rho\lambda}(\underbrace{\nabla_\mu g_{\alpha\rho}}_{0} + \Gamma^{\tau}_{\mu\alpha}\, g_{\tau\rho} + \Gamma^{\tau}_{\mu\rho}\, g_{\alpha\tau})\Gamma_{\lambda\sigma\nu}$$

$$+ g^{\rho\lambda}(\underbrace{\nabla_\nu g_{\alpha\rho}}_{0} + \Gamma^{\tau}_{\nu\alpha}\, g_{\tau\rho} + \Gamma^{\tau}_{\nu\rho}\, g_{\alpha\tau})\Gamma_{\lambda\sigma\mu} + \dots$$

• $\partial_\mu \delta_\nu^{\rho} = \partial_\mu (g_{\nu\lambda}\, g^{\rho\lambda})$

$\Rightarrow g_{\nu\lambda}\, \partial_\mu g^{\rho\lambda} = -g^{\rho\lambda}\, \partial_\mu g_{\nu\rho}$

• $\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu}\, g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho}\, g_{\nu\lambda} = 0$

$$= \partial_\mu \Gamma_{\alpha\sigma\mu} - \partial_\nu \Gamma_{\alpha\sigma\mu} + \overbrace{\Gamma_{\alpha\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}}^{1} \underbrace{- \Gamma_{\alpha\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}}_{2} - \Gamma^{\tau}_{\mu\alpha}\Gamma_{\tau\sigma\nu} \underbrace{- g^{\rho\lambda}\Gamma_{\alpha\mu\rho}\Gamma_{\lambda\sigma\nu}}_{\substack{\Gamma^{\rho}_{\sigma\nu}\Gamma_{\alpha\mu\rho} \\ 1}} + \Gamma^{\tau}_{\nu\alpha}\Gamma_{\lambda\sigma\mu}$$

$$+ \underbrace{g^{\rho\lambda}\Gamma_{\alpha\nu\rho}\Gamma_{\lambda\sigma\mu}}_{\Gamma^{\lambda}_{\sigma\mu}\Gamma_{\alpha\nu\lambda}}$$

Thus

$$R_{\alpha\sigma\mu\nu} = \partial_\mu \Gamma_{\alpha\sigma\nu} - \partial_\nu \Gamma_{\alpha\sigma\mu} - g_{\tau\rho}(\Gamma^{\rho}_{\mu\alpha}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\tau}_{\sigma\mu})$$

using $\quad g_{\mu\tau}\left\{\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho})\right\}$

$$\Gamma_{\tau\nu\rho} = \frac{1}{2}(\partial_\nu g_{\tau\rho} + \partial_\rho g_{\tau\nu} - \partial_\tau g_{\nu\rho})$$

Thus

$$R_{\alpha\sigma\mu\nu} = \frac{1}{2}\partial_\mu(\partial_\sigma g_{\alpha\nu} + \partial_\nu g_{\alpha\sigma} - \partial_\alpha g_{\sigma\nu}) - \frac{1}{2}\partial_\nu(\partial_\sigma g_{\alpha\mu} + \partial_\mu g_{\alpha\sigma} - \partial_\alpha g_{\mu\sigma})$$

$$- g_{\tau\rho}(\Gamma^{\rho}_{\mu\alpha}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\tau}_{\sigma\mu})$$

$$= \frac{1}{2}(\partial_\mu \partial_\sigma g_{\alpha\nu} - \partial_\mu \partial_\alpha g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\alpha\mu} + \partial_\nu \partial_\alpha g_{\mu\sigma}) - g_{\tau\rho}(\Gamma^{\rho}_{\mu\alpha}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\tau}_{\sigma\mu})$$

Thus

- $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$
- $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
- $R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 = R_{\mu[\nu\rho\sigma]} = \frac{1}{3!}(\underline{R_{\mu\nu\rho\sigma}} + \underline{R_{\mu\sigma\nu\rho}} + \underline{R_{\mu\rho\sigma\nu}}$
  $- \underline{R_{\mu\rho\nu\sigma}} - \underline{R_{\mu\sigma\rho\nu}} - \underline{R_{\mu\nu\sigma\rho}}) = 0$

Components of the Riemann tensor

$$\underbrace{R_{\mu\nu}}_{\frac{d(d-1)}{2}}\underbrace{{}_{\rho\sigma}}_{\frac{d(d-1)}{2}} \qquad A_{\mu\nu\,\rho\sigma}$$

$$\frac{n(n+1)}{2} = \frac{d(d-1)}{2}\left(\frac{d(d-1)}{2} + 1\right) - \frac{1}{4!} d(d-1)(d-2)(d-3) = \frac{1}{12} d^2(d^2-1)$$

• Bianchi identity

$$\nabla_\mu R^{\rho}{}_{\nu\sigma\lambda} + \nabla_\lambda R^{\rho}{}_{\nu\mu\sigma} + \nabla_\sigma R^{\rho}{}_{\nu\lambda\mu} = 0$$

$$g'_{\mu\nu}(x') = \eta_{\mu\nu}, \quad \partial_\mu g'_{\nu\rho}(x)\Big|_{x=x'} = 0$$

$x^\mu(x) = x^\mu(x')$

$$= a^\mu{}_\nu x'^\nu + \frac{1}{2!} a^\mu{}_{\nu\rho} x'^\nu x'^\rho + \frac{1}{3!} a^\mu{}_{\nu\rho\sigma} x'^\nu x'^\rho x'^\sigma + \dots \quad : \text{Taylor series}$$

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

and

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + \partial_\alpha g_{\mu\nu}(0) x^\alpha + \frac{1}{2} \partial_\alpha \partial_\beta g_{\mu\nu}(0) x^\alpha x^\beta +$$

Then

$$g'_{\mu\nu}(x') = \left(a^\rho_\mu + a^\rho_{\mu\alpha_1} x'^{\alpha_1} + \frac{1}{2} a^\rho_{\mu\alpha_1\alpha_2} x'^{\alpha_1} x'^{\alpha_2} + \dots\right)\left(a^\sigma_\nu + a^\sigma_{\nu\beta_1} x'^{\beta_1} + \frac{1}{2} a^\sigma_{\nu\beta_1\beta_2} x'^{\beta_1} x'^{\beta_2} + \dots\right)$$
$$\left(g_{\rho\sigma}(0) + \partial_\gamma g_{\rho\sigma}(0)\left[a^\gamma_\tau x'^\tau + a^\gamma_{\tau_1\tau_2} x'^{\tau_1} x'^{\tau_2} \dots\right]\right.$$
$$+ \partial_\tau \partial_\gamma g_{\rho\sigma}(0)\left[a^\tau_{\beta_1} x'^{\beta_1} + a^\tau_{\beta_1\beta_2} x'^{\beta_1} x'^{\beta_2}\right]\left[a^\gamma_{\delta_1} x'^{\delta_1} + a^\gamma_{\delta_1\delta_2} x'^{\delta_1} x'^{\delta_2}\right]$$

i-) $\underbrace{g'_{\mu\nu}(0)}_{\text{10 elements}} = a^\rho_\mu \, a^\sigma_\nu \, g_{\rho\sigma}(0)$

$a^\mu{}_\nu \rightarrow 16$
- $\rightarrow$ 10 to fix $g'_{\mu\nu}(0) = \eta_{\mu\nu}$
- $\rightarrow$ 6 are the Lorentz Transformation

ii-) $\underbrace{\underbrace{\partial'_\lambda}_{4} \, \underbrace{g'_{\mu\nu}}_{10}}_{40}(0) = a^\rho_\mu \, a^\sigma_\nu \, a^\gamma_\lambda \, \partial_\gamma g_{\rho\sigma}(0) + \left[a^\rho_\mu \, a^\sigma_{\nu\lambda} + a^\rho_{\mu\lambda} \, a^\sigma_\nu\right] g_{\rho\sigma}(0)$

$a^\mu{}_{\nu\rho} \rightarrow$ 40 components $\quad \partial'_\lambda g'_{\mu\nu}(0) = 0$

$4\binom{4+5}{2} = 40$

iii-) $\underbrace{\underbrace{\partial'_{\lambda_1} \partial'_{\lambda_2}}_{10} \, \underbrace{g_{\mu\nu}}_{10}}_{100}(0) \xleftarrow{\text{doesnt match}} a^\mu{}_{\nu\rho\sigma} \rightarrow$ 80 components

It doesnt enough make it zero

$\Gamma = 0$ (Christoffels vanished)

Thus, in Riemann normal coordinates, we have

$$\nabla_\lambda R_{\mu\nu\rho\sigma} = \frac{1}{2} \partial_\lambda\left(\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\nu\sigma} + \partial_\nu \partial_\rho g_{\mu\sigma} - \partial_\nu \partial_\sigma g_{\mu\rho}\right)$$

$\nabla_\sigma R_{\mu\nu\lambda\rho} =$

Note $\nabla_\mu V_\nu = V_{\nu;\mu}$

$+ \nabla_\rho R_{\mu\nu\sigma\lambda} =$

$= 0 \quad = R_{\mu\nu[\rho\sigma;\lambda]} = 0$

Ricci Tensor:

$R^\lambda{}_{\mu\lambda\nu} = R_{\mu\nu}$ with $R_{\mu\nu} = R_{\nu\mu}$

since $R^\lambda{}_{\mu\lambda\nu} = R_{\lambda\nu}{}^\lambda{}_\mu = R^\lambda{}_{\nu\lambda\mu} = R_{\nu\mu}$

Ricci Scalar

$$g^{\mu\nu} R_{\mu\nu} = R$$

Let $\rho = \sigma$

$$\nabla_\mu \underbrace{R^\rho{}_{\nu\rho\lambda}}_{R_{\nu\lambda}} + \nabla_\lambda \underbrace{R^\rho{}_{\nu\mu\rho}}_{-\nabla_\lambda R^\rho{}_{\nu\rho\mu}} + \nabla_\rho R^\rho{}_{\nu\lambda\mu} = 0$$

$$-\nabla_\lambda R_{\nu\mu}$$

$$\Rightarrow \left\{ \nabla_\rho R^\rho{}_{\nu\lambda\mu} = -\nabla_\mu R_{\nu\lambda} + \nabla_\lambda R_{\nu\mu} \right\} g^{\mu\nu}$$

$$\nabla_\rho R^\rho{}_\lambda = -\nabla_\mu R^\mu{}_\lambda + \partial_\lambda R$$

$$\nabla_\rho \left(\boxed{R^{\rho\lambda} - \frac{1}{2} g^{\rho\lambda} R} \right) = 0 \rightarrow \text{Einstein Tensor}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$2 \nabla_\rho R^\rho{}_\lambda = \nabla_\lambda R$$

$$g^{\lambda\sigma} \quad \nabla_\rho R^\rho{}_\lambda - \frac{1}{2} \nabla_\lambda R = 0$$

$$\nabla_\rho R^{\rho\sigma} - \frac{1}{2} \nabla_\lambda g^{\lambda\sigma} R = 0$$

$$\nabla_\rho R^{\rho\sigma} - \frac{1}{2} \nabla_\rho g^{\rho\sigma} R = 0$$

$$\nabla_\rho \underbrace{\left(R^{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} R \right)}_{G^{\rho\sigma}} = 0$$

$$\nabla^\mu G_{\mu\nu} = 0 \rightarrow \text{identically zero}$$

Parallel Transportation and the meaning of curvature

Flat Space: $\quad \dfrac{dV^\mu(x)}{d\lambda} = 0$

$$V = \tilde{V}^\mu e_\mu$$

Let $\tilde{V}^\mu = \dfrac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu$

then

$$0 = \frac{d}{d\lambda}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu \right) = \frac{dx^\rho}{d\lambda} \frac{\partial^2 \tilde{x}^\mu}{\partial x^\rho \partial x^\nu} V^\nu + \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\nu}}_{\nu \to \sigma} \frac{dV^\nu}{d\lambda}$$

then

$$0 = \frac{dV^\sigma}{d\lambda} + \frac{dx^\rho}{d\lambda} \underbrace{\frac{\partial x^\sigma}{\partial \tilde{x}^\mu} \frac{\partial^2 \tilde{x}^\mu}{\partial x^\rho \partial x^\nu}}_{\Gamma^\sigma_{\rho\nu}\ ???} V^\nu = \frac{dx^\rho}{d\lambda} \underbrace{\left(\partial_\rho V^\sigma + \Gamma^\sigma_{\rho\nu} V^\nu \right)}_{\nabla_\rho V^\sigma}$$

So parallel transportation

$$\boxed{\frac{DV^\sigma}{D\lambda} = \frac{dx^\rho}{d\lambda} \nabla_\rho V^\sigma = 0}$$

Consider a displacement, by parallel transport, along an infinitesmall segment of the curve $x^\mu(\lambda)$

$$\left[0 = \frac{dV^\sigma}{d\lambda} + \frac{dx^\rho}{d\lambda} \Gamma^\sigma_{\rho\nu} V^\nu \right] d\lambda$$

$$0 = \underbrace{\frac{dV^\sigma}{d\lambda}\delta\lambda}_{\delta V^\sigma} + \underbrace{\frac{dx^\rho}{d\lambda}\delta\lambda}_{\delta x^\rho} \Gamma^\sigma_{\rho\nu} V^\nu \Rightarrow \boxed{\delta V^\mu = -\Gamma^\mu_{\nu\rho} V^\rho \delta x^\nu}$$

$$\Delta V^\mu = \oint_C \delta V^\mu(x) = -\oint_C \Gamma^\mu_{\nu\rho}(x) V^\rho(x)\, dx^\nu$$

$$V^\mu(x) = V^\mu(0) - \Gamma^\mu_{\nu\rho}(0) V^\rho(0)\, x^\nu$$

$$\Gamma^\mu_{\nu\rho}(x) = \Gamma^\mu_{\nu\rho}(0) + \partial_\gamma \Gamma^\mu_{\nu\rho}(0) x^\gamma + \ldots$$

Thus

$$\Delta V^\mu = -\oint \left(\Gamma^\mu_{\nu\rho}(0) + \partial_\gamma \Gamma^\mu_{\nu\rho}(0) x^\gamma + \ldots\right)\left(V^\rho(0) - \Gamma^\rho_{\alpha\beta} V^\beta(0) x^\alpha + \ldots\right) dx^\nu$$

$$= -\Gamma^\mu_{\nu\rho}(0) V^\rho(0) \underbrace{\oint_C dx^\nu}_{0} - \partial_\gamma \Gamma^\mu_{\nu\rho} V^\rho(0) \oint_C x^\gamma dx^\nu + \Gamma^\mu_{\nu\rho}(0) \Gamma^\rho_{\gamma\beta} V^\beta \underbrace{\oint_C x^\gamma dx^\nu}_{= -\oint_C x^\nu dx^\gamma + \cancel{x^\gamma x^\nu}}$$

$$= -\frac{1}{2}\underbrace{\left[\partial_\gamma \Gamma^\mu_{\nu\rho}(0) - \partial_\nu \Gamma^\mu_{\gamma\rho}(0) + \Gamma^\mu_{\nu\lambda}(0)\Gamma^\lambda_{\gamma\rho}(0) - \Gamma^\mu_{\gamma\lambda}(0)\Gamma^\lambda_{\nu\rho}(0)\right]}_{R^\mu{}_{\rho\gamma\nu}} V^\rho \oint x^\gamma dx^\nu$$

Geodesic Deviation

Schwarzschild: $ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2$

ϕ

$g \sim \phi$

$\Gamma \sim F$

$R \sim$

- *Timelike-Curves*

Geodesic equation

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau}$$

$$= u^\nu \nabla_\nu u^\mu = \frac{dx^\nu}{d\tau}\left(\partial_\nu u^\mu + \Gamma^\mu_{\nu\rho} u^\rho\right) = \frac{Du^\mu}{D\tau}$$

$$0 = \frac{d^2 \delta x^\mu}{d\tau^2} + \partial_\sigma \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} \delta x^\sigma + 2\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{d}{d\tau} \delta x^\rho$$

$$= \frac{d^2 z}{d\tau^2} + \partial_\sigma \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + 2\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dz^\rho}{d\tau} = 0$$

$$a^\mu = \frac{D^2 z^\mu}{D\tau^2} = u^\nu \nabla_\nu \left(\frac{Dz^\mu}{D\tau}\right)$$

$$\frac{Dz^\mu}{D\tau} = u^\nu \nabla_\nu z^\mu = \frac{dx^\nu}{d\tau}\left(\partial_\nu z^\mu + \Gamma^\mu_{\nu\rho} z^\rho\right) = \frac{dz^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau} z^\rho$$

$$= \frac{dx^\nu}{d\tau}\left(\partial_\nu\left(\frac{Dz^\mu}{D\tau}\right) + \Gamma^\mu_{\nu\rho}\frac{Dz^\rho}{D\tau}\right)$$

$$= \frac{dx^\nu}{d\tau}\left[\partial_\nu\left(\frac{dz^\mu}{d\tau} + \Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\tau} z^\rho\right) + \frac{dx^\nu}{d\tau}\Gamma^\mu_{\nu\rho}\left(\frac{dz^\rho}{d\tau} + \Gamma^\rho_{\sigma\lambda}\frac{dx^\sigma}{d\tau} z^\lambda\right)\right.$$

$$-\partial_\sigma \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} z^\sigma$$

$$-2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dz^\rho}{d\tau} = \frac{d^2 z^\mu}{d\tau^2} + \partial_\sigma \Gamma^\mu_{\lambda\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau} z^\rho + \Gamma^\mu_{\lambda\rho}\frac{d^2 x^\lambda}{d\tau^2} z^\rho + \underline{\Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dz^\rho}{d\tau}}$$

$$+ \underline{\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dz^\rho}{d\tau}} + \Gamma^\mu_{\nu\alpha}\Gamma^\alpha_{\sigma\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} z^\lambda$$

$$\frac{d^2 x^\lambda}{d\tau^2} \to -\Gamma^\mu_{\lambda\rho}\Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} z^\rho$$

$$\frac{D^2 z^\mu}{D\tau^2} = a^\mu = \left(\partial_\sigma \Gamma^\mu_{\lambda\rho} - \partial_\rho \Gamma^\mu_{\lambda\sigma} + \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\lambda\rho} - \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\sigma\lambda}\right)\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau} z^\rho$$

$$= R^\mu{}_{\lambda\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau} z^\rho$$

Geodesic deviation equation

Flat space $R^\mu{}_{\nu\rho\sigma} = 0 \Rightarrow$ no geodesic deviation

$\hookrightarrow$ non-local effects (tidal effects)

Geodesic motion in Newtonian limit

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

- $\frac{dx^i}{d\tau} \ll 1 (c)$; $v \ll c$
- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (weak gravity) $\rightarrow \delta_\mu^\rho = g_{\mu\nu} g^{\nu\rho}$
- $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$$= (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - h^{\nu\rho})$$

$$= \delta_\mu^\rho + \underbrace{h_{\mu\nu}\eta^{\nu\rho}}_{h_\mu^\rho} - \underbrace{\eta_{\mu\nu}h^{\nu\rho}}_{h_\mu^\rho} + \underbrace{\mathcal{O}(h^2)}_{\text{neglect}}$$

$$d\tau^2 = dt^2 - dx^i dx_i$$

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \cancel{\left(\frac{dx_i}{d\tau}\right)\left(\frac{dx^i}{d\tau}\right)} \qquad dt \sim d\tau$$

- $\frac{d^2 x^i}{dt^2} + \Gamma^i_{00} = 0$
- $\frac{\partial g_{\mu\nu}}{\partial t} = 0$ (Metrik statik)

$$\Gamma^i_{00} = -\frac{1}{2}\delta^{ij}\partial_j h_{00} = -\frac{1}{2}\partial^i h_{00}$$

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}\left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}\right)$$

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i \underbrace{h_{00}}_{h_{00}=-2\phi} \rightarrow \frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi$$

T^{ij} — i: i sabit yüzeyinden geçen; j: j yönündeki akı

$$ds^2 = -(1+2\phi)dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$$

Newton Theory

$$\nabla^2\phi = 4\pi G\rho$$

- $\nabla^\mu T_\mu{}^\nu = 0$: Energy-momentum tensor is divergence free

- $\partial^\mu T_\mu{}^\nu = 0$

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu$$

$$S = \int d^dx\, \mathcal{L}(\phi_a, \partial_\mu\phi_a) \qquad (a=1,\ a=\mu,\ a=\mu\nu ..)$$

a → Some arbitrary label

i.) $\phi(x) \longrightarrow \phi(x') = \phi(x+\xi^\mu) = \phi(x) + \xi^\mu\partial_\mu\phi$

$$\delta\phi = \xi^\mu\partial_\mu\phi$$

$$\delta S = \int d^dx\, \delta\mathcal{L} = \int d^dx\, \xi^\mu\partial_\mu\mathcal{L}$$

$$\delta S = \int d^dx\, \delta\mathcal{L} = \int d^dx\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\,\partial_\mu\phi_a\right]$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu\delta\phi_a = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right] - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a$$

$$= \int d^dx\left\{\underbrace{\cancel{\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]}}_{\text{eom}}\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\right\}$$

$$\Rightarrow \partial_\mu\left[\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - \xi^\mu\mathcal{L}}_{J^\mu}\right] = 0$$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - \xi^\mu\mathcal{L} \quad \text{with } \partial^\mu J_\mu = 0$$

$$\phi_a(x) \longrightarrow \phi_a(x') = \phi_a(x+\xi) = \phi_a(x) + \xi^\mu \partial_\mu \phi$$

$$\delta\phi_a(x) = \xi^\nu \partial_\nu \phi_a(x)$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \xi^\nu \partial_\nu \phi_a - \xi^\mu \mathcal{L}$$

$$J^\mu = \xi^\nu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \right)}_{T^\mu{}_\nu} \Rightarrow \partial_\mu T^\mu{}_\nu = 0$$

Belinfante – Rosenfeld Tensor (symmetrized)

$$\nabla^\mu G_{\mu\nu} = 0$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$g_{\mu\nu} \sim \phi$$

$$\Gamma^\mu_{\nu\rho} \sim \partial^i \phi$$

$$R_{\mu\nu\rho\sigma} \sim \nabla^2 \phi$$

$$G_{\mu\nu} = \lambda T_{\mu\nu}$$

which satisfies

Diffeomorphism invariance $\rightarrow$ $\nabla^\mu G_{\mu\nu} = 0$: identical zero

and

$\nabla^\mu T_{\mu\nu} = 0$; when eom hold

- $g^{\mu\nu}(G_{\mu\nu} = \lambda T_{\mu\nu})$

$$g^{\mu\nu} G_{\mu\nu} = \underbrace{g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)}_{R - \frac{1}{2} 4R}$$

$$\lambda g^{\mu\nu} T_{\mu\nu} = -R$$

Newtonian limit the point-like particle

$$T_{\mu\nu} = \rho\, \delta_\mu{}^0 \delta_\nu{}^0 \delta(x)$$

$$g^{\mu\nu} \sim \eta^{\mu\nu}$$

$$-R = \lambda g^{\mu\nu} T_{\mu\nu} = -\lambda \rho \qquad R \approx \lambda \rho$$

Thus

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \lambda T_{\mu\nu}$$

$$R_{00} + \frac{1}{2} \lambda \rho \approx \lambda \rho$$

$$R_{00} \approx \frac{1}{2} \lambda \rho$$

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\rho\nu} + \Gamma\Gamma - \Gamma\Gamma$$

$\mathcal{O}(h^2) \sim$ neglect

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \cancel{\partial_0 \Gamma^{i}_{j0}}$$

$$R_{00} = R^{\mu}{}_{0\mu0} = \cancel{R^{0}{}_{000}} + R^{i}{}_{0i0} = \partial_i \Gamma^{i}_{00} = \vec{\nabla}^2 \phi$$

$$R_{00} \approx \frac{1}{\lambda} \partial \rho \qquad \lambda = 8\pi G$$

Schwarzschild Solution

Birkoff theorem

$$ds^2 = -V(r,t)\,dt^2 + W(r,t)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$\dot{} = \frac{d}{dt} \qquad {}' = \frac{d}{dr}$$

Then

$$R_{tt} = \frac{V'}{rW} - \frac{V'^2}{4VW} - \frac{V'W'}{4W^2} + \frac{V''}{2W} + \frac{\dot{V}\dot{W}}{4VW} + \frac{\dot{W}^2}{4W^2} - \frac{\ddot{W}}{2W}$$

$$R_{tr} = \frac{\dot{W}}{rW}$$

$$R_{rr} = \frac{W'^2}{4V^2} + \frac{V'}{rW} + \frac{V'W'}{4VW} - \frac{V''}{2V} - \frac{\dot{V}\dot{W}}{4VW} - \frac{\dot{W}^2}{4VW} + \frac{\ddot{W}}{2V}$$

$$R_{\theta\theta} = 1 - \frac{1}{W} - \frac{rV'}{2VW} + \frac{rW'}{2W^2}$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

Süreklilik denklemi
local olarak falan filan
sınav sorusu

Then

i-) $R_{tr} = 0 \Rightarrow W = W(r)$

ii-) $\frac{W}{V} R_{tt} - R_{rr} = \frac{1}{r}\left(\frac{V'}{V} + \frac{W'}{W}\right)$

$$= \frac{1}{r}\frac{d}{dr} \ln(V.W) = 0$$

$$G_{\mu\nu} = 0 \rightarrow \lambda T_{\mu\nu} \rightarrow R = -\lambda \cancel{g^{\mu\nu} T_{\mu\nu}} = 0$$

$$G_{\mu\nu} = 0 \rightarrow R_{\mu\nu} - \frac{1}{2}\cancel{g_{\mu\nu} R}$$

$$G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$$

Thus

$\ln(V.W) =$ some arbitrary function of t

$$V = \frac{g(t)}{W(r)}$$

Thus

$$ds^2 = -\frac{g(t)}{W(r)}dt^2 + W(r)dr^2 + \underbrace{r^2 d\Omega_2^2}$$

Line element of unit 2-sphere

$$dt' = \sqrt{g(t)}\,dt$$

$$ds^2 = \frac{-dt^2}{W(r)} + W(r)dr^2 + r^2 d\Omega_2^2$$

$$R_{\theta\theta} = 1 - \frac{1}{W} + \frac{rW'}{W^2} = 0 \Rightarrow \frac{1}{W(r)} = 1 + \frac{c}{r}$$

some arbitrary constant

$$ds^2 = -\left(1+\frac{c}{r}\right)dt^2 + \frac{1}{\left(1+\frac{c}{r}\right)}dr^2 + r^2 d\Omega_2^2$$

c: determined by Newtonian limit

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)dt^2 + \left(1-\frac{2GM}{rc^2}\right)dr^2 + r^2 d\Omega_2^2$$: Schwarzschild solution