Geometric Formulation of Hamiltonian Mechanics

a-) Symplectic Geometry : Works for autonomous system : $H \neq H(t)$

Definition : Let M^{2n} be an even-dimensional manifold. A symplectic structure on M is a closed non-degenerate 2-form

i-) $d\omega = 0$: closed

ii-) Non degenerate $\forall X \neq 0$ there exist ($\exists$) Y $\omega(X,Y) \neq 0$: matrix with rank 2

The pair (M^{2n}, ω) is called symplectic manifold.

- Define Ω as an inverse of ω

$$\Omega^{\alpha\beta} \omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

: since ω is non-degenerate, this is unique. So, ω defines an isomorphism between TM and T^*M

This allows us to "identify vectors with covectors".

$\omega : TM \longrightarrow T^*M$, $V^{\alpha} \rightarrow V_{\beta} = V^{\alpha}\omega_{\alpha\beta}$ or $V \rightarrow \omega(V, \cdot) = V \lrcorner \omega$

$\Omega : T^*M \rightarrow TM$, $\eta_{\alpha} \rightarrow \eta^{\beta} = \Omega^{\beta\alpha}\eta_{\alpha}$ or $\eta \rightarrow \Omega . \eta$

This is, we can rise or lower indices, similar to the metric but where metric is symmetric, ω and Ω are anti-symmetric.

- Given a function f on M, ω defines a Hamiltonian vector field X_f

$$\frac{d}{dt_f} = X_f = \Omega . df \qquad X_f^{\alpha} = \Omega^{\alpha\beta}\partial_{\beta} f$$

$$X_f \lrcorner \omega = -df \qquad X^{\alpha}\omega_{\alpha\gamma} = \Omega^{\alpha\beta}\partial_{\beta} f \,\omega_{\alpha\gamma} = \partial_{\beta} f \underbrace{\Omega^{\alpha\beta}\omega_{\alpha\gamma}}_{= -\Omega^{\beta\alpha}\omega_{\alpha\gamma} = -\delta^{\beta}_{\gamma}}$$

$$= -\partial_{\gamma} f = -df$$

Theorem: A Hamiltonian vector field preserves symplectic structure

$\mathcal{L}_{X_f} \omega = 0$: Hamiltonian vector fields which means vector fields which are generated by functions on the phase space, they behave nicely wrt ω.

Proof: $\mathcal{L}_{X_f}\omega = \cancel{X_f \cdot \underbrace{d\omega}_{0}} + \underbrace{d(\underbrace{X_f \cdot \omega}_{-df})}_{0} = 0$

Consequence : (Liouville's Theorem)

A Hamiltonian vector flow preserves the volume element on M

$$\epsilon = \omega^{\wedge n} = \frac{1}{n!} \underbrace{\omega \wedge \omega \dots \wedge \omega}_{n \text{ times}} \quad : 2n\text{-form} \qquad \omega \wedge \omega = 4\text{-form}$$

$$\mathcal{L}_{X_f}\epsilon = 0$$

$$\mathcal{L}_{X_f}\epsilon = \frac{1}{n!} \underbrace{\mathcal{L}_{X_f}\omega}_{0} \underbrace{\wedge \dots \wedge \omega}_{(n-1) \text{ times}} + \omega \wedge \underbrace{\mathcal{L}_{X_f}\omega}_{0} \wedge \dots + \dots = 0$$

• Given 2 functions f, g on M, ω defines their Poisson bracket by

$$\{f,g\} = df \cdot \Omega \cdot dg = -\omega(X_f, X_g)$$

Satisfies properties of P.B. Linearity, antisymmetry, Leibniz property

* Interesting is Jacobi identity $\leftrightarrow d\omega = 0$

One can also get

* $X_{\{f,g\}} = [X_f, X_g]$

• Darboux Theorem : Since ω is antisymmetric, closed and non-degenerate : Canonical form

± 1?

There exist a coordinate system on M

$$X^\alpha = (q^i, p_i) \qquad \Omega^{\alpha\beta} = \begin{array}{cc} & \begin{array}{cc} q_1 & p_1 \end{array} \\ \begin{array}{c} q_1 \\ p_1 \end{array} & \left(\begin{array}{cccc} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \emptyset \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & \ddots & \\ \emptyset & & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{array}\right) \end{array}$$

$$\Rightarrow \omega_{\alpha\beta} = \begin{array}{cc} & \begin{array}{cc} q_1 & p_1 \end{array} \\ \begin{array}{c} q_1 \\ p_1 \end{array} & \left(\begin{array}{cccc} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & \emptyset \\ & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & \\ & & \ddots & \\ \emptyset & & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} \end{array}\right) \end{array}$$

$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \ldots$

$\omega = dp_i \wedge dq^i$ These do not mean vector or 1-form, p and q can be thought as coordinates

Consequence

$$\epsilon = \underbrace{\omega \wedge \omega \ldots \wedge \omega}_{n} = \frac{1}{n!}\omega^{\wedge n} = dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 \ldots \wedge dp_n \wedge dq^n$$

is a volume form on M

* $\int f \hat{=} \int f \epsilon = \int f \, dp_1 dq^1 \ldots dp_n dq^n$

• The poisson brackets take the canonical form

$$\{f,g\} = df \cdot \Omega \cdot dg = \partial_\alpha f \, \Omega^{\alpha\beta} \, \partial_\beta g = \Omega^{\alpha\beta} \partial_\alpha f \, \partial_\beta g = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

$$= df \cdot X_g = X_g(f) = \frac{df}{dt_g} = \{f,g\}$$

$$= -\{g,f\} = -X_f(g) = -\frac{dg}{dt_f} \Rightarrow X_f = \frac{d}{dt_f} = \{\ ,f\}$$

$$X_f = \{\ ,f\} = \underbrace{\frac{\partial f}{\partial p_i}}_{\text{component}} \underbrace{\frac{\partial}{\partial q^i}}_{\text{basis}} - \underbrace{\frac{\partial f}{\partial q^i}}_{\text{comp}} \underbrace{\frac{\partial}{\partial p_i}}_{\text{basis}} = X_f^\mu \partial_\mu$$

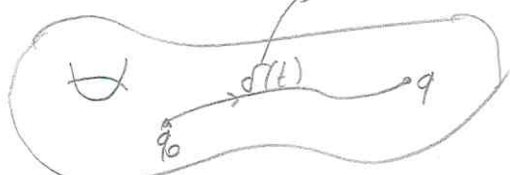
and $\{g,f\} = X_f g$ Canonical P.B.

b-) Hamiltonian Mechanics.

① Basic theory

Configuration space C for a system with n dof is a manifold of dimension n, equipped with local coordinates $(q^1 \dots q^n)$

trajectory $\simeq$ Time evolution



Phase space = Cotangent bundle T^*C. It has dimension $2n$ and local coordinates $(q^1 \dots q^n, p^1 \dots p^n)$

- Cartan's 1-form $\quad \theta \in T^*(T^*C) =$ 1-form space over phase space

$$\theta = p_i dq^i$$
$$\omega = d\theta = dp_i \wedge dq^i \quad : \text{Natural symplectic structure on } T^*C$$

- The dynamics is defined by specifying Hamiltonian $H : T^*C \to \mathbb{R}$

$$H = H(q^i, p_i) \quad , H \neq H(t) : \text{Autonomous system}$$

This defines dynamical Hamiltonian vector field

$$X_H = \frac{d}{dt_H} = \frac{d}{dt} = \Omega \cdot dH = \{\ , H\}$$

This field generates its integral curves

$$\gamma(t) = (q^i(t), p_i(t))$$

$$X_H = \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i} = \{\ , H\} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

$$\boxed{\frac{\partial H}{\partial p_i} = \dot q^i \ , \quad \frac{\partial H}{\partial q^i} = -\dot p_i} \quad \text{Hamilton's equations} \to \text{Determine } \gamma(t)$$

- Finally, people interested in $\sigma(t)$ rather than $\gamma(t)$

$$\sigma(t) = \pi(\gamma(t))$$

- $\mathcal{L}_{X_H}\omega = 0$: since X_H is a Hamiltonian vector flow (Liouville's theorem)
- Legendre transformation $L \longleftrightarrow H$ is a map $TC \longleftrightarrow T^*C$ that identifies $\dot q^j \longleftrightarrow p_j$ according to

$$p_j = \frac{\partial L}{\partial \dot q^j} \ , \quad \dot q^j = \frac{\partial H}{\partial p_j} \ , \quad H = \dot q^j p_j - L \Big|_{\dot q^j = \dot q^j(q^i, p_i)}$$

$$\begin{tikzcd}
 & \mathbb{R} & \\
(q^j, \dot q^j) = z \in TC \arrow[ur, "L"] \arrow[rr, leftrightarrow, "\text{Legendre}", "\text{Trfs}"'] \arrow[dr, "\pi"'] & & T^*C \ni z^* = (q^j, p_j) \arrow[ul, "H"'] \arrow[dl, "\pi"] \\
 & C \ni Q = (q^j) &
\end{tikzcd}$$

② Canonical transformation

$Q^j = Q^j(q,p)\ ,\ P_j = P_j(q,p)$

is canonical if it preserves the canonical form of ω

$\omega = dp_j \wedge dq^j = dP_j \wedge dQ^j$

For example: $n=1$

$$\omega = dP \wedge dQ = \left(\frac{dP}{dq}dq + \frac{dP}{dp}dp\right) \wedge \left(\frac{dQ}{dq}dq + \frac{dQ}{dp}dp\right)$$

$$= \underbrace{\left(\frac{dP}{dq}\frac{dQ}{dp} - \frac{dQ}{dq}\frac{dP}{dp}\right)}_{\{Q,P\}=1;\ \text{fundamental brackets is the condition}} dq \wedge dp = dq \wedge dp$$

$\theta = p_i dq^i\ ,\ d\theta = \omega$

$\tilde{\theta} = P_i dQ^i\ ,\ d\tilde{\theta} = \omega$

$0 = d(\underbrace{\theta - \tilde{\theta}}_{dF})$ locally there exists a generating function $F = F(q,p)$

$p_i dq^i - P_i dQ^i = dF$

We can take $F_1(q^i, Q^i) = F(q^i, P_i(q^j, Q^j))$

then $p_i dq^i - P_i dQ^i = \frac{\partial F_1}{\partial q^i} dq^i + \frac{\partial F_1}{\partial Q^i} dQ^i$

$p_i = \frac{\partial F_1}{\partial q^i}\ ,\ P_i = -\frac{\partial F_1}{\partial Q^i}$, $H' = H$: there is no time dependency

③ Symmetries

Noether's Theorem (V4: Hamiltonian version)

Let Y be a vector field in phase space $Y \in T^*(TC)$ such that

$\mathcal{L}_Y \omega = 0 = \mathcal{L}_Y H$ (symmetry)

$\exists\ I$ such that $\frac{dI}{dt} = 0 = \mathcal{L}_{X_H} I$

Proof: $\mathcal{L}_Y \omega = 0 = Y \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{Y \lrcorner \omega}_{-dI\ \text{locally}})$

There exist I s.t. $Y \lrcorner \omega = -dI$ $\qquad Y = Y_I$ is a Hamiltonian vector field corresponding to I.

- $0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = Y_I H = \{H, I\}$ $\longrightarrow$ Lie derivative acting on a function

$= -\{I, H\} = -\frac{dI}{dt} = 0$

This is on the level of phase space and hence is more general than usually presented. We can distinguish two cases based on the canonical projection $\pi^*: T(T^*C) \longrightarrow TC$ of the vector field Y:

$\pi^*(y)$ → vector field which lives on C, symmetry of C --- isometry
↘ Not well defined on C... dynamical (hidden) symmetries

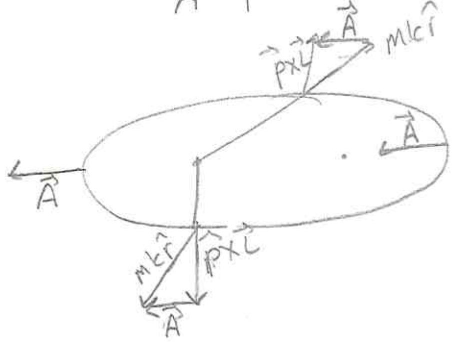
<u>Example</u> Kepler problem

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

We have isometries : stationary → E : energy
spherical symmetry → $\vec{L}$: Angular momentum

However, there also exist an additional conserved vector called the <u>Laplace-Runge-Lenz</u> vector

$$\vec{A} = \vec{p}\times\vec{L} - mk\hat{r}$$



- This is an example of dynamical symmetry (no associated cyclic coordinate can be found).
- Note that there are two constraints

$\vec{A}\cdot\vec{L} = 0$, $A^2 = m^2k^2 + 2mEL^2$

$\{\underset{1}{E}, \underset{3}{\vec{L}}, \underset{3}{\vec{A}}\}$ $= 7-2 = 5$ independent conserved quantities

$n=3$ dof we need for complete integrability
n independent integrals of motion
If you have more than n, only n of them in involution
<u>$2n-1$ in Kepler</u>, ↪ <u>"maximally superintegrable"</u>

④ <u>Nambu Mechanics</u>

- <u>Hamiltonian mechanics</u>
  - Even dimensional phase space
  - ω, 2-form
    $\{f,g\} = \Omega(df, dg)$: Binary operation
  - Dynamics are described with single Hamiltonian

i-) why do we restrict to even-dimensional space, can not we have e.g. (q^i, p_i, r^i) $i=1..n$? $3n$

ii-) why single function H is enough

Postulated existence of a <u>Nambu tensor</u> η: non degenerate p-form such that $D\eta = 0$

This defines a <u>generalized Poisson bracket</u>

$$\{f_1, f_2 .., f_p\} = \eta(X_{f_1}, X_{f_2} .. X_{f_p}) = \eta(df_1 .. df_p)$$

Contact Geometry

Definition: A vector X for which $\omega(X,Y)=0$ for all Y is called <u>null vector</u> of 2-form ω.

$X \lrcorner\, \omega = 0$: X is null vector

$\omega . X = 0$: X eigenvector $\qquad \omega . u = \lambda u$

Definition A 2-form ω is <u>non-singular</u> when the vector space generated by its null vectors has <u>minimal</u> possible dimension

$$\begin{cases} \dim = 0 & \text{even dimensions: } M^{2n} \\ \dim = 1 & \text{odd dimensions: } M^{2n+1} \end{cases}$$

1-dimensional space of null vectors

Definition: when ω is closed and non-singular, the pair (M^{2n+1}, ω) is called <u>contact geometry</u>

Remarks

i-) This is an odd dimensional version of symplectic geometry

ii-) ω is generated by contact 1-form θ:

$$\omega = d\theta$$

and determines a unique <u>vortex direction</u> (null direction) X:

$$X \lrcorner\, \omega = 0$$

Specifically, <u>Reeb vector</u> is a normalized vortex direction : $\theta \lrcorner X = 1$

iii-) (interesting branch of mathematics the so called special <u>Riemannian manifolds</u>

$$\begin{cases} \text{Kähler manifolds: } (M^{2n}, \text{ Kahler 2-form } \omega) \\ \text{Sasaki manifolds: } (M^{2n+1}, \text{ Reeb vector } X) \end{cases}$$

iv) $\mathcal{L}_X \omega = 0$

$$\mathcal{L}_X \omega = X \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{X \lrcorner\, \omega}_{0}) = 0$$

v-) <u>Darboux Theorem</u> Since ω is closed and non-singular, there exists a coordinate system $X^\alpha = (q^j, p_j, t)$ such that

$$\omega = dp_i \wedge dq^i$$

$$X = \frac{\partial}{\partial t}$$

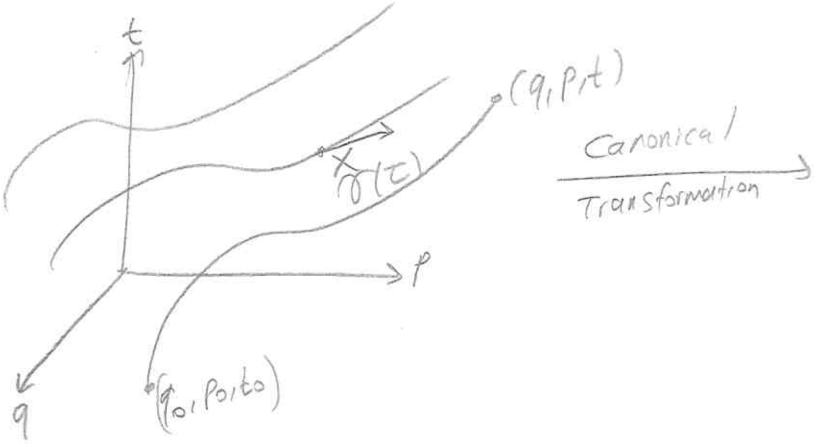
$$\omega_{\alpha\beta} = \begin{array}{c|c} & q_1 \quad p_1 \quad \cdots \quad t \\ \hline \begin{array}{c} q_1 \\ p_1 \\ \vdots \\ t \end{array} & \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \phi \\ & \phi & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ & & & \boxed{0} \end{pmatrix} \end{array}$$

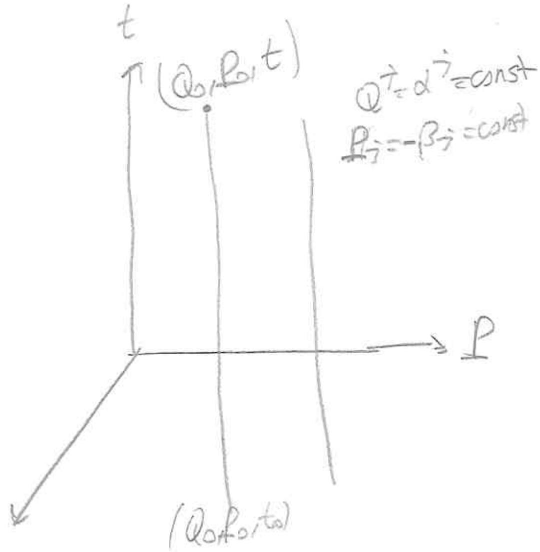
$$H' = H + \frac{\partial F_1}{\partial t}, \quad p_j = \frac{\partial F_1}{\partial q^j}, \quad P_j = -\frac{\partial F_1}{\partial Q^j}$$

Hamilton-Jacobi Theory Revised

Idea: Let's make a special canonical transformation so that (Q^j, P_j, t) are Darboux coordinates for ω, that is

$$\omega = dP_j \wedge dQ^j, \quad H' = 0, \quad X = \frac{\partial}{\partial t}$$





We have seen previously that such a canonical transformation is generated by $F = S$ where

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) = 0$$

Summary

Classical mechanics finds natural description in terms of symplectic and contact geometries.

	Hamiltonian symplectic (2n)	Hamiltonian contact (2n+1)	Lagrangian symplectic
Manifold M	P.S. T^*C	Extended P.S. $T^*C \times \mathbb{R}$	Velocity P.S.
Dynamics	$H = H(q, p)$	$H = H(q, p, t)$	$L = L(q, \dot{q})$
1-form	Cartan $\theta = p_i dq^i$	contact $\theta_H = p_i dq^i - H dt$	Lagrange $\theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i$
2-form	symplectic $\omega = d\theta$, $d\omega = 0$	$\omega_H = d\theta_H$	Lagrange symplectic 2-form $\omega_L = d\theta_L$
Eqn for dynamical field X	X_H $X_H \lrcorner\, \omega_H = -dH$	$\tilde{X}_H$ $\tilde{X}_H \lrcorner\, \omega_H = 0$	$X_L : \mathcal{L}_{X_L} \theta = dL$ or $X_L \lrcorner\, \omega_L = -dE$