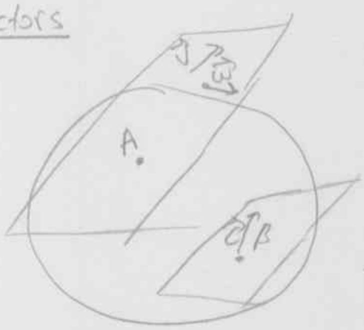
# Fundamentals

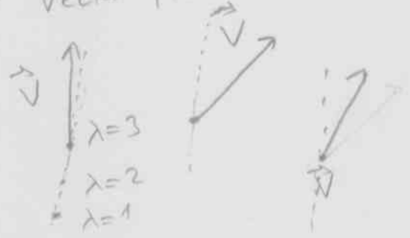
(Diff geo)


## Vectors




Vector live at $T_A$ = tangent space attached to A
vectors at different points live in distinct tangent spaces
distinct vector spaces

Vector field $\vec{V} \leftrightarrow$ integral curves




integral curves which $\vec{V}$ is tangents

$\hookrightarrow X^\alpha = X^\alpha(\lambda)$

$V^\alpha = \dfrac{dX^\alpha}{d\lambda}$ ; $X^\alpha(\lambda=0)$ initial condition

$\dfrac{df}{d\lambda} = \partial_\alpha f \, V^\alpha$ , $f = f(X^\alpha)$

## Covectors (dual vectors, one-forms)

linear functions of vectors $\longrightarrow$ scalars $\qquad w_\alpha V^\alpha = \text{scalar}$

## Tensors

Multilinear functions of vectors and covectors $\longrightarrow$ scalars $\qquad T^\alpha{}_\beta \, w_\alpha V^\beta = \text{scalar}$

## Metric

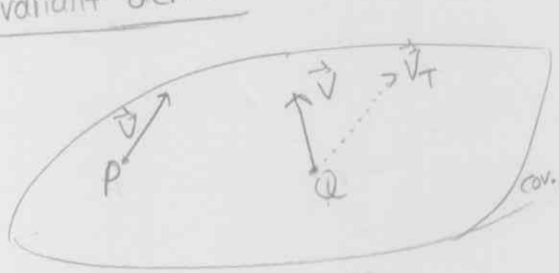
$g_{\alpha\beta} V^\alpha V^\beta$ = squared length of $V^\alpha$ (negative, positive, zero)

inverse = $g^{\alpha\beta}$ $\qquad g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma$ , $\quad g^{\alpha\beta} w_\alpha w_\beta$ = squared length of $w_\alpha$

$V_\alpha = g_{\alpha\beta} V^\beta$

$w^\alpha = g^{\alpha\beta} w_\beta$

## Covariant derivative

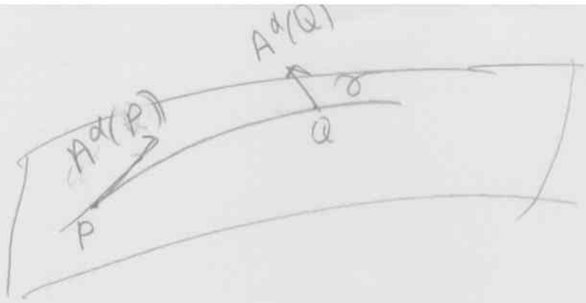



Connection $\Gamma$ – rule to take a vector at Q and bring it to P : parallel transport

new structure

cov. derivative : $\vec{V}(Q) - \vec{V}_T(Q)$

$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma_\alpha{}^\beta{}_\gamma V^\gamma$




$A^\alpha(Q)\, B^\beta(P)$ , $A^\alpha(Q) - A^\alpha(P)$ are not defined as tensorial operations.

Consider a curve $\gamma$, its tangent vector $u^\alpha$ and a vector field $A^\alpha$ defined in the neighbourhood of $\gamma$. Let point $P$ on the curve have coordinates $x^\alpha$ and $Q$ have coordinates $x^\alpha + dx^\alpha$

$$dA^\alpha \stackrel{\sim}{=} A^\alpha(Q) - A^\alpha(P) = A^\alpha(x^\beta + dx^\beta) - A^\alpha(x^\beta)$$
$$= \cancel{A^\alpha(x^\beta)} + \partial_\mu A^\alpha(x^\beta) dx^\mu - \cancel{A^\alpha(x^\beta)}$$
$$= A^\alpha{}_{,\beta} dx^\beta$$

To check the tensoriality

$$\frac{\partial A^{\alpha'}}{\partial x^{\beta'}} = \frac{\partial}{\partial x^{\beta'}} \frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha(x^\beta) = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \partial_{\beta'} A^\alpha + \frac{\partial^2 x^{\alpha'}}{\partial x^\alpha \partial x^{\beta'}} A^\alpha$$

; There is an extra $\frac{\partial x}{\partial x}$ term but due to $\beta$ contraction it gives $\delta$ and output is thi

which is not tensorial operation

The derivative should have the form

$$DA^\alpha = A_T^\alpha(P) - A^\alpha(P)$$

where $A_T^\alpha(P)$ is the vector transported from $Q$ to $P$

$\delta A^\alpha = A_T^\alpha(P) - A^\alpha(Q)$ is also not a tensorial operation

$$DA^\alpha = dA^\alpha + \delta A^\alpha \quad \text{where}$$

D: covariant differentiation

$$= \cancel{A^\alpha(Q)} - A^\alpha(P) + A_T^\alpha(P) - \cancel{A^\alpha(Q)}$$

$$DA^\alpha = A_T^\alpha(P) - A^\alpha(P)$$

We demand that $\delta A^\alpha$ be linear in both $A^\mu$ and $dx^\beta$ so that

$$\delta A^\alpha = \Gamma^\alpha_{\mu\beta} A^\mu dx^\beta$$

for some (non tensorial) field $\Gamma^\alpha_{\mu\beta}$ called the connection

$$DA^\alpha = \partial_\beta A^\alpha dx^\beta + \Gamma^\alpha_{\mu\beta} A^\mu dx^\beta$$

, dividing through by $d\lambda$

$$\frac{DA^\alpha}{d\lambda} = \partial_\beta A^\alpha u^\beta + \Gamma^\alpha_{\mu\beta} A^\mu u^\beta \Rightarrow \frac{DA^\alpha}{d\lambda} = \nabla_\beta A^\alpha u^\beta$$

where $u^\beta$ is tangent vector

and $\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha_{\mu\beta} A^\mu$ : Covariant derivative of the vector $A^\alpha$

$$\nabla_\beta A^\alpha = A^\alpha{}_{;\beta} \,, \quad \frac{DA^\alpha}{d\lambda} \equiv \nabla_u A^\alpha$$

$$\Gamma^{\alpha}_{\mu\beta} A^{\mu} = \nabla_\beta A^\alpha - \partial_\beta A^\alpha$$

$$\Gamma^{\alpha'}_{\mu'\beta'} A^{\mu'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \nabla_\beta\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha\right) - \frac{\partial x^\beta}{\partial x^{\beta'}}\partial_\beta\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha} A^\alpha\right)$$

$$= \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\left(\cancel{\partial_\beta A^\alpha} + \Gamma^\alpha_{\beta\gamma}A^\gamma\right) - \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\alpha}A^\alpha - \cancel{\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\partial_\beta A^\alpha}$$

$$\Gamma^{\alpha'}_{\mu'\beta'}\frac{\partial x^{\mu'}}{\partial x^\mu}A^\mu = \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^\alpha_{\beta\mu}A^\mu - \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta\partial x^\mu}A^\mu$$

$$\Gamma^{\alpha'}_{\mu'\beta'}\frac{\partial x^{\mu'}}{\partial x^\mu} = \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^\alpha_{\beta\mu} - \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta\partial x^\mu}$$

$$\Gamma^{\alpha'}_{\mu'\beta'} = \underbrace{\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\mu}{\partial x^{\mu'}}\Gamma^\alpha_{\beta\mu}}_{\checkmark} - \underbrace{\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}}}_{\times}$$

- Connection is not a tensor
- $D$ must obey the Liebnitz rule.
- $D=d$ for a scalar

$$D(A^\alpha P_\alpha) \equiv d(A^\alpha P_\alpha) = (DA^\alpha)P_\alpha + A^\alpha DP_\alpha$$

$$= (\partial_\beta A^\alpha dx^\beta)P_\alpha + A^\alpha(\partial_\beta P_\alpha dx^\beta)$$

$$\frac{DP_\alpha}{d\lambda} = \nabla_\beta P_\alpha dx^\beta$$

- $\frac{DT^{\mu\ldots\nu}{}_{\sigma\ldots\rho}}{d\lambda} = \nabla_\alpha T^{\mu\ldots\nu}{}_{\sigma\ldots\rho}\, u^\alpha$ where $u^\alpha$ is the tangent vector

Extend to other tensors $\longrightarrow$ Leibnitz

scalar: $\nabla_\alpha f = \partial_\alpha f$

$$\nabla(A^{\cdots}_{\cdot\cdot}\, B^{\cdots}_{\cdot\cdot}) = (\nabla A^{\cdots}_{\cdot\cdot})\, B^{\cdots}_{\cdot\cdot} + A^{\cdots}_{\cdot\cdot}(\nabla B^{\cdots}_{\cdot\cdot})$$

covector: $\nabla_\alpha(\omega_\beta V^\beta) = \partial_\alpha(\omega_\beta V^\beta) = (\partial_\alpha \omega_\beta)V^\beta + \omega_\beta\, \partial_\alpha V^\beta$

$$= (\nabla_\alpha \omega_\beta)V^\beta + \omega_\beta(\nabla_\alpha V^\beta)$$

$$(\nabla_\alpha \omega_\beta)V^\beta = -\omega_\beta(\underline{\partial_\alpha V^\beta} + \Gamma^\beta_{\alpha\gamma} V^\gamma) + (\partial_\alpha \omega_\beta)V^\beta + \underline{\omega_\beta(\partial_\alpha V^\beta)}$$

$$(\nabla_\alpha \omega_\beta)\underline{\underline{V^\beta}} = (\partial_\alpha \omega_\beta - \omega_\gamma \Gamma^\gamma_{\alpha\beta})\underline{\underline{V^\beta}}$$

$$\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\gamma_{\alpha\beta}\,\omega_\gamma$$

<u>Connection in GR</u>

- symmetric $\Gamma^\gamma_{\beta\alpha} = \Gamma^\gamma_{\alpha\beta}$
- metric compatible $\nabla_\alpha g_{\beta\gamma} = 0$

$\Big\}$ $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma})$ "Christoffel symbols"

<u>Covariant derivation along a curve</u> $V^\alpha$: vector field

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\gamma$$

$$\frac{DV^\beta}{d\lambda} = u^\alpha(\nabla_\alpha V^\beta) = \frac{dx^\alpha}{d\lambda}\nabla_\alpha V^\beta$$

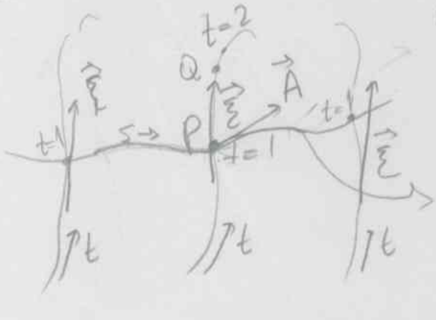
$$= \frac{dx^\alpha}{d\lambda}\left(\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\gamma} V^\gamma\right) = \frac{dV^\beta}{d\lambda} + \Gamma^\beta_{\alpha\gamma} u^\alpha V^\gamma$$


$-\; x^\alpha = x^\alpha(\lambda)$

$$u^\alpha = \frac{dx^\alpha}{d\lambda}$$

<u>Lie Derivative</u>

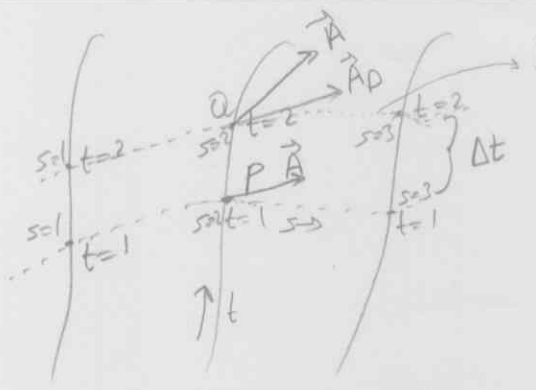
- without $\Gamma \longrightarrow$ more primitive
- two vector fields, $\vec{\xi}$, $\vec{A}$




Integral curves of $\vec{\xi}$, $x^\alpha(t)$

$$\xi^\alpha = \frac{dx^\alpha}{dt}$$

Integral curve of $\vec{A}$ through $P$ $-\; z^\alpha(s)$, $A^\alpha = \dfrac{dz^\alpha}{ds}$




Lie derivative $\mathcal{L}_\Sigma A^\alpha \Big|_P \equiv \lim_{\Delta t \to 0} \dfrac{\left(A^\alpha(Q) - A^\alpha_D(Q)\right)}{\Delta t}$

$$\mathcal{L}_\Sigma A^\alpha = \Sigma^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \Sigma^\alpha$$
$$= \Sigma^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta \Sigma^\alpha \quad : \Gamma \text{ cancel out}$$
$$= -\mathcal{L}_A \Sigma^\alpha$$

Scalars

$$\mathcal{L}_\Sigma f \equiv \Sigma^\alpha \partial_\alpha f$$

Covectors

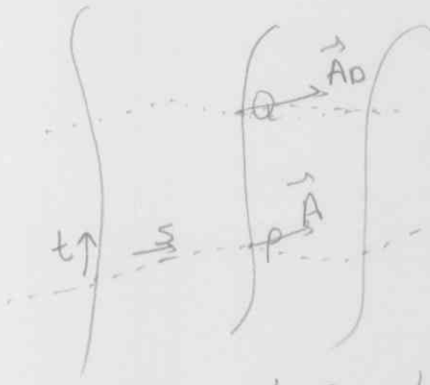
$$\mathcal{L}_\Sigma (\omega_\alpha V^\alpha) = \Sigma^\beta \partial_\beta (\omega_\alpha V^\alpha) = (\Sigma^\beta \partial_\beta \omega_\alpha) V^\alpha + \omega_\alpha (\Sigma^\beta \partial_\beta V^\alpha)$$
$$= (\mathcal{L}_\Sigma \omega_\alpha) V^\alpha + \omega_\alpha \mathcal{L}_\Sigma V^\alpha$$

$$(\mathcal{L}_\Sigma \omega_\alpha) V^\alpha = -\omega_\alpha (\cancel{\Sigma^\beta \partial_\beta V^\alpha} - V^\beta \partial_\beta \Sigma^\alpha) + (\Sigma^\beta \partial_\beta \omega_\alpha) V^\alpha + \omega_\alpha (\cancel{\Sigma^\beta \partial_\beta V^\alpha})$$

$$(\mathcal{L}_\Sigma \omega_\alpha) V^\alpha = (\omega_\beta \partial_\alpha \Sigma^\beta) V^\alpha + (\Sigma^\beta \partial_\beta \omega_\alpha) V^\alpha$$
$$\mathcal{L}_\Sigma \omega_\alpha = \omega_\beta \partial_\alpha \Sigma^\beta + \Sigma^\beta \partial_\beta \omega_\alpha \quad //$$
$$= \omega_\beta \nabla_\alpha \Sigma^\beta + \Sigma^\beta \nabla_\beta \omega_\alpha \quad //$$

Tensors

$$\mathcal{L}_\Sigma T^{\alpha\beta} = \Sigma^\gamma \partial_\gamma T^{\alpha\beta} - T^{\gamma\beta} \partial_\gamma \Sigma^\alpha - T^{\alpha\gamma} \partial_\gamma \Sigma^\beta$$
$$\mathcal{L}_\Sigma T_{\alpha\beta} = \Sigma^\gamma \partial_\gamma T_{\alpha\beta} + T_{\gamma\beta} \partial_\alpha \Sigma^\gamma + T_{\alpha\gamma} \partial_\beta \Sigma^\gamma$$
$$\mathcal{L}_\Sigma T^\alpha{}_\beta = \Sigma^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma \Sigma^\alpha + T^\alpha{}_\gamma \partial_\beta \Sigma^\gamma$$




Describe all curves as $X^\alpha(t,s)$

s is fixed, t varies $\rightarrow$ integral curves of $\Sigma$ $\rightarrow$ $\Sigma^\alpha = \left(\frac{\partial X^\alpha}{\partial t}\right)_s$

s vary, t is fixed $\rightarrow$ Dragged curves of $\vec{A}$ $\rightarrow$ $A_D^\alpha = \left(\frac{\partial X^\alpha}{\partial s}\right)_t$

At P, $X^\alpha = 0$, $t=0, s=0$

$$X^\alpha(t,s) = a^\alpha t + b^\alpha s + \frac{1}{2} c^\alpha t^2 + d^\alpha s t + \frac{1}{2} e^\alpha s^2 + O(3)$$

$$\Sigma^\alpha = \left(\frac{\partial X^\alpha}{\partial t}\right)_s = a^\alpha + c^\alpha t + d^\alpha s + O(2)$$

$$A_D^\alpha = \left(\frac{\partial X^\alpha}{\partial s}\right)_t = b^\alpha + e^\alpha s + d^\alpha t + O(2)$$

$$A^\alpha = A_P^\alpha + p^\alpha t + q^\alpha s + O(2)$$ : No relation with $X^\alpha(t,s)$

when $t=0$, $A^\alpha = A_D^\alpha$
integral curves of A = Dragged curve

$$A_P^\alpha + q^\alpha s = b^\alpha + e^\alpha s \qquad A_P^\alpha = b^\alpha \qquad q^\alpha = e^\alpha$$

$$A^\alpha = b^\alpha + p^\alpha t + e^\alpha s$$

$$\mathcal{L}_\Sigma A^\alpha = \frac{A^\alpha(t) - A_D^\alpha(t)}{t} = \frac{\cancel{b^\alpha} + p^\alpha t + \cancel{e^\alpha s} - \cancel{b^\alpha} - \cancel{e^\alpha s} - d^\alpha t}{t} = p^\alpha - d^\alpha$$

- $\Sigma^\beta \partial_\beta A^\alpha = \left(\frac{\partial X^\beta}{\partial t}\right)_s \partial_\beta A^\alpha = \left(\frac{\partial A^\alpha}{\partial t}\right)_s \Big|_{P,\, s=0} = p^\alpha$

- $A^\beta \partial_\beta \Sigma^\alpha = \underline{A_D^\beta \partial_\beta \Sigma^\alpha} + \underbrace{(A^\beta - A_D^\beta)}_{O(t^1)} \partial_\beta \Sigma^\alpha$

$$= \left(\frac{\partial X^\beta}{\partial s}\right)_t \partial_\beta \Sigma^\alpha$$

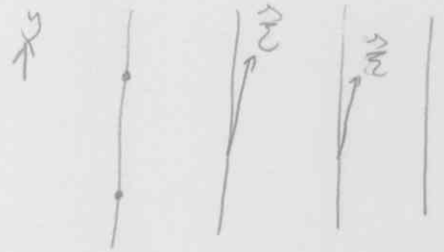
$$= \lim_{t\to 0} \left(\frac{\partial \Sigma^\alpha}{\partial s}\right)_{t=0} + \cancel{\lim_{t\to 0} (p^\alpha - d^\alpha) t \; \partial_\beta \Sigma^\alpha}$$

$$A^\beta \partial_\beta \Sigma^\alpha = d^\alpha$$

$$\mathcal{L}_\Sigma A^\alpha = p^\alpha - d^\alpha = \Sigma^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta \Sigma^\alpha$$

# Killing vectors

In some coordinates $(t,x,y,z)$, suppose that $\partial_y g_{\mu\nu} \overset{*}{=} 0$ → in the coordinate system we chose




$\xi^\mu \overset{*}{=} (0,0,1,0)$

$$\mathcal{L}_\xi g_{\mu\nu} = \underbrace{\xi^\lambda \partial_\lambda g_{\mu\nu}}_{\substack{\partial_y g_{\mu\nu} \\ \overset{*}{=} 0}} + g_{\lambda\nu} \underbrace{\partial_\mu \xi^\lambda}_{0} + g_{\mu\lambda} \underbrace{\partial_\nu \xi^\lambda}_{0}$$

$\mathcal{L}_\xi g_{\mu\nu} \overset{*}{=} 0$ $\qquad$ $\mathcal{L}_\xi g_{\mu\nu} = 0$ in all coordinate systems

$\xi^\alpha$ = killing vector

$$0 = \mathcal{L}_\xi g_{\alpha\beta} = \xi^\gamma \underbrace{\nabla_\gamma g_{\alpha\beta}}_{=0} + g_{\gamma\beta} \nabla_\alpha \xi^\gamma + g_{\alpha\gamma} \nabla_\beta \xi^\gamma$$
metric compatibility

$$= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$$

$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$ : Killing's equation

$\nabla_{(\alpha} \xi_{\beta)} = 0$

$\nabla_\alpha \xi_\beta$ = anti-symmetric

Example: Minkowski metric

$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

10 $\begin{cases} 4 \text{ translational Killing vectors} \\ 3 \text{ rotational KV} \\ 3 \text{ boost KV} \end{cases}$

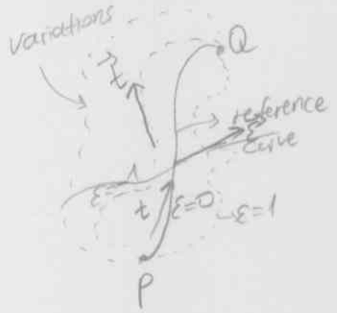
Example: spherical symmetry, static

$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$

$\xi^\alpha_{(t)} = (1,0,0,0)$ $\qquad$ $\xi^\alpha_{(1)} = (0,0,\sin\phi, \cot\theta\cos\phi)$

$\xi^\alpha_{(\phi)} = (0,0,0,1)$ $\qquad$ $\xi^\alpha_{(2)} = (0,0,-\cos\phi, \cot\theta\sin\phi)$

# Geodesics




A timelike geodesic extremizes proper time between two (neighbourhood) points

2-parameter family of curves: $x^\alpha(t,\varepsilon) = \underbrace{x^\alpha(t) + \varepsilon\,\delta x^\alpha(t) + O(\varepsilon^2)}$ Standard way but we will do it in a different way

$t$ vary, $\varepsilon$ fixed $\rightarrow$ motion along each curve

$\varepsilon$ vary, $t$ fixed $\rightarrow$ motion across each curves

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_\varepsilon \qquad \mathcal{L}_\varepsilon t^\alpha = 0 = \mathcal{L}_t \varepsilon^\alpha$$

$$\cdot\ \varepsilon^\beta \nabla_\beta t^\alpha = t^\beta \nabla_\beta \varepsilon^\alpha$$

$$\varepsilon^\alpha = \left(\frac{\partial x^\alpha}{\partial \varepsilon}\right)_t$$

$$\mathcal{L}_\varepsilon t^\alpha = \varepsilon^\beta \partial_\beta t^\alpha - t^\beta \partial_\beta \varepsilon^\alpha$$
$$= \left(\frac{\partial x^\beta}{\partial \varepsilon}\right)\left(\frac{\partial t^\alpha}{\partial x^\beta}\right) - \left(\frac{\partial x^\beta}{\partial t}\right)\left(\frac{\partial \varepsilon^\alpha}{\partial x^\beta}\right)$$
$$= \left(\frac{\partial t^\alpha}{\partial \varepsilon}\right) - \left(\frac{\partial \varepsilon^\alpha}{\partial t}\right) = \frac{\partial}{\partial \varepsilon}\frac{\partial x^\alpha}{\partial t} - \frac{\partial}{\partial t}\frac{\partial x^\alpha}{\partial \varepsilon} = 0$$

For all curves

$t(P) = 0$

$t(Q) = 1$


$$d\tau^2 = -ds^2 = -g_{\alpha\beta}\,dx^\alpha dx^\beta \qquad d\tau\text{: proper time}$$
$$d\tau = \sqrt{-g_{\alpha\beta}\,dx^\alpha dx^\beta}$$
$$= \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}\,dt = \sqrt{-g_{\alpha\beta}t^\alpha t^\beta}\,dt$$

$$\Delta\tau = \int_P^Q \sqrt{-g_{\alpha\beta}t^\alpha t^\beta}\,dt$$

$$S = -\int_0^1 \sqrt{-g_{\alpha\beta}t^\alpha t^\beta}\,dt \quad ; \quad L = -\sqrt{-g_{\alpha\beta}t^\alpha t^\beta} \Rightarrow L = L(g_{\alpha\beta}, t^\alpha)$$

$$\delta S = S(\varepsilon) - S(0) = \varepsilon \left.\frac{\partial S}{\partial \varepsilon}\right|_{\varepsilon=0} = \varepsilon \int_0^1 \frac{\partial L}{\partial \varepsilon}\,dt$$

$$\frac{\partial L}{\partial \varepsilon} = \varepsilon^\mu \partial_\mu L = \varepsilon^\mu \nabla_\mu L = \varepsilon^\mu\left(\frac{\partial L}{\partial g_{\alpha\beta}}\cancel{\nabla_\mu g_{\alpha\beta}} + \frac{\partial L}{\partial t^\alpha}\nabla_\mu t^\alpha\right)$$
$$= \underbrace{\frac{\partial L}{\partial t^\alpha}}_{p_\alpha}\varepsilon^\mu \nabla_\mu t^\alpha = p_\alpha\, t^\mu \nabla_\mu \varepsilon^\alpha$$

$$\frac{\partial L}{\partial \varepsilon} = p_\alpha t^\mu \nabla_\mu \varepsilon^\alpha = p_\alpha \frac{D\varepsilon^\alpha}{dt}$$

$$\varepsilon^{-1}\delta S = \int_0^1 p_\alpha \frac{D\varepsilon^\alpha}{dt}\,dt = \int \left(\underbrace{\frac{D}{dt}}_{D=d}\underbrace{(p_\alpha \varepsilon^\alpha)}_{\text{scalar}} - \varepsilon^\alpha \frac{Dp_\alpha}{dt}\right)dt$$

$$\varepsilon^{-1}\delta S = \underbrace{p_\alpha \varepsilon^\alpha \Big|}_{0} - \int \varepsilon^\alpha \frac{Dp_\alpha}{dt}\, dt = 0$$

All of the variations begin at P and end at Q

Rules of variation: $\varepsilon^\alpha$ is arbitiary between P and Q, but zero at P and Q

$$\frac{Dp_\alpha}{dt} = 0 \quad \text{(geodesic equation)}$$

$$p_\alpha = \frac{\partial L}{\partial t^\alpha} \; ; \; L = -\sqrt{-g_{\alpha\beta}t^\alpha t^\beta}$$

$$L = -(-g_{\mu\nu}t^\mu t^\nu)^{1/2}$$

$$p_\alpha = \frac{\partial L}{\partial t^\alpha} = -\frac{1}{2}(-g_{\mu\nu}t^\mu t^\nu)^{-1/2}\cdot(-g_{\mu\nu})\Big(\underbrace{\frac{\partial t^\mu}{\partial t^\alpha}}_{\delta^\mu_\alpha} t^\nu + t^\mu \underbrace{\frac{\partial t^\nu}{\partial t^\alpha}}_{\delta^\nu_\alpha}\Big)$$

$$= \frac{1}{2}\frac{1}{(-L)}\left(g_{\alpha\nu}t^\nu + g_{\mu\alpha}t^\mu\right)$$

$$p_\alpha = -\frac{1}{L} t_\alpha$$

$$p^\alpha = -\frac{1}{L} t^\alpha$$

$$\frac{Dp^\alpha}{dt} = 0 \longrightarrow \frac{D}{dt}\left(\frac{t^\alpha}{(-L)}\right) = \left(\frac{1}{-L}\right)\frac{Dt^\alpha}{dt} - \left(\frac{1}{(-L^2)}\right)\frac{d(-L)}{dt} t^\alpha = 0$$

$$\frac{Dt^\alpha}{dt} = \underbrace{\frac{1}{(-L)}\frac{d(-L)}{dt}}_{K} t^\alpha$$

$$\frac{Dt^\alpha}{dt} = K t^\alpha$$

Affine variation, we choose $t \equiv$ proper time $\tau$

$$L = -1,\; K = 0$$

$$L = -\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} = -\sqrt{\frac{d\tau^2}{d\tau^2}} = -1$$

$$\boxed{\frac{D}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) = 0}$$ geodesic equation (proper time parametrization)

Killing's eqn: $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$

Geodesic eqn:

$$\begin{cases} t^\alpha = \dfrac{dx^\alpha}{dt} \longrightarrow \dfrac{Dt^\alpha}{dt} = K t^\alpha \\[2ex] u^\alpha = \dfrac{dx^\alpha}{d\tau} \longrightarrow \dfrac{Du^\alpha}{d\tau} = 0 \; ; \; \tau \to \underbrace{a\tau + b}_{\text{affine parameter}} \end{cases}$$

: leaves unchanged the eqn

$$\frac{Du^\alpha}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$$

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

## Conserved quantities

Assume: geodesic: $\frac{Du^\alpha}{d\tau} = 0$

$u^\alpha$, $\uparrow \tau$

Killing vector $\xi^\alpha$

Conserved quantity: $c = u_\alpha \xi^\alpha$

$$\frac{dc}{d\tau} = 0$$

$$\frac{Dc}{d\tau} = \frac{D(u_\alpha \xi^\alpha)}{dt} = \underbrace{\frac{Du_\alpha}{d\tau}}_{0} \xi^\alpha + u_\alpha \frac{D\xi^\alpha}{d\tau} = u^\alpha \frac{D\xi_\alpha}{d\tau}$$

$$= \underbrace{u^\alpha u^\beta}_{\text{symmetric}} \underbrace{\nabla_\beta \xi_\alpha}_{\text{antisymmetric}} = 0$$

$$= \frac{1}{2} u^\alpha u^\beta \nabla_\beta \xi_\alpha + \frac{1}{2} u^\beta u^\alpha \nabla_\alpha \xi_\beta = \frac{1}{2} u^\alpha u^\beta (\underbrace{\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta}_{0}) = 0$$

Example:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

$$\xi^\alpha_{(t)} = (1,0,0,0) \quad , \quad \xi^\alpha_{(\phi)} = (0,0,0,1)$$

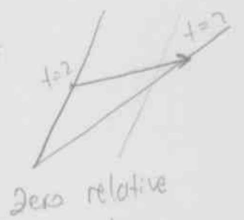
$$\tilde{E} \equiv -u_\alpha \xi^\alpha_{(t)} \equiv \frac{\text{Energy at infinity}}{\text{mass}}$$

$$\tilde{L} = u_\alpha \xi^\alpha_{(\phi)} \equiv \frac{\text{Angular momentum at infinity}}{\text{mass}}$$

## Curvature

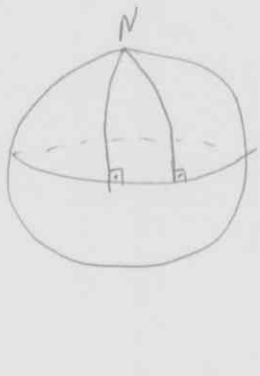
$$[\nabla_\alpha, \nabla_\beta] A^\mu = \underbrace{R^\mu{}_{\nu\alpha\beta}}_{\text{Riemann curvature tensor}} A^\nu$$

flat space:




- zero relative acceleration
- non zero relative velocity

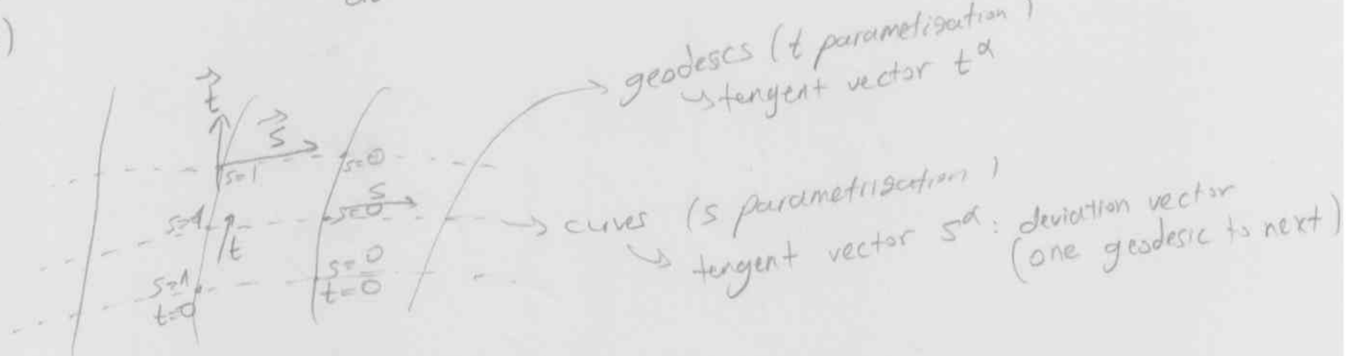
sphere:




- non zero relative acceleration
- non zero relative velocity

Geometrical set up :

geodesic vector field $t^\alpha$; $\frac{Dt^\alpha}{dt} = 0$ ; $t^\beta \nabla_\beta t^\alpha = 0$
(affine)




two parameter family of curves

$x^\alpha(t,s)$

s fixed, t varies → geodesic $t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_s$

t fixed, s varies → cross curves $s^\alpha = \left(\frac{\partial x^\alpha}{\partial s}\right)_t$

$\mathcal{L}_s t^\alpha = 0 = \mathcal{L}_t s^\alpha$

$s^\beta \nabla_\beta t^\alpha = t^\beta \nabla_\beta s^\alpha$

relative seperation between geodesics : $s^\alpha$

relative velocity between geodesics : $\frac{Ds^\alpha}{dt} = t^\beta \nabla_\beta s^\alpha$

relative acceleration between geodesics : $\underbrace{\frac{D}{dt}\left(\frac{Ds^\alpha}{dt}\right)}_{\frac{D^2 s^\alpha}{dt}} = t^\gamma \nabla_\gamma (t^\beta \nabla_\beta s^\alpha)$

$$\frac{D^2 s^\alpha}{dt^2} = t^\gamma \nabla_\gamma (s^\beta \nabla_\beta t^\alpha) = (t^\gamma \nabla_\gamma s^\beta) \nabla_\beta t^\alpha + t^\gamma s^\beta \nabla_\gamma \nabla_\beta t^\alpha$$

$$= \underbrace{(t^\gamma \nabla_\gamma s^\beta) \nabla_\beta t^\alpha}_{②} + t^\gamma s^\beta \underbrace{(\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma) t^\alpha}_{R^\alpha{}_{\mu\gamma\beta} t^\mu} + \underbrace{t^\gamma s^\beta \nabla_\beta \nabla_\gamma t^\alpha}_{①}$$

$$① = s^\beta \nabla_\beta (\underbrace{t^\gamma \nabla_\gamma t^\alpha}_{\frac{Dt^\alpha}{dt} = 0}) - (s^\beta \nabla_\beta t^\gamma) \nabla_\gamma t^\alpha$$

$$\left.\begin{aligned} ① &= -(s^\beta \nabla_\beta t^\gamma) \nabla_\gamma t^\alpha \\ ② &= (t^\gamma \nabla_\gamma s^\beta) \nabla_\beta t^\alpha = (s^\gamma \nabla_\gamma t^\beta) \nabla_\beta t^\alpha \end{aligned}\right\} ① + ② = 2$$

gravity ⟷ Riemann

$$\frac{D^2 s^\alpha}{dt^2} = R^\alpha{}_{\mu\gamma\beta} t^\mu t^\gamma s^\beta = -R^\alpha{}_{\mu\beta\gamma} t^\mu s^\beta t^\gamma$$

· Curved manifold is locally flat

Pick a point $P$ in space time

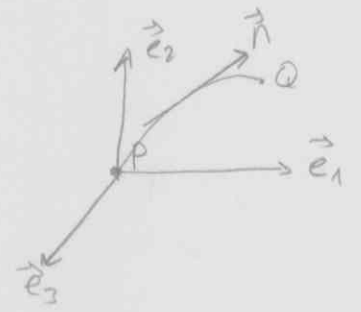
There exist coordinates $X^\mu$ such that $g_{\mu\nu}(P) \overset{*}{=} \eta_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu}(P) \overset{*}{=} 0 \longleftrightarrow \partial_\lambda g_{\mu\nu}(P) \overset{*}{=} 0$$

$$\partial_\lambda \partial_\rho g_{\mu\nu}(P) \overset{*}{\neq} 0 \leftrightarrow R^\lambda{}_{\rho\mu\nu}(P) \overset{*}{\neq} 0$$

1- Construct orthonormal basis $e^\alpha_{(\mu)}$ at $P$

$$g_{\alpha\beta}\, e^\alpha_{(\mu)}\, e^\beta_{(\nu)} = \eta_{\mu\nu}$$




2- Select the <u>unique</u> geodesic that relates $Q$ to $P$

Tangent vector $n^\alpha$. $(g_{\alpha\beta}\, n^\alpha n^\beta = \pm 1)$

3- Decompose tangent vector in orthonormal basis

$$n^\alpha = \underbrace{n^{(\mu)}}_{\text{coefficients}} e^\alpha_{(\mu)}$$

4- $X^\mu_Q = \left(\begin{array}{c}\text{proper time or distence between} \\ P \text{ and } Q\end{array}\right) . \; n^{(\mu)}$

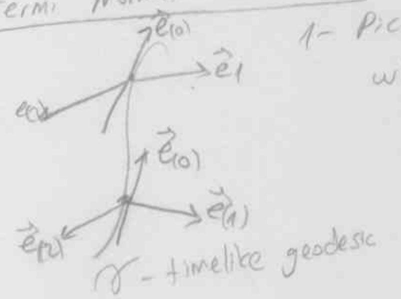
$$\rightarrow g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} \underbrace{R_{\mu\lambda\nu\rho}(P)}_{1/R^2} \underbrace{X^\lambda X^\rho}_{\sim s^2} + O(s^3/R^3)$$

$R$: Radius of the curvature

$$\underset{x=0}{g_{\mu\nu}(P)} = \eta_{\mu\nu}$$

$$\partial_\lambda g_{\mu\nu}(P) \underset{x=0}{=} 0 \longleftrightarrow \Gamma^\lambda_{\mu\nu}(P) = 0$$

Fermi Normal Coordinates

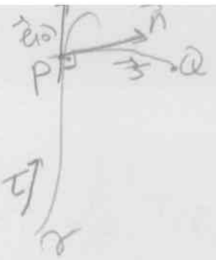



1- Pick a point on $\gamma$, construct an orthonormal basis with $e^\alpha_{(0)} = u^\alpha$

$$g_{\alpha\beta}\, e^\alpha_{(\mu)}\, e^\beta_{(\nu)} = \eta_{\mu\nu}$$

2- Define $e^\alpha_{(\mu)}$ at any point on $\gamma$ by parallel transport

$$\frac{D e^\alpha_{(\mu)}}{dt} = u^\beta \nabla_\beta e^\alpha_{(\mu)} = 0$$




3- we are gonna select the unique geodesic that links [illegible]
crosses $\gamma$ orthogonally.

on $\gamma$, $g_{\alpha\beta}\, n^\alpha e^\beta_{(0)} = 0$ : $\vec{e}_{(0)} \cdot \vec{n} = 0$ due to orthogonality

4- Decomposition of the tangent vector $\vec{n}$ into the orthonormal basis that lives at that point of encounter

$n^\alpha = n^{(j)} e^\alpha_{(j)}$ (that excludes $e^\alpha_{(0)}$ because of the (3-))

Fermi coordinates: $t_Q$ = proper time at intersection point P

$\tau(P) = t_Q$

$$X^j_Q = \left(\begin{matrix}\text{proper distance from} \\ \text{P to Q}\end{matrix}\right) n^{(j)}$$

$$g_{tt} = -1 - R_{tptq}(t)\, X^p X^q + O(s^3/R^3)$$

$$g_{tj} = \frac{2}{3} R_{jpqt}(t)\, X^p X^q + O(s^3/R^3)$$

$$g_{jk} = \delta_{jk} - \frac{1}{3} R_{jpkq}\, X^p X^q + O(s^3/R^3)$$

$$g_{\mu\nu}(\gamma) = \eta_{\mu\nu}$$

$$\Gamma^\lambda_{\mu\nu}(\gamma) = 0$$

2-) $t^\alpha = \frac{dx^\alpha}{d\lambda}$ $\quad u^\alpha = \frac{dx^\alpha}{d\lambda^*}$

$\frac{Dt^\alpha}{d\lambda} = K t^\alpha$ $\quad \frac{Du^\alpha}{d\lambda^*} = 0$

$$u^\alpha = \frac{d\lambda}{d\lambda^*}\frac{dx^\alpha}{d\lambda} = \frac{d\lambda}{d\lambda^*} t^\alpha$$

$$\frac{Du^\alpha}{d\lambda^*} = \frac{d\lambda}{d\lambda^*}\frac{Du^\alpha}{d\lambda} = \frac{d\lambda}{d\lambda^*}\frac{D}{d\lambda}\left(\frac{d\lambda}{d\lambda^*}t^\alpha\right)$$

$$= \frac{d\lambda}{d\lambda^*}\left(\underbrace{\frac{Dt^\alpha}{d\lambda}}_{Kt^\alpha}\frac{d\lambda}{d\lambda^*} + \frac{D}{d\lambda}\left(\frac{d\lambda}{d\lambda^*}\right)t^\alpha\right) = 0$$

$$K\cancel{t^\alpha}\underbrace{\frac{d\lambda}{d\lambda^*}}_{f} + \frac{D}{d\lambda}\left(\frac{d\lambda}{d\lambda^*}\right)\cancel{t^\alpha} = 0$$

$D=d$ for scalar functions

$$\frac{D}{d\lambda}f = -Kf$$

$$\int\frac{df}{f} = \int -K\,d\lambda \qquad f = e^{-\int K(\lambda)d\lambda} = \frac{d\lambda}{d\lambda^*}$$

$$\frac{d\lambda^*}{d\lambda} = e^{\int K(\lambda)d\lambda}$$

3-)

a-) $\varepsilon = -t_\alpha t^\alpha$

$$\frac{D\varepsilon}{d\lambda} = -2t_\alpha\frac{Dt^\alpha}{d\lambda} = -2t_\alpha K t^\alpha = +2\varepsilon K$$

$D=d$ for scalar

$$\frac{d\varepsilon}{d\lambda} = +2\varepsilon K$$

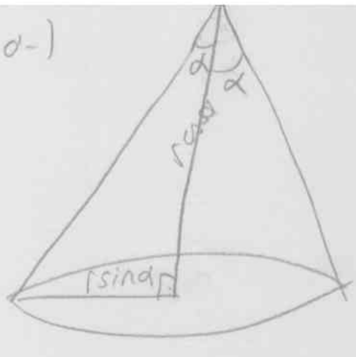
$$\boxed{\varepsilon = \varepsilon_0 e^{+2\int\varepsilon(\lambda)K(\lambda)d\lambda}}$$

b-) $p = \xi_\alpha t^\alpha$

$$\frac{D}{d\lambda}(\underbrace{\xi_\alpha t^\alpha}_{p}) = \frac{D(\xi_\alpha)}{d\lambda}t^\alpha + \xi_\alpha\frac{Dt^\alpha}{d\lambda} = \cancel{\underbrace{t^\alpha t^\beta}_{\text{symmetric}}\underbrace{\nabla_\beta\xi_\alpha}_{\text{anti symmetric}}} + \xi_\alpha K t^\alpha = pK$$

$$p = p_0 e^{\int K d\lambda}$$

a-)




$z = r\cos\alpha$

$\rho = r\sin\alpha$

$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$ : cylinder metric

$d\rho = dr\ \sin\alpha$

$dz = dr\ \cos\alpha$

$ds^2 = \underline{dr^2}\ \sin^2\alpha + r^2\sin^2\alpha\ d\phi^2 + \underline{dr^2\cos^2\alpha}$

$= dr^2 + r^2\sin^2\alpha\ d\phi^2$

$ds^2 = dr^2 + \underline{\underline{(r\sin\alpha)}}^2 d\phi^2$

b-)
c-) $\left.\begin{array}{l} x = r\cos\phi \\ y = r\sin\phi \end{array}\right\} \longrightarrow dx^2 + dy^2 = dr^2 + \underline{\underline{r}}^2 d\phi^2$

$\phi' = \phi\sin\alpha$

$x = r\cos\phi'$

$y = r\sin\phi'$

$dx = dr\cos\phi' - r\sin\alpha\ \sin\phi'\ d\phi$

$dy = dr\ \sin\phi' + r\ \sin\alpha\ \cos\phi' d\phi$

$dx^2 + dy^2 = dr^2 + r^2\sin^2\alpha\ d\phi^2$ //

$0 \leq \phi \leq 2\pi \rightarrow \quad 0 \leq \phi' \leq 2\pi\sin\alpha$

covers the whole plane

$0 \leq \underbrace{\phi - \phi'}_{\Delta\phi} \leq 2\pi(1-\sin\alpha)$

There is a missing part $\Delta\phi = 2\pi(1-\sin\alpha) = \beta$