Perimeter Institute for Theoretical Physics

PSI Study Text

Theoretical Mechanics

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Abstract

This text provides a concise introduction to *differential geometry* and its applications to *theoretical mechanics*. A number of sources stated below have been used.

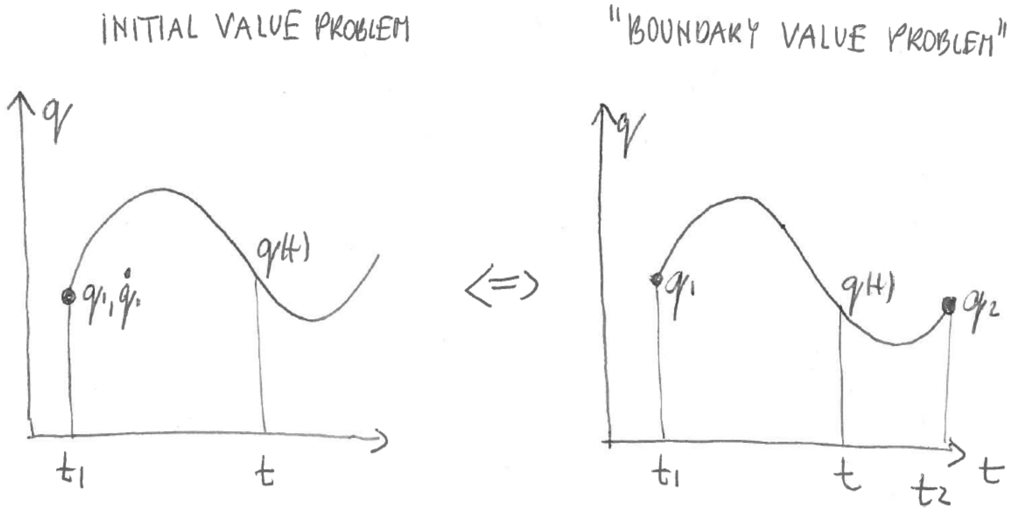
- Classic books on classical mechanics:
  - L.D. Landau and E.M. Lifshitz: Mechanics [1]
  - V.I. Arnold: Mathematical methods of classical mechanics [2]
  - H. Goldstein: Classical Mechanics [3]
- Also useful:
  - M. Göckeler and T. Schücker, Differential geometry, gauge theories, and gravity [4]
  - D. Tong, Classical dynamics [5]
  - P.A.M. Dirac: Lectures on quantum mechanics (reference on constraints) [6]
  - J. Podolsky, Theoretical mechanics in a language of differential geometry (in Czech) [7]
  - B. Zwiebach, First course in string theory [8]

Contents

Chapter 1: Lagrangian mechanics

1.1 Hamilton's principle of least action

- Let $q^I = q^I(t)$, $I = 1, \ldots, n$ be the *generalized coordinates* (length, angle,...), where n is the number of true degrees of freedom.[1]
- Then we can think of a dynamical problem in two equivalent ways as i) "*Initial value problem*" and as ii) "*Boundary value problem*", see figure.



Note that the latter is more natural in quantum mechanics and is also in the heart of formulation of the loop quantum gravity.

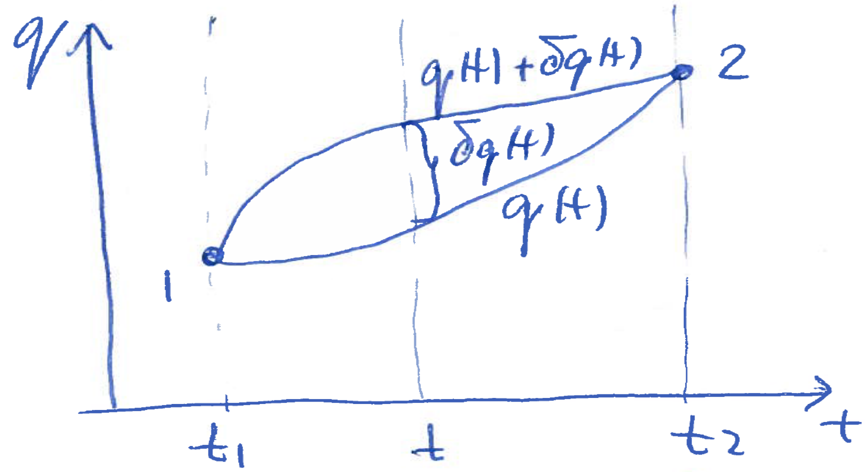
Principle of least action [W.R. Hamilton (1805–1865)]. Motions of the mechanical system in time interval $t \in (t_1, t_2)$ coincide with the extremals of the action functional

$$\boxed{S = S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt\,,} \tag{1.1}$$

where the Lagrangian L is some concrete function characterizing the system.

[1] We automatically employ the "Occam's razor principle" and describe the system with a 'clever choice' of (as little as possible) generalized coordinates, eliminating so all the constraints. For example, in the case of a pendulum, we rather choose a description in terms of one angle variable φ than cartesian coordinates supported by a constraint $x^2 + y^2 = l^2$.

- To derive the *equations of motion (EOM)* we consider *'fixed end points'*: $\delta q(t_1) = 0 = \delta q(t_2)$, as displayed in the figure,



while we seek to find an extremum of the action. That is, the true trajectory $q(t)$ is such that for any $\delta q(t)$ we have

$$\delta S \equiv S[q(t) + \delta q(t)] - S[q(t)] = 0\,. \tag{1.2}$$

Let us stress that t is fixed in a variation δ. So we have

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q\Big) dt \\ &= \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q}\Big)\Big]\delta q dt + \Big[\frac{\partial L}{\partial \dot q}\delta q\Big]_{t_1}^{t_2}, \end{aligned} \tag{1.3}$$

where we have integrated by parts and used the fact that

$$\delta \frac{d}{dt} = \frac{d}{dt}\delta\,, \tag{1.4}$$

valid for any function $f = f(q, \dot q, t)$, as obvious from the picture. Throwing away the boundary term (it vanishes due to the fixed endpoint condition) and imposing the principle of least action, we thus recovered the *Euler–Lagrange (E-L)equations*

$$\boxed{\frac{\partial L}{\partial q^I} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q^I}\Big) = 0\,,} \tag{1.5}$$

restoring the index I for a set of generalized coordinates.

- The following three remarks are in order:

  1. For $L = L(q, \dot q, t)$, the Euler–Lagrange equations (1.5) are *second-order* (in time) equations of motion. Note that this is no longer true when higher time derivatives of q are included in L, see tutorial.

2. One can show, see, e.g., excellent book by Landau and Lifshitz [1], that for conservative mechanical systems, we have

$$L = T - V\,, \tag{1.6}$$

where T stands for the kinetic and V for the potential energy of the system.

3. L is not defined uniquely: we have a freedom of adding a total time derivative of an arbitrary function of time and coordinates. Namely

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d\Lambda(q,t)}{dt}\,, \tag{1.7}$$

gives the same EOM.

To prove the third statement we either plug directly new L' to (E-L) equations and keep differentiating, or more simply, invoke the Principe of least action, writing

$$S' = \int L'dt = \int Ldt + \int \frac{d\Lambda}{dt}dt = S + [\Lambda]_{t_1}^{t_2} \quad \Rightarrow \quad \delta S' = \delta S + \left[\frac{\partial \Lambda}{\partial q}\delta q\right]_{t_1}^{t_2} = \delta S\,, \tag{1.8}$$

as we have fixed endpoints.

1.2 Integrals of motion

- Solving (E-L) equations is not an easy task. People usually use one (or a combination of) the following three approaches:

  i) *numerical* integration
  ii) *approximation* methods
  iii) *analytical* methods: rare case of integrable systems.

  As we shall see later on, iii) is only possible when a sufficient number of (nice) integrals of motion is present.

- **Definition.** *Integral (constant) of motion is a function $I = I(q, \dot{q}, t)$ such that*

$$\boxed{\frac{dI}{dt} = 0} \tag{1.9}$$

  for any $q(t)$ solving EOM.

- It was proven by a brilliant mathematician E. Noether (1882–1935) that integrals of motion are in one-to-one correspondence with certain symmetries of the system. (If interested see a recent reprint [9] of the original paper.) In order to formulate this theorem let us introduce the following terminology:

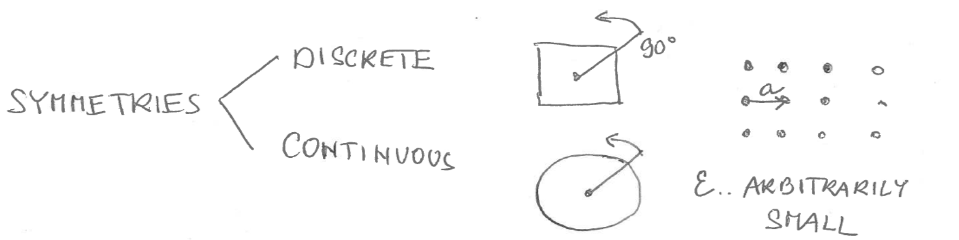
1. A statement is valid *on shell* means that it is valid provided EOM are satisfied; whereas *off-shell* means it is valid irrespective of EOM.
2. *Types of symmetries.* Consider for concreteness a transformation (note that $\tilde{\delta}$ is different from field variation δ.)

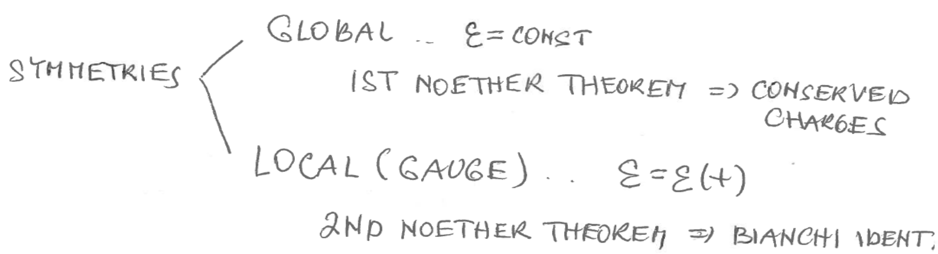
$$\boxed{t \to t' = t + \tilde{\delta} t\,, \quad q \to q'(t') = q(t) + \tilde{\delta} q(t)\,,} \tag{1.10}$$

and write

$$\tilde{\delta} q = \epsilon \Delta q\,, \quad \tilde{\delta} t = \epsilon \Delta t\,. \tag{1.11}$$

Here, Δq stands for the "generator" (that defines the action) and ϵ stands for the parameter (that determines how big the action is). The we distinguish the following types of symmetries:





In this module we shall only discuss (various versions of) the First Noether's theorem.

- Using this terminology we have the following:

> **Noether's theorem (Version 1).** *For every global continuous symmetry of the system, there is a corresponding integral of motion.*

Note that this theorem "*geometrizes physics*", that is, it connects a geometrical property of the configuration space with a property of the motion, namely, the existence of an integral of motion. Let us provide two simple examples:

1. Let Lagrangian L is not an explicit function of t, i.e., $L \neq L(t)$. Then the following quantity:[2]

$$\boxed{E = \frac{\partial L}{\partial \dot{q}^I}\dot{q}^I - L\,,} \tag{1.12}$$

is an integral of motion called the *generalized energy.* The proof of this is very simple, consider

$$\frac{dE}{dt} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big)\dot{q} - \frac{\partial L}{\partial q}\dot{q} = 0\,,$$

provided the (E-L) equations are satisfied.

2. Let q be a *cyclic coordinate*, that is a coordinate such that $L \neq L(q)$. Then the following quantity:

$$\boxed{p = \frac{\partial L}{\partial \dot{q}}} \tag{1.13}$$

is an integral of motion called the *generalized momentum.*

- **Noether theorem (Version 2: explicit version).** *Let $\tilde{\delta}$ be a global continuous symmetry, i.e., off-shell we find $\tilde{\delta}t, \tilde{\delta}q$, s.t.,*

$$\tilde{\delta}S = 0 \quad \Rightarrow \quad I = \frac{\partial L}{\partial \dot{q}}\tilde{\delta}q + \Big(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\Big)\tilde{\delta}t \tag{1.14}$$

is an (on-shell) integral of motion.

Ex: i) $(\tilde{\delta}t, \tilde{\delta}q) = \varepsilon(1,0)$ ... time translation $\Leftrightarrow$ I energy

ii) $(\tilde{\delta}t, \tilde{\delta}q) = \varepsilon(0,1)$ ... space translation $\Leftrightarrow$ I momentum

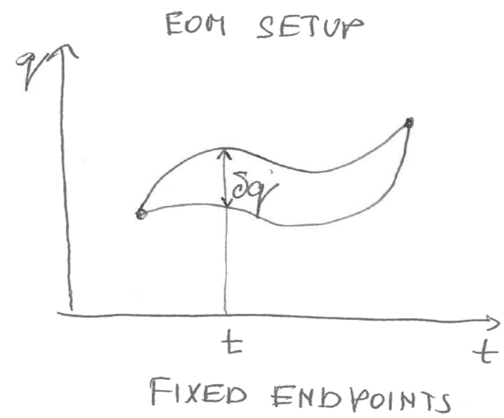
} spacetime homogeneity

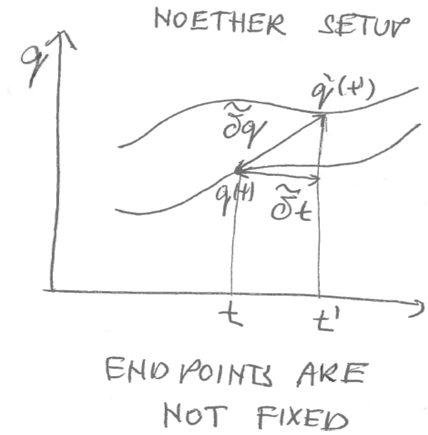
iii) $(\tilde{\delta}t, \tilde{\delta}\varphi) = \varepsilon(0,1)$ ... space rotation $\Leftrightarrow$ I ang. m. ... isotropy

Proof:

- In order to prove this theorem, we have to understand first the difference between the variations δ and $\tilde{\delta}$, see figure:

[2] Throughout the text we use the Einstein summation convention: sum over an index that repeats twice. For example $\frac{\partial L}{\partial \dot{q}^I}\dot{q}^I = \sum_{I=1}^{n}\frac{\partial L}{\partial \dot{q}^I}\dot{q}^I$.





$$
\begin{aligned}
\tilde{\delta} dt &= dt' - dt = d\tilde{\delta}t = \frac{d\tilde{\delta}t}{dt}dt \,, \\
\tilde{\delta} q(t) &\equiv q'(t') - q(t) = q'(t) + \tilde{\delta}t\frac{dq'(t)}{dt} + \cdots - q(t) \\
&= \delta q(t) + \tilde{\delta}t\frac{dq'(t)}{dt} = \delta q(t) + \tilde{\delta}t\frac{dq(t)}{dt} \,,
\end{aligned}
$$

using the Taylor expansion in the first step and the fact that the second term is already small and trajectories q' an q are infinitesimally close in the last step. More generally, we have

$$
\boxed{\tilde{\delta} = \delta + \tilde{\delta}t\frac{d}{dt}} \tag{1.15}
$$

for any function of generalized coordinates and time $f(q, \dot{q}, t)$.

– Using the fact that we have a symmetry and employing the identity (1.15), we have[3]

$$
\begin{aligned}
\tilde{\delta} S &= \int \left[\tilde{\delta} L dt + L\tilde{\delta} dt\right] = 0 \\
&= \int \left[\delta L + \tilde{\delta}t\frac{dL}{dt} + L\frac{d\tilde{\delta}t}{dt}\right] dt = \int \left[\delta L + \frac{d}{dt}\left(L\tilde{\delta}t\right)\right] dt \\
&= \left|\delta L = \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial q}\delta q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{\partial L}{\partial q}\delta q\right| \\
&= \int \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q + L\tilde{\delta}t\right) + \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q\right\} dt \,.
\end{aligned} \tag{1.16}
$$

[3] We stress again that the variation δ is in Noether's setup no longer fixed at the endpoints. That is why in the derivation of Noether's theorems we no longer throw away the boundary terms.

The second term vanishes on-shell, while the first yields that

$$I = \frac{\partial L}{\partial \dot{q}} \delta q + L\tilde{\delta} t \tag{1.17}$$

is an integral of motion, that is for a fixed trajectory we have $I|_{t_1} = I_{t_2}$. Expressing finally I in terms of $\tilde{\delta} q$ variations we recover the statement of the theorem.

- **Noether theorem (Version 3: sneaky recipe).**

  i) Observe that S is invariant under a global continuous transformation (characterized by a constant parameter ϵ): $\tilde{\delta} S = 0$.

  ii) Promote ϵ to $\epsilon(t)$ with fixed endpoints. Then we must have

  $$\boxed{\tilde{\delta} S = \int dt \dot{\epsilon} I\,.} \tag{1.18}$$

  (Which is the most general form linear in ϵ one can write.)

  iii) Integrating by parts we find

  $$\tilde{\delta} S = -\int dt \frac{dI}{dt} \epsilon = 0\,, \tag{1.19}$$

  where the last equality follows from the fact that ϵ is an "arbitrary variation δq" and it is valid on-shell. (We used the action principle.) This then implies that $dI/dt = 0$. So I is an on-shell integral of motion and can be read off from expression (1.18).

We shall look at explicit examples in tutorial. Note also that (as we derived above)

$$\tilde{\delta} S = \delta S + \int \frac{d}{dt}\left(L\tilde{\delta} t\right) dt\,, \tag{1.20}$$

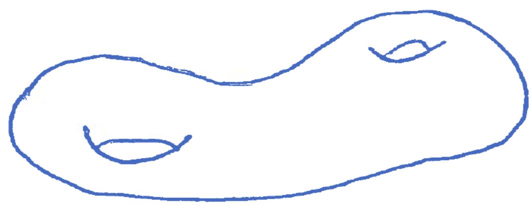
and so the two variations are equal up to a total derivative. This immediately implies that on-shell we have $\delta S = 0 = \tilde{\delta} S$. This justifies equality (1.19).

I refer you to [10] for a discussion of the sneaky recipe and 2nd Noether's theorem in the case of *higher-derivative actions*.

Chapter 2: Introduction to differential geometry

2.1 Manifolds and tensors

- Manifolds.



An n-dimensional manifold has a local differential structure of $\mathbb{R}^n$, but not necessarily its global properties (not necessarily embedded in higher-dimensional Euclidean space). Slightly more precisely:

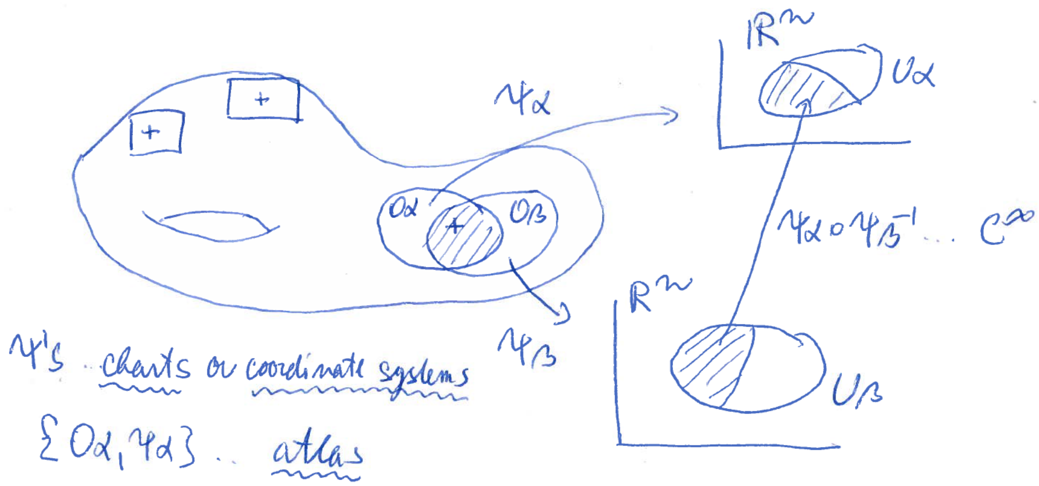
Definition. An n-dimensional *manifold* M is a 'set of points' together with a collection of subsets $\{O_\alpha\}$ satisfying:

i) Each $p \in M$ lies in at least one O_α, i.e., $\{O_\alpha\}$ cover M.

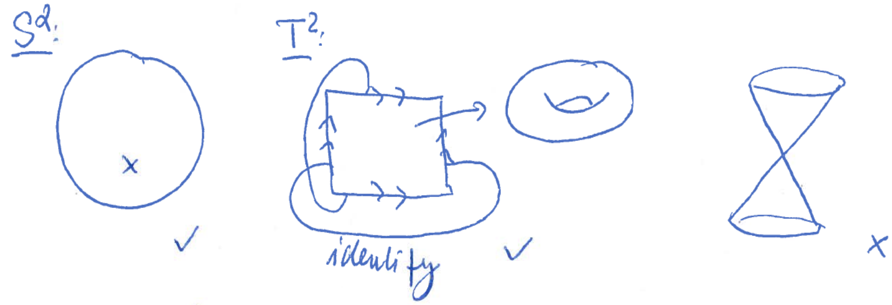
ii) For each α, there is 1-1, onto, map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$ (a union of open balls).

iii) If any two sets of O_α and O_β overlap, $O_\alpha \cap O_\beta \neq 0$, the maps $\psi_\beta \circ \psi_\alpha^{-1}$ is C^∞.

In other words: "a manifold is made of pieces that look like open subsets of $\mathbb{R}^n$ which are sewn together smoothly".

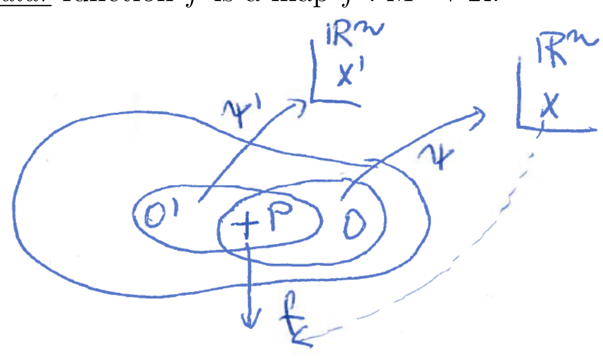


Examples:



Note. GR (particles) need manifolds. String theory works fine on orbifolds (e.g. $x \sim -x$, i.e. $\mathbb{R}^1/\mathbb{Z}_2$: fundamental domain has boundary...singular in GR.)

- A *scalar* function f is a map $f : M \to \mathbb{R}$.

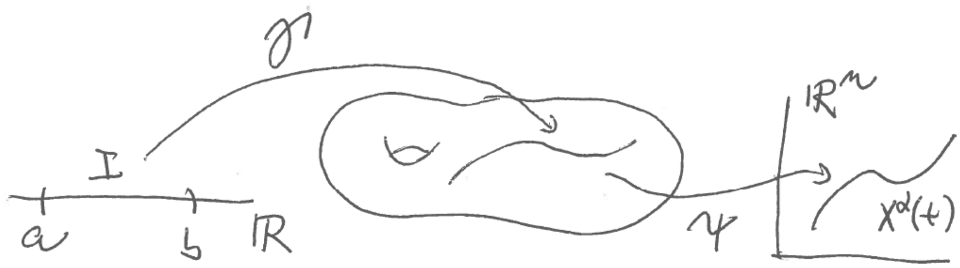


In this definition we exploit the coordinate map ψ associated with the manifold. That is, the function is defined by first going to $\mathbb{R}^n$ and then by defining a function from $\mathbb{R}^n$ to $\mathbb{R}$. If this definition is to be any good, under a change of coordinates we must have: $f(p) = f(x(p)) = f'(x'(p))$, giving

$$\boxed{f'(x') = f(x)} \tag{2.1}$$

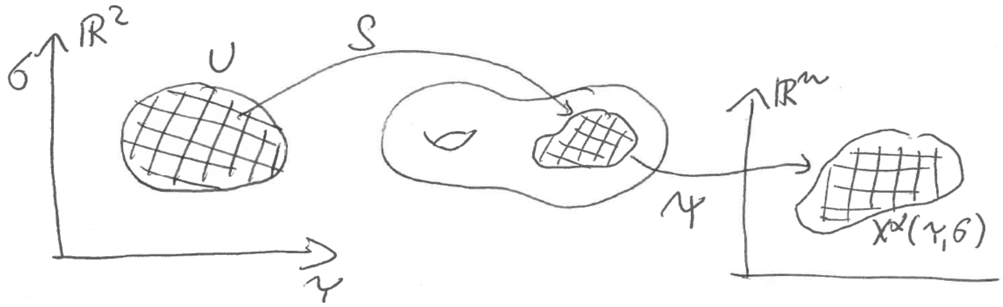
as a transformation rule for scalar functions. An example of a scalar function on a manifold is intrinsic curvature scalar, or "temperature on Earth".

- A *curve* γ on M is a map $\gamma : I \subset \mathbb{R}^1 \to M$, s.t., $(\psi_\alpha \circ \gamma)(t) = [x^1(t), \ldots, x^n(t)]$ are smooth.



E.g. river, trajectory (worldline) of a particle.

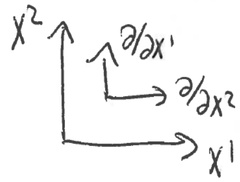
Similarly a *surface* S is a map $S : U \subset \mathbb{R}^2 \to M$ such that $(\psi_\alpha \circ S)(\tau, \sigma) = [x^1(\tau, \sigma), \ldots, x^n(\tau, \sigma)]$ are smooth.



E.g. lake surface, worldsheet of a string.

- A *tangent vector* is associated with "direction of a derivative at a point".

in $\mathbb{R}^n$: $v^\mu = (v^1, \ldots, v^n) \leftrightarrow$ directional derivative op. $v^\mu \frac{\partial}{\partial x^\mu}$



$$\hat{v} f = v^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$$

Characterized by linearity & Leibnitz rule.

Definition. Let $\mathfrak{F}$ be a collection of C^∞ scalar functions. A *tangent vector* V at point $p \in M$ is a map $V : \mathfrak{F} \to \mathbb{R}$ that is

linear: $V(af + bg) = aV(f) + bV(g)$ for all $a, b \in \mathbb{R}$.

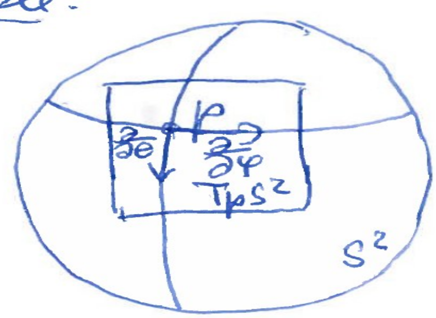
obeys Leibniz rule: $V(fg) = f(p)V(g) + g(p)V(f)$.

Theorem. The set of tangent vectors at p forms a *tangent vector space* T_pM which has the same dimensionality as M, with coordinate basis $\frac{\partial}{\partial x^\mu}$. Any vector V can be expressed in the form

$$\boxed{V = V^\mu \frac{\partial}{\partial x^\mu}\,,} \tag{2.2}$$

where V^μ are vector components.

- Example:



Under a transformation of coordinates (using the chain rule) we have

$$V = V^\mu(x)\frac{\partial}{\partial x^\mu} = V^\mu(x)\frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = V'^\nu(x')\frac{\partial}{\partial x'^\nu}\,,$$

and hence:

$$\boxed{V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu}V^\mu(x)\,.} \tag{2.3}$$

This is how components of a vector transform under change of coordinates. (Note that vector itself is a geometric object and remains invariant under such a change.)

- We now want to extend these notions to the 'whole manifold'.

Definition. A *tangent vector field* V is defined as $\{V|_p \in T_pM$ for all $p \in M$: $V(f)$ is smooth$\}$. A *tangent bundle* $TM = \bigcup_p T_pM$.

Note that TM has local coordinates (x^μ, V^ν).

- **Definition.** A *cotangent vector (1-form)* ω at a point $p \in M$ is a map $\omega : T_pM \to \mathbb{R}$. 1-forms form a *cotangent vector space* T^*_pM with coordinate basis dx^μ defined by
$$\boxed{dx^\mu\Big(\frac{\partial}{\partial x^\nu}\Big) = \delta^\mu_\nu\,.} \tag{2.4}$$
One can write
$$\boxed{\omega = \omega_\mu dx^\mu\,.} \tag{2.5}$$
Changing coordinates we have
$$\omega = \omega_\mu(x)dx^\mu = \omega_\mu(x)\frac{\partial x^\mu}{\partial x'^\nu}dx'^\nu = \omega'_\nu(x')dx'^\nu\,, \tag{2.6}$$
that is
$$\boxed{\omega'_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu}\omega_\mu(x)\,.} \tag{2.7}$$
This is how components of a covector transform under a change of coordinates.[1]

- A *cotangent bundle* $T^*M = \bigcup_p T^*_pM$. It has local coordinates (x^μ, ω_ν). Tangent and cotangent bundles are specific examples of *fibre bundles*.

- Canonical projection. Note that $V \in TM$ uniquely defines: i) $p \in M$ and $v \in T_pM$. This allows one to define *canonical projection* π:
$$\pi : TM \to M : \pi(V) = p\,. \tag{2.8}$$
In local coordinates $V \in TM$ has components $(x^1, \ldots, x^n, v^1, \ldots, v^n)$. We then have
$$\pi(x^1, \ldots, x^n, v^1, \ldots v^n) = (x^1, \ldots, x^n)\,. \tag{2.9}$$
Similarly for T^*M.

- Tensor = "product of vectors and forms, has components $T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}$."
Definition. A *tensor of type (k,l) of rank $(k+l)$* is a multilinear map
$$T : \underbrace{T^*_p \times \cdots \times T^*_p}_{k\text{-times}} \times \underbrace{T_p \times \cdots \times T_p}_{l\text{-times}} \to \mathbb{R}\,. \tag{2.10}$$
So the tensor "eats vectors and forms." Tensor field extends this notion to the whole manifold. Its components transform as
$$T'^{\alpha\ldots}{}_{\mu\ldots}(x') = \underbrace{\frac{\partial x'^\alpha}{\partial x^\delta}\ldots}_{k}\underbrace{\frac{\partial x^\kappa}{\partial x'^\mu}\ldots}_{l}T^{\delta\ldots}{}_{\kappa\ldots}(x)\,. \tag{2.11}$$

[1]Note that the 'transformation matrix' for a covector $A^\alpha_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}$ is an inverse of the transformation matrix for the vector $B^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$: $B^\alpha_\beta A^\beta_\gamma = \delta^\alpha_\gamma$.

For example: Tensor of type $(2,1)$ is of rank 3 and writes as

$$T = T^{\alpha\beta}{}_{\gamma}\partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma \,. \tag{2.12}$$

Here $T^{\alpha\beta}{}_{\gamma}$ are the components of T and $\partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma$ is its basis. The components are nothing else then the tensor evaluated on the corresponding basis vectors:

$$T^{\kappa\delta}{}_{\iota} = T(dx^\kappa, dx^\delta, \partial_{x^\iota}) \,. \tag{2.13}$$

Indeed, we have

$$T(dx^\kappa, dx^\delta, \partial_{x^\iota}) = T^{\alpha\beta}{}_{\gamma} dx^\kappa(\partial_{x^\alpha}) dx^\delta(\partial_{x^\beta}) dx^\gamma(\partial_{x^\iota}) = T^{\alpha\beta}{}_{\gamma}\delta^\kappa_\alpha \delta^\delta_\beta \delta^\gamma_\iota = T^{\kappa\delta}{}_{\iota} \,. \tag{2.14}$$

By linearity we then have

$$T(\omega, \nu, W) = T(\omega_\alpha dx^\alpha, \nu_\beta dx^\beta, W^\gamma \partial_{x^\gamma}) = \omega_\alpha \nu_\beta W^\gamma T(dx^\alpha, dx^\beta, \partial_{x^\gamma}) = T^{\alpha\beta}{}_{\gamma}\omega_\alpha \nu_\beta W^\gamma \,, \tag{2.15}$$

which is a scalar function.

- Cook book (tensor algebra). Let T, S be two tensors of rank t and s, then

  i) $T + S$ is a tensor (if they are the same type).

  ii) *tensor product* $\otimes$ "creates bigger tensors", namely, $T \otimes S$ is a tensor of rank $(t+s)$. For example, considering the above tensor T, together with a 1-form $S = S_\delta dx^\delta$, we have

  $$T \otimes S = \underbrace{T^{\alpha\beta}{}_{\gamma} S_\delta}_{(T\otimes S)^{\alpha\beta}{}_{\gamma\delta}} \partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma \otimes dx^\delta \,. \tag{2.16}$$

  iii) *contraction* $\cdot$ "creates smaller tensors". Each contraction 'connects' the corresponding vector and covector basis, reducing the rank of a tensor by two. For example, considering the above (2,1) tensor T, we can contract the first and the last index, obtaining a $(1,0)$ tensor T_{contr}, given by

  $$T_{contr} = T^{\alpha\beta}{}_{\gamma}\partial_{x^\beta} \underbrace{dx^\gamma(\partial_{x^\alpha})}_{\delta^\gamma_\alpha} = T^{\alpha\beta}{}_{\alpha}\partial_{x^\beta} = T^\beta_{contr}\partial_{x^\beta} \,. \tag{2.17}$$

  One can also contract indices of two tensors, creating a tensor $T \cdot S$ of rank $(t + s - 2 \times \#_{\text{contractions}})$. E.g.

  $$(T \cdot S)^{\alpha\ldots\mu\ldots}{}_{\gamma\ldots\nu\ldots} = T^{\kappa\delta\alpha\ldots}{}_{\gamma\ldots} S^{\mu\ldots}{}_{\kappa\delta\nu\ldots} \,. \tag{2.18}$$

  Here indices that are summed over, namely κ and δ are called *dummy indices* whereas the 'surviving indices' $(\alpha, \ldots)$ are called *free indices*.

  iv) Tensors are invariant objects. In particular, if a tensor is zero in one coordinate system, it is zero in all coordinate systems.

2.2 Lie derivative

- Differentiation of tensors on M is problematic, c.f.,

$$\left.\frac{df}{dt}\right|_{t_0} = \lim_{s\to 0} \frac{f(t_0+s) - f(t_0)}{s} .$$

  On M, one might want to replace t_0 by p. However how to add s to p? And how to compare a vector at $p+\delta p$ to a vector at p when they live on a different space?

- To resolve these issues one needs an additional structure. We have 3 standard possibilities:

  i) *Lie derivative* (vector field U).

  ii) *Exterior derivative* (forms).

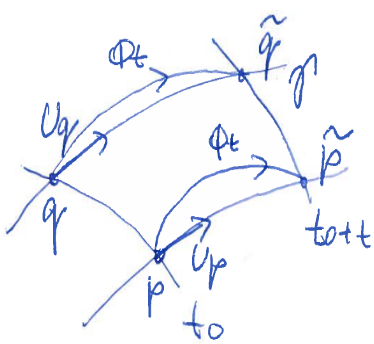
  iii) *Covariant derivative* (connection Γ).

- A vector field U defines its *integral curves* on M (their tangent vector coincides with U_p for all $p \in M$):

proof: in coordinates x^μ

$$\frac{dx^\mu}{dt} = U^\mu(x) \quad \text{has unique sol.}$$

  This defines a map $\phi_t : M \to M$ by $\phi_t : \gamma(t_0) \to \gamma(t_0+t)$.

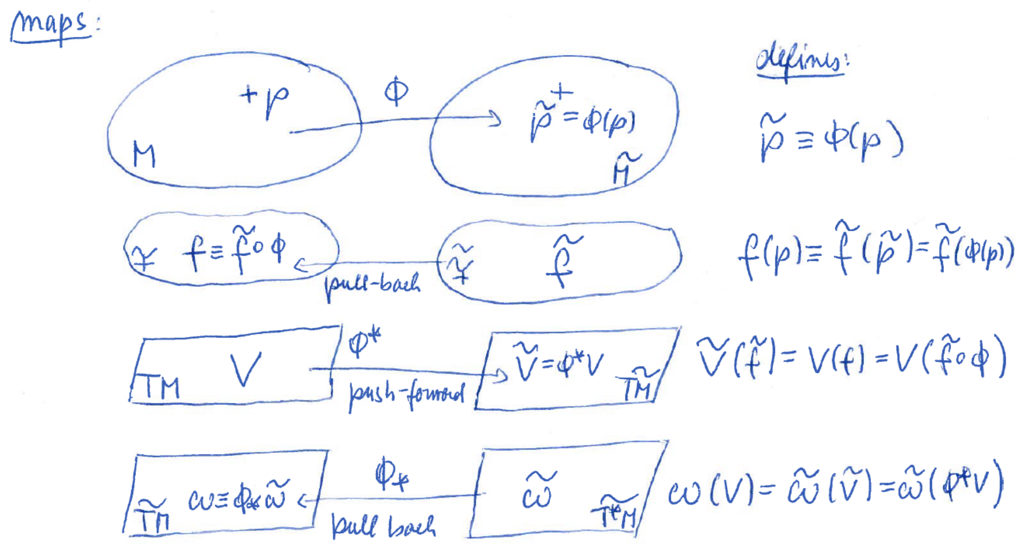


ϕ_t: • continuous

• $\phi_0 = Id$, $\phi_{t+s} = \phi_t \circ \phi_s$, $\phi_{-t} = \phi_t^{-1}$

defines 1-parametric group (Lie) of diffeomorphisms

- Maps between manifolds. Let M and $\tilde{M}$ be two manifolds and $\phi : M \to \tilde{M}$ a smooth map. The we find the following induced maps:



Note that for a given type of the object, the map always works only one way (one cannot define it other way). However, a very special case happens when ϕ is a diffeomorphism.

- A *diffeomorphism* is a map $\phi : M \to \tilde{M}$: 1-1, onto, ϕ^{-1} is smooth. In this case, one can use ϕ^{-1} to define above maps *both ways*. In particular one can define an "overloaded operation":

$$\phi^* : \text{tensors on } M \to \text{tensors on } \tilde{M}\,. \tag{2.19}$$

- **Definition.** Let ϕ_t be a 1-parametric group of diffeomorphisms generated by a vector field U. Then the *Lie derivative* $\mathcal{L}_U$ w.r.t. U is defined as

$$\boxed{\mathcal{L}_U T\big|_p = \lim_{t\to 0} \frac{T|_p - \phi_t^* T|_p}{t}\,.} \tag{2.20}$$

"WHAT WAS THERE MINUS WHAT I TRANSPORTED THERE".

Example:

$$\mathcal{L}_U f = \lim_{t\to 0} \frac{f(t_0) - \tilde{f}(t_0)}{t} = \Big|\tilde{f}(t_0) = f(t_0 - t)\Big| = \frac{df}{dt} = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu} = U^\mu \frac{\partial f}{\partial x^\mu} = U(f)\,, \tag{2.21}$$

where we have used the definition of the integral curve.

- Properties:

  i) $\mathcal{L}_U$ maps (r, s) tensors to (r, s) tensors.

  ii) $\mathcal{L}_U$ is linear and preserves contraction.

  iii) Leibnitz: $\mathcal{L}_U(T \otimes S) = (\mathcal{L}_U T) \otimes S + T \otimes (\mathcal{L}_U S)\,.$

iv) We have the following expressions for the Lie derivative of a function f and a vector V:

$$\mathcal{L}_U f = Uf = U^\mu \frac{\partial f}{\partial x^\mu}\,, \quad \mathcal{L}_U V = [U,V] = UV - VU\,,$$

where the latter is called a *Lie bracket*.

v) For components of a general tensor we then find

$$\mathcal{L}_U T^{\alpha\dots}{}_{\beta\dots} = U^\gamma \frac{\partial}{\partial x^\gamma} T^{\alpha\dots}{}_{\beta\dots} - T^{\gamma\dots}{}_{\beta\dots} \frac{\partial}{\partial x^\gamma} U^\alpha + \dots + T^{\alpha\dots}{}_{\gamma\dots} \frac{\partial}{\partial x^\beta} U^\gamma\,. \quad (2.22)$$

The explicit expressions for the Lie derivative are easily obtained by using the *passive* rather than the *active* approach to diffeomorphisms. In the passive approach we simply interpret the associated 'map' as a coordinate transformation.

- Symmetries. The Lie derivative plays a key role for defining symmetries. Namely, it may happen that for a given tensor field T one can find such a vector field U so that

$$\mathcal{L}_U T^{\alpha\dots}{}_{\beta\dots} = 0\,. \quad (2.23)$$

This means that vector U describes a special direction in the manifold along which the tensor T 'stays the same'—it describes a *symmetry* of a given tensor field T. You may be familiar with a particular case of symmetries of the metric, $\mathcal{L}_U g_{\alpha\beta} = 0$, called *isometries*. We shall see some other examples soon in this module.

2.3 Differential forms

- **Definition.** A *differential p-form* ω is a totally antisymmetric tensor of type $(0,p)$, that is,

$$\omega_{\alpha_1\dots\alpha_p} = \omega_{[\alpha_1\dots\alpha_p]} = \frac{1}{p!}\sum_{\mathrm{perm}\pi} \mathrm{sign}(\pi)\omega_{\alpha_{\pi(1)}\dots\alpha_{\pi(p)}}\,. \quad (2.24)$$

Hence, a differential form is antisymmetric under exchange of any 2 indices. We shall denote Λ^p_x a vector space of p-forms at x. One can show that it has a dimensionality $dim\Lambda^p_x = \binom{n}{p}$.

- **Definition.** A *wedge product* $\wedge : \Lambda^p_x \times \Lambda^q_x \to \Lambda^{p+q}_x$:

$$(\omega\wedge\nu)_{\alpha_1\dots\alpha_p\beta_1\dots\beta_q} = \frac{(p+q)!}{p!q!}\omega_{[\alpha_1\dots\alpha_p}\nu_{\beta_1\dots\beta_q]}\,. \quad (2.25)$$

That is, $\omega\wedge\nu$ is a $(p+q)$-form. It obeys

$$\omega\wedge\nu = (-1)^{pq}\nu\wedge\omega\,. \quad (2.26)$$

Since dx^α is a coordinate basis of 1-forms, general p-form can be written as

$$\boxed{\omega = \frac{1}{p!}\omega_{\alpha_1\dots\alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \,.} \tag{2.27}$$

- For any vector V, we define an <u>*inner derivative*</u> $i_V : \Lambda^p \to \Lambda^{p-1}$:

$$i_V\omega = V \lrcorner\, \omega = V \cdot \omega = \omega(V, \dots) : \qquad (V \lrcorner\, \omega)_{\alpha_1 \dot{\alpha}_{p-1}} = V^\beta \omega_{b\alpha_1\dots\alpha_{p-1}} \,. \tag{2.28}$$

<u>Properties of inner derivative:</u>

i) i_V is linear

ii) i_V is linear in $V : i_{fV+gW} = fi_V + gi_W$.

iii) graded Leibnitz rule: For $\omega \in \Lambda^p$ we have

$$i_V(\omega \wedge \nu) = (i_V\omega) \wedge \nu + (-1)^p \omega \wedge i_V \nu \,. \tag{2.29}$$

iv)

$$i_v i_W + i_W i_V = 0 \quad \text{spec.} \quad i_V^2 = 0 \,. \tag{2.30}$$

- **Definition.** <u>*Exterior derivative*</u> $d : \Lambda^p \to \Lambda^{p+1}$ is defined as follows:

i) On a function f we have $d : f \to df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$.

ii) On a p-form ω we then have

$$d : \omega \to d\omega = \frac{1}{p!} d\omega_{\alpha_1\dots\alpha_p} \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \,. \tag{2.31}$$

That is $(d\omega)_{\alpha_1\dots\alpha_{p+1}} = (p+1)\partial_{[\alpha_1}\omega_{\alpha_2\dots\alpha_{p+1}]}$.
Note that we have $d^2 = 0$. Conversely, a p-form α is called <u>*closed*</u> when $d\alpha = 0$. It is called <u>*exact*</u> when $\alpha = d\beta$. Any closed form α can be *locally* written as $\alpha = d\beta$ but not *globally*.[2]

- <u>*Cartan's lemma.*</u> For a vector field V and a p-form ω, we have the following identity:

$$\boxed{\mathcal{L}_V\omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega) \,.} \tag{2.32}$$

In particular, this implies that

$$\mathcal{L}_V df = d\mathcal{L}_V f \,. \tag{2.33}$$

[2] Dimension of a vector space of closed p-forms modulo the exact p-forms equals p-th <u>*Betti number*</u> of the manifold and is a topological quantity.

- Integration. A p-form ω can be integrated over a p-dimensional (sub)manifold. Writing $\omega = f dx^1 \wedge \dots dx^p$ we then define

$$\int_{O_p} \omega = \int_{\psi(O_p)} f dx^1 \dots dx^p \quad \text{where r.h.s. is defined as Lebesgue integral.} \tag{2.34}$$

Note that this definition is independent of coordinates, as we have

$$\omega = f' dx'^1 \wedge \dots \wedge dx'^p \,, \quad f' = f \det\Big(\frac{\partial x^\mu}{\partial x'^\nu}\Big)\,. \tag{2.35}$$

Stokes theorem. The following identity is valid

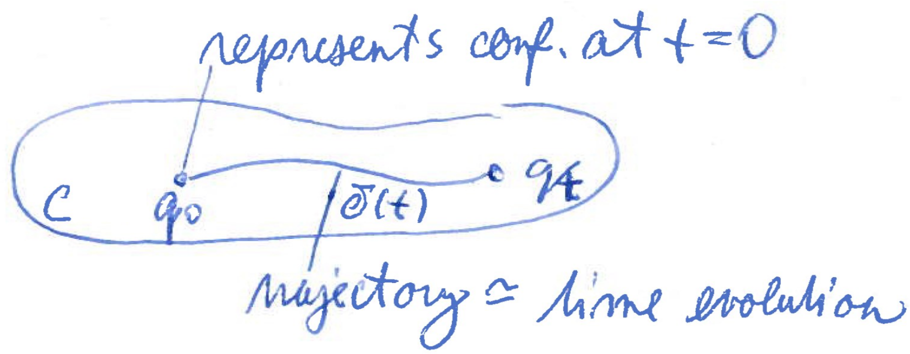
$$\boxed{\int_\Omega d\omega = \int_{\partial\Omega} \omega\,.} \tag{2.36}$$

Many of the differential vector identities you know are a special case of this beautiful theorem.

If you want to know more about differential geometry, I refer you to an excellent (but concise) book [4].

2.4 Geometric formulation of Lagrangian mechanics

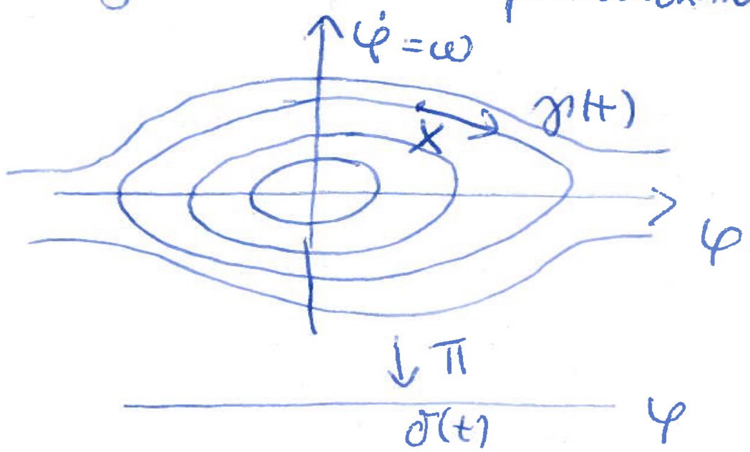
- Basic definitions. A *configuration space* C with n dof is a manifold of dimension n, equipped with local coordinates $(q^1, \dots, q^n)$:



'Points' of $\delta(t)$ describe a "photo of the system" at a given time.

A *velocity phase space* is a tangent bundle over configuration space, TC. It has a dimensions $2n$ and generalized coordinates $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$.

e.g. mathematical pendulum



'Points' of $\gamma(t)$ describe a "physical state of the system" at a given time.

A *Lagrangian* $L : TC \to \mathbb{R}$ determines the dynamics of the system. The dynamics is equally encoded in the *dynamical vector field* $X \in T(TC)$ which determines integral curves $\gamma(t)$ on TC. Our aim is to find system's trajectory:

$$\boxed{\delta(t) = \pi(\gamma(t))\,,} \tag{2.37}$$

where π is the corresponding canonical projection to the configuration space.

- Determining the dynamical field. The dynamical field X generates integral curves and hence
$$X = \frac{d}{dt} = \frac{dq^j(t)}{dt}\frac{\partial}{\partial q^j} + W^j \frac{\partial}{\partial \dot{q}^j}\,, \quad W^j = \frac{d\dot{q}^j}{dt}\,. \tag{2.38}$$
We have an extra condition: tangent to $\delta(t)$ is a velocity:
$$\boxed{\frac{dq^j(t)}{dt} = \dot{q}^j\,.} \tag{2.39}$$
So, our dynamical field X is a special field of the type
$$\boxed{X = \dot{q}^j \frac{\partial}{\partial q^j} + W^j(q^i, \dot{q}^i)\frac{\partial}{\partial \dot{q}^j}\,.} \tag{2.40}$$

Definition. We define the following quantities: *Lagrange 1-form:* θ, *Lagrange* symplectic 2-form: ω, and *Lagrange energy:* E, given by

$$\boxed{\theta = \frac{\partial L}{\partial \dot{q}^j} dq^j}\,, \quad \boxed{\omega = d\theta}\,, \quad \boxed{E = \frac{\partial L}{\partial \dot{q}^j}\dot{q}^j - L}\,. \tag{2.41}$$

Theorem. The physical state of the Lagrangian system described by the Lagrangian L is determined from integral curves of X, where X is given by

$$\mathcal{L}_X\theta = dL\,. \tag{2.42}$$

Proof: want to show equivalence to Euler–Lagrange equations. Using the definition of θ we have

$$\begin{aligned}\mathcal{L}_X\theta &= \underbrace{\Big(\mathcal{L}_X\frac{\partial L}{\partial \dot q^j}\Big)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^j}\right)=\frac{\partial L}{\partial q^j}} dq^j + \frac{\partial L}{\partial \dot q^j}\underbrace{\mathcal{L}_X(dq^j)}_{d(\mathcal{L}_X q^j)=d\dot q^j}\\ &= \frac{\partial L}{\partial q^j}dq^j + \frac{\partial L}{\partial \dot q^j}d\dot q^j = dL\,,\end{aligned} \tag{2.43}$$

where in the first step we have used the (E-L) equations and in the second Cartan's lemma.

Corollary. Equivalently, we may write

$$\boxed{X\lrcorner\,\omega = -dE\,.} \tag{2.44}$$

Proof: Using the expression for θ, (2.41), and X, (2.40), we find

$$X\lrcorner\,\theta = \dot q^i\frac{\partial L}{\partial \dot q^i}\,.$$

So, using Cartan's lemma, we have

$$X\lrcorner\,\omega = X\lrcorner\,d\theta = \mathcal{L}_X\theta - d(X\lrcorner\,\theta) = dL - d(X\lrcorner\,\theta) = -dE\,,$$

upon the use of definition of E.

Formal solution for X: Define Ω as the "inverse of ω": $\omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta^\gamma_\alpha$. The Corollary reads $X^\alpha\omega_{\alpha\beta} = (dE)_\beta$. Let us multiply both sides by $\Omega^{\beta\gamma}$. Then we find

$$X^\gamma = (-dE)_\beta\Omega^{\beta\gamma} \quad\Leftrightarrow\quad \boxed{X = -dE\cdot\Omega\,.} \tag{2.45}$$

- Conservation laws. Let the system admits: i) Lagrangian L ii) dynamical vector field X and the associated diffeomorphism ϕ_t^X, and iii) extra field Z together with the associated diffeomorphism ϕ_ϵ^Z. We then have

Noether's theorem (Version 4: Lagrangian formulation). Let $\mathcal{L}_Z L = 0$. Then $I = Z\lrcorner\,\theta$ is an integral of motion, i.e., $\mathcal{L}_X I = 0\,.$

Proof. One has

$$\mathcal{L}_X I = \mathcal{L}_X(Z \rfloor \theta) = \mathcal{L}_X Z \rfloor \theta + \underbrace{Z \rfloor \mathcal{L}_X \theta}_{Z \rfloor dL = \mathcal{L}_Z L = 0} = [X, Z] \rfloor \theta \,. \tag{2.46}$$

The fact that the last expression vanishes is left for you to prove during long Canadian winter evenings.

Note that this formulation naturally works on the velocity phase space. When we want to describe symmetries of the configuration space, the following elaborate construction is necessary. Let Z be a natural extension of Y that generates *point transformations* on C:

$$\begin{aligned} Y: &\quad q^j \to q^j_\epsilon = \phi_\epsilon(q^j) \quad \text{i.e. } Y^i = \frac{dq^i_\epsilon}{d\epsilon}\Big|_{\epsilon=0} \\ Z: &\quad (q^j, \dot{q}^j) \to (q^j_\epsilon, \dot{q}^j_\epsilon) = (\phi_\epsilon(q^j), \phi^*_\epsilon(\dot{q}^j)) \,, \end{aligned} \tag{2.47}$$

that is

$$Z = Y^i \frac{\partial}{\partial q^i} + \dot{Y}^i \frac{\partial}{\partial \dot{q}^i} \,, \quad \dot{Y}^i = \frac{dY^i(q^l)}{dt} = \frac{\partial Y^i}{\partial q^l} \dot{q}^l \,. \tag{2.48}$$

The fact that the restriction to “Point transformations” is ‘artificial’ and nowhere needed in the formulation of Noether’s theorem, hints on the fact that one can have more general phase space symmetries. We shall return to this topic in the next section.

Chapter 3: Hamiltonian mechanics: basic theory

3.1 Hamilton's canonical equations

W.R. Hamilton formulated his equations in 1834 (at the age of 29)—time is clicking guys.

- *canonical momentum*:

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}^j}\,.} \tag{3.1}$$

Configuration space
≡ "photo" of a system
(q^j) .. dim $\underline{n}$

×

phase space
describes physical states
(q^j, p_j) .. dim $= 2n$

- *Hamiltonian* = Legendre transform of the Lagrangian:

$$\boxed{H(q^i, p_i, t) = p_j \dot{q}^j - L\Big|_{\dot{q}^j = \dot{q}^j(q^i, p_i, t)}\,.} \tag{3.2}$$

Theorem. The system of n (E-L) equations is equivalent to the system of $2n$ first-order Hamilton's equations

$$\boxed{\dot{p}_j = -\frac{\partial H}{\partial q^j}\,, \quad \dot{q}^j = \frac{\partial H}{\partial p_j}\,.} \tag{3.3}$$

Proof: i) Direct calculation: for example, calculate $\frac{\partial H}{\partial p_j}$, using definition of H, (3.2), and (E-L) equations.

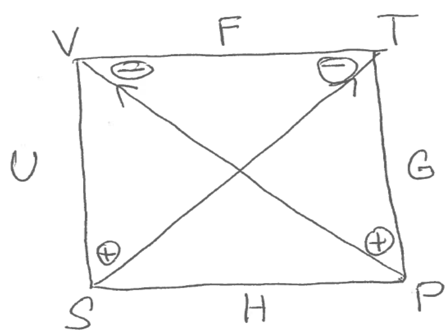
ii) variational principle:

$$\begin{aligned} 0 &= \delta S = \delta \int_1^2 \big(p_i \dot{q}^i - H\big) dt = \int_1^2 \Big(p_i \frac{d}{dt}\delta q^i + \dot{q}^i \delta p_i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i\Big) dt \\ &= \underbrace{\big[p_i \delta q^i\big]_1^2}_{0} + \int_1^2 \Big[\Big(\dot{q}^i - \frac{\partial H}{\partial p_i}\Big)\delta p_i - \Big(\dot{p}_i + \frac{\partial H}{\partial q^i}\Big)\delta q^i\Big] dt\,, \end{aligned}$$

from where the statement follows.

Note that we have $\frac{dH}{dt} = \frac{\partial H}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$, upon using the canonical equations. So if $H \neq H(t)$, then $H = H(q(t), p(t)) = const.$ (Compare this to the generalized energy studied in Lagrangian formalism.)

- Intermezzo: Legendre transform in thermodynamics



① TD potentials are total differentials when expressed in their variables

$dU = TdS - PdV$

(signs ⊕ ⊖ near potentials) [↗↖ carry the same info]

② Legendre transform $U(V,S) \to F(V,T)$

$F = U - TS$, $T = \frac{\partial U}{\partial S}$ (signs ⊕ ⊖ near potentials)

③ Maxwell's relations

$\left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial T}{\partial P}\right)_S$

proof: $dH = TdS + VdP$ $\quad \frac{\partial^2 H}{\partial S \partial P} = \frac{\partial T}{\partial P}\Big|_S = \frac{\partial V}{\partial S}\Big|_P$

3.2 Poisson brackets

Definition. Let f, g be two phase space functions. Their *canonical Poisson bracket* is a new function on phase space, defined by

$$\boxed{\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,.} \tag{3.4}$$

Properties:

i) Fundamental brackets are given by

$$\boxed{\{q^i, p_j\} = \delta^i_j\,, \quad \{q^i, q^j\} = 0\,, \quad \{p_i, p_j\} = 0\,.} \tag{3.5}$$

The fundamental brackets play an important role in the transition to quantum mechanics. Namely, one promotes (q^i, p_j) to operators $(\hat{q}^i, \hat{p}_j)$ and replaces Poisson brackets $\{\,,\,\}$ with commutators $[\,,\,]$, obtaining in particular $[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j$.

ii) antisymmetry: $\{f, g\} = -\{g, f\}\,.$

iii) linearity: $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$ for $\alpha, \beta \in \mathbb{R}\,.$

iv) Leibnitz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g\,.$

v) Jacobi identity: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0\,.$

Remarks:

- Poisson brackets define a Lie algebra, properties i)–v) can be used as a definition. (You will play with this ala Feynman in your tutorial.)

- Using the Poisson brackets, the time evolution is written as

$$\boxed{\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,.} \tag{3.6}$$

Proof of this statement is very simple:

$$\frac{df}{dt} = \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t}\,,$$

upon using the Hamilton's equations of motion, (3.3).

- An integral of motion I obeys

$$\boxed{\{I, H\} = 0\,.} \tag{3.7}$$

Note that having two integrals of motion, I_1 and I_2,

$$I_3 = \{I_1, I_2\} \tag{3.8}$$

is also an integral of motion. You shall show this in one of your tutorials.

3.3 Canonical transformations

- The strongest point of Hamiltonian mechanics is a possibility to exploit the freedom in its description encoded in the so called canonical transformations.
  Definition. A *canonical transformation* is such a transformation of phase space coordinates
  $$\boxed{Q^j = Q^j(q^i, p_i, t)\,, \quad P_j = P_j(q^i, p_i, t)} \tag{3.9}$$
  that preserves the form of Hamilton's equations. That is, one can find a 'new Hamiltonian' $H' = H'(Q^j, P_j, t)$ such that
  $$\frac{\partial H'}{\partial P_j} = \dot{Q}^j\,, \quad \frac{\partial H'}{\partial Q^j} = -\dot{P}_j\,. \tag{3.10}$$

- Note that not every transformation is canonical! For example, consider a free particle $H = \frac{p^2}{2m}$ and a transformation $Q = \sqrt{q}, P = p$. Guessing $H' = \frac{P^2}{2m}$, we have
  $$\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq \dot{Q}\,.$$

- Canonical transformations are generated by *generating functions* of the type
  $$F = F(old, new)\,. \tag{3.11}$$
  Since *old* $\in (q,p)$ and *new* $\in (Q,P)$ we have 4 possibilities (which are generically connected by Legendre transformations).
  For example, consider $F = F(q, Q, t)$. Then we find
  $$\begin{aligned} S &= \int \Bigl(p\dot{q} - H - \frac{dF}{dt}\Bigr)dt = \int \Bigl[\dot{q}\Bigl(p - \frac{\partial F}{\partial q}\Bigr) - \frac{\partial F}{\partial Q}\dot{Q} - \Bigl(H + \frac{\partial F}{\partial t}\Bigr)\Bigr]dt \\ &= \int \bigl(P\dot{Q} - H'(P,Q,t)\bigr)dt\,. \end{aligned} \tag{3.12}$$
  In the first line we exploited the freedom in Lagrangian formulation of mechanics (to get the same EOM), and in the last we expressed what we WANT TO achieve in order to get the same form of Hamilton's equations (3.10). So we arrived at the requirements
  $$\boxed{p = \frac{\partial F}{\partial q} \quad P = -\frac{\partial F}{\partial Q}\,, \quad H' = H + \frac{\partial F}{\partial t}\,.} \tag{3.13}$$
  For a given F, these will give (Q, P, H'). Note that if we want to go the other direction, we have the following integrability condition:
  $$\frac{\partial^2 F}{\partial Q \partial q} = \frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q} = \frac{\partial^2 F}{\partial q \partial Q}\,.$$
  Remarks:

i) Poisson brackets are invariant w.r.t. canonical transformations, i.e.,

$$\{f,g\}_{q,p} = \{f,g\}_{Q,P}\,. \tag{3.14}$$

Hint on proof: Following [1] think of function g as some Hamiltonian: $g \equiv H$, generating "time flow t". Then it is intuitively obvious that

$$\frac{\text{“}df\text{''}}{dt} = \{f,g\}$$

is independent of coordinates we use to describe our system.

ii) (Q,P) are new canonical coordinates and hence

$$\{Q^i,P_j\} = \delta^i_j\,,\quad \{Q^i,Q^j\} = 0\,,\quad \{P_i,P_j\} = 0\,. \tag{3.15}$$

Note that when the Poisson brackets are calculated in terms of (q,p), these give conditions for the transformation to be canonical.

iii) "Time evolution = change of coordinates". Or more precisely:

$$\boxed{\text{Time evolution} = \text{canonical transformation generated by } F = -S,}$$

where S is the so called Hamilton's function.
Proof. The key point of the proof is to understand what we mean by S.
Definition. A *Hamilton's function* is an "action understood as a function of coordinates and time". In order to obtain this function we i) solve the Euler–Lagrange equations for q, expressing it in terms of the boundary value data: $q = q(t_1,q_1,t_2,q_2,t)$. ii) Once this solution is known, we plug it back to L, and integrate L over t to get

$$S(t_1,q_1,t_2,q_2) = \int_{t_1}^{t_2} L(t_1,q_1,t_2,q_2,t)dt\,. \tag{3.16}$$

Since $Ldt = p_i dq^i - Hdt$, it can be shown that the total differential of S reads [1]

$$dS = \big(p_i dq^i - Hdt\big)_{\text{final}} - \big(p_i dq^i - Hdt\big)_{\text{initial}}\,. \tag{3.17}$$

Identifying $t = t_2$ and $q = q_2$, the last equality implies two important relations that lead to the Hamilton–Jacobi equation (derived in a different fashion in the next section):

$$\boxed{\frac{\partial S}{\partial q^i} = p_i\,,\quad \frac{\partial S}{\partial t} = -H\,.} \tag{3.18}$$

Note also that, upon $q_0 = q_1$ and $t_0 = t_1$ we have $\frac{\partial S}{\partial q_0^i} = -p_{0i}\,,\quad \frac{\partial S}{\partial t_0} = H_0\,.$

Final step of the proof is to realize that the time evolution can be written as a coordinate transformation

$$q(t) = q(q_0, p_0, t)\,, \quad p(t) = p(q_0, p_0, t)\,, \tag{3.19}$$

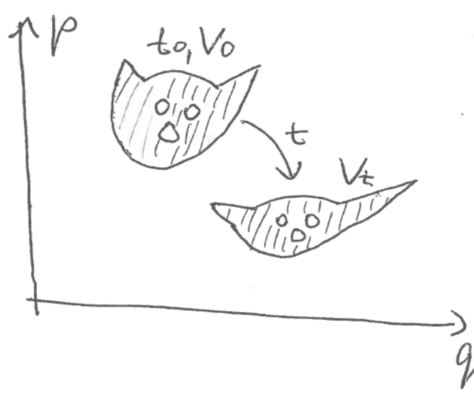
where (q_0, p_0) are the "old coordinates" and (q, p) are the "new ones". Integrating (3.17) we then find

$$\int \left[pdq - Hdt\right] = \int \left[p_0 dq_0 - H_0 dt - \frac{d(-S)}{dt} dt\right]. \tag{3.20}$$

Hence, the transformation is canonical, generated by $-S = -S(q_0, q, t)$ generating function.

iv) We also have the following theorem important for statistical physics:

Liouville's theorem. Volume of the phase space remains invariant under canonical transformations. In particular, time evolution preserves the phase space volume.



proof: $(n=1)$

$$V_t = \int dQ\,dP = \int |\mathrm{Jac}|\,dq\,dp$$

$$\mathrm{Jac} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \{Q, P\} = 1$$

$$= \int dq\,dp = V_0$$

3.4 Hamilton–Jacobi theory

- We have seen that a canonical transformation can be generated by $F = F(q^j, Q^j, t)$ provided

$$p_j = \frac{\partial F}{\partial q^j}\,, \quad P_j = -\frac{\partial F}{\partial Q^j}\,, \quad H' = H + \frac{\partial F}{\partial t}\,, \tag{3.21}$$

while

$$\frac{\partial H'}{\partial P_j} = \dot{Q}^j\,, \quad \frac{\partial H'}{\partial Q^j} = -\dot{P}_j\,. \tag{3.22}$$

[It preserves the form not the explicit expression!]

- Ingenious idea: Let's find a special generating function, call it S, such that the new Hamiltonian $H'=0$. Then we can immediately solve the Hamilton's equations (3.22), to get

$$Q^j=\alpha^j=const.\,,\quad P_j=-\beta_j=const. \tag{3.23}$$

The middle relation in (3.21) then implies

$$P_j=-\frac{\partial S}{\partial Q^j}\quad\Leftrightarrow\quad \beta_j=\frac{\partial S(q^i,\alpha^i,t)}{\partial\alpha^j}\,. \tag{3.24}$$

This is an implicit solution—the explicit one is found by inversion:

$$q^i=q^i(t,\alpha^j,\beta_j) \tag{3.25}$$

and we are done!

So how to find the miraculous S? The remaining equations in (3.21) now read

$$p_j=\frac{\partial S}{\partial q^j}\,,\quad H'=0=H+\frac{\partial S}{\partial t}\,, \tag{3.26}$$

or

$$\boxed{\frac{\partial S}{\partial t}+H(q^i,\frac{\partial S}{\partial q^i},t)=0\,.} \tag{3.27}$$

This is the famous *Hamilton–Jacobi equation*. It comprises one PDE for $S=S(q^j,t)$ [which is an action expressed as a function of coordinates, c.f., (3.18)].

To solve this equation we are interested in "*complete integral*" $S(q^j,\alpha^j,t)$ (a solution that depends on n constants α^j) rather than a *general integral* (arbitrary function).
A useful trick is to try *separation of variables*.[1]

$$\begin{aligned} H\neq H(t) \quad&\Rightarrow\quad S(q^j,t)=S_0(q^j)-Et\,,\\ H\neq H(q_c) \quad&\Rightarrow\quad S(q^j,t)=\alpha_c q_c+S_c(q^1,\dots,\hat{q}_c,\dots,q^n,t)\,. \end{aligned} \tag{3.28}$$

One should always try *additive separation ansatz*:

$$S(q^j,t)=S_1(q^1)+\dots+S_n(q^n)\,. \tag{3.29}$$

If this works we won the lottery :)

- Example: free fall in homogeneous gravitational field:

$$H=\frac{p^2}{2m}+V(x)\,,\quad V=-mgx\,. \tag{3.30}$$

[1]Be aware that this is a coordinate-dependent statement: it may happen that in some coordinates a complete separation of variables is possible whereas in other coordinates it is not.

The Hamilton–Jacobi equation for $S = S(x,t)$ now reads

$$\frac{1}{2m}\Big(\frac{\partial S}{\partial x}\Big)^2 + V + \frac{\partial S}{\partial t} = 0\,. \tag{3.31}$$

Since $H \neq H(t)$ we can try $S = S_0(x) - Et$ and get

$$\frac{1}{2m}\Big(\frac{dS_0}{dx}\Big)^2 + V - E = 0 \quad \Rightarrow \quad S_0 = \int \sqrt{2m(E-V)}dx\,. \tag{3.32}$$

Performing the integral for our V, we get

$$S = \frac{1}{3gm^2}\Big[2m(E+mgx)\Big]^{3/2} - Et\,. \tag{3.33}$$

Using this we then set

$$\frac{\partial S}{\partial E} = \beta = \frac{1}{mg}\sqrt{2m(E-V)} - t \tag{3.34}$$

Inversion of this finally gives

$$x = \frac{g}{2}(t+\beta)^2 - \frac{E}{mg}\,. \tag{3.35}$$

Note that the meaning of constant β (connected with energy) is just to set the initial time.

- Connection to QM. Recall that the Schrodinger equation reads

$$i\hbar\frac{\partial \psi}{\partial t} = \hat{H}\psi\,, \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(x)\,, \quad \hat{p} = -\hbar\nabla\,. \tag{3.36}$$

Trick is now to use the *geometric optics (WKB) approximation:*

$$\boxed{\psi = \psi_0 e^{\frac{i}{\hbar}S(x,t)}\,.} \tag{3.37}$$

So we get

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \underbrace{\Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\psi}_{\Big(\frac{(\nabla S)^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V\Big)\psi} = \hat{H}\psi\,. \tag{3.38}$$

Taking now the $\hbar \to 0$ limit, we get

$$\frac{(\nabla S)^2}{2m} + V + \frac{\partial S}{\partial t} = 0\,, \tag{3.39}$$

which is the Hamilton–Jacobi equation.

3.5 Integrable systems

- **Definition.** The dynamical system with n degrees of freedom ($2n$-dimensional phase space) is *completely (Liouville) integrable*, if it possess n independent conserved quantities $F_i(q,p) = f_i$, $\{H, F_i\} = 0$, that are in *involution*: $\{F_i, F_j\} = 0\ \forall i, j$.

Liouville's theorem. The solution of equations of motion of a completely integrable system can be obtained by "*quadrature*, that is by a finite number of algebraic operations and integrations.

- Remarks.

  i) Note that to integrate $2n$ ODE's we must know $2n$ integrals of motion. For a given canonical system it is sufficient to have just half, that is n. This is possible because each integral can be "used twice", i.e., reduces the order of the system by two!

  Analogous statement in QFT. In any gauge theory we have:

  $$\#(\text{true dof}) = \#(\text{apparent dof}) - 2 \times (\text{dof of gauge function})\,. \tag{3.40}$$

  Examples: i) Electromagnetism is described by the potentials $(\varphi, \vec{A})$, that is 4 apparent dof. The following gauge transformation

  $$\varphi \to \varphi + \partial_t \Lambda\,, \quad \vec{A} \to \vec{A} + \vec{\nabla}\Lambda \tag{3.41}$$

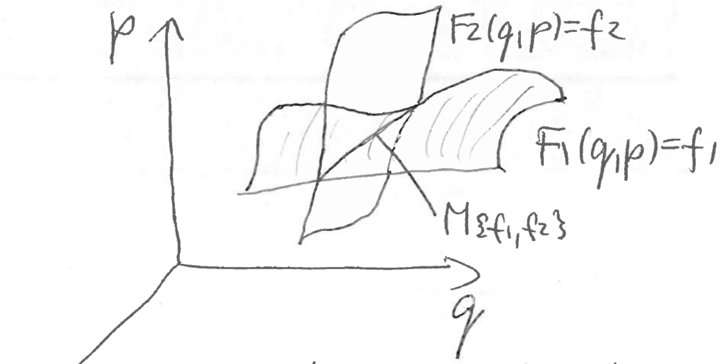
  leaves observable fields $\vec{E}$ and $\vec{B}$ invariant. That is, the gauge function Λ has one dof. This means that the elctromagnetic field has $4 - 2 \times 1 = 2$ true degrees of freedom. Of course, these correspond to two possible polarizations of the photon.

  ii) Gravity is described by 4×4 symmetric matrix, metric $g_{\mu\nu}$ which has 10 apparent dof. Gauge transformations

  $$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{3.42}$$

  leave field strength invariant. Here ξ_μ is the gauge functions and has 4 dof. So the true number of degrees of freedom is $10 - 2 \times 4 = 2$. This correspond to two polarizations of the graviton.

  ii) Independence. Each integral defines a hypersurface in phase space, dynamical trajectories must remain in this surface:



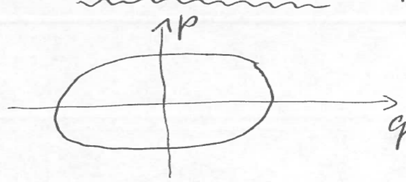
<u>indep</u>: hyperplanes are "never tangent"
(global ≃ foliate space time)

$M_{\{f\}}$ given by $\{F_i = f_i\}$ has dimension n.

iii) One cannot have more than n independent integrals of motion that are in involution. Otherwise, the Poisson bracket would be degenerate. This implies that $H = H(F_i)$.

iv) Under a suitable global hypothesis, $M_{\{f\}}$ is an n-dimensional <u>*tori*</u> T_n.

<u>Ex:</u> harmonic oscillator $H = \frac{1}{2}(p^2 + \omega^2 q^2)$



phase space is fibred into ellipses $H = E$ .... T_1
(except $(0,0)$ .. stationary point)

- introduce $p = \rho \cos\theta$, $q = \frac{\rho}{\omega} \sin\theta$

"action–angle var."

$\Rightarrow$ motion $= \{ \rho = \sqrt{2E}\,,\ \theta = \omega t + \theta_0 \}$

- generalization $H = \frac{1}{2} \sum_i \underbrace{(p_i^2 + \omega_i^2 q_i^2)}_{F_i} \qquad (*)$

F_i n conserved quantities in involution

$M_f \equiv \{F_i = f_i\}$ is T_n

"all integrable systems look like $(*)$"

v) One can show that whenever the Hamilton–Jacobi completely separates, the motion is completely integrable.

vi) For small perturbations of integrable systems, the (Liouville) tori exist almost everywhere. This is a content of the Kolmogorov–Arnold–Moser:

<u>*KAM theorem*</u> and is connected to beautiful field of deterministic chaos which we have no time to study.

<u>Idea of the proof of integrability theorem.</u>

– We want to find a canonical transformation $(q^i, p_i) \to (F^i, \psi_i)$, where F^i are our conserved quantities. If we succeed, then

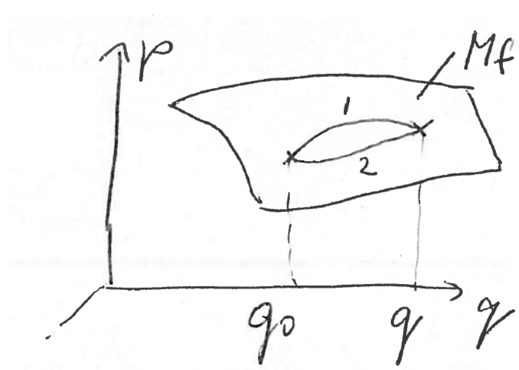
$$\begin{aligned} \dot{F}^i &= \{H, F^i\} = 0\,, \\ \dot{\psi}_i &= \{H, \psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i = \Omega_i(F_j) \ \ldots \text{constant in time.} \end{aligned} \tag{3.43}$$

This then means that we get a solution

$$F^i(t) = F^i(0)\,, \quad \psi_i(t) = \psi_i(0) + \Omega_i t\,. \tag{3.44}$$

– To construct this let's find the corresponding generating function S. We have $M_{\{f\}} = \{F^i(p,q) = f^i\}$. In principle we can invert this, to get $p_i = p_i(f,q)$ on $M_{\{f\}}$ and can define

$$\boxed{S(q,F) = \int_{q_0}^{q} p_i dq^i\,.} \tag{3.45}$$



If such an integral exists (see figure) then (c.f. (3.13))

$$\psi_j = -\frac{\partial S}{\partial F^j} \tag{3.46}$$

gives the desired canonical transformation. [Note that $\frac{\partial S}{\partial q^i} = p_i$ is automatically satisfied.]

– One finally needs to show that S is well defined, that is integral (3.45) is independent of integration path. One can show that is is exactly equivalent to the requirement that F^i are in involution: $\{F^i, F^j\} = 0$.

– So we have obtained the solution of EOM by one integral (3.45) and some algebraic operations (needed to express p as function of q and F).

3.6 Constraints

- Let not all q^i, p_i be independent, i.e., at any instant of time we have a *constraint*[2]

$$\boxed{\phi(q^i, p_i) = 0\,.} \tag{3.47}$$

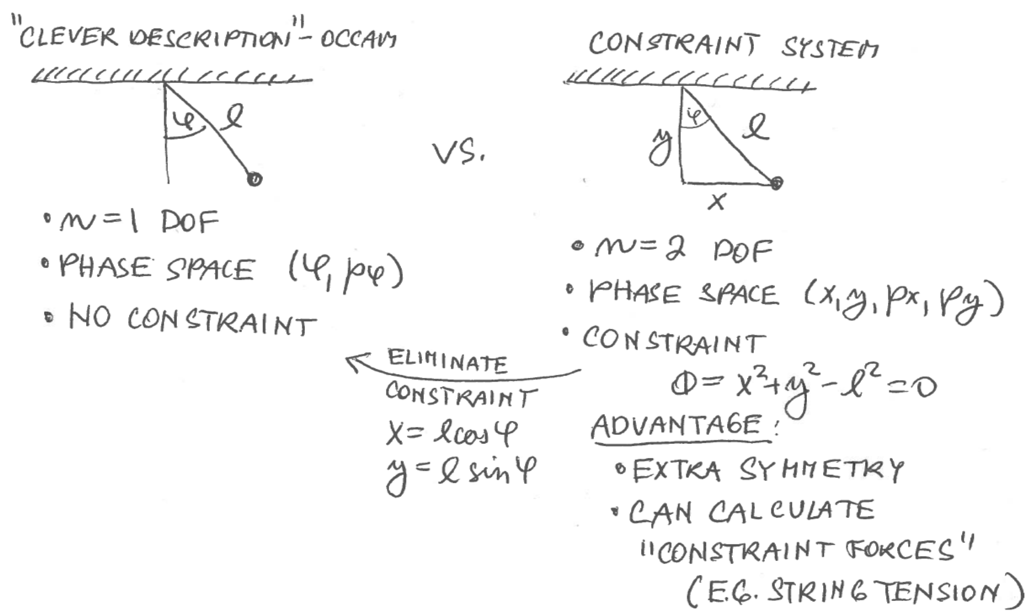
- Any phase space function f can be understood as a "Hamiltonian" and generates a corresponding *flow* (in fictitious time λ)

$$\frac{dq^i}{d\lambda} = \{q^i, f\}\,, \quad \frac{dp_i}{d\lambda} = \{p_i, f\}\,. \tag{3.48}$$

  Specifically, a conserved quantity obeys $\{I, H\} = -\{H, I\} = 0$. This means that Hamiltonian H is unaffected by the flow generated by I. This corresponds to a symmetry in the system.

  Constraints are 'conserved quantities' and imply the presence of symmetries in physical systems.

- Standard example: Mathematical pendulum



- To deal with constraints one uses the method of *Lagrange multipliers*. Namely in the presence of m number of constraints ϕ_J, we define the *total Hamiltonian* according to

$$\boxed{H_T = H + \sum_{J=1}^{m} \lambda_J \phi_J\,.} \tag{3.49}$$

[2]See Dirac's book [6] for various types of constraints.

Here, $\lambda_J(t)$ are (unknown) Lagrange multipliers. The equations of motion are then derived w.r.t. this total Hamiltonian.

People consider 2 possibilities:

– We can enlarge the phase space even further and include λ_J as new coordinates, to get a system with $(n+m)$ dof. We then vary the corresponding action w.r.t. λ_J as well, to get $(n+m)$ Euler–Lagrange equations, out of which m are the on-shell constraints, for $(n+m)$ unknowns. Let's demonstrate this in Lagrangian formalism. Identifying

$$L_T = L - \lambda_J \phi_J \tag{3.50}$$

as the total Lagrangian, we have

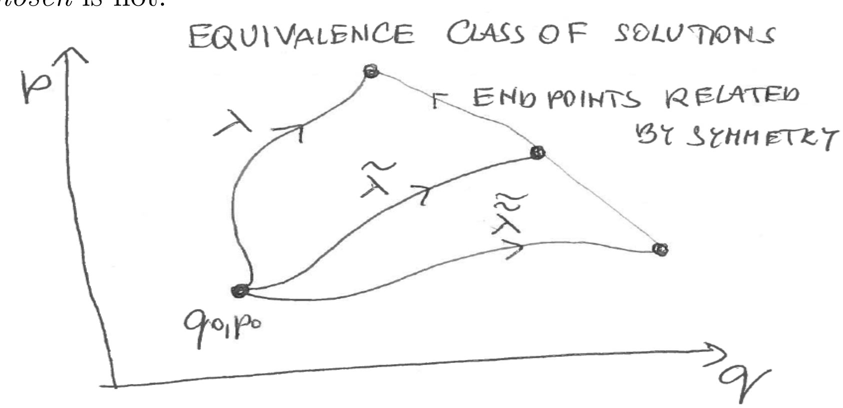
$$\delta S = \int (\delta L + \lambda_J \delta\phi_J + \phi_J \delta\lambda_J) dt = \int \Big[\Big(\frac{\partial L_T}{\partial q^i} - \frac{d}{dt}\Big(\frac{\partial L_T}{\partial \dot q^i}\Big)\Big)\delta q^i + \phi_J \delta\lambda_J\Big] dt\,, \tag{3.51}$$

up to vanishing boundary terms. So we got the standard Euler–Lagrange equations for the total Lagrangian L_T, together with the on-shell constraints:

$$\frac{\partial L_T}{\partial q^i} - \frac{d}{dt}\Big(\frac{\partial L_T}{\partial \dot q^i}\Big) = 0\,, \quad \phi_J = 0\,. \tag{3.52}$$

We can use these to solve for q^i as well as λ_J. Note also that proceeding to the Hamiltonian picture, we find that $p_{\lambda_J} = 0$ and so H_T is given by equation (3.49) above.

– The second possibility is to keep λ_J as 'arbitrary functions of time'. In this case we have n Euler–Lagrange equations for q^i which however depend on λ_j. So in this situation the solution to equations of motion following from H_T provided initial data (q_0, p_0) depends on m arbitrary Lagrange multipliers. This implies that the evolution is not unique, but the end configurations are related by a symmetry of the system. This of course does not mean that the underlying physics is not unique! It is unique but the *description we have chosen* is not!



- One typical origin of constraints is the transition between Lagrangian and Hamiltonian descriptions: constraints emerge whenever the defining equation for canonical momenta
$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{3.53}$$
does not have an inversion. This happens when $\det A_{ij}$, where $A_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$, is equal to zero. One can then write
$$\dot{q}^i = \dot{q}^i(q^j, p_j, \lambda^J)\,, \tag{3.54}$$
where λ^J are arbitrary and we have that many of them as the number of zero eigenvalues of A_{ij}.

- Toy example. Let's consider the motion of free particle from a perspective of an observer with "lousy clock". Let T denotes an inertial time and t the lousy time. Starting from the Lagrangian written in terms of the lousy time and performing the standard Legendre transformation we find that the corresponding Hamiltonian vanishes:
$$\boxed{H = 0\,.} \tag{3.55}$$
This is a characteristic of a *totally constraint system.* At the same time we find a constraint (see Tutorial)
$$\phi = p_T + \frac{p_x^2}{2m}\,. \tag{3.56}$$
So there is no 'time evolution' and the only flow is w.r.t. the constraint. (Total Hamiltonian is non-trivial and determines this flow.) Despite the lack of true time evolution, *relational predictions* are still possible for this type of a system.

  Interestingly, this toy example is a prototype of what happens for gauge theories and in particular what happens in gravity.

Chapter 4: Geometric formulation of Hamiltonian mechanics

4.1 Symplectic geometry

Definition Let M^{2n} be an even-dimensional manifold. A *symplectic structure* on M is a closed non-degenerate 2-form ω, that is,

$$\begin{aligned} &i) \quad \text{closed:} \quad d\omega = 0\,, \\ &ii) \quad \text{non-degenerate:} \quad \forall X \neq 0\ \exists Y \text{ such that } \quad \omega(X,Y) \neq 0\,. \end{aligned}$$

The pair (M^{2n}, ω) is called a *symplectic manifold*.

Remarks:

i) The very existence of a non-degenerate 2-form already implies that $dim M = 2n$.

ii) Define Ω as an "inverse" of ω:

$$\Omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,. \tag{4.1}$$

Since ω is non-degenerate, this is unique. So, ω defines an isomorphism between TM and T^*M as follows:

$$\begin{aligned} \omega: \quad & TM \to T^*M: \quad X \to \omega(X,\cdot) = X \lrcorner\, \omega \quad \text{or} \quad X^\alpha \to X^\alpha \omega_{\alpha\beta} = X_\beta\,. \\ \Omega: \quad & T^*M \to TM: \quad \eta \to \Omega\cdot\eta \quad \text{or} \quad \eta_\alpha \to \Omega^{\beta\alpha}\eta_\alpha = \eta^\beta\,. \end{aligned} \tag{4.2}$$

That is, we can *rise* or *lower* indices, similar to what you may be used to do with metric.

iii) Given a function f on M, ω defines a *Hamiltonian vector field*

$$\boxed{X_f = \Omega \cdot df} \quad \Leftrightarrow \quad \boxed{X_f \lrcorner\, \omega = -df\,.} \tag{4.3}$$

To see this, let's use coordinates. Then we have $X^\alpha_f = \Omega^{\alpha\beta}\partial_\beta f$. Multiplying by $\omega_{\alpha\gamma}$ we then have

$$X^\alpha \omega_{\alpha\gamma} = \underbrace{\Omega^{\alpha\beta}\omega_{\alpha\gamma}}_{-\delta^\beta_\gamma}\partial_\beta f = -\partial_\gamma f\,.$$

Theorem. A Hamiltonian vector field preserves the symplectic structure:

$$\mathcal{L}_{X_f}\omega = 0\,. \tag{4.4}$$

Proof: Using Cartan's lemma and the second expression in (4.3), we have

$$\mathcal{L}_{X_f}\omega = X_f \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{X_f \lrcorner\, \omega}_{-df}) = 0\,.$$

Corollary (Liouville's theorem): A Hamiltonian vector flow preserves the volume element

$$\epsilon = \omega^{\wedge n} = \frac{1}{n!}\underbrace{\omega\wedge\omega\wedge\cdots\wedge\omega}_{n\,\text{times}}$$

(and hence also volume) on M.

Proof. We have

$$\mathcal{L}_{X_f}\epsilon = \frac{1}{n!}\mathcal{L}_{X_f}\left(\omega\wedge\cdots\wedge\omega\right) \propto \omega^{\wedge(n-1)}\wedge\mathcal{L}_{X_f}\omega = 0\,,$$

where in the second step we have used the Leibnitz property of the Lie derivative.

iv) Given 2 functions f, g on M, ω defines their *Poisson bracket* by

$$\boxed{\{f,g\} = df\cdot\Omega\cdot dg = -\omega(X_f, X_g)\,.} \tag{4.5}$$

Properties of ω guarantee linearity, antisymmetry, Leibnitz properties of the bracket, the Jacobi identity is equivalent to the fact that $d\omega = 0$ (do it if you don't believe). We also find identities like

$$X_{\{f,g\}} = [X_f, X_g]\,. \tag{4.6}$$

v) *Darboux theorem:* Since ω is closed and non-degenerate, there exists a coordinate system $x^\alpha = (q^i, p_j)$ s.t. we have

$$\Omega^{\alpha\beta} = \begin{array}{cc} & \begin{array}{cc} q^1 & p_1 \end{array} \\ \begin{array}{c} q^1 \\ p_1 \end{array} & \begin{pmatrix} \begin{matrix}0&1\\-1&0\end{matrix} & & & \emptyset \\ & \begin{matrix}0&1\\-1&0\end{matrix} & & \\ & & \ddots & \\ \emptyset & & & \begin{matrix}0&1\\-1&0\end{matrix} \end{pmatrix} \end{array} \Leftrightarrow \omega_{\alpha\beta} = \begin{pmatrix} \begin{matrix}0&-1\\1&0\end{matrix} & \emptyset \\ \emptyset & \begin{matrix}0&-1\\1&0\end{matrix} \end{pmatrix}$$

That is

$$\boxed{\omega = dp_i\wedge dq^i = dp_1\wedge dq^1 + \cdots + dp_n\wedge dq^n\,.} \tag{4.7}$$

Using these coordinates it is

- obvious that

$$\epsilon = \omega^{\wedge n} = dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 \wedge \cdots \wedge dp_n \wedge dq^n \tag{4.8}$$

  is a volume form on M. I remind that (see Eq. (2.34))

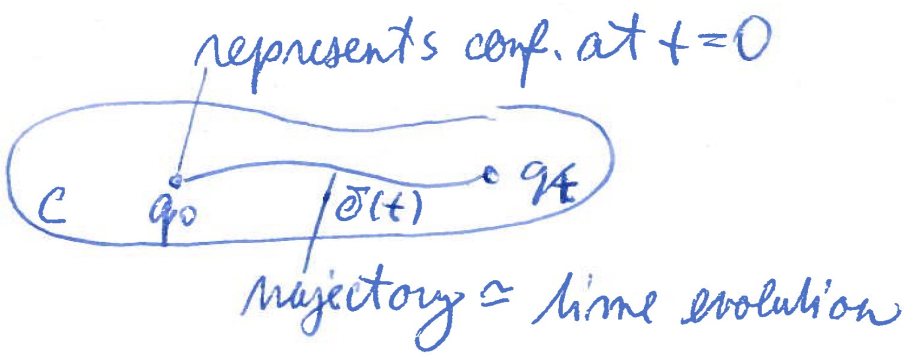
$$\int f = \int f\epsilon = \int f dp_1 \ldots dp_n dq^1 \ldots dq^n \, . \tag{4.9}$$

- The Poisson bracket takes the <u>*canonical form*</u>

$$\{f, g\} = \Omega^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \, . \tag{4.10}$$

4.2 Hamiltonian mechanics

- <u>Basic definitions.</u>
  <u>*Configuration space*</u> C for a system with n dof is a manifold of dimension n, equipped with coordinates $(q^1, \ldots, q^n)$:



  'Points' of $\delta(t)$ describe a "photo of the system" at a given time.

  <u>*Phase space*</u> = cotangent bundle T^*C. It has a dimension $2n$ and coordinates $(p_1, \ldots, p_n, q^1, \ldots, q^n)$.

  <u>*Cartan 1-form*</u> $\theta \in T^*(T^*C)$:

$$\boxed{\theta = p_j dq^j} \tag{4.11}$$

  defines a natural <u>*symplectic structure*</u> on T^*C:

$$\boxed{\omega = d\theta = dp_j \wedge dq^j \, .} \tag{4.12}$$

  [This is obviously non-degenerate and closed.]

  The dynamics is defined by specifying <u>*Hamiltonian*</u> $H : T^*C \to \mathbb{R}$

$$H = H(p_1, \ldots, p_n, q^1, \ldots, q^n) \, . \tag{4.13}$$

Note that there is no explicit dependence on t here! (For time-dependent Hamiltonians one has to use the contact geometry description, discussed in Sec. 4.3.)

Hamiltonian determines the *dynamical Hamiltonian vector field* X_H on the phase space, given by

$$\boxed{X_H = \Omega \cdot dH = \{\cdot, H\}\,.} \quad \Leftrightarrow \boxed{X_H \lrcorner\, \omega = -dH} \tag{4.14}$$

This field then generates its *integral curves* $\gamma(t) = (p_j(t), q^j(t))$:

$$X_H = \frac{d}{dt} = \frac{dq^j}{dt}\frac{\partial}{\partial q^j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j}\,. \tag{4.15}$$

At the same time we have

$$X_H = \{\cdot, H\} = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}\,, \tag{4.16}$$

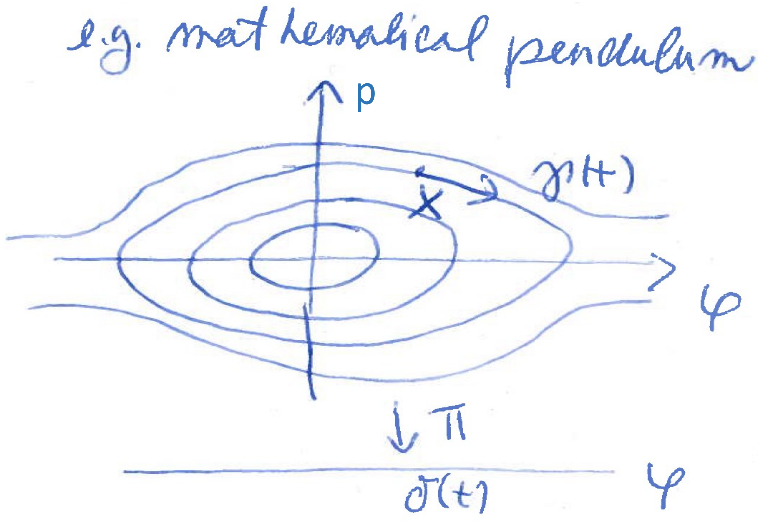
using the expression for the Poisson bracket in canonical coordinates (4.10). By comparing the two we hence derived *Hamilton's equations*:[1]

$$\boxed{\dot q^j = \frac{\partial H}{\partial p_j}\,, \quad \dot p_j = -\frac{\partial H}{\partial q^j}\,.} \tag{4.17}$$

These determine the trajectory $\delta(t)$, given as a canonical projection of integral curves $\gamma(t)$:

$$\boxed{\delta(t) = \pi(\gamma(t))\,.} \tag{4.18}$$

This is illustrated on the following example for a mathematical pendulum:



'Points' of $\gamma(t)$ describe a "physical state of the system" at a given time.

[1] Note that if coordinates are not canonical, one obtains more complicated Hamilton's equations. This can be "mimicked" by performing a non-canonical transformation to a system written originally in canonical coordinates. This is in direct analogy to what happens with second Newton's law in the non-inertial frame (we have Coriolis force and other crap).

- Two remarks: i) Note that the *Legendre transform* $L \leftrightarrow H$ is a map: $TC \leftrightarrow T^*C$ that identifies $\dot{q}^j \leftrightarrow p_j$ according to[2]

$$p_j = \frac{\partial L}{\partial \dot{q}^j}\,, \quad \dot{q}^j = \frac{\partial H}{\partial p_j}\,, \quad H = \dot{q}^j p_j - L\Big|_{\dot{q}^j = \dot{q}^j(q^i, p_i)}\,. \tag{4.19}$$

Namely, we have the following picture:

$$\begin{tikzcd}
 & \mathbb{R} & \\
(q^j, \dot{q}^j) = z \in TC \arrow[ur, "L"] \arrow[dr, "\pi"'] \arrow[rr, leftrightarrow, "\text{Legendre}", "\text{transf.}"'] & & T^*C \ni z^* = (q^j, p_j) \arrow[ul, "H"'] \arrow[dl, "\pi"] \\
 & C \ni Q = (q^j) &
\end{tikzcd}$$

ii) Since X_H is a Hamiltonian vector flow, we have $\mathcal{L}_{X_H}\omega = 0$ and hence also the *Liouville's theorem*.

- Canonical transformations. A transformation

$$Q^j = Q^j(q^i, p_i)\,, \quad P_j = P_j(q^i, p_i)\,, \tag{4.20}$$

is *canonical* if it preserves the canonical form of ω, i.e.,

$$\boxed{dp_j \wedge dq^j = dP_j \wedge dQ^j\,.} \tag{4.21}$$

For example: $n = 1$.

$$\omega = dP \wedge dQ = \Big(\frac{\partial P}{\partial q}dq + \frac{\partial P}{\partial p}dp\Big) \wedge \Big(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\Big) = \{Q, P\}_{q,p} dp \wedge dq\,. \tag{4.22}$$

So we get a condition that $\{Q, P\} = 1$.

Having two Cartan 1-forms

$$\theta = p_j dq^j\,, \quad \omega = d\theta \quad \text{and} \quad \tilde{\theta} = P_j dQ^j\,, \quad \omega = d\tilde{\theta}\,, \tag{4.23}$$

we have

$$d(\theta - \tilde{\theta}) = 0 \quad \Rightarrow \quad \theta - \tilde{\theta} = dF \quad \text{locally}\,. \tag{4.24}$$

This means that there exists a *generating function* $F = F(q, p)$ s.t.

$$\boxed{p_j dq^j - P_j dQ^j = dF\,.} \tag{4.25}$$

[2]More precisely, any function $f : TC \to \mathbb{R}$ can be used to identify TC with T^*C as follows. It defines a map $\Lambda_f : TC \to T^*C$ mapping $(x, v) \to (x, p)$ by $p(v) = \frac{d}{dt}f(x, w + tv)|_{t=0}$, i.e., $p = \frac{\partial f}{\partial \dot{q}^i} dq^i$.

For example, assume that in a neighborhood of some (q_0, p_0) we can take (Q, q) as new independent coordinates $\left[\det \frac{\partial(Q,q)}{\partial(q,p)} \neq 0\right]$, i.e., we can write

$$F_1(q^j, Q^j) = F(q^j, p_i(Q^j, q^j)) \,. \tag{4.26}$$

Then (4.25) gives

$$p_j dq^j - P_j dQ^j = \frac{\partial F_1}{\partial q^j} dq^j + \frac{\partial F_1}{\partial Q^j} dQ^j \,, \tag{4.27}$$

that is

$$\boxed{p_j = \frac{\partial F_1}{\partial q^j} \,, \quad P_j = -\frac{\partial F_1}{\partial Q^j} \,.} \tag{4.28}$$

Similarly, one could take $F_2(q^j, P_j) = F + P_j Q^j$, and other possibilities.

- <u>Symmetries.</u> Note that

$$\begin{aligned} \{f, g\} &= df \cdot \Omega \cdot dg = df \cdot X_g = \mathcal{L}_{X_g} f \equiv \frac{df}{dt_g} \\ &= -\{g, f\} = -\mathcal{L}_{X_f} g \equiv -\frac{dg}{dt_f} \,. \end{aligned} \tag{4.29}$$

Noether's theorem (Version 4: Hamiltonian version).

$$\text{Let } Y \text{ s.t. } \mathcal{L}_Y \omega = 0 \text{ and } \mathcal{L}_Y H = 0 \quad \Rightarrow \quad \exists\, I \text{ s.t. } \frac{dI}{dt} = \mathcal{L}_{X_H} I = 0 \,. \tag{4.30}$$

<u>Proof.</u>

$$\mathcal{L}_Y \omega = 0 = Y \lrcorner \underbrace{d\omega}_{0} + d(\underbrace{Y \lrcorner \omega}_{-dI \text{ locally}}) \tag{4.31}$$

That is there exists I s.t. $Y = Y_I$ is a Hamiltonian vector field corresponding to I. We then have

$$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\} = -\{I, H\} = -\frac{dI}{dt} = 0 \,. \tag{4.32}$$

Note that the notion of symmetry is at the level of *phase space* and hence is more general than usually presented. We can distinguish two cases based on the canonical projection $\pi^* : T(T^*C) \to TC$ of the vector field Y:

$$\pi^*(Y) = \begin{cases} \text{vector field on } C & \underline{\textit{isometry}} \\ \text{not well defined on } C & \underline{\textit{dynamical (hidden) symmetry}} \end{cases} \tag{4.33}$$

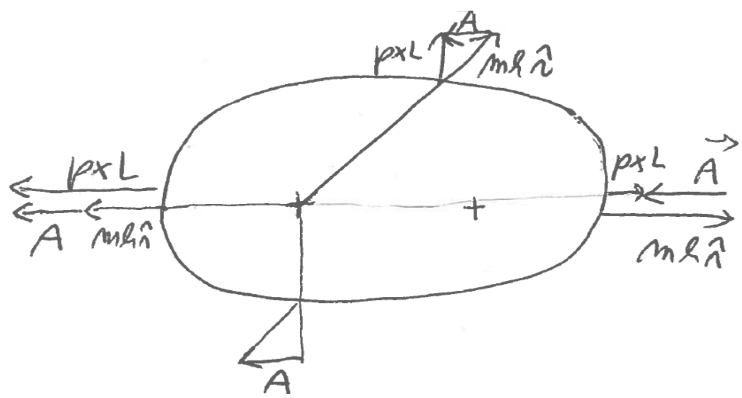
- Example: Let us consider the *Kepler problem*, that is a motion in the central force

$$\vec{F} = -\frac{k}{r^2}\hat{r}\,. \tag{4.34}$$

We then find the following isometries: energy E and angular momentum $\vec{L}$. (These are related to the fact that problem is stationary and spherically symmetric.) However, there also exists an additional conserved vector

$$\boxed{\vec{A} = \vec{p}\times\vec{L} - mk\hat{r}\,,} \tag{4.35}$$

called the *Laplace–Runge–Lenz vector*:[3]



This is an example of dynamical symmetry (no associated cyclic coordinate can be found). One can show that this is related to hidden $SO(4)$ symmetry (free particle on a 3-sphere). Note also that we have two following constraints:

$$\vec{A}\cdot\vec{L} = 0\,,\quad A^2 = m^2k^2 + 2mEL^2\,. \tag{4.36}$$

So the set $\{E, \vec{L}, \vec{A}\}$ gives $7-2=5$ integrals of motion, making the motion maximally superintegrable.[4] Interestingly, similar symmetry exists for rotating black holes in full General Relativity; if interested, see [11].

4.3 Contact geometry

In the symplectic geometry approach to Hamiltonian mechanics we were able to discuss "time-independent" Hamiltonians. In order to 'include time' as well as to get a better understanding of the Hamilton–Jacobi theory we have to study an "odd-dimensional cousin" of symplectic geometry, called *contact geometry*.

[3] The existence of this quantity is bound to the particular choice of Kepler force (4.34) and it is no longer present in general central force problem.

[4] A system with n dof is called *superintegrable* if it admits more than n functionally independent integrals of motion out of which n are in involution. (I remind that since Poisson brackets are non-degenerate one can have at most n quantities in involution.) *Maximally superintegrable* system is a superintegrable system with maximum number of independent integrals of motion, that is $2n-1$.

- **Definition.** A vector X for which $\omega(X,Y)=0\ \forall Y$ is called a <u>*null vector*</u> of the 2-form ω.

  Definition. A 2-form ω is <u>*non-singular*</u> when the vector space generated by its null vectors has minimal possible dimension, i.e.,

$$\begin{cases} dim = 0 & \text{even dimensions: } M^{2n} \\ dim = 1 & \text{odd dimensions: } M^{2n+1} \end{cases} \tag{4.37}$$

  Definition. When ω is closed and non-singular, the pair (M^{2n+1},ω) is called the <u>*contact geometry*</u>.

- <u>Remarks.</u>

  i) This is an "odd-dimensional version of symplectic geometry".

  ii) ω is generated by <u>*contact 1-form*</u> θ:

$$\boxed{\omega = d\theta} \tag{4.38}$$

  and determines a unique <u>*vortex direction*</u> X:

$$\boxed{X \lrcorner\, \omega = 0\,.} \tag{4.39}$$

  Specifically, <u>*Reeb vector*</u> is a normalized vortex direction: $\theta(X)=1$.

  iii) Interesting branch of mathematics studies the so called <u>*special Riemannian*</u> <u>manifolds</u>:

$$\begin{cases} \text{Kahler manifolds :} & (M^{2n}, \text{Kahler 2-form } \omega) \\ \text{Sasaki manifolds :} & (M^{2n+1}, \text{Reeb vector } X) \end{cases} \tag{4.40}$$

  iv) Note that

$$\boxed{\mathcal{L}_X\omega = 0\,.} \tag{4.41}$$

  The proof of this statement is very simple:

$$\mathcal{L}_X\omega = X\lrcorner \underbrace{d\omega}_{0} + d(\underbrace{X\lrcorner\,\omega}_{0}) = 0\,. \tag{4.42}$$

  v) <u>*Darboux theorem.*</u> Since ω is closed and non-singular, there exists a coordinate system $x^\alpha = (q^j, p_j, t)$ such that

$$\omega_{\alpha\beta} = \begin{array}{cc} & \begin{matrix} q^1 & p_1 & & q^n & p^n & t \end{matrix} \\ \begin{matrix} q^1 \\ p_1 \\ \\ \\ t \end{matrix} & \begin{pmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & \\ & & & \boxed{0} \end{pmatrix} \end{array}$$

Note the presence of extra zero! That is, ω and the vortex direction take the following form:

$$\boxed{\omega = dp_i \wedge dq^i\,, \quad X = \frac{\partial}{\partial t}\,.} \tag{4.43}$$

vi) One can define a <u>*Lagrange bracket*</u> by requiring

$$[X_f, X_g] = X_{\{f,g\}_L}\,. \tag{4.44}$$

- <u>Hamiltonian mechanics revised.</u>

Definition. <u>*Extended phase space*</u> $M^{2n+1} = T^*C \times \mathbb{R}$ with local coordinates $(p_1, \dots, p_n, q^1, \dots, q^n, t)$.

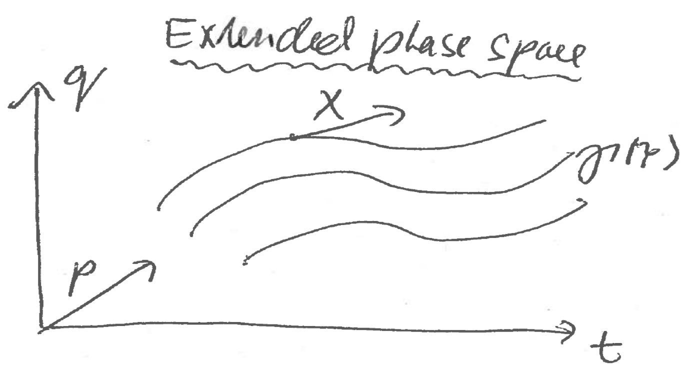
A <u>*Hamiltonian function*</u> $H = H(p_i, q^i, t)$, defines the following <u>*contact 1-form*</u>:

$$\boxed{\theta = p_i dq^i - H dt} \quad \Rightarrow \quad \boxed{\omega = d\theta = dp_i \wedge dq^i - dH \wedge dt\,.} \tag{4.45}$$

Hence (M, ω) is a <u>*contact manifold*</u>.

The corresponding vortex field is (please verify)

$$X = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j} + \frac{\partial}{\partial t}\,. \tag{4.46}$$



Therefore, its integral curves $\gamma(\tau) = (q^j(\tau), p_j(\tau), t(\tau))$ (see figure) determined from

$$X \equiv \frac{d}{d\tau} = \dot{q}^j \frac{\partial}{\partial q^j} + \dot{p}_j \frac{\partial}{\partial p_j} + \dot{t}\frac{\partial}{\partial t}\,, \tag{4.47}$$

are, upon the identification $\tau = t$, integral curves of canonical equations

$$\boxed{\dot{p}_j = -\frac{\partial H}{\partial q^j}\,, \quad \dot{q}^j = \frac{\partial H}{\partial p_j}\,.} \tag{4.48}$$

So, X determines time evolution of any function f according to

$$\boxed{\mathcal{L}_X f = X(f) = \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,,} \tag{4.49}$$

where $\{\cdot,\cdot\}$ stands for the standard Poisson bracket. In particular, integral of motion obeys $X(I) = 0$.

Definition. A transformation

$$Q^j = Q^j(q^i, p_i, t)\,, \quad P_j = P_j(q^i, p_i, t)\,. \tag{4.50}$$

is *canonical* if it preserves the form of ω. That is there exists a new Hamiltonian H' such that[5]

$$\omega = dp_j \wedge dq^j - dH \wedge dt = dP_j \wedge dQ^j - dH' \wedge dt\,. \tag{4.51}$$

Note that a diffeomorphism ϕ_X generated by X gives a canonical transformation since $\mathcal{L}_X\omega = 0$ and hence $\phi_X^*\omega = \omega$.

Having 2 different contact forms

$$\theta = p_j dq^j - H dt\,, \quad \omega = d\theta \quad \text{and} \quad \tilde\theta = P_j dQ^j - H' dt\,, \quad \omega = d\tilde\theta\,, \tag{4.52}$$

we find

$$\theta - \tilde\theta = dF\,, \tag{4.53}$$

where F is a generating function. Choosing for example $F = F(q^j, Q^j, t)$ we then have

$$p_j dq^j - H dt - P_j dQ^j + H' dt = \frac{\partial F}{\partial q^j} dq^j + \frac{\partial F}{\partial Q^j} dQ^j + \frac{\partial F}{\partial t} dt\,, \tag{4.54}$$

which gives

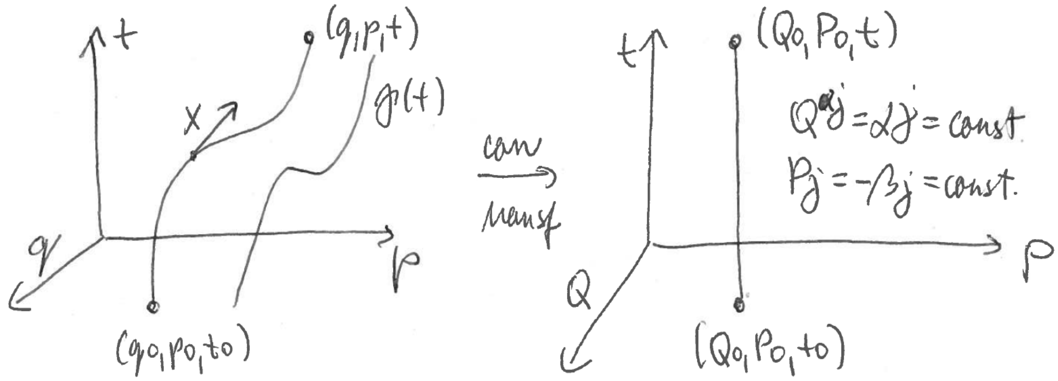
$$\boxed{H' = H + \frac{\partial F}{\partial t}\,, \quad p_j = \frac{\partial F}{\partial q^j}\,, \quad P_j = -\frac{\partial F}{\partial Q^j}\,,} \tag{4.55}$$

and similarly for other types of generating functions.

- The Geometrical picture of Hamilton–Jacobi theory
  Idea: Let's make a special canonical transformation so that (Q^j, P_j, t) are *Darboux coordinates* for ω, that is

$$\omega = dP_j \wedge dQ^j\,, \quad H' = 0\,, \quad X = \frac{\partial}{\partial t}\,. \tag{4.56}$$

[5] Can I also change time?



We have seen previously that such a canonical transformation is generated by $F = S$ where

$$\frac{\partial S}{\partial t} + H(q^i, \frac{\partial S}{\partial q^i}, t) = 0\,. \tag{4.57}$$

If geometrical formulation of classical mechanics caught your interest, I refer you to Arnold's book [2] for more details and results.

Chapter 5: Instead of conclusions

5.1 Nambu mechanics

- In Hamiltonian mechanics, the phase space is even-dimensional, equipped with a symplectic 2-form ω (which is non-degenerate and closed). This defines Poisson bracket
$$\{f, g\} = \Omega(df, dg)\,, \tag{5.1}$$
which is a binary operation. Lie algebra properties of Poisson brackets follow from properties of ω (antisymmetry $\leftrightarrow$ $\omega_{\alpha\beta} = \omega_{[\alpha\beta]}$, Leibnitz $\leftrightarrow$ derivative, Jacobi $\leftrightarrow$ $d\omega = 0$).

  Dynamics is encapsulated in a single scalar function—Hamiltonian H:
$$\frac{df}{dt} = \{f, H\}\,. \tag{5.2}$$

- Nambu mechanics (1973). Nambu [12] asked the following questions: i) Why do we restrict to even-dimensional phase space? Cannot we have e.g. (q^i, p_i, r^i) $i = 1, \ldots, n$?
ii) Why 1 function is sufficient to describe dynamics?

  Postulated existence of a *Nambu tensor* η: non-degenerate, p-form s.t. $D\eta = 0$. This defines a *generalized Poisson bracket*
$$\{f_1, \ldots, f_p\} = \eta(X_{f_1}, X_{f_2}, \ldots, X_{f_p})\,. \tag{5.3}$$
The dynamics is then defined by prescribing $(p-1)$ Hamiltonians $H_1, \ldots, H_{p-1}$:
$$\frac{df}{dt} = \{f, H_1, \ldots, H_{p-1}\}\,. \tag{5.4}$$
Interestingly, the properties of η guarantee that we still have Liouville's theorem, canonical transformations,. . . [1]

[1] I have the following question to you: I can set:
$$\{f, H_1, \ldots, H_{p-1}\} = \eta^{i_1 \ldots i_p} \frac{\partial f}{\partial x^{i_1}} \frac{\partial H_1}{\partial x^{i_2}} \ldots \frac{\partial H_{p-1}}{\partial x^{i_p}} = V^{i_1} \frac{\partial f}{\partial x^{i_1}}\,, \quad V^{i_1} \equiv \eta^{i_1 \ldots i_p} \frac{\partial H_1}{\partial x^{i_2}} \ldots \frac{\partial H_{p-1}}{\partial x^{i_p}}\,. \tag{5.5}$$
Is it possible to write $V^i = \Omega^{ij} \frac{\partial H}{\partial x^j}$ and hence transform this to the Hamiltonian picture? Also, can we get odd-dimensional phase space in the Hamiltonian picture, by simply ignoring one of the momenta?

Example: 3d space with coordinates (x^1, x^2, x^3) and $\eta = \epsilon$:

$$\{f, g, h\} = \epsilon^{ijk}\partial_i f \partial_j g \partial_k h \,. \tag{5.6}$$

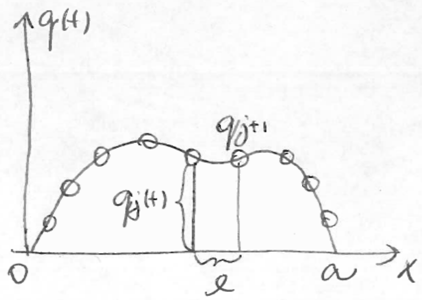
At this point there are applications of Nambu mechanics to superintegrable systems and brane dynamics (talk to one of your external examiners D. Minic).

5.2 Transverse vibrations of a string: example of a field theory

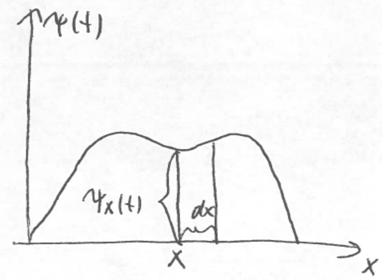
- Let us consider the following procedure going from a *discrete system* of N identical beeds hanging on a rope to *continuum*:

DISCRETE SYSTEM CONTINUUM

- $\underline{N}$ identical beeds, separated $\underline{l}$, with mass $\underline{m}$
- let's label them by 'position' $j = 1,..,N$



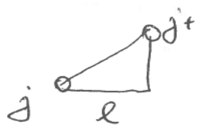
$\xrightarrow[N\to\infty]{l\to 0}$



$q_j(t) \to \psi(x,t)$ .. 'Field'

- $T_{kin} = \frac{1}{2}\sum_{j=1}^{N} m\,\dot{q}_j^{\,2}$ $\qquad T_{kin} = \frac{1}{2}\int_0^a \mu\, dx\, \dot{\psi}^2$ ← mass density

$V = \sum_{j=1}^{N} T(\Delta l)_j$ ← harmonic forces; T ← string tension $\qquad V = \frac{1}{2}\int_0^a T\, \psi_x^2\, dx$



$$\Delta l = \sqrt{l^2 + (q_{j+1} - q_j)^2} - l \simeq \frac{1}{2} l \left(\frac{q_{j+1} - q_j}{l}\right)^2$$

$$V = \frac{1}{2} l \sum_{j=1}^{N} T \left(\frac{q_{j+1} - q_j}{l}\right)^2$$

$$\boxed{L = T_{kin} - V = \frac{1}{2}\int_0^a \left(\mu\dot{\psi}^2 - T\psi_x^2\right) dx}$$

Writing the action in terms of <u>*Lagrangian density*</u> $\mathcal{L}$

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^a dx \mathcal{L}(\psi, \dot{\psi}, , \psi_x, t, x)\,, \tag{5.7}$$

we recover

$$\boxed{\mathcal{L} = \frac{1}{2}\mu\dot{\psi}^2 - \frac{1}{2}T\psi_x^2\,.} \tag{5.8}$$

Note that t, x in $\mathcal{L}$ are just 'labels'. A dynamical variable that changes is the value of the field at a given point.

- <u>Action principle.</u> Let us denote

$$\boxed{p^t = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\,, \quad p^x = \frac{\partial \mathcal{L}}{\partial \psi_x} = -T\psi_x\,.} \tag{5.9}$$

In order to impose the actio principle, $\delta S = 0$, we calculate

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt \int_0^a \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}}_{p^t} \delta\dot{\psi} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_x}}_{p^x} \delta\psi_x + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{0} \delta\psi \Big] \\
&= \Big| \delta\dot{\psi} = \frac{\partial}{\partial t}\delta\psi\,, \quad \delta\psi_x = \frac{\partial}{\partial x}\delta\psi \Big| \\
&= -\int_{t_1}^{t_2} dt \int_0^a dx \big[\dot{p}^t + \partial_x p^x\big]\delta\psi + \int_0^a \big[p^t\delta\psi\big]_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \big[p^x\delta\psi\big]_0^a dt\,.
\end{aligned} \tag{5.10}$$

We got three terms that must vanish independently!

  – The first term describes a motion of the string for $x \in (0, a)$ and $t \in (t_1, t_2)$. Since $\delta\psi$ is not restricted here we must have

$$\dot{p}^t + \partial_x p^x = 0 \quad \Leftrightarrow \quad \boxed{\psi_{xx} - \frac{1}{v^2}\ddot{\psi} = 0\,, \quad v = \sqrt{T/\mu}\,.} \tag{5.11}$$

  – The second term is determined by the the string configuration at t_1 and t_2 and corresponds to <u>*initial data*</u>

$$\delta\psi(t_1, x) = 0\,, \quad \delta\psi(t_2, x) = 0\,. \tag{5.12}$$

  – The last term is related to the evolution of endpoints $\psi(t, 0)$ and $\psi(t, a)$ and represents the <u>*boundary conditions*</u>. Two types of boundary conditions are often prescribed:
    i) <u>Dirichlet</u> boundary conditions describe *fixed endpoints*:

$$\delta\psi|_a = 0 = \delta\psi|_0 \quad \Leftrightarrow \quad \boxed{\dot{\psi}|_0 = 0 = \dot{\psi}|_a\,.} \tag{5.13}$$

ii) Neumann boundary conditions describe *free* to move endpoints (no friction). This means that $\delta\psi|_0$ and $\delta\psi|_a$ ar unconstrained but

$$p^x|_a = 0 = p^x|_0 \quad \Leftrightarrow \quad \boxed{\psi_x|_0 = 0 = \psi_x|_a\,.} \tag{5.14}$$

- Remarks.
  - Momentum carried by string.

$$\boxed{P = \int_0^a \mu\dot{\psi}dx = \int_0^a p^t dx\,.} \tag{5.15}$$

For Neumann boundary conditions P is conserved, whereas with Dirichlet conditions momentum exchanges with the 'wall" holding the fixed endpoints. In string theory, the endpoints of open strings may be attached to *D-branes* that can exchange momentum with the string. For more details see e.g. [8].

  - Relativistic field theory. Fields live in spacetime $\varphi(t,\vec{x})$. The actions takes the form

$$S = \int d^4x \mathcal{L}(\varphi, \partial_\mu\varphi) \tag{5.16}$$

and its variation leads to Euler–Lagrange equations of the type:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0\,.} \tag{5.17}$$

This is where your QFT will start.

5.3 Summary

- Classical mechanics finds natural description in terms of symplectic and contact geometries.

	Hamiltonian symplectic (2n)	Hamiltonian contact (2n+1)	Lagrangian (symplectic)
Manifold M	phase space T^*C	extended phase space $T^*C\times\mathbb{R}$	velocity phase space TC
dynamics	Hamiltonian $H(q,p)$	Ham $H(q,p,t)$	Lagrangian $L(q,\dot q)$
1-form	Cartan $\theta = p_i dq^i$	contact $\theta_H = p_i dq^i - H dt$	Lagrange $\theta_L = \frac{\partial L}{\partial \dot q^i} dq^i$
2-form	symplectic (non-deg, $d\omega=0$) $\omega = d\theta$	(non-sing., $d\omega_H=0$) $\omega_H = d\theta_H$	Lagrange sympl. 2-form $\omega_L = d\theta_L$
EOM for dyn. field X	X_H $X_H \lrcorner d\omega = -dH$	$\tilde X_H$ $\tilde X_H \lrcorner \omega_H = 0$	$\tilde X_L$: $\mathcal{L}_{X_L}\theta_L = dL$ or $X_L \lrcorner \omega_L = -dE$ $E = \frac{\partial L}{\partial \dot q^j}\dot q^j - L$ Lagrange e.

Similar framework works in thermodynamics, see e.g., [13] and references therein. Namely

$$dE = TdS - PdV \tag{5.18}$$

defines a contact 1-form.

There is a generalization of Hamiltonian mechanics in terms of *Nambu mechanics* waits for you to find applications!

- Differential geometry has many applications in theoretical physics:

$$\text{applications}\begin{cases}\text{GR} & \text{(Einstein reluctant at first)}\\ \text{Mechanics, TDs} & \\ \text{Gauge theories} & \text{(gauge potentials as connections)}\\ \text{String theory} & \\ \text{Mathematical physics} & \text{(Clifford algebra, special manifolds,...)}\end{cases}$$

- Your immediate future:

  i) Quantum mechanics: $(q^i, p_i) \to [\hat q^i, \hat p_i]$.

  i) QFT: Coordinates $q(t)$ are replaced by fields $\phi(\vec x, t)$. The Lagrangian $L = L(q, \dot q, t)$ is replaced by the Lagrangian density: $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, x^\mu)$, and

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q}\Big) = 0 \quad \to \quad \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Big) = 0\,. \tag{5.19}$$

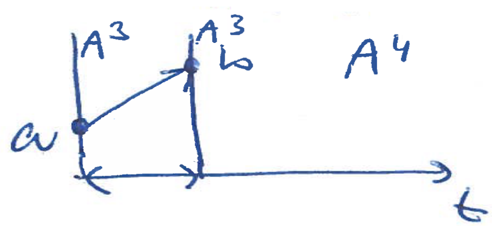
i) GR: In classical mechanics the central object is a symplectic 2-form ω which is non-degenerate, antisymmetric, and $d\omega = 0$.
In GR the central object is a *metric* g that describes the gravitational field. g is non-degenerate, symmetric, and obeys $\nabla g = 0$.

GOOD LUCK!

Appendix A: Newtonian mechanics

Following [2], let us review the Newtonian formulation of mechanics.

- Experimental facts:
  - Spacetime: 1+3 dimensions: 1 time and 3 Euclidean
  - Galileo's principle of relativity: There exists a coordinate, called *inertial* such that: i) All law of nature at all moments of time are the same in all inertial coordinate systems.
    ii) Coordinate system in uniform rectilinear motion w.r.t. inertial one are themselves inertial.
  - *Newton's principe of determinism:* Initial state of a mechanical system (positions and velocities at some moment of time) *uniquely* determines its motion. (This in particular implies that one need not to know the initial acceleration.)
- Newton's equations.
  - *Galilean space:* "$\mathbb{R} \times E^3$"
    The universe is a four-dimensional affine space A^4, $a \in A^4$ is a world point or an event. Time $t : \mathbb{R}^4 \to \mathbb{R}$. Time interval is denoted by $t(b-a)$, see figure.



    If $t(b-a) = 0$ we say that the two events are simultaneous. A space of simultaneous events is E^3. Distance between simultaneous events is $\rho(a,b), \quad a,b \in A^3$.
  - Galilena group: preserves a structure of the Galilean space. Elements of Galilean transformation preserve intervals of time and distance between simultaneous events.

E.g: $g_1(t,x) = (t, x+vt)$ ... 3

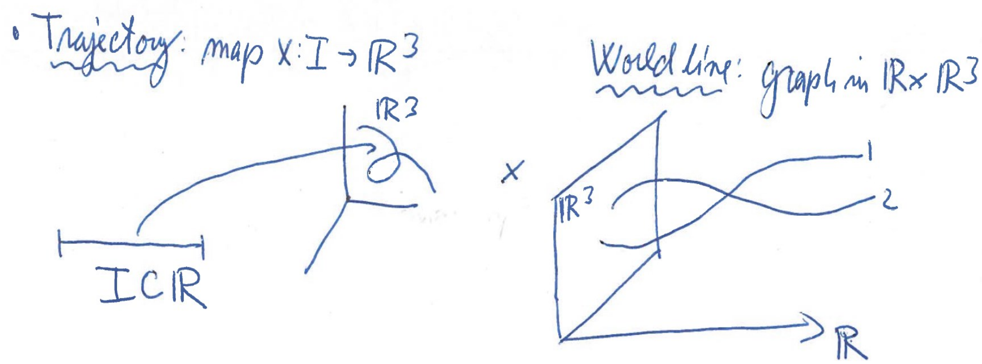
$g_2(t,x) = (t+s, x+y)$ ... 4

$g_3(t,x) = (t, Gx)$ ... 3

orthogonal — 10 params

$\forall g$ can be uniquely written as composition of g_1, g_2, g_3.

Let M be a set. A map $\varphi : M \to \mathbb{R} \times \mathbb{R}^3$ is called galilean coordinates on M. φ_2 moves uniformly w.r.t. φ_1 when $\varphi_1 \circ \varphi_2^{-1} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ is a Galilena transformation. (φ_1 and φ_2 give M the same Galilean structure.)



Given n points: configuration space is $\underbrace{\mathbb{R}^3 \otimes \cdots \otimes \mathbb{R}^3}_{n\times}$. Motion: $x : \mathbb{R} \to \mathbb{R}^N$, $N = 3n$.

– Newton's equations. Newton's determinism principle implies that $(x_0, \dot{x}_0)$ determine the motion. In particular, this is true for the acceleration. This means that there exists a function $F : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ such that

$$\boxed{\ddot{x} = F(x, \dot{x}, t)\,.} \tag{A.1}$$

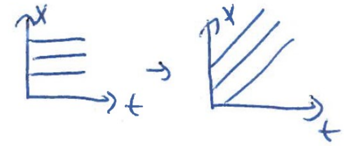
Theory of ODEs implies that there exists a unique solution for a given $x_0, \dot{x}_0$ and F.[1] F defines the system and is determined experimentally.

– Principle of relativity imposes a series of conditions on the form of r.h.s. (written in inertial coordinate system). "If we subject the worldlines of a mechanical system to a Galilean transformation, we obtain worldlines of the same system (with new initial conditions)."
For example. The laws of nature remain constant w.r.t. translations in time.

[1] This is provided a sufficient 'smoothness', see for example *Newton's dome* where a point particle stands on the top of a slope obeying $h = \frac{2}{3} g r^{3/2}$ and leads to a non-analytical force of the type $F = \sqrt{r}$.

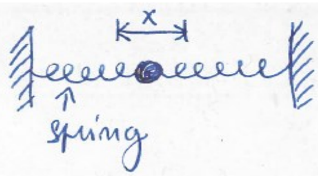
Therefore, if $x = \varphi(t)$ is a solution, so is $x = \varphi(t+s)$.

$\Rightarrow \quad \ddot{x} = F(x, \dot{x}, \cancel{t})$



Similarly, *isotropy* of the space implies that $F(Gx, G\dot{x}) = GF(x, \dot{x})$ and *homogeneity* and *uniform motion* implies that $F = F(x_j - x_k, \dot{x}_j - \dot{x}_k)$.
Exercise. Show that mechanical system consisting of one point has zero acceleration in an inertial frame. This statement is known as *Newton's 1st law.*[2]

- Example of a mechanical system.



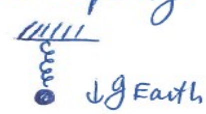
Experiment shows $\ddot{x} = -\alpha^2 x$

2 bodies: under the same extension of a spring

$\frac{\ddot{x}_1}{\ddot{x}_2} \overset{\text{exp.}}{=}$ FIXED independent of x, depend only on the bodies themselves.

$\Rightarrow$ define ratio of masses: $\frac{\ddot{x}_1}{\ddot{x}_2} = \frac{m_2}{m_1}$.

- take unit mass $\equiv$ mass of some fixed body, e.g. 1 liter of water
- by experience $m > 0$.
- $m\ddot{x}$ does not depend on the body ... characteristic of the extension of the spring ... force
- unit of force "Newton" (1 liter of water on a spring at the surface of the Earth $\equiv f = 9.8N$)



Instead of developing this formalism we rather switch to more universal and effective description of Lagrangian and Hamiltonian mechanics studied in the main text.

[2]Proof: $F \neq F(t, x, \dot{x})$ and invariant under rotations.

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