Hamiltonian Mechanics : Basic Theory

a-) Hamilton's Canonical equations

- Canonical momentum : $p_i = \frac{\partial L}{\partial \dot{q}^i}$

- Configuration space ≃ "photo of the system"
  (q^i) ... dim n ; n dof

- phase space : Describes physical states
  $(q^i, \underset{\text{form}}{p_i})$ ... dim $2n$; $2n$ dof , $2n$ coordinates (cotangent bundle)

(- velocity phase space
  $(q^i, \underset{\text{vector}}{\dot{q}^i})$ ... dim $2n$; $2n$ dof : $2n$ coordinates (tangent bundle)

- Hamiltonian governs the dynamics
  $H(q,p,t) = p_i \dot{q}^i - L$, $\dot{q} = \dot{q}(q,p,t)$

Theorem: The system of n $(E\text{-}L)^{2^{\text{nd}}\text{ order}}$ is equivalent to the system of $2n$ first order Hamilton's equations

$$\dot{p}_j = -\frac{\partial H}{\partial q^j} \quad , \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

Proof: Variational principle

$$\delta S = 0 = \delta \int L dt = \int \delta (p_i \dot{q}^i - H) dt$$

$$= \int \left(\delta p_i \dot{q}^i + \underset{\substack{\text{integrate}\\ \text{by part}}}{p_i \delta \dot{q}^i} - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt = 0$$

$$= \int \delta p_i \underbrace{\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right)}_{0} dt + \int \delta q^i \underbrace{\left(-\dot{p}_i - \frac{\partial H}{\partial q^i}\right)}_{0} dt = 0$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

Remark: $\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} = \cancel{\frac{\partial H}{\partial q}\frac{\partial H}{\partial p}} - \cancel{\frac{\partial H}{\partial p}\frac{\partial H}{\partial q}} + \frac{\partial H}{\partial t}$

$= \frac{\partial H}{\partial t}$

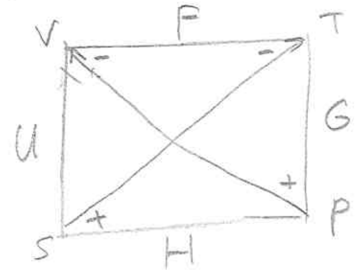
If $H \neq H(t) \Rightarrow H$: conserved energy

- Legendre Transformation

$L \leftrightarrow H$

$$\mathcal{T}(TM) \longleftrightarrow \mathcal{T}(T^{+}M)$$

- Magic Square in TD



① TD potentials are total differentials in terms of their variables

$dU = TdS - PdV \quad , \quad U = U(S,V)$

Ex: $G(T,P)$, $dG = -SdT + VdP$

② Legendre Transformation

Ex: $F = U - TS \quad , \quad T = \frac{\partial U}{\partial S}$

$U = H - PV$

$G = F + PV$

$G = H - TS$

③ Maxwell's relations

$$\left.\frac{\partial V}{\partial S}\right)_P = \left.\frac{\partial T}{\partial P}\right)_S$$

$$\left.\frac{\partial V}{\partial T}\right)_P = -\left.\frac{\partial S}{\partial P}\right)_T$$

Proof:

$$dH = TdS + VdP$$

$$dH = \left.\frac{\partial H}{\partial S}\right)_P dS + \left.\frac{\partial H}{\partial P}\right)_S dP$$

$$T = \left.\frac{\partial H}{\partial S}\right)_P \quad , \quad V = \left.\frac{\partial H}{\partial P}\right)_S$$

$$\left.\frac{\partial T}{\partial P}\right)_S = \left.\frac{\partial V}{\partial S}\right)_P \quad //$$

6-) Poisson Brackets

Definition : For two phase space functions f and g. Their canonical poisson bracket is a new phase-space function.

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

Properties

i-) Fundamental brackets

$$\{q^i, q^j\} = \{p_i, p_j\} = 0$$

$$\{q^i, p_j\} = \delta^i_j$$

$$QM : \{,\} \rightarrow [,] = i\hbar\{,\}$$

ii-) Anti-symmetry

$$\{f,g\} = -\{g,f\}$$

iii-) Linearity

$$\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$$

iv-) Liebniz

$$\{fg,h\} = f\{g,h\} + \{f,h\}g$$

v-) Jacobi

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

- $$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}$$

$$= \{f,H\} + \frac{\partial f}{\partial t}$$

- Integral of motion

$$\frac{dI}{dt} = 0 = \{I,H\}$$

Note that having two integrals of motion, I_1 and I_2 $I_3 = \{I_1, I_2\}$ is also an integral of motion.

C-) Canonical Transformations

Definition: A canonical transformation

$$Q^j = Q^j(q^i, p_i, t) \quad , \quad P_j = P_j(q^i, p_i, t)$$

is such transformation so that it preserves the form of the Hamilton's equations. That is, one can find a "new Hamiltonian"

$$H' = H'(Q^j, P_j, t).$$

$$\frac{\partial H'}{\partial P_j} = \dot{Q}^j \quad , \quad \frac{\partial H'}{\partial Q^j} = -\dot{P}_j$$

Ex: Not every transformation is canonical

Free particle

$$H = \frac{p^2}{2m} \quad , \quad Q = \sqrt{q} \, , \; P = p \qquad \text{is this canonical ?}$$

guessing $H' = \frac{P^2}{2m}$

$$\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq \dot{Q} \qquad \times$$

- Canonical transformations are generated by generating functions of the type

$$F = F(\text{old}, \text{new})$$

Since old $\in (q, p)$ and new $\in (Q, P)$. we have 4 possibilities (related by Legendre transformation)

Adding a total derivative to the Lagrangian does not change the eom.

Ex: $F = F(q, Q, t)$

$$S = \int dt \left(p\dot{q} - H - \frac{dF}{dt}\right) = \int \Big[\dot{q}\underbrace{\left(p - \frac{\partial F}{\partial q}\right)}_{0} \underbrace{- \frac{\partial F}{\partial Q}}_{P}\dot{Q} - \underbrace{\left(H + \frac{\partial F}{\partial t}\right)}_{H'}\Big] dt$$

$$\overset{\text{want}}{=} \int \left(P\dot{Q} - H'(P, Q, t)\right) dt$$

$$P = -\frac{\partial F}{\partial Q} \quad , \quad p = \frac{\partial F}{\partial q} \quad , \quad H' = H + \frac{\partial F}{\partial t}$$

$$\frac{\partial P}{\partial q} = -\frac{\partial p}{\partial Q}$$

Remarks:

1-) P.B. are invariant w.r.t canonical transformation

$$\{f,g\}_{q,p} = \{f,g\}_{Q,P}$$

$\frac{df}{dt} = \{f,H\}$ when $g \equiv H$, time evolution does not depend coordinates

ii-) Transformation is canonical iff fundamental brackets

$$\{Q^i, P_j\} = \delta^i{}_j \quad , \quad \{Q^i, Q^j\} = \{P_i, P_j\} = 0$$

iii-) Time evolution = Canonical transformation generated by $F = -S$, where S is called Hamilton's function

Hamilton's function = "Action" understood as a function of coordinates and time"

In order to obtain this function

i-) Solve the (E-L) equations for q, expressing it in terms of the boundary value data: $q = q(t_1, q_1, t_2, q_2, t)$

ii-) Once this solution is known, we plug it back to L, and integrate L over t to get

$$S(t_1, q_1, t_2, q_2) = \int_{t_1}^{t_2} L(t_1, q_1, t_2, q_2, t)\, dt \quad : \text{On shell fixed trajectory}$$

Since $L dt = p_i dq^i - H dt$

$$dS = L_f - L_i = (p_i \underline{\underline{dq^i}} - H \underline{dt})_{final} - (p_i dq^i - H dt)_{initial} \quad (3.17)$$

Identifying $t = t_2$, $q = q_2 \Rightarrow q = q(t_0, q_0, t)$

$$dS = \left.\frac{\partial S}{\partial t_1}\right) dt_1 + \left.\frac{\partial S}{\partial q_1}\right) dq_1 + \left.\frac{\partial S}{\partial t}\right) \underline{dt} + \left.\frac{\partial S}{\partial q}\right) \underline{\underline{dq}}$$

leads us to Hamilton-Jacobi equation

$$\frac{\partial S}{\partial q^i} = p_i \quad , \quad \frac{\partial S}{\partial t} = -H$$

Note also that, upon $q_0 = q_1$, $t_0 = t_1$

$$\frac{\partial S}{\partial q_0^i} = -p_{0i} \quad , \quad \frac{\partial S}{\partial t_0} = H_0$$

Solution: Time evolution can be written as coordinate transformation

$$q(t) = q(q_0, p_0, t)$$
$$p(t) = p(q_0, p_0, t)$$

where (q_0, p_0) are the "old coordinates" and (q, p) are the "new ones".

Integrating (3.17)

$$\int \underbrace{(p\,dq - H\,dt)}_{New} = \int \Big[\underbrace{p_0\,dq_0 - H_0 dt}_{old} - \frac{d(-S)}{dt}\,dt \Big] \qquad -S = -S(q_0, q, t)\ )$$

$F = -S$ generates time evolution ...

Liouville's Equation : Volume of the phase space remains invariant under canonical transformations. In particular, time evolution preserves the phase space volume.

Proof: $(n=1)$

$$V_t = \int dQ\,dP = \int \underbrace{|\text{Jacobian}|}_{J}\,dq\,dp$$

$$J = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \{Q, P\} = 1$$

d-) Hamilton - Jacobi Theory

Idea: Find a special generating function $\underset{S}{\overset{F}{\shortparallel}}$, so that new Hamiltonian

$H' = 0$

$$\dot{Q}^j = \frac{\partial H'}{\partial P_j} = 0 \Rightarrow Q^j = \alpha^j = \text{constant}$$

$$\dot{P}_j = -\frac{\partial H'}{\partial Q^j} = 0 \Rightarrow P_j = -\beta_j \quad \text{constant}$$

$$P_j = -\beta = -\frac{\partial S(q, Q, t)}{\partial Q^j} = \frac{\partial S(q, \alpha, t)}{\partial \alpha^j} \Rightarrow \beta_j = \frac{\partial S(q, \alpha, t)}{\partial \alpha^j}$$

This is an implicit solution, the explicit one is found by inversion

$$q^i = q^i(t, \alpha^j, \beta_j)$$

$$H' = H + \frac{\partial F}{\partial t} = 0 \Rightarrow \boxed{\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) = 0}$$

HJ - Theory
one - PDE for $S(q,t)$

$$p_j = \frac{\partial S}{\partial q^j}$$

- Solve (H-J) equations interested in the solutions such that $S(q, \underbrace{\alpha}_{n \text{ integration constant}}, t)$: Type of solutions which depends on n integration constants are 'complete integral'
- A trick for finding this is to use additive seperation of variables
  $S(q,t) = S_1(q^1) + S_2(q^2) \ldots S_n(q^n) + S_t(t)$ → If $H \neq H(t)$, when S put into the HJ we get $S_t(t) = -Et$

$H \neq H(t) \quad S = \underset{t\text{-independent}}{S_0} - Et$

$H \neq H(x) \quad S = X C_x + \underset{x\text{-independent}}{S_0}$

Ex: Free fall in homogeneous gravitational field

$$H = \frac{p^2}{2m} - mgX \quad , V(x) = -mgX$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) + \frac{\partial}{\partial t}(S_0 - Et) = 0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) - E + \cancel{\frac{\partial S_0}{\partial t}} = 0$$

$$\left(\frac{\partial S}{\partial x}\right)^2 = 2m(E-V) \Rightarrow S_0 = \int dx \sqrt{2m(E-V)}$$

$$S_0 = \int dx \sqrt{2m(E+mgx)}$$, performing the integral

$$S = \frac{1}{3gm^2}\left[2m(E+mgx)\right]^{3/2} - Et$$

$\beta = \frac{\partial S}{\partial \alpha}$, β and α are constant

set $\alpha = E$, $\beta = \frac{\partial S}{\partial E} = \frac{1}{mg}\sqrt{2m(E-V)} - t$

Inversion of this gives

$$\boxed{x = \frac{g}{2}(\beta+t)^2 - \frac{E}{mg}}$$

β can be found using initial conditions
$t_0 = \beta$ in this case

Connection to QM

- Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = \frac{p^2}{2m}\psi + V\psi \quad , p = -i\hbar\nabla$$

- WKB (Geometric optics) Approximation

$$\psi = \psi_0 e^{\frac{i}{\hbar}S(x,t)}$$

LHS: $i\cancel{\hbar}\psi \frac{i}{\cancel{\hbar}}\frac{\partial S}{\partial t}$

RHS: $-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[\left(\frac{i}{\cancel{\hbar}}\frac{\partial S}{\partial x}\right)\psi\right] = \left(\frac{(\nabla S)^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V\right)\psi$

Taking now the $\hbar \to 0$ limit

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0$$ which is the (H-J) equation

e-) Integrable Systems

Definition: The dynamical system with n dof ($2n$-dimensional phase space) is completely (Liouville) integrable, if it posses n independent conserved quantities $F_i(q,p) = f_i$, $\{H, F_i\} = 0$, that are in involution: $\{F_i, F_j\} = 0$ for all i, j

- Liouville's Theorem: The solution of eom of a completely integrable system can be obtained by quadrature, that is finite numbers of integrations and algebraic operations

Proof is constructive and tells how knowing $\{F_i\}$, I can find the solution by one integration and some inversions

- Interesting observation is that I need only n (not $2n$). Each integral of motion can be used twice.

Similar statement in field theory: In any gauge theory

$$\text{Number of True dof} = \text{Number of apparent dof} - 2\left(\begin{array}{c}\text{Number of dof}\\ \text{of gauge function}\end{array}\right)$$

EX: EM described in terms of $(\varphi, \vec{A})$

$$\varphi \to \varphi - \partial_t \Lambda \qquad \Rightarrow E, B \text{ are invariant}$$
$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda \qquad \text{where } \Lambda \text{ is arbitrary function}$$

Λ: gauge function and has one dof (scalar)

4: apparent $(\underset{1}{\varphi}, \vec{A}_3)$ $\qquad$ # True dof $= 4 - 2 = 2 \to$ polarization of photon

1: gauge function

EX: Gravity

$g_{\mu\nu}$ 4x4 symmetric matrix

$\frac{4 \cdot 5}{2} = 10$ apparent

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

ξ → gauge function with 4 dof

True dof $= 10 - 8 = 2 \to$ polarization of graviton

f-) Constraints

- Let not all q^i, p_i be independent, that is at any instant of time we have a constraint

$$\phi(q^i, p_i) = 0$$

- Remark: Any phase space function f can be understood as "Hamiltonian" and generates the corresponding "flow"

$$\frac{dq^i}{dt_f} = \{q^i, f\}_{\text{"}H\text{"}} \quad , \quad \frac{dp_i}{dt_f} = \{p_i, f\}$$

specifically, a conserved quantity obeys $\{I, H\} = -\{H, I\} = 0$. This means that Hamiltonian is invariant under the flow generated by I... symmetry

- Time evolution of $I = 0$, so it is conserved
- Time evolution generated by I leaves the Hamiltonian invariant.

Constraints are "conserved quantities" and imply the presence of symmetries in physical systems.

- Standart example

"clever description" (Occam)

- $n = 1$ dof
- $2n = 2$ phase space (φ, p_φ)
- No constraint

vs

Constraint description

- $n = 2$ dof
- phase space (x, y, p_x, p_y)
- Constraint $\phi = x^2 + y^2 - l^2 = 0$

Eliminate constraint

$x = l\cos\varphi$

$y = l\sin\varphi$

- Advantages
  - Extra symmetry
  - Can calculate "force of constraint" (Tension of string T)
- To deal with constraints one uses the method of Lagrange multipliers. Namely in the presence of m number of constraints ϕ_j, we define total Hamiltonian

$$H_T = H + \sum_{j=1}^{m} \lambda_j(t)\phi_j \qquad \lambda_j(t): \text{Lagrange multipliers}$$

The eom are then derived wrt this total Hamiltonian

2 possibilities

① Enlarge phase space even further

Consider $\lambda_j(t)$ as new coordinate

We have to vary wrt λ_j as well

$(\underbrace{n}_{\text{dimension}} + \underbrace{m}_{\text{constraint}})$ E-L equations

$\phi_j = 0$, $j = 1 \dots m$: on shell constraints

$+ \, n$ modified (E-L) for q^j

$L_T = L + \lambda_j \phi_j$

$\delta L + \delta\lambda_j \phi_j + \lambda_j \delta\phi_j$

$$\delta S = \int \left[\delta L + \delta\lambda_j \phi_j + \frac{d}{dt}\left(\frac{\partial L}{\partial q}\right)\delta q\right]$$

$$= \int \left[\frac{\partial L}{\partial q}\delta q + \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right)} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q\right] + \phi_j \delta\lambda_j$$

$$+ \lambda_j \left[\frac{\partial \phi}{\partial q}\delta q + \cancel{\frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot q}\delta q\right)} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot q}\right)\delta q\right]$$

$$= \int \left[\frac{\partial (L + \lambda\phi)}{\partial q} - \frac{d}{dt}\left(\frac{\partial (L+\lambda\phi)}{\partial \dot q}\right)\right]\delta q + \phi_j \delta\lambda_j$$

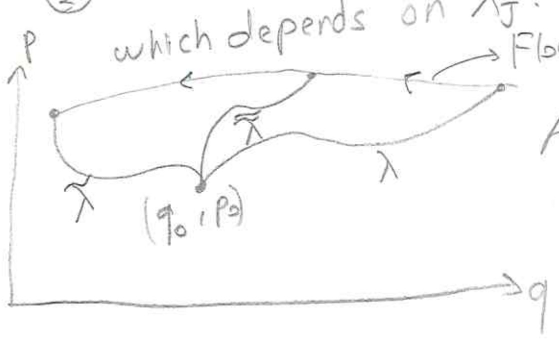
$$= \int \left[\frac{\partial L_T}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot q}\right)\right]\delta q + \phi_j \delta\lambda_j \quad \text{up to vanishing boundary terms.}$$

together with on-shell constraint

$$\frac{\partial L_T}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot q}\right) = 0 \ , \quad \phi_j = 0$$

We can use these to solve for q^i as well as λ_j. Note also that proceeding to the Hamiltonian picture, we find that $p_{\lambda_j} = 0$ and so H_T is given by $H_T = H + \lambda_j \phi_j$. In this case we have n (E-L) for q^j

② Keep $\lambda_j(t)$ as <u>arbitrary function</u>. Time evolution is <u>not unique</u>. which depends on λ_j. Flow generated by constraint



All the endpoints are physically equivalent (related by the symmetry)

• Remark: One source of constraints is a Legendre transformation as $L \to H$

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \xrightarrow{\text{Invert}} \dot{q}^i = \dot{q}^i(q^j, p_j, t)$$

may not exist

Infact

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$$ ... if this matrix has zero eigen values ($\det A = 0$)

one can write $\dot{q}^i = \dot{q}^i(q^k, p_k, \lambda^j, t)$ $\quad j = 1 \ldots m$ where m is the number of zero valued eigen values

Toy example: Consider the motion of free particle but using "lousy time"

$$L = \frac{1}{2} m \dot{x}^2$$

$T = T(t)$: inertial time and t is lousy time

Starting from the L written in terms of the lousy time and performing the standard Legendre transformation we find that corresponding Hamiltonian vanishes: $H = 0$: does not generate dynamics

This is a characteristic of a totally constraint system

$$H_T = \lambda(t) \phi$$

So there is no "time evolution" and the only flow is wrt the constraint. (Total Hamiltonian is non trivial and determines this flow). Despite the lack of a true time evolution, relational predictions are still possible for this type of system.

1-)

a-) $\underbrace{\{\{I_1, I_2\}, H\}}_{0} + \underbrace{\{\underbrace{\{I_2, H\}}_{0}, I_1\}}_{0} + \underbrace{\{\underbrace{\{H, I_1\}}_{0}, I_2\}}_{0} = 0$

To get the zero $\{I_1, I_2\}$ must give a conserved charge I_3 and $\{I_3, H\} = 0$

b-) $L_i = \epsilon_{ijk} q^j p^k$

$\{q_i, L_j\} = \{q_i, \epsilon_{jmn} q^m p^n\} = \epsilon_{jmn} q^m \delta_i^{\ n} = \epsilon_{jmi} q^m$

$\{p_i, L_j\} = \{p_i, \epsilon_{jmn} q^m p^n\} = -\epsilon_{jmn} p^n \delta_i^{\ m} = -\epsilon_{jin} p^n = \epsilon_{ijn} p^n$

- $\{L_i, L_j\} = \{\epsilon_{imn} p^m q^n, L_j\} = \epsilon_{imn}\left(p^m \{q^n, L_j\} + \{p^m, L_j\} q^n\right)$

$= \epsilon_{imn}\left(p^m \epsilon_{jl}^{\ \ n} q^l + \epsilon^m_{\ jl} p^l q^n\right)$

$= \epsilon_{imn} \epsilon_{jl}^{\ \ n} p^m q^l + \epsilon_{imn} \epsilon^m_{\ jl} p^l q^n$

$= (\delta_{ij} \delta_{ml} - \delta_{il} \delta_{mj}) p^m q^l - (\delta_{ij} \delta_{nl} - \delta_{il} \delta_{nj}) p^l q^n$

$= \delta_{ij} \cancel{p_m} q^m - p_j q_i - \cancel{\delta_{ij} p_n} q^n + p_i q_j = p_i q_j - p_j q_i = \underline{\epsilon_{ijk} L^k}$

- $\{\vec{L}^2, L_i\} = \{L_j L^j, L_i\} = L_j \{L^j, L_i\} + \{L_j, L_i\} L^j$

$= L^j \{L_j, L_i\} + \{L_j, L_i\} L^j$

$= L^j \epsilon_{jik} L^k + \epsilon_{jik} L^k L^j$

$= \epsilon_{jik} L^j L^k + \epsilon_{jik} L^k L^j = (\epsilon_{jik} + \epsilon_{kij}) L^j L^k$

$= \epsilon_{jik} L^j L^k + \epsilon_{kij} L^j L^k =$ $-\epsilon_{ijk} - \epsilon_{ikj}$ $-\cancel{\epsilon_{ijk}} + \cancel{\epsilon_{ijk}}$ $= 0$

c-) If L_i, L_j are conserved, so is L_k //

2-)

2.1) a-) • $[f(x), g(x)] = [f(0) + f'(0).x + f''(0)\frac{x^2}{2} \dots, g(0) + g'(0)x + \dots] = 0$

• $[p, f(x)] = \sum_{n=0} \frac{f^{(n)}(0)}{n!} [p, x^n]$

$= -\sum \frac{f^{(n)}(0)}{n!} [x^n, p] = -\sum \frac{f^{(n)}(0)}{\cancel{n!}} \cancel{n} x^{n-1} \underbrace{[x, p]}_{1}$

$(n-1)!$

$n \to n+1$

$= -\frac{\partial}{\partial x} \sum \frac{f^{(n)}(0)}{n!} x^n = -\frac{\partial f}{\partial x}$

• $[p_a, f(x)] = -\underline{\partial_a f}$

b-) $[x_a, H] = \dot{x}_a = \frac{p_a}{m} = \frac{p_b}{m}\delta_{ab} = \frac{p_b}{m}[x_a, p_b] = [x_a, \frac{p^2}{2m}]$

$[x_a, V(x)] = 0 \quad \to \quad [x_a, H] = [x_a, \underline{\frac{p^2}{2m} + V(x)}] = \dot{x}_a$

c-) $\dot{p}_a = [p_a, H] = [p_a, \frac{p^2}{2m} + V(x)] = [p_a, V(x)] = -\frac{\partial V}{\partial x^a}$ //

2.2

a-) $[[p_a, p_b], x_c] + [[p_b, x_c], p_a] + [[x_c, p_a], p_b] = 0$

$\underbrace{[\underbrace{[p_a, p_b]}_{F_{ab}(x)}, x_c]}_{0} = -[[p_b, x_c], p_a] - [[x_c, p_a], p_b]$

$= [[x_c, p_b], p_a] - [[x_c, p_a], p_b]$

when $a \leftrightarrow b$, F_{ab} is antisymmetric

b-) $F_{[ab, c]} = [[p_a, p_b], p_c] + [[p_b, p_c], p_a] + [[p_c, p_a], p_b] = 0$

$A = A(x)$

$F_{ab} = [p_a, p_b] = \partial_a A_b - \partial_b A_a$

$= -[p_a, A_b] + [p_b, A_a]$

c-) $\pi_a = p_a + A_a$

• $[x_a, x_b] = 0$, • $[x_a, \pi_b] = [x_a, p_b + A_b(x)] = [x_a, p_b] = \delta_{ab}$

• $[\pi_a, \pi_b] = [p_a + A_a(x), p_b + A_b(x)] = [p_a, A_b] + [A_a, p_b]$

$= [p_a, A_b] - [p_b, A_a]$

$= -\partial_a A_b + \partial_b A_a = -F_{ab}(x) = 0$

d-) $[x_a, H] = \dot{x}_a = \frac{p_a}{m} =$

$= [x_a, \frac{p^2}{2m} + V(x)] = [x_a, \frac{(\pi - A)^2}{2m} + V(x)]$

$H = \frac{(\pi - A)^2}{2m} + V(x)$

e-) $\dot{p}_a = [p_a, H] = [p_a, \frac{p_b p^b}{2m} + V(x)] = \frac{1}{m} F_{ab} p^b - \frac{\partial V}{\partial x^a}$ $F =$

$\dot{p}_a = F_{ab} \dot{x}^b \underbrace{- \frac{\partial V}{\partial x^a}}_{qE} \iff \frac{dp}{dt} = q(E + u \times B)$

$F_{0i} = -E_i$

$F_{ij} = \epsilon_{ijk} B^k$

On the other hand this is not general case

$$\frac{dx}{dt} = \lambda \frac{p_x}{m}$$
$$\frac{dT}{dt} = \lambda$$

$$\Rightarrow \quad \frac{dx}{dT} = \frac{p_x}{m} \quad \Rightarrow x(T) = \frac{p_x}{m}(T - T_0) + x_0$$

That is we have recovered the Newtonian mechanics using H_{tot} by going back to the inertial time T

2-) $L = \frac{1}{2} m\dot{q}^2$

$$S = \int dt \left(\frac{1}{2} m\dot{q}^2\right)$$

EoM: $m\ddot{q} = 0 \Rightarrow \ddot{q} = 0 \qquad \dot{q} = A \Rightarrow q(t) = At + B$

$t = t_0,\ q = q_0 \Rightarrow q_0 = At_0 + B \qquad q_1 - q_0 = A(t_1 - t_0)$

$t = t_1,\ q = q_1 \Rightarrow q_1 = At_1 + B$

$$A = \left(\frac{q_1 - q_0}{t_1 - t_0}\right) \qquad B = \frac{q_0 t_1 - t_0 q_1}{t_1 - t_0}$$

$$S(q_0, q_1, t_0, t_1) = \int dt \left(\frac{1}{2} m A^2\right) = \int_{t_0}^{t_1} dt \left(\frac{q_1 - q_0}{t_1 - t_0}\right)^2 \frac{m}{2}$$
$$= (t_1 - t_0)\frac{m}{2}\left(\frac{q_1 - q_0}{t_1 - t_0}\right)^2$$

$$S = S(q_0, q_1, t_0, t_1) = \frac{1}{2} m \frac{(q_1 - q_0)^2}{t_1 - t_0}$$

b-) $$\frac{\partial S}{\partial q_1} = p_1 = m\frac{(q_1 - q_0)}{t_1 - t_0} = p$$

$$\frac{\partial S}{\partial q_0} = -p_0 = -\frac{m(q_1 - q_0)}{t_1 - t_0} = -p$$

$p_1 = p_0 = p$ as we expected due to q being a cyclic coordinate, momentum has conserved

$$\frac{\partial S}{\partial t_1} = -E = -\frac{1}{2}\frac{m(q_1 - q_0)^2}{(t_1 - t_0)^2} \Rightarrow E = \frac{1}{2} m \left(\frac{q_1 - q_0}{t_1 - t_0}\right)^2$$

c-) $L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2$

$\ddot{q} + \omega^2 q = 0 \Rightarrow q(t) = A\cos\omega t + B\cos\omega t$

$$A = \frac{q_0 \sin(\omega t_1) - q_1 \sin(\omega t_0)}{\sin(\omega t_1)\cos(\omega t_0) - \sin(\omega t_0)\cos(\omega t_1)}$$

$$B = \frac{-q_0\cos(\omega t_1) + q_1\cos(\omega t_0)}{\sin(\omega t_1)\cos(\omega t_0) - \sin(\omega t_0)\cos(\omega t_1)}$$

$$S(q_0, q_1, t_0, t_1) = m\omega\left(\frac{(q_0^2 + q_1^2)\cos[\omega(t_1 - t_0)] - 2q_0 q_1}{2\sin[\omega(t_1 - t_0)]}\right)$$

1-)

a-) $S=\int\left(\frac{1}{2}m\left(\frac{dX}{dT}\right)^2\right)dT$ when using a normal T and $T=T(t)$ where t is lousy time

$$\frac{d}{dT}=\frac{dt}{dT}\frac{d}{dt}=\frac{1}{\dot{T}}\frac{d}{dt}$$

$$dT=\frac{dT}{dt}dt=\dot{T}dt \qquad S=\int\left(\frac{1}{2}m\frac{1}{\dot{T}^2}\dot{x}^2\right)\dot{T}dt=\int\left(\frac{1}{2}m\frac{\dot{x}^2}{\dot{T}}\right)dt$$

X and T dynamical variable

$$\frac{dX}{dT}=\frac{dt}{dT}\frac{dx}{dt}=\frac{1}{\dot{T}}\dot{x}$$

b-) We already did it for $T(t)\to t$, it is also valid vice versa

c-) $P_x=\frac{\partial L}{\partial\dot{x}}=\frac{m\dot{x}}{\dot{T}}$, $P_T=\frac{\partial L}{\partial\dot{T}}=-\frac{m\dot{x}^2}{2\dot{T}^2}$

and following constraint would be

$$\frac{1}{2m}\Big(\underbrace{\frac{m\dot{x}}{\dot{T}}}_{P_x}\Big)^2=\frac{P_x^2}{2m} \Rightarrow \phi(x,t)=\frac{P_x^2}{2m}+P_T=0$$

on the other hand

$$H=P_x\dot{x}+P_T\dot{T}-L=\frac{m\dot{x}^2}{\dot{T}}-\frac{m\dot{x}^2}{2\dot{T}}-\frac{m\dot{x}^2}{2\dot{T}}=0$$

Hence we have no <u>time evolution</u>. This means that evolution in lousy time t does not mean anything. Introduction of arbitrary possible clocks has made all possible trajectories equivalent.

d-) Let's now find the true dynamics

$\lambda(t)$: Lapse function

$$H_{Tot}=H+\lambda(t)\phi$$

$$\dot{x}=\{x,H_{tot}\}=\{x,\lambda\frac{P_x^2}{2m}\}=\frac{\lambda}{2m}\cdot 2P_x=\frac{\lambda P_x}{m}$$

$$\dot{P}_x=\{P_x,H_{tot}\}=0$$

$$\dot{T}=\{T,H_{tot}\}=\lambda$$

$$\dot{P}_T=\{P_T,H_{tot}\}=0$$

In particular, when $\lambda=$ constant (when the clock is not that much lousy)

$$\frac{dx}{dt}=\lambda\frac{P_x}{m} \Rightarrow x(t)=\lambda\frac{P_x}{m}(t-t_0)+x_0 \ : \text{ Newtonian equation (almost)}$$

1-) $\frac{\partial S}{\partial t} + H(q^i, \frac{\partial S}{\partial q^i}, t) = 0$

a-)

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) - V(r)$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \ , \quad P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi}$$

$$H = P_i\dot{q}^i - \mathcal{L} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{P_\varphi^2}{2mr^2\sin^2\theta} + V(r)$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right) + V(r) = 0$$

b-) Since $H \neq H(t)$ and $H \neq H(\varphi)$ (it may or may not work)

$$S = -Et + L\varphi + S_0(r,\theta)$$

$$S = -Et + L\varphi + R(r) + \Theta(\theta)$$

$$\frac{1}{2m}\left(R'^2 + \frac{\Theta'^2}{r^2} + \frac{L^2}{r^2\sin^2\theta}\right) + V - E = 0$$

$$\underbrace{\frac{r^2 R'^2}{2m} + (V-E)r^2}_{r \text{ dependent}} = \underbrace{-\frac{L^2}{2m\sin^2\theta} - \frac{\Theta'^2}{2m}}_{\theta \text{ dependent}} = -K \quad , K > 0$$

$$\frac{dR}{dr} = \sqrt{-\frac{2mK}{r^2} + (E-V)2m} \Rightarrow R = \int dr \sqrt{(E-V)2m - \frac{2mK}{r^2}}$$

$$\frac{d\Theta}{d\theta} = \sqrt{+K2m - \frac{L^2}{\sin^2\theta}} \Rightarrow \Theta = \int d\theta \sqrt{2mK - \frac{L^2}{\sin^2\theta}}$$

c-) The three corresponding integrals of motion E, L, K have the meaning of energy, z-component of the angular momentum, and total angular momentum L^2. The actual solution of equation motions is obtained by setting

$$t_0 = \frac{\partial S}{\partial E} \ , \quad \varphi_0 = \frac{\partial S}{\partial L} \ , \quad \alpha = \frac{\partial S}{\partial K}$$