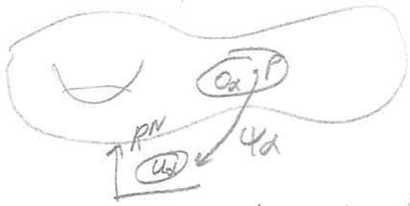
Introduction to Differential Geometry

a-) Manifolds and Tensors

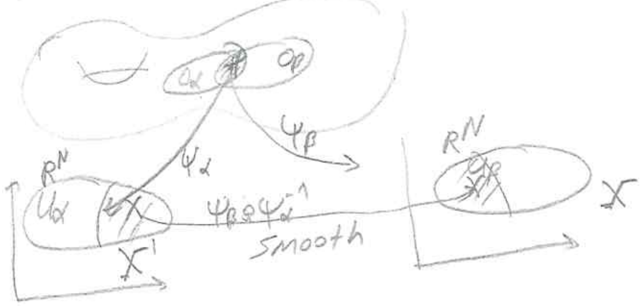


Definition: A N-dimensional Manifold M is a "set of points" together with a collection of subsets $\{O_\alpha\}$ satisfying:

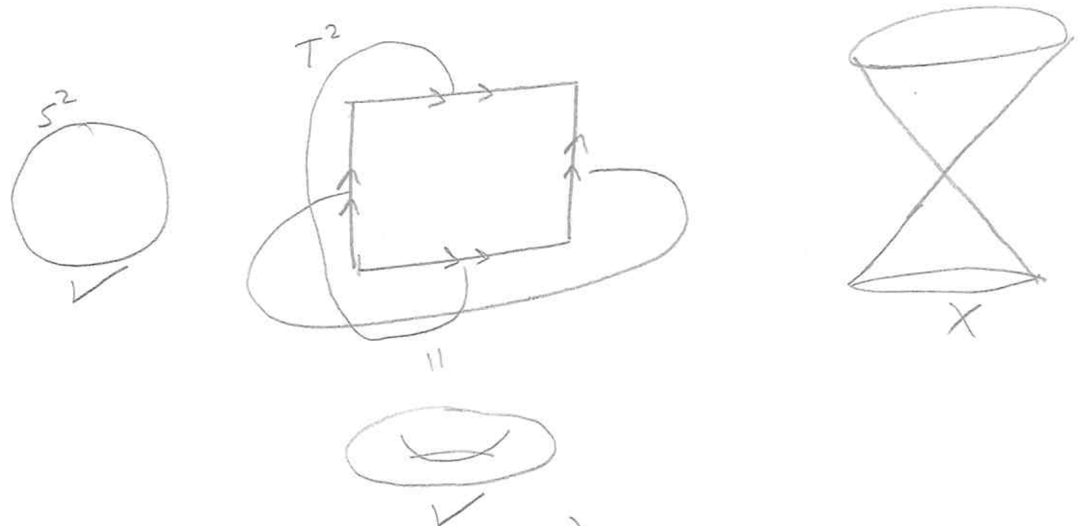
i-) Each $p \in M$ lies in at least one O_α $\{O_\alpha\}$ cover all M

ii-) For each α, there is 1-1, onto map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of R^N

iii-) If any two sets O_α and O_β overlap, $O_\alpha \cap O_\beta \neq 0$, then the map $\psi_\beta \circ \psi_\alpha^{-1}$ is smooth (C^∞)



Examples:

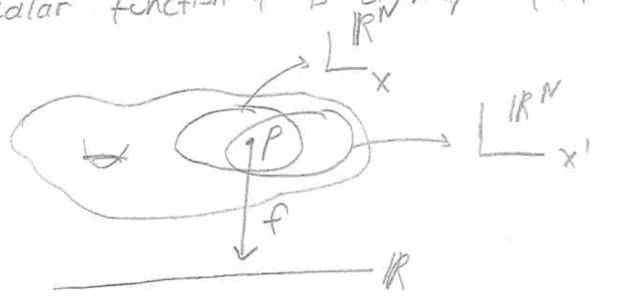


ψ's ... charts (coordinate system - X)

$\{O_\alpha, \psi_\alpha\}$ ... Atlas

Point particles behave nicely on manifolds. (needs)

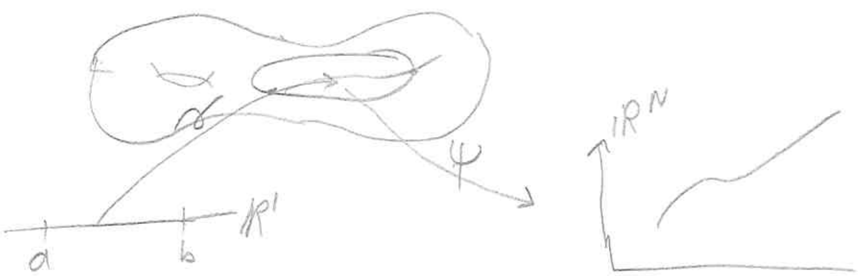
- A scalar function f is a map $f: M \to \mathbb{R}$



$f(p) = f(x(p)) = f(x'(p))$

$f(x) = f'(x')$

- A curve γ on M is a map $\gamma: I \subset \mathbb{R}^1 \to M$ such that $(\psi \circ \gamma)(t) = [x^1(t), x^2(t) \dots x^N(t)]$ is smooth



<u>Ex</u>: River, trajectory of particle

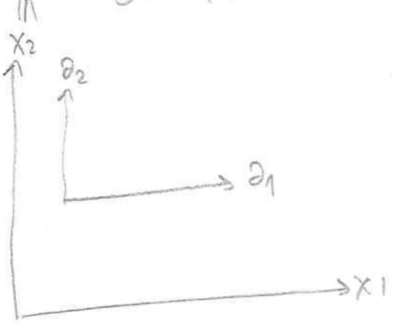
- A surface S is a map $S: U \subset \mathbb{R}^2 \to M$

$(\psi \circ S)(\tau, \sigma) = [x^1(\tau,\sigma), x^2(\tau,\sigma) \dots, x^N(\tau,\sigma)]$

<u>Ex:</u> cake, worldsheet of a string

- A <u>tangent vector</u> is associated with "directional derivative at a point"

in $\mathbb{R}^N$ $v^\mu = (v^1, v^2 \dots v^N)$ $\longleftrightarrow$ Directional derivative operator $v^\mu \partial_\mu = \hat{v}$



$\hat{v}f = v^\mu \partial_\mu f \in \mathbb{R}$ at a point

Characterized by linearity and Liebniz rule

Definition: Let $\mathcal{F}$ be a collection of smooth (C^∞) scalar functions. A tangent vector V at a point $p \in M$ is a map $V: \mathcal{F} \longrightarrow \mathbb{R}$ so that

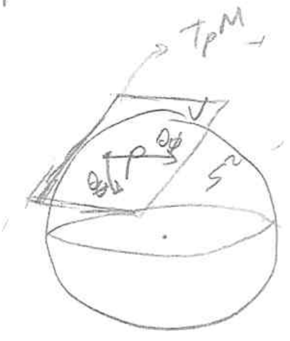
i-) Linear: $V(af + bg) = aV(f) + bV(g) \quad a, b \in \mathbb{R}$

ii-) Leibniz: $V(fg) = f(p)V(g) + V(f)g(p)$

Theorem: The set of tangent vectors at p, forms a tangent vector space T_pM of dim N (same as manifold), with coordinates basis ∂_μ. Any vector

$$\underbrace{V}_{} = \underbrace{V^\mu}_{\text{components}} \underbrace{\partial_\mu}_{\text{basis}}$$

Ex:



Using the chain rule, the components of the vector change of coordinates as follows:

$$V = V^\mu(x)\,\partial_\mu = \underbrace{V^\mu(x)\,\frac{\partial x'^\nu}{\partial x^\mu}}_{V'^\nu(x')}\,\frac{\partial}{\partial x'^\nu} = V'^\nu(x')\,\partial_\nu$$

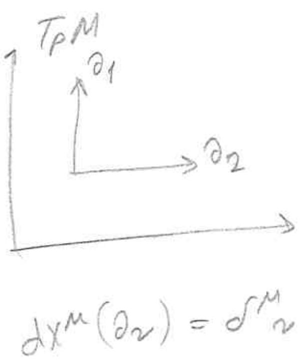
$$V'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\mu}\,V^\mu(x)$$

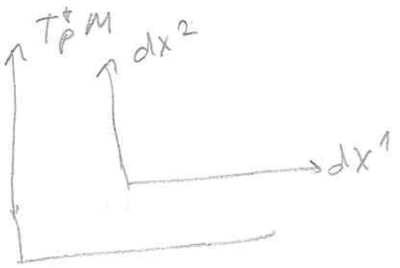
Vector remains invariant, while components and basis changes

- A tangent vector field V is defined as $\{V|_p \in T_pM$ for all $p \in M$: $V(f)$ is smooth$\}$

A tangent bundle $TM = \bigcup_p T_pM$. TM has local coordinates (x^μ, V^ν)

Definition: A cotangent vector (1-form) ω at a point $p \in M$ is a map $\omega: T_pM \longrightarrow \mathbb{R}$. Have coordinate basis in cotangent vector space T_p^*M: dx^μ





$$dx^\mu(\partial_\nu) = \delta^\mu{}_\nu$$

$$\omega = \underbrace{\omega_\mu}_{\text{components}} \underbrace{dx^\mu}_{\text{basis}}$$

Change coordinates

$$\omega = \omega_\mu(x)\,dx^\mu = \underbrace{\omega_\mu(x)\,\frac{\partial x^\mu}{\partial x'^\nu}}_{\omega'_\nu(x')}\,dx'^\nu$$

$$\omega'_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu}\,\omega_\mu(x) \quad \text{: covector}$$

$T^*M = \bigcup_p T_p^*M$: cotangent bundle

(Ex of Fibre bundles)

(x^μ, ω_ν)

Canonical Projection

$V \in TM \quad (x^\mu, V^\nu)$

$\pi : TM \to M \qquad \pi(V) = p$

$\pi(x^1, x^2 \ldots x^N, V^1, V^2 \ldots V^N) \longrightarrow (x^1, x^2 \ldots x^N)$

Tensor $\simeq$ Product of vectors and forms

Definition: A tensor of type (k, l) and rank $k+l$ is a multilinear map

how many co-vector to eat which is a vector

vice-versa

$$T : \underbrace{T_p^* \times T_p^* \cdots T_p^* M}_{k} \times \underbrace{T_p \times T_p \cdots T_p M}_{l} \longrightarrow \mathbb{R}$$

$$T'^{\alpha \ldots}{}_{\mu \ldots}(x') = \underbrace{\frac{\partial x'^\alpha}{\partial x^\sigma} \cdots}_{k} \underbrace{\frac{\partial x^\rho}{\partial x'^\mu} \cdots}_{l} T^{\sigma \ldots}{}_{\rho \ldots}(x)$$

Ex: Metric g $(0,2)$ tensor

$g(V, W) \to \mathbb{R}$

Tensor Algebra

i-) One can add two tensors of the same type.

ii-) Tensor product $T \otimes S$ .. $(k+s)$ rank tensor

iii-) Contraction $T^{\alpha\beta}{}_\gamma \xrightarrow{\text{contraction}} T^{\alpha\beta}{}_\alpha = V^\beta$

free index

Dummy indices

Ex: Type $(2,1)$ rank 3

$$T = \underbrace{T^{\alpha\beta}{}_\gamma}_{\text{components}} \underbrace{\partial x_\alpha \otimes \partial x_\beta \otimes dx^\gamma}_{\text{basis}}$$

Ex: $T^{\mu\nu}{}_\sigma = T(dx^\mu, dx^\nu, \partial_\sigma)$

$$= T^{\alpha\beta}{}_\gamma \underbrace{dx^\mu(\partial_\alpha)}_{\delta^\mu_\alpha} \underbrace{dx^\nu(\partial_\beta)}_{\delta^\nu_\beta} \underbrace{\partial_\sigma dx^\gamma}_{\delta^\gamma_\sigma} = T^{\mu\nu}{}_\sigma$$

Ex: $T(\omega, v, W) = T(\omega_\mu dx^\mu, v_\nu dx^\nu, W^\alpha \partial_\alpha)$

$$= \omega_\mu v_\nu W^\alpha \underbrace{T(dx^\mu, dx^\nu, \partial_\alpha)}_{T^{\mu\nu}{}_\alpha}$$

$$= T^{\mu\nu}{}_\alpha \, \omega_\mu v_\nu W^\alpha$$

Ex:

$$S = S_\mu dx^\mu$$

$$T \otimes S = \underbrace{T^{\alpha\beta}{}_\gamma S_\mu}_{(T\otimes S)^{\alpha\beta}{}_{\gamma\mu}} \quad \partial_\alpha \otimes \partial_\beta \otimes dx^\gamma \otimes dx^\mu$$

Ex: $T_{contraction} = T^{\alpha\beta}{}_\gamma \, \partial_\alpha \otimes \underbrace{\partial_\beta \otimes dx^\gamma}_{\delta^\gamma{}_\beta}$

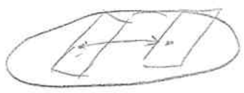
$$= T^{\alpha\beta}{}_\beta \, \partial_\alpha$$

$$= (T_{contraction})^\alpha \, \partial_\alpha$$

6-) Lie Derivative

Its problem when we want to differentiate

Ex: $\frac{df}{dt} = \lim_{s \to 0} \frac{f(s+t) - f(t)}{s}$



on a manifold this can not be done unless we have additional structure

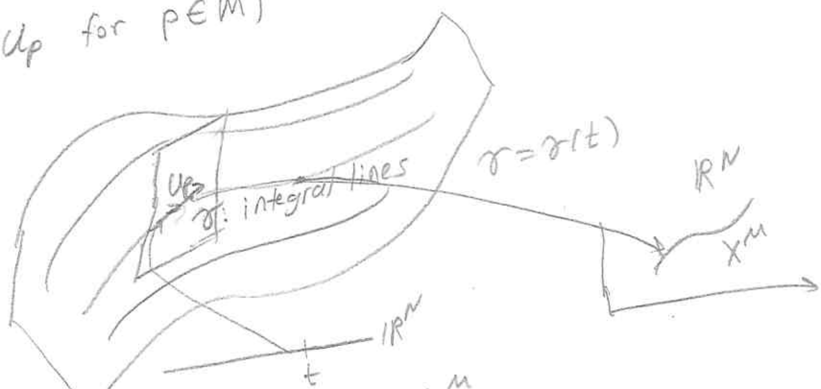
3 standart possibilities

i-) Lie derivative (Vector field U)

ii-) Exterior derivative : doesnt need extra structure but it only works for special tensors (differential forms)

iii-) Covariant derivative (Connection $\Gamma^{\alpha}{}_{\beta\gamma}$)

- A vector field U defines integral curves on M (Tangent vector coincides with U_p for $p \in M$)

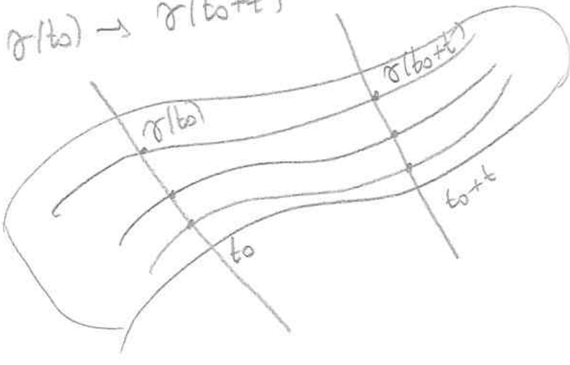


Proof: Choose coordinate X^μ

$$\frac{dX^\mu}{dt} = U^\mu(x)$$ solution X^μ always exists

This defines a map $\phi_t : M \to M$

By $\phi_t : \gamma(t_0) \to \gamma(t_0+t)$



ϕ_t is really nice

- continious in t

$\phi_0 = I$, $\phi_{t+s} = \phi_t \circ \phi_s$, $\phi_{-t} = \phi_t^{-1}$

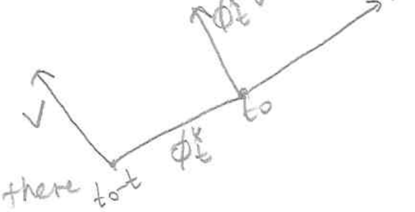
Defines 1- parametric group (Lie) of diffeomorphisms.

Definition: Diffeomorphism $\phi : M \to \tilde{M}$: 1-1, onto, ϕ and ϕ^{-1} are smooth

We can define an induced map

$\phi^* :$ Tensor on $M \longrightarrow$ Tensor on $\tilde{M}$

Definition: Let ϕ_t be a 1-parametric group of diffeomorphisms generated by u, then the lie derivative $\mathcal{L}_u$ is defined



$$\mathcal{L}_u T|_p = \lim_{t \to 0} \frac{T|_p - \phi_t^* T|_p}{t}$$

what was there — what I carried there

Ex: $\mathcal{L}_u f = \lim_{t \to 0} \frac{f(t_0) - \tilde{f}(t_0)}{t}$; $\tilde{f}(t_0) = f(t_0 - t)$

$$= \lim_{t \to 0} \frac{f(t_0) - \cancel{f(t_0)} + t \frac{d}{dt} f(t_0)}{t} = \frac{df(t_0)}{dt}$$

$\mathcal{L}_u f = \frac{df(t_0)}{dt}$: Lie derivative of a function is just vector field acting on a function.

Properties

i-) $\mathcal{L}_u$ maps (k,l) tensors to (k,l) tensors

ii-) $\mathcal{L}_u$ is linear and preserves contraction

iii-) Leibniz $\mathcal{L}_u (T \otimes S) = (\mathcal{L}_u T) \otimes S + T \otimes (\mathcal{L}_u S)$

iv-) $\mathcal{L}_u f = U(f)$, $\mathcal{L}_u V = [U, V] = UV - VU$

$$\mathcal{L}_u T^\alpha{}_\beta = U^\gamma \partial_\gamma T^\alpha{}_\beta - T^\gamma{}_\beta \partial_\gamma U^\alpha + T^\alpha{}_\gamma \partial_\beta U^\gamma$$

Symmetries

$\mathcal{L}_u T = 0$

u determines a special direction so that "T does not change"

special: $\mathcal{L}_u g = 0$: isometry

c-) Differential Forms

Definition: A differential p-form ω is a totally anti-symmetric tensor of type $(0,p)$

$$\omega_{\alpha_1 \dots \alpha_p} = \omega_{[\alpha_1 \dots \alpha_p]} = \frac{1}{p!} \sum_{\text{perm } \pi} \text{sign}(\pi)\, \omega_{\alpha_{\pi(1)} \dots \alpha_{\pi(p)}}$$

Ex: $\omega_{[\alpha\beta]} = \frac{1}{2} (\omega_{\alpha\beta} - \omega_{\beta\alpha})$

Hence, a differential form is anti-symmetric under exchange of any two indices

We shall denote Λ^p_x a vector space of p-forms at x.

Choosing p different indices out of n: $\dim \Lambda^p_x = \binom{n}{p}$

Definition: A wedge product $\wedge : \Lambda^p_x \times \Lambda^q_x = \Lambda^{p+q}_x$

$$(\omega \wedge \nu)_{\alpha_1 \dots \alpha_p \beta_1 \dots \beta_q} = \frac{(p+q)!}{p!\, q!} \omega_{[\alpha_1 \dots \alpha_p} \nu_{\beta_1 \dots \beta_q]}$$

$$(\omega \wedge \nu) = (-1)^{pq} (\nu \wedge \omega)$$

In coordinate basis dx^α

$$\omega = \frac{1}{p!} \omega_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$$

- For any vector field V we define an inner derivative

$i_V : \Lambda^p \to \Lambda^{p-1}$

$i_V \omega = V \lrcorner \omega = V \cdot \omega = \omega(\underbrace{V, \cdot, \dots, \cdot}_{p-1 \text{ empty slot}})$; $(V \lrcorner \omega)_{\alpha_1 \dots \alpha_{p-1}} = V^\beta \omega_{\beta \alpha_1 \dots \alpha_{p-1}}$

Properties

i-) i_V is linear, linear in V: $i_{fV + gW} = f i_V + g i_W$

ii-) Leibniz: $i_V(\omega \wedge \nu) = (i_V \omega) \wedge \nu + (-1)^p \omega \wedge (i_V \nu)$: graded Liebniz rule

iii-) $i_V i_W + i_W i_V = 0$: especially $i_V^2 = 0$

$V^{(\alpha} V^{\beta)} \omega_{\alpha\beta \dots} = 0$

Definition: Exterior derivative $d: \Lambda^r \to \Lambda^{r+1}$ defined as follows

i-) On a function f we have $d: f \to df = \partial_\alpha f \, dx^\alpha$

ii-) On a p-form we then have

$$d: \omega \to d\omega = \frac{1}{p!} d\omega_{\alpha_1 \dots \alpha_p} \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$$

That is $(d\omega)_{\alpha_1 \dots \alpha_{p+1}} = (p+1) \, \partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{p+1}]}$

Note that $d^2 = 0$; $\partial_{[\alpha} \partial_\beta \omega_{\dots]} = 0$

Definition: A p-form α is closed when $d\alpha = 0$. It is exact when $\alpha = d\beta$. Any closed form α can be written as $\alpha = d\beta$ locally but not globally (condition of closed implies exact)

Exact implies closed.

- Cartan's Lemma: For a vector field V and a p-form ω, we have the following identity: $\mathcal{L}_V \omega = V \lrcorner d\omega + d(V \lrcorner \omega)$

In particular, this implies that

$$\mathcal{L}_V df = V \lrcorner \underbrace{d^2 f}_{0} + d(V \lrcorner df) = d(V.f) = d\mathcal{L}_V f \quad : \text{check HW 2.2}$$

$$=$$

- Integration of forms: A p-form ω can be integrated over a p-dimensional (sub) manifold

$$\omega = f \, dx^1 \wedge \dots \wedge dx^p$$

$$\int_{O_p} \omega = \int_{\psi(O_p)} f \, dx^1 \dots dx^p \quad : \text{where rhs is defined as Lebesgue integral standart integral}$$

Note that this definition is independent of coordinates, as we have

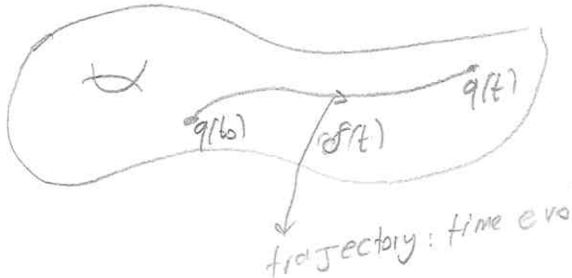
$$\omega = f' dx'^1 \wedge dx'^2 \dots \wedge dx'^p \quad , \; f' = f \det\left(\frac{\partial x^\mu}{\partial x'^\nu} \right)$$

- Stokes Theorem:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

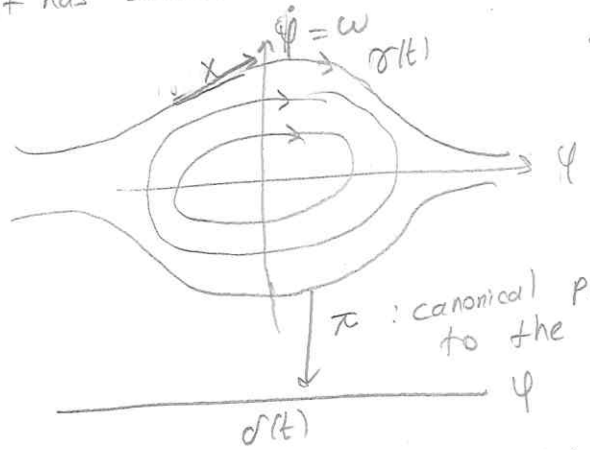
d-) Hints on Geometric Formulation of Lagrangian Mechanics

- A <u>configuration space</u> C with n degree of freedom is a manifold of dimension n, equipped with local coordinates $(q_1, \dots q^n)$



"Points" of $\delta(t)$ describe a "photo of the system" at a given time

- A <u>velocity phase space</u> is a <u>tangent bundle</u> over configuration space <u>(TC)</u>. It has dimensions $2n$ and generalized coordinates $(q^1, \dots q^n, \dot{q}^1 \dots \dot{q}^n)$



- "Points" of $\gamma(t)$ describe a "physical state of the system" at a given time

A Lagrangian $L : TC \to \mathbb{R}$ determines the dynamics of the system. The dynamics is equally encoded in <u>dynamical vector field X</u> $\in T(TC)$ which determines integral curves $\gamma(t)$ on TC. Our aim is to find system's trajectory

$$\delta(t) = \pi(\gamma(t))$$

where π is the corresponding canonical projection to the configuration space

- <u>Determining the dynamical field</u>: The dynamical field X generates integral curves and hence

$$X = \frac{d}{dt} = \underbrace{\frac{dq^j(t)}{dt}}_{\dot{q}^j(t)} \frac{\partial}{\partial q^j} + \frac{d\dot{q}^j(t)}{dt} \frac{\partial}{\partial \dot{q}^j} \quad \in \underline{T(TC)}$$

Tangent space of the tangent bundle over configuration space

$\dot{q}^j(t)$ $\Rightarrow$ tangent to $\delta(t)$ is a velocity

Definition: Lagrange 1-form: $\theta = \frac{\partial L}{\partial \dot{q}^j} dq^j$

Lagrange symplectic 2-form: $\omega = d\theta$

Lagrange energy: $E = \frac{\partial L}{\partial \dot{q}^j} \dot{q}^j - L$

Theorem: The physical state of the lagrangian system described by the Lagrangian L is determined from integral curves of X, where X is given by

- $\mathcal{L}_X \theta = dL$ (Euler-Lagrange equation)

Equation for X

Proof: $$\mathcal{L}_X \theta = \underbrace{\mathcal{L}_X \Big(\overbrace{\frac{\partial L}{\partial \dot{q}^j}}^{f} \Big)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = \frac{\partial L}{\partial q^j}} dq^j + \frac{\partial L}{\partial \dot{q}^j} \underbrace{\mathcal{L}_X(dq^j)}_{d(\mathcal{L}_X q^j) = d\dot{q}^j} \text{ : Cartan's Lemma}$$

$$\mathcal{L}_X \theta = \frac{\partial L}{\partial q^j} dq^j + \frac{\partial L}{\partial \dot{q}^j} d\dot{q}^j = dL$$

Corrolary, Equivalently, we may write

- $X \circ \omega = X \lrcorner \, \omega = -dE$

$$\mathcal{L}_X \theta = X \lrcorner \underbrace{d\theta}_{\omega} + d(X \lrcorner \, \theta) \Rightarrow X \lrcorner \, \omega = \underbrace{\mathcal{L}_X \theta}_{dL} - d(X \lrcorner \, \theta)$$

$X \lrcorner \, \theta$: X is a vector field and θ is one form, so X will eat the one-form

since $\theta = A_j dq^j$ we just need $X = B^i \partial_i$

$$X \lrcorner \, \theta = \left(\dot{q}^j \frac{\partial}{\partial q^j}\right) \cdot \left(\frac{\partial L}{\partial \dot{q}^j} dq^i \right) = \dot{q}^j \frac{\partial L}{\partial \dot{q}^i} \delta^i_j = \dot{q}^j \frac{\partial L}{\partial \dot{q}^j}$$

$$X \lrcorner \, \omega = dL - d\left(\dot{q}^j \frac{\partial L}{\partial \dot{q}^j}\right) = -dE$$

Formal solution for X: Define Ω as the "inverse of ω": $\omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta_\alpha^\gamma$.

The corrolary reads $X^\alpha \omega_{\alpha\beta} = (-dE)_\beta$. Let us now multiply both sides by $\Omega^{\beta\gamma}$

$$X^\gamma = (-dE)_\beta \Omega^{\beta\gamma} \Leftrightarrow X = -dE \cdot \Omega$$

Conservation Laws : Let the system admits

i-) Lagrangian

ii-) Dynamical vector field X and associated diffeomorphism ϕ_t^X

iii-) Extra field Z together with associated diffeomorphism ϕ_t^Z

We then have

Noether's Theorem (V4: Lagrangian Formulation)

Let $\mathcal{L}_Z L = 0$, then $I = Z \lrcorner \theta$ is an integral of motion, i.e. $\mathcal{L}_X I = 0$

* Proof: one has

$$\mathcal{L}_X I = \mathcal{L}_X (Z \lrcorner \theta) = (\mathcal{L}_X Z) \lrcorner \theta + \underbrace{Z \lrcorner (\mathcal{L}_X \theta)}_{Z \lrcorner dL = \mathcal{L}_Z L = 0} = (\mathcal{L}_X Z) \lrcorner \theta = [X, Z] \lrcorner \theta$$

Note that this formulation naturally works on the velocity phase space. When we want to describe symmetries of the configuration space, the following elaborate construction is necessary.

Let Z be a natural extension of y that generates point transformations on C

$$y: q^j \to q^j_\epsilon = \phi_\epsilon(q^j) \quad \text{i.e.} \quad y^i = \left.\frac{dq^i_\epsilon}{d\epsilon}\right|_{\epsilon=0}$$

$$Z: (q^j, \dot{q}^j) \to (q^j_\epsilon, \dot{q}^j_\epsilon) = (\phi_\epsilon(q^j), \phi^*_\epsilon(\dot{q}^j))$$

that is

$$Z = y^i \frac{\partial}{\partial q^i} + \dot{y}^i \frac{\partial}{\partial \dot{q}^i}, \quad \dot{y}^i = \frac{dy(q^\ell)}{dt} = \dot{q}^\ell \frac{\partial y^i}{\partial q^\ell}$$

The fact that the restriction to "point transformations" is 'artificial' and nowhere needed in the Noether's theorem, hints on the fact that one can have more general phase space symmetry.

b-)

$$f'(x') = f(x), \quad V'^{\mu'}(x') = \underbrace{\frac{\partial x'^{\mu'}}{\partial x^{\nu}}}_{A^{\mu'}{}_{\nu}} V^{\nu}(x), \quad w'_{\mu'}(x') = \underbrace{\frac{\partial x^{\nu}}{\partial x'^{\mu'}}}_{B^{\nu}{}_{\mu'}} w_{\nu}(x)$$

$$(AB)^{\mu'}{}_{k'} = A^{\mu'}{}_{\nu} B^{\nu}{}_{k'} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{k}} = \delta^{\mu}_{k}$$

$A^{\mu}{}_{\nu}$, $B^{\nu}{}_{\mu}$ are transformation matrices

$$(BA)^{\mu}{}_{k} = B^{\mu}{}_{\nu} A^{\nu}{}_{k} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{k}} = \delta^{\mu}_{k}$$

$$AB = BA = I$$

The transformations of vectors and 1-forms are inverse

c-)

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$ $\Rightarrow$ $r = (x^2+y^2)^{1/2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta + \cancel{\frac{\partial z}{\partial x}\partial_z} = \frac{x}{r}\partial_r - \frac{y}{x^2}\cdot\frac{1}{1+\frac{y^2}{x^2}}\partial_\theta = \cos\theta\,\partial_r - \frac{y}{x^2+y^2}\partial_\theta$$

$$= \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta$$

$$\partial_y = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta$$

$$\partial_z = \partial_z$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\theta}d\theta = \cos\theta\,dr - r\sin\theta\,d\theta$$

$$dy = \sin\theta\,dr + r\cos\theta\,d\theta$$

$$dz = dz$$

- $f = \underbrace{x^2+y^2}_{r^2} + z^2 = r^2+z^2$

- $V = x\partial_x + y\partial_y + z\partial_z = (r\cos\theta)\cdot\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right) + (r\sin\theta)\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right) + z\partial_z$

$$= r\cos^2\theta\,\partial_r - \cancel{\sin\theta\cos\theta\,\partial_\theta} + r\sin^2\theta\,\partial_r + \cancel{\sin\theta\cos\theta\,\partial_\theta} + z\partial_z$$

$$= r\,\partial_r + z\,\partial_z$$

$$V^{\mu}(x,y,z) \longleftrightarrow V'^{\mu}(x') = (r, 0, z)$$

- $w = f(y\,dx - x\,dy) = -(r^2+z^2)\,r^2\,d\theta$

$$w_{\mu}(x) = (fy, -fx, 0) \longleftrightarrow w_{\mu'}'(x') = (0, -(r^2+z^2)r^2, 0)$$

$g = dx \otimes dx + dy \otimes dy$ ...

$= (\cos\theta dr - r\sin\theta d\theta) \otimes (\cos\theta dr - r\sin\theta d\theta) + (\sin\theta dr + r\cos\theta d\theta) \otimes (\ldots)$

$= \cos^2\theta dr \otimes dr - r\sin\theta\cos\theta d\theta \otimes dr - r\sin\theta\cos\theta d\theta \otimes dr + r^2\sin^2\theta d\theta \otimes d\theta$

$+ \sin^2\theta dr \otimes dr$ ..

$= dr \otimes dr + r^2 d\theta \otimes d\theta + dz \otimes dz$

d-) $dx^\mu \partial_{x^\nu} = \delta^\mu_\nu \Rightarrow dx\, \partial_x = 1 = dy\, \partial_y = dz\, \partial_z$

$dx\, \partial_y = 0$ ..

- $(V.\omega) = (x\partial_x + y\partial_y + z\partial_z).f(y\,dx - x\,dy)$
  $= f(xy) - f(yx) + 0 = 0$
- $(V \otimes \omega)^x{}_y = V^x \otimes \omega_y = x(-fx) = -fx^2$
- $(V \otimes \omega)^y{}_x = V^y \otimes \omega_x = y(fy) = fy^2$
- $f \otimes \omega \otimes \omega = f^3(y\,dx - x\,dy) \otimes (y\,dx - x\,dy)$
  $= f^3(y^2 dx \otimes dx - yx\, dx \otimes dy - xy\, dy \otimes dx + x^2 dy \otimes dy)$
- $V.g = g(V,\cdot) = (x\partial_x + y\partial_y + z\partial_z).(dx \otimes dx + dy \otimes dy + dz \otimes dz)$
  $= x\,dx + y\,dy + z\,dz$

e-) A tangent vector V_γ to a curve γ at a point $p \in M$ is a map $V_\gamma : F \to \mathbb{R}$ defined as

$$V_\gamma(f) = \frac{df}{dt} = \frac{df(\gamma(t))}{dt} = \lim_{s \to 0} \frac{f(\gamma(t+s)) - f(\gamma(t))}{s} = \frac{dx^\mu}{dt}\frac{df}{dx^\mu}$$

where the last expression is valid in a given coordinate system x^μ where γ is parameterized as $x^\mu(t)$

E^3

cyclindrical: $x^\mu = (r,\theta,z) = (R, t, 0)$, $t \in (0, 2\pi)$

cartesian: $x^\mu = (x, y, z) = (R\cos t, R\sin t, 0)$, $t \in (0, 2\pi)$

The corresponding tangent vector field is

cyclindrical: $V_\gamma = \frac{dx^\mu}{dt}\partial_\mu = 0.\partial r + 1\partial\theta + 0 = \partial\theta$

cartesian : $V_\gamma = \frac{dx^\mu}{dt}\partial_\mu = -R\sin t\,\partial_x + R\cos t\,\partial_y$

b-) • $d^2\omega = 0$, can be calculated for $d^2 f = 0$, too

$$d^2 f = d(df) = d(\partial_\mu f) \wedge dx^\mu$$
$$= \underbrace{\partial_\mu \partial_\nu f}_{\text{symmetric}} \, \underbrace{dx^\mu \wedge dx^\nu}_{\text{Anti-symmetric}} = 0$$

• $$d(\omega \wedge \nu) = \frac{1}{p!q!} d(\omega_{\mu_1 \dots \mu_p} \nu_{\nu_1 \dots \nu_q}) \, dx^{\mu_1} \wedge \dots dx^{\mu_p} \wedge dx^{\nu_1} \wedge \dots dx^{\nu_q}$$
$$= \frac{1}{p!q!} \left[(d\omega_{\mu_1 \dots \mu_p}) \nu_{\nu_1 \dots \nu_q} + \omega_{\mu_1 \dots \mu_p} (d\nu_{\nu_1 \dots \nu_q}) \right] dx^{\mu_1} \wedge \dots dx^{\mu_p} \wedge dx^{\nu_1} \wedge \dots dx^{\nu_q}$$

Indices must match side by side

$$= \frac{1}{p!q!} \left[(\partial_\alpha \omega_{\mu_1 \dots \mu_p}) \nu_{\nu_1 \dots \nu_q} \right] dx^\alpha \wedge dx^{\mu_1} \dots dx^{\mu_p} \wedge dx^{\nu_1} \wedge \dots dx^{\nu_q}$$
$$+ \frac{1}{p!q!} \left[\omega_{\mu_1 \dots \mu_p} (\partial_\alpha \nu_{\nu_1 \dots \nu_q}) \right] \underbrace{dx^\alpha \wedge dx^{\mu_1} \dots dx^{\mu_p}}_{\text{Jumps } p \text{ times}} \wedge dx^{\nu_1} \wedge \dots dx^{\nu_q}$$

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^p \, \omega \wedge d\nu$$

c-) • $$(\omega \wedge \nu) = \frac{1}{p!q!} \omega_{\alpha_1 \dots \alpha_p} \nu_{p+1 \dots p+q} \, dx^{\alpha_1} \wedge \dots dx^{\alpha_p} \wedge dx^{\alpha_{p+1}} \wedge \dots dx^{\alpha_{p+q}}$$
$$= \frac{1}{(p+q)!} (\omega \wedge \nu)_{\alpha_1 \dots \alpha_p \dots \alpha_{p+q}} \, dx^{\alpha_1} \wedge \dots \cancel{dx^{\alpha_p}} \wedge dx^{\alpha_{p+1}} \wedge \dots dx^{\alpha_{p+q}}$$

$$(\omega \wedge \nu)_{\alpha_1 \dots \alpha_{p+q}} = \frac{(p+q)!}{p!q!} \cdot \omega_{[\alpha_1 \dots \alpha_p} \nu_{\alpha_{p+1} \dots \alpha_{p+q}]}$$

• $$d\omega = \frac{1}{(p+1)!} (d\omega)_{\alpha_1 \dots \alpha_{p+1}} \, dx^{\alpha_1} \wedge \dots dx^{\alpha_{p+1}} = \frac{1}{p!} \underbrace{d(\omega_{\alpha_1 \dots \alpha_p})}_{\partial_\mu (\omega_{\alpha_1 \dots \alpha_p}) dx^\mu} dx^{\alpha_1} \wedge \dots dx^{\alpha_p}$$ then $\mu \to \alpha_1$, $\alpha_1 \to \alpha_2$, $\alpha_p \to \alpha_{p+1}$.

$$(d\omega)_{\alpha_1 \dots \alpha_{p+1}} = \underbrace{\frac{(p+1)!}{p!}}_{(p+1)} \left(\partial_{[\alpha_1} \omega_{\alpha_2 \dots \alpha_{p+1}]} \right)$$

• $$(V \lrcorner \omega) = \frac{1}{1!p!} (V^\rho \omega_{\beta \alpha_1 \dots \alpha_{p-1}}) \, \overbrace{\partial_\rho \lrcorner dx^\beta}^{\delta^\beta_\rho} \wedge dx^{\alpha_1} \wedge \dots dx^{\alpha_{p-1}}$$
$$= \frac{1}{(p-1)!} (V \lrcorner \omega)_{\alpha_1 \dots \alpha_{p-1}} \, dx^{\alpha_1} \wedge \dots dx^{\alpha_{p-1}}$$

$$(V \lrcorner \omega)_{\alpha_1 \dots \alpha_{p-1}} = \underbrace{\frac{(p-1)!}{p!}}_{1/p} (V^\beta \omega_{\beta \alpha_1 \dots \alpha_{p-1}})$$