Supergravity Solution Manual

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This manual provides clear, step-by-step solutions to key exercises and derivations encountered in the study of Supergravity. Intended as a companion to standard textbooks, it aims to support graduate students and researchers by offering detailed calculations, conceptual insights, and helpful commentary on this advanced field unifying supersymmetry and general relativity.

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1 Chapter 1

1.1

To verify that the translation map

$$\phi^i(x) \to \phi'^i(x) = \phi^i(x+a)$$

leaves the action invariant, where a^μ is a constant vector, we substitute the transformed field into the action. The action for a scalar field is given by

$$S[\phi^i(x)] = \int d^D x, \mathcal{L}(x) = -\frac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i(x)\partial_\nu\phi^i(x) + m^2\phi^i(x)\phi^i(x)\right].$$

After applying the transformation, the action becomes

$$\begin{aligned} S[\phi'^i(x)] &= -\frac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi'^i(x)\partial_\nu\phi'^i(x) + m^2\phi'^i(x)\phi'^i(x)\right] \\ &= -\frac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i(x+a)\partial_\nu\phi^i(x+a) + m^2\phi^i(x+a)\phi^i(x+a)\right]. \end{aligned}$$

Since a^μ is a constant vector, we can perform a change of variables $y = x + a$, for which $d^D y = d^D x$. Applying this change of variables to the action, we obtain

$$\begin{aligned} S[\phi'^i(x)] &= -\frac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i(x+a)\partial_\nu\phi^i(x+a) + m^2\phi^i(x+a)\phi^i(x+a)\right] \\ &= -\frac{1}{2}\int d^D y \left[\eta^{\mu\nu}\partial_\mu\phi^i(y)\partial_\nu\phi^i(y) + m^2\phi^i(y)\phi^i(y)\right]. \end{aligned}$$

Relabeling the integration variable $y \to x$, we see that

$$\boxed{S[\phi'^i(x)] = S[\phi^i(x)]}$$

Therefore, the action is invariant under global spacetime translations.

1.2

The action with the Lagrangian density of interest is given by

$$S = \int d^D x\,\mathcal{L} \;=\; -\frac{1}{2}\int d^D x \left[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i \;+\; m^2\phi^i\phi^i\right] \tag{1}$$

where the repeated index i is also summed in addition to the usual Einstein summation convention. Let us first consider the transformation given in **Exercise 1.1**, namely the spacetime trasnlation is given by

$$\phi^i \to \phi'^i(x) = \phi^i(x+a)\,. \tag{2}$$

In order to see this transformation more explicitly, we are going to Fourier expand the field around x. We are going to do this up to first order in a, since it suffices to show that the Lagrangian density will not remain invariant. The higher order derivatives will also be unrelated to the term that will be considered, so this verification will be well defined.

$$\phi'^i(x) = \phi^i(x+a) = \phi^i(x) + a^\mu \partial_\mu \phi^i(x) + \mathcal{O}(a^2)\,. \tag{3}$$

Under this transformation we have

$$\mathcal{L}' = \mathcal{L}[\phi'] = -\frac{1}{2}\int d^Dx\big[\,\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i + m^2\phi^i\phi^i \\ + a^\sigma\left(\eta^{\mu\nu}\partial_\sigma\partial_\mu\phi^i\,\partial_\nu\phi^i + \eta^{\mu\nu}\partial_\mu\phi^i\,\partial_\nu\partial_\sigma\phi^i + 2\,m^2\phi^i\,\partial_\sigma\phi^i\right) + \mathcal{O}(a^2)\big]$$

Hence one can see that the given Lagrangian denstiy fails to remain invariant under the spacetime translation. Let us now consider $SO(N)$ symmetry, which acts as follows

$$\phi(x) \to \phi'(x) = R^i{}_j\,\phi^j(x)\,.$$

First we must note that we are going to use $\phi^i\delta_{ij}\phi^j$ for summation convention in latin indices in order to see more clearly. After this, all one need to do is realize that the rotation matrices do not get affected by derivatives. After this we simply have

$$\phi^i\phi^i = \phi^i\delta_{ij}\phi^j \to \phi'^i\delta_{ij}\phi'^j = R^i{}_k\phi^k(x)\,\delta_{ij}\,R^j{}_l\phi^l(x) = \phi^k\delta_{kl}\phi^l$$

since

$$R^i{}_k\,\delta_{ij}\,R^j{}_l = \delta_{kl}\,.$$

And hence this verifies that the given Lagrangian density remains invariant under $SO(N)$ internal symmetry.

1.3

i-)

$$\Lambda^\mu_\rho\eta_{\mu\nu}\Lambda^\nu_\sigma = \eta_{\rho\sigma}$$

To write down above equation in a compact form, each index must match with the next index, by taking the transpose of the first Λ, it gives us

$$(\Lambda^T)^\mu_\rho\eta_{\mu\nu}\Lambda^\nu_\sigma = \eta_{\rho\sigma}$$

and its compact form is

$$\Lambda^T\eta\Lambda = \eta$$

We know that inverse of the metric $\eta_{\mu\nu}$ is $\eta^{\mu\nu}$. Minkowskian metric is (-,+,+,+ ...) in our convention. So $\eta^2 = 1$.

By multiplying above equation from the left side with η^{-1} we get

$$\eta^{-1}\Lambda^T\eta\Lambda = 1$$

$$\eta^{-1}\Lambda^T\eta = \Lambda^{-1}$$

$$\eta^{\mu\nu}\Lambda^\sigma_\nu\eta_{\sigma\rho} = (\Lambda^{-1})^\mu_\rho$$

multiplying the last equation with $\eta_{\mu\tau}$ we get,

$$\Lambda^\sigma_\tau\eta_{\sigma\rho} = \eta_{\mu\tau}(\Lambda^{-1})^\mu_\rho$$

$$\Lambda_{\rho\tau} = (\Lambda^{-1})_{\tau\rho}$$

ii-) Multiplying the answer of the **i** by $\eta^{\mu\rho}$ we get

$$\Lambda_{\rho\tau} = (\Lambda^{-1})_{\tau\rho}$$
$$\Lambda^{\mu}_{\tau} = (\Lambda^{-1})^{\mu}_{\tau}$$

iii-)

$$x'^{\mu} = (\Lambda^{-1})^{\mu}_{\nu} x^{\nu}$$
$$x^{\mu} = \Lambda^{\mu}_{\nu} x'^{\nu}$$

are given in the book. Multiplying the first equation with $\eta_{\mu\rho}$ we get

$$x'_{\rho} = (\Lambda^{-1})_{\rho\nu} x^{\nu} = (\Lambda^{-1})^{\nu}_{\rho} x_{\nu}$$

and using the result of **ii**

$$x'_{\rho} = \Lambda^{\nu}_{\rho} x_{\nu}$$

we can change the position of the Λ because what we are changing is a number not the matrices itself.

$$x'_{\rho} = (\Lambda^{-1})^{\nu}_{\rho} x_{\nu} = x_{\nu} \Lambda^{\nu}_{\rho}$$

1.4

In any Lie Algebra it is possible to write the commutator of two basis elements as

$$[X_a, X_b] = f_{ab}{}^{c} X_c$$

for the Lorentz algebra the expression above can be written as

$$\left[m_{[\mu\nu]}, m_{[\rho\sigma]}\right] = f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} m_{[\kappa\tau]}$$

So equation 1.34 becomes

$$f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} m_{[\kappa\tau]} = \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]}$$

We know that $m_{[\alpha\beta]} = \frac{1}{2}(m_{\alpha\beta} - m_{\beta\alpha})$. We can convert this equation to

$$m_{[\mu\sigma]} = \frac{1}{2}(\delta^{\kappa}_{\mu}\delta^{\tau}_{\sigma} - \delta^{\tau}_{\mu}\delta^{\kappa}_{\sigma}) m_{\kappa\tau}$$

If we combine the expressions we found, we get the following.

$$\begin{aligned} f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} m_{[\kappa\tau]} &= \frac{1}{2}\left[\eta_{\nu\rho}(\delta^{\kappa}_{\mu}\delta^{\tau}_{\sigma} - \delta^{\tau}_{\mu}\delta^{\kappa}_{\sigma}) m_{\kappa\tau} - \eta_{\mu\rho}(\delta^{\kappa}_{\nu}\delta^{\tau}_{\sigma} - \delta^{\tau}_{\nu}\delta^{\kappa}_{\sigma}) m_{\kappa\tau} - \eta_{\nu\sigma}(\delta^{\kappa}_{\mu}\delta^{\tau}_{\rho} - \delta^{\tau}_{\mu}\delta^{\kappa}_{\rho}) m_{\kappa\tau} + \eta_{\mu\sigma}(\delta^{\kappa}_{\nu}\delta^{\tau}_{\rho} - \delta^{\tau}_{\nu}\delta^{\kappa}_{\rho}) m_{\kappa\tau}\right] \\ &= \frac{1}{2}\left[\eta_{\nu\rho}(\delta^{\kappa}_{\mu}\delta^{\tau}_{\sigma} - \delta^{\tau}_{\mu}\delta^{\kappa}_{\sigma}) - \eta_{\mu\rho}(\delta^{\kappa}_{\nu}\delta^{\tau}_{\sigma} - \delta^{\tau}_{\nu}\delta^{\kappa}_{\sigma}) - \eta_{\nu\sigma}(\delta^{\kappa}_{\mu}\delta^{\tau}_{\rho} - \delta^{\tau}_{\mu}\delta^{\kappa}_{\rho}) + \eta_{\mu\sigma}(\delta^{\kappa}_{\nu}\delta^{\tau}_{\rho} - \delta^{\tau}_{\nu}\delta^{\kappa}_{\rho})\right] m_{\kappa\tau} \end{aligned}$$

Equation 1.30 shows that $m_{\mu\nu} = -m_{\nu\mu}$ it is easy to see that $m_{\mu\nu} = m_{[\mu\nu]}$, Thus we can simplify $m_{\kappa\tau}$ from each side of the equation. So the expression becomes

$$\begin{aligned} f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} &= \frac{1}{2}\left[\eta_{\nu\rho}(\delta^\kappa_\mu\delta^\tau_\sigma - \delta^\tau_\mu\delta^\kappa_\sigma) - \eta_{\mu\rho}(\delta^\kappa_\nu\delta^\tau_\sigma - \delta^\tau_\nu\delta^\kappa_\sigma) - \eta_{\nu\sigma}(\delta^\kappa_\mu\delta^\tau_\rho - \delta^\tau_\mu\delta^\kappa_\rho) + \eta_{\mu\sigma}(\delta^\kappa_\nu\delta^\tau_\rho - \delta^\tau_\nu\delta^\kappa_\rho)\right] \\ &= \eta_{\nu\rho}\delta^\kappa_{[\mu}\delta^\tau_{\sigma]} - \eta_{\mu\rho}\delta^\kappa_{[\nu}\delta^\tau_{\sigma]} - \eta_{\nu\sigma}\delta^\kappa_{[\mu}\delta^\tau_{\rho]} + \eta_{\mu\sigma}\delta^\kappa_{[\nu}\delta^\tau_{\rho]} \\ &= \eta_{\nu\rho}\delta^\kappa_{[\mu}\delta^\tau_{\sigma]} - \eta_{\mu\rho}\delta^\kappa_{[\nu}\delta^\tau_{\sigma]} - \eta_{\nu\sigma}\delta^\kappa_{[\mu}\delta^\tau_{\rho]} + \eta_{\mu\sigma}\delta^\kappa_{[\nu}\delta^\tau_{\rho]} \end{aligned}$$

To pass from the explicit form of the structure constants (as in equation 1.34) to the compact form (1.35), notice that each term can be rewritten as a combination of Kronecker deltas and metric tensors, with antisymmetrization in the relevant pairs of indices. By expanding the antisymmetrization in (1.35), you recover all terms present in the explicit commutator. The antisymmetrized notation provides a manifestly covariant and compact way to encode all index symmetries, with a factor of 8 from the total number of antisymmetrizations over three index pairs. The explicit and compact forms are equivalent, and the compact form is especially useful for group-theoretic manipulations. Thus the expression becomes,

$$\boxed{f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} = 8\eta_{[\nu|[\rho}\delta^{[\kappa}_{\mu]}\delta^{\tau]}_{\sigma]}}$$

1.5

First of all, let us clarify the fact that the $\Lambda \in SO^+(D-1,1)$ since the Jacobian that will appear due to component change must be 1. As an extra one can see this Jacobian term using forms, it follows as

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D} = \epsilon^{\mu_1\cdots\mu_D} d^D x$$

then by realizing

$$d(x'^\mu) = d(\Lambda^\mu{}_\nu x^\nu) = \Lambda^\mu{}_\nu \, d(x^\nu)\,,$$

we have

$$dx'^{\mu_1} \wedge \cdots \wedge dx'^{\mu_D} = \epsilon^{\nu_1\cdots\nu_D}\Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_D}{}_{\nu_D} d^D x = det(\Lambda)\epsilon^{\mu_1\cdots\mu_D} d^D x = \epsilon^{\mu_1\cdots\mu_D} d^D x'\ ,$$

hence by comparison one can see the Jacobi determinant arising. Also note that from

$$\Lambda^\mu{}_\rho \eta_{\mu\nu}\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

This identity defines the Lorentz group $O(D-1,1)$, whose elements preserve the Minkowski metric. From this, one can show that the determinant of Λ satisfies

$$\det(\Lambda)^2 = 1 \quad \Rightarrow \quad \det(\Lambda) = \pm 1,$$

so the Lorentz group includes both proper and improper transformations. However, to ensure invariance of the action and orientation-preserving coordinate transformations, we restrict to the **proper Lorentz group**, where $\det(\Lambda) = +1$.

Now consider the transformation of the kinetic term. Recall that under the transformation

$$\phi'^i(x) = \phi^i(\Lambda x),$$

we have:

$$\partial_\mu \phi'^i(x) = \frac{\partial}{\partial x^\mu}\phi^i(\Lambda x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial \phi^i(x')}{\partial x'^\rho} = \Lambda^\rho{}_\mu \partial'_\rho \phi^i(x').$$

Then the kinetic term transforms as:

$$\begin{aligned}\eta^{\mu\nu}\partial_\mu\phi'^i(x)\,\partial_\nu\phi'^i(x) &= \eta^{\mu\nu}\Lambda^\rho{}_\mu\partial'_\rho\phi^i(x')\cdot\Lambda^\sigma{}_\nu\partial'_\sigma\phi^i(x') \\ &= \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu\eta^{\mu\nu}\,\partial'_\rho\phi^i(x')\partial'_\sigma\phi^i(x') \\ &= \eta^{\rho\sigma}\,\partial'_\rho\phi^i(x')\partial'_\sigma\phi^i(x') \quad \text{(by Lorentz invariance of } \eta^{\mu\nu}\text{)}\end{aligned}$$

Thus, the kinetic term is also invariant and since only dummy variables of integral change without any additional or missing terms action remains invariant.

1.6

i-) It is given that

$$L_{[\rho\sigma]} = x_\rho\partial_\sigma - x_\sigma\partial_\rho$$

To compute the commutator of the given differantial operator one should check for the appendix $\partial_\mu x_\nu = \eta_{\mu\nu}$.
The question asks for

$$[L_{[\mu\nu]}, L_{[\rho\sigma]}] = L_{[\mu\nu]}L_{[\rho\sigma]} - L_{[\rho\sigma]}L_{[\mu\nu]}$$

Lets compute the first term, do not forget to multiply the commutator with a test function 'f' to not to have any vanishing term due to derivatives. The first term is

$$L_{[\mu\nu]}L_{[\rho\sigma]}f = (x_\mu\partial_\nu - x_\nu\partial_\mu)(x_\rho\partial_\sigma - x_\sigma\partial_\rho)f$$

$$= (x_\mu\partial_\nu - x_\nu\partial_\mu)(x_\rho\partial_\sigma f - x_\sigma\partial_\rho f)$$

$$= x_\mu(\eta_{\nu\rho}\partial_\sigma f + x_\rho\partial_\nu\partial_\sigma f - \eta_{\nu\sigma}\partial_\rho f - x_\sigma\partial_\nu\partial_\rho f)$$

$$-x_\nu(\eta_{\mu\rho}\partial_\sigma f + x_\rho\partial_\mu\partial_\sigma f - \eta_{\mu\sigma}\partial_\rho f - x_\sigma\partial_\mu\partial_\rho f)$$

and the second term can be found by changing the indices as $\mu \leftrightarrow \rho$ and $\nu \leftrightarrow \sigma$. Do not forget to multiply the second term with a minus because of the commutator.
The terms which includes two derivatives cancel out because partial derivatives commute. Finally we get

$$[L_{[\mu\nu]}, L_{[\rho\sigma]}] = \eta_{\nu\rho}x_\mu\partial_\sigma - \eta_{\nu\sigma}x_\mu\partial_\rho - \eta_{\mu\rho}x_\nu\partial_\sigma + \eta_{\mu\sigma}x_\nu\partial_\rho$$

$$-\eta_{\sigma\mu}x_\rho\partial_\nu + \eta_{\sigma\nu}x_\rho\partial_\mu + \eta_{\rho\mu}x_\sigma\partial_\nu - \eta_{\rho\nu}x_\sigma\partial_\mu$$

Using the given definition of the L, and remembering the metric is symmetric, lets match the terms.
1. with 8.
4. with 5.
2. with 6.
3. with 7.
and we get

$$[L_{[\mu\nu]}, L_{[\rho\sigma]}] = \eta_{\nu\rho}L_{[\mu\sigma]} + \eta_{\mu\sigma}L_{[\nu\rho]} - \eta_{\nu\sigma}L_{[\mu\rho]} - \eta_{\mu\rho}L_{[\nu\sigma]}$$

What we found is in the same form as (1.34) in the book.
ii-) Book defines the transformation of the scalar field in (1.37).
It is also given as $\Lambda^\mu_\nu = \delta^\mu_\nu + \lambda^\mu_\nu$

$$\phi'(x^\mu) = \phi(\Lambda x) = \phi(x^\mu + \lambda^{\mu\nu}x_\nu) = \phi(x^\mu) + \lambda^{\rho\sigma}x_\sigma\partial_\rho\phi(x^\mu)$$

To anti-symmetrize the $\lambda^{\rho\sigma}x_\sigma\partial_\rho$ one can divide by 2 and get

$$\lambda^{\rho\sigma}x_\sigma\partial_\rho = \frac{1}{2}\lambda^{\rho\sigma}x_\sigma\partial_\rho + \frac{1}{2}\lambda^{\rho\sigma}x_\sigma\partial_\rho$$

and takes the second term's indices $\rho \leftrightarrow \sigma$ and $\sigma \leftrightarrow \rho$. And gets the second term as

$$\frac{1}{2}\lambda^{\sigma\rho}x_\rho\partial_\sigma = -\frac{1}{2}\lambda^{\rho\sigma}x_\rho\partial_\sigma$$

$$\begin{aligned}
\phi(x^\mu + \lambda^{\mu\nu}x_\nu) &= \phi(x^\mu) + \frac{1}{2}\lambda^{\rho\sigma}(x_\sigma\partial_\rho - x_\rho\partial_\sigma)\phi(x^\mu) \\
&= \phi(x^\mu) - \frac{1}{2}\lambda^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\phi(x^\mu) \\
\phi(x^\mu + \lambda^{\mu\nu}x_\nu) &= \phi(x^\mu) - \frac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}\phi(x^\mu)
\end{aligned}$$

1.7

In equation 1.40 it's shown that $U(\Lambda) \equiv e^{-\frac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}}$. For an infinitesimal transformation:

$$\phi'(x) = U(\Lambda)\phi^i(x) \approx \phi^i(x) - \frac{1}{2}\lambda^{\mu\nu}L_{[\mu\nu]}\phi^i(x)$$

So the product of $U(\Lambda_1)$ and $U(\Lambda_2)$ becomes

$$\begin{aligned}
U(\Lambda_1)U(\Lambda_2)\phi^i(x) &\approx (1 - \frac{1}{2}\lambda_1^{\rho\sigma}L_{[\rho\sigma]})(1 - \frac{1}{2}\lambda_2^{\mu\nu}L_{[\mu\nu]})\phi^i(x) \\
&= (1 - \frac{1}{2}\lambda_1^{\rho\sigma}L_{[\rho\sigma]} - \frac{1}{2}\lambda_2^{\mu\nu}L_{[\mu\nu]} + \frac{1}{4}\lambda_2^{\mu\nu}\lambda_1^{\rho\sigma}L_{[\rho\sigma]}L_{[\mu\nu]})\phi^i(x)
\end{aligned}$$

We can apply same think to find $U(\Lambda_2)U(\Lambda_1)$

$$U(\Lambda_2)U(\Lambda_1)\phi^i(x) = (1 - \frac{1}{2}\lambda_1^{\rho\sigma}L_{[\rho\sigma]} - \frac{1}{2}\lambda_2^{\mu\nu}L_{[\mu\nu]} + \frac{1}{4}\lambda_1^{\rho\sigma}\lambda_2^{\mu\nu}L_{[\mu\nu]}L_{[\rho\sigma]})\phi^i(x)$$

Thus the commutator of $U(\Lambda_1)$ and $U(\Lambda_2)$ can be found as

$$\begin{aligned}
[U(\Lambda_2), U(\Lambda_1)]\,\phi^i(x) &= [U(\Lambda_2)U(\Lambda_1) - U(\Lambda_1)U(\Lambda_2)]\,\phi^i(x) \\
&= \frac{1}{4}\lambda_2^{\rho\sigma}\lambda_1^{\mu\nu}L_{[\mu\nu]}L_{[\rho\sigma]}\,\phi^i(x) - \frac{1}{4}\lambda_1^{\mu\nu}\lambda_2^{\rho\sigma}L_{[\rho\sigma]}L_{[\mu\nu]}\,\phi^i(x) \\
&= \frac{1}{4}\lambda_2^{\rho\sigma}\lambda_1^{\mu\nu}\left(L_{[\mu\nu]}L_{[\rho\sigma]} - L_{[\rho\sigma]}L_{[\mu\nu]}\right)\phi^i(x) \\
&= \frac{1}{4}\lambda_2^{\rho\sigma}\lambda_1^{\mu\nu}[L_{[\rho\sigma]}, L_{[\mu\nu]}]\,\phi^i(x)
\end{aligned}$$

We know that in group theory, the product of two infinitesimal transformations gives another infinitesimal transformation, with parameters adding and including commutators at quadratic order. So,

$$[L_{\mu\nu}, L_{\rho\sigma}] = f_{[\mu\nu][\rho\sigma]}{}^{[\alpha\beta]}L_{[\alpha\beta]}$$

Therefore,

$$[U(\Lambda_2), U(\Lambda_1)]\,\phi^i(x) = \frac{1}{4}\lambda_2^{\mu\nu}\lambda_1^{\rho\sigma}f_{[\mu\nu][\rho\sigma]}{}^{[\alpha\beta]}L_{[\alpha\beta]}\phi^i(x)$$

This is precisely the form of another infinitesimal Lorentz transformation, with new "parameters"

$$\Omega^{\alpha\beta} = \frac{1}{2}\lambda_2^{\mu\nu}\lambda_1^{\rho\sigma} f^{[\alpha\beta]}_{[\mu\nu][\rho\sigma]}$$

Thus,

$$\boxed{[U(\Lambda_2), U(\Lambda_1)]\, \phi^i(x) = \frac{1}{2}\Omega^{\alpha\beta} L_{[\alpha\beta]}\phi^i(x)}$$

1.8

Since this operator is differential operator it satisfies Leibniz rule which will result in

$$J_{[\rho\sigma]}(V^\mu W_\mu) = W_\nu J_{[\rho\sigma]} V^\mu + V^\mu J_{[\rho\sigma]} W_\mu = L_{[\rho\sigma]}(V^\mu W_\mu) + W_\mu\, m^{\mu}_{[\rho\sigma]\ \nu} V^\nu + V^\mu\, m_{[\rho\sigma]\mu}{}^{\nu} W_\nu$$

By changing the dummy indices and using the $m^{\mu}_{[\rho\sigma]\ \nu} = -\, m_{[\rho\sigma]\nu}{}^{\mu}$ we have

$$W_\mu\, m^{\mu}_{[\rho\sigma]\ \nu} V^\nu + V^\mu\, m_{[\rho\sigma]\mu}{}^{\nu} W_\nu = m^{\mu}_{[\rho\sigma]\ \nu}(V^\mu W_\mu - V^\mu W_\mu) = 0\,.$$

Hence, only

$$J_{[\rho\sigma]}(V^\mu W_\mu) = L_{[\rho\sigma]}(V^\mu W_\nu)$$

remains, and this concludes the proof.

1.9

i-) We are asked for to prove

$$U((\Lambda)^{-1}a)U(\Lambda) = U(\Lambda)U(a)$$

It can be understood by checking the equation (1.43) which is

$$\phi(\Lambda_1\Lambda_2 x) = U(\Lambda_2)U(\Lambda_1)\phi(x)$$

What we are asked to prove can be done easily by acting the LHS and RHS to a ϕ.
Lets do it for LHS

$$U((\Lambda)^{-1}a)U(\Lambda)\phi(x) = U(\Lambda)\phi(x + \Lambda^{-1}a)$$
$$= \phi(\Lambda x + \Lambda\Lambda^{-1}a) = e^{-\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}}\phi(\Lambda x + a)$$

What we get from the LHS is actually is the same as the equation (1.55) which is the RHS in this case.

ii-) Similar argument goes for the second part of the question but lets take it easy and show it for 2 arbitrary Poincare transformation.

$$U(a_1, \Lambda_1)U(a_2, \Lambda_2) = U(\Lambda_1)U(a_1)U(\Lambda_2)U(a_2)$$

when it acts on $\phi(x)$ we get

$$U(\Lambda_1)U(a_1)U(\Lambda_2)U(a_2)\phi(x) = U(a_1)U(\Lambda_2)U(a_2)\phi(\Lambda_1 x)$$
$$= U(\Lambda_2)U(a_2)\phi(\Lambda_1 x + a_1) = U(a_2)\phi(\Lambda_2\Lambda_1 x + \Lambda_2 a_1)$$
$$= \phi(\Lambda_2\Lambda_1 x + \Lambda_2 a_1 + a_2) = U(\Lambda_2 a_1 + a_2, \Lambda_2\Lambda_1)\phi(x)$$

The questions are

$$U(a)\phi(\Lambda' x + b) = \phi(\Lambda' x + \Lambda' a + b)$$
$$U(\Lambda)\phi(\Lambda' x + b) = \phi(\Lambda' \Lambda x + b)$$

First equation can be written as

$$U(a)U(\Lambda')U(b)\phi(x)$$

which means $a_2 = b, \Lambda_2 = \Lambda'$ and $a_1 = a, \Lambda_1 = 0$. $\Lambda_1 = 0$ is no transformation. So it gives

$$U(a)U(\Lambda')U(b)\phi(x) = \phi(\Lambda' x + \Lambda' a + b)$$

and the second one can be written as

$$U(\Lambda)U(\Lambda')U(b)\phi(x) = \phi(\Lambda' \Lambda x + b)$$

1.10

i-) We know that the generators can be written as

$$J_{[\mu\nu]} = (x_\mu \partial_\nu - x_\nu \partial_\mu)$$

Let $f(x)$ be any function. We want to compute:

$$[J_{[\mu\nu]}, J_{[\rho\sigma]}]\, f(x) = J_{[\mu\nu]} J_{[\rho\sigma]} f(x) - J_{[\rho\sigma]} J_{[\mu\nu]} f(x)$$

Lets solve each term one by one. First of all apply $J_{[\mu\nu]}$ to $J_{[\rho\sigma]} f(x)$:

$$\begin{aligned} J_{[\mu\nu]}(J_{[\rho\sigma]} f(x)) &= (x_\mu \partial_\nu - x_\nu \partial_\mu)\, [(x_\rho \partial_\sigma - x_\sigma \partial_\rho) f(x)] \\ &= \big[x_\mu \partial_\nu (x_\rho \partial_\sigma - x_\sigma \partial_\rho) f(x) - x_\nu \partial_\mu (x_\rho \partial_\sigma - x_\sigma \partial_\rho) f(x)\big] \end{aligned}$$

If we expand the derivatives:

$$\begin{aligned} x_\alpha \partial_\beta (x_\gamma \partial_\delta f) &= x_\alpha \left(\delta_{\beta\gamma} \partial_\delta f + x_\gamma \partial_\beta \partial_\delta f\right) \\ &= x_\alpha \delta_{\beta\gamma} \partial_\delta f + x_\alpha x_\gamma \partial_\beta \partial_\delta f \end{aligned}$$

Therefore,

$$\begin{aligned} J_{[\mu\nu]} J_{[\rho\sigma]} f(x) = \Big\{ \; & x_\mu \delta_{\nu\rho} \partial_\sigma f(x) + x_\mu x_\rho \partial_\nu \partial_\sigma f(x) - x_\mu \delta_{\nu\sigma} \partial_\rho f(x) - x_\mu x_\sigma \partial_\nu \partial_\rho f(x) \\ & - x_\nu \delta_{\mu\rho} \partial_\sigma f(x) - x_\nu x_\rho \partial_\mu \partial_\sigma f(x) + x_\nu \delta_{\mu\sigma} \partial_\rho f(x) + x_\nu x_\sigma \partial_\mu \partial_\rho f(x) \Big\} \end{aligned}$$

Now we can write the commutation. Subtracting the two results, most double-derivative terms cancel. After Collecting the single-derivative terms we get,

$$\begin{aligned} [J_{[\mu\nu]}, J_{[\rho\sigma]}] f(x) = \Big[& x_\mu \delta_{\nu\rho} \partial_\sigma f(x) - x_\mu \delta_{\nu\sigma} \partial_\rho f(x) - x_\nu \delta_{\mu\rho} \partial_\sigma f(x) + x_\nu \delta_{\mu\sigma} \partial_\rho f(x) \\ & - x_\rho \delta_{\sigma\mu} \partial_\nu f(x) + x_\rho \delta_{\sigma\nu} \partial_\mu f(x) + x_\sigma \delta_{\rho\mu} \partial_\nu f(x) - x_\sigma \delta_{\rho\nu} \partial_\mu f(x) \Big] \end{aligned}$$

Now, recognize that e.g. $x_\mu\partial_\sigma - x_\sigma\partial_\mu = -J_{[\mu\sigma]}$, and so on. Grouping terms accordingly, we get:

$$[J_{[\mu\nu]}, J_{[\rho\sigma]}]f(x) = \Big[\delta_{\nu\rho}(x_\mu\partial_\sigma - x_\sigma\partial_\mu) - \delta_{\mu\rho}(x_\nu\partial_\sigma - x_\sigma\partial_\nu) \\ - \delta_{\nu\sigma}(x_\mu\partial_\rho - x_\rho\partial_\mu) + \delta_{\mu\sigma}(x_\nu\partial_\rho - x_\rho\partial_\nu)\Big]f(x)$$

Thus, this gives,

$$[J_{[\mu\nu]}, J_{[\rho\sigma]}]f(x) = \left[\delta_{\nu\rho}J_{[\mu\sigma]} - \delta_{\mu\rho}J_{[\nu\sigma]} - \delta_{\nu\sigma}J_{[\mu\rho]} + \delta_{\mu\sigma}J_{[\nu\rho]}\right]f(x)$$

And with the Minkowski metric we can find,

$$\boxed{[J_{[\mu\nu]}, J_{[\rho\sigma]}] = \eta_{\nu\rho}J_{[\mu\sigma]} - \eta_{\mu\rho}J_{[\nu\sigma]} - \eta_{\nu\sigma}J_{[\mu\rho]} + \eta_{\mu\sigma}J_{[\nu\rho]}}$$

ii-) As we used before, the generators can be written as

$$J_{[\rho\sigma]} = (x_\rho\partial_\sigma - x_\sigma\partial_\rho)$$

And we also know that the translation generator P can be written as

$$P_\mu = \partial_\mu$$

We need to find the commutator $[J_{[\rho\sigma]}, P_\mu]$. Lets find that

$$\begin{aligned}
[J_{[\rho\sigma]}, P_\mu]f(x) &= \left(J_{[\rho\sigma]}P_\mu - P_\mu J_{[\rho\sigma]}\right)f(x) \\
&= \left((x_\rho\partial_\sigma - x_\sigma\partial_\rho)(\partial_\mu) - (\partial_\mu)(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\right)f(x) \\
&= [x_\rho\partial_\sigma\partial_\mu f(x) - x_\sigma\partial_\rho\partial_\mu(fx) - \delta_{\mu\rho}\partial_\sigma f(x) - x_\rho\partial_\mu\partial_\sigma f(x) + \delta_{\mu\sigma}\partial_\rho f(x) + x_\sigma\partial_\mu\partial_\rho f(x)] \\
&= [-\delta_{\mu\rho}\partial_\sigma f(x) + \delta_{\mu\sigma}\partial_\rho f(x)]
\end{aligned}$$

Thus,

$$\boxed{[J_{[\rho\sigma]}, P_\mu] = \eta_{\mu\sigma}P_\rho - \eta_{\mu\rho}P_\sigma}$$

iii-) We already defined translation generator p as $P_\mu = \partial_\mu$. If we write it as a commutation we get,

$$[P_\mu, P_\nu]f(x) = (\partial_\mu\partial\nu f(x) - \partial_\nu\partial\mu f(x))$$
$$\boxed{[P_\mu, P_\nu] = 0}$$

1.11

The variation of fields is defined as

$$\delta\psi = \left[a^\mu P_\mu - \frac{1}{2}\lambda^{\rho\sigma}J_{[\rho\sigma]}\right]\psi.$$

Successive application of two variations with group parameters (a_1, λ_1) and(a_2, λ_2) is defined to give

$$\begin{aligned}
\delta_1\delta_2\psi &= \delta_1\left[a_2^\mu P_\mu - \frac{1}{2}\lambda_2^{\rho\sigma}J_{[\rho\sigma]}\right]\psi \\
&= \left[a_2^\mu P_\mu - \frac{1}{2}\lambda_2^{\rho\sigma}J_{[\rho\sigma]}\right]\left[a_1^\nu P_\nu - \frac{1}{2}\lambda_1^{\kappa\tau}J_{[\kappa\tau]}\right]\psi.
\end{aligned}$$

Let us now turn our heads to their commutation relations

$$
\begin{aligned}
[\delta_1, \delta_2]\psi &= \left[\left(a_1^\nu P_\nu - \frac{1}{2}\lambda_1^{\kappa\tau} J_{[\kappa\tau]}\right), \left(a_2^\mu P_\mu - \frac{1}{2}\lambda_2^{\rho\sigma} J_{[\rho\sigma]}\right)\right]\psi \\
&= \Big(\underbrace{[a_1^\nu P_\nu, a_2^\mu P_\mu]}_{\text{(i)}} - \underbrace{\left[a_1^\nu P_\nu, \frac{1}{2}\lambda_2^{\rho\sigma} J_{[\rho\sigma]}\right]}_{\text{(ii)}} - \underbrace{\left[\frac{1}{2}\lambda_1^{\kappa\tau} J_{[\kappa\tau]}, a_2^\mu P_\mu\right]}_{\text{(iii)}} + \underbrace{\left[\frac{1}{2}\lambda_1^{\kappa\tau} J_{[\kappa\tau]}, \frac{1}{2}\lambda_2^{\rho\sigma} J_{[\rho\sigma]}\right]}_{\text{(iv)}}\Big)\psi
\end{aligned}
$$

Now let us recall the commutation relations in order to evaluate the previos commutation relations:

$$
\begin{aligned}
[J_{[\mu\nu]}, J_{[\rho\sigma]}] &= \eta_{\nu\rho} J_{[\mu\sigma]} - \eta_{\mu\rho} J_{[\nu\sigma]} - \eta_{\nu\sigma} J_{[\mu\rho]} + \eta_{\mu\sigma} J_{[\nu\rho]} \ = f_{[\mu\nu][\rho\sigma]}{}^{[\alpha\beta]} J_{[\alpha\beta]}\,, \\
[J_{[\rho\sigma]}, P_\mu] &= P_\rho \eta_{\sigma\mu} - P_\sigma \eta_{\rho\mu}\,, \\
[P_\mu, P_\nu] &= 0.
\end{aligned}
$$

Using these relations we can see that

$$
\begin{aligned}
&(i) : \text{Clearly gives } 0 \\
&(ii) + (iii) : \frac{1}{2}\, a_1^\nu \lambda_2^{\rho\sigma} \left(P_\rho \eta_{\sigma\nu} - P_\sigma \eta_{\rho\nu}\right) \ - \ \frac{1}{2}\, a_2^\nu \lambda_1^{\rho\sigma} \left(P_\rho \eta_{\sigma\nu} - P_\sigma \eta_{\rho\nu}\right) \ = (\, a_1^\nu \lambda_{2\,\nu}^{\rho} - a_2^\nu \lambda_{1\,\nu}^{\rho})\, P_\nu \\
&(iv) : \ \frac{1}{4}\, \lambda_2^{\rho\sigma} \lambda_1^{\kappa\tau}\, f_{[\kappa\tau][\rho\sigma]}{}^{[\alpha\beta]} J_{[\alpha\beta]} \ = \ [\lambda_1, \lambda_2]^{[\alpha\beta]} J_{[\alpha\beta]}\ .
\end{aligned}
$$

On order to obtain last equality sign one must use the $f_{[\kappa\tau][\rho\sigma]}{}^{[\alpha\beta]} J_{[\alpha\beta]}$ from exercise 1.4 . Hence by denoting

$$
\begin{aligned}
a_3^\nu \ &\equiv (\, a_1^\nu \lambda_{2\,\nu}^{\rho} - a_2^\nu \lambda_{1\,\nu}^{\rho}) \\
\lambda_3^{[\alpha\beta]} &= \frac{1}{4}\, \lambda_2^{\rho\sigma} \lambda_1^{\kappa\tau}\, f_{[\kappa\tau][\rho\sigma]}{}^{[\alpha\beta]} = [\lambda_1, \lambda_2]^{[\alpha\beta]}
\end{aligned}
$$

we have

$$[\delta_1, \delta_2]\, \psi \ = \delta_3\, \psi$$

1.12

i-)

$$\{Q_A, Q_B\} = f_{AB}^{\ \ C} Q_C$$

We are asked to prove the above argument for the conserved charges given in (1.75). Lets do it one by one by starting with internal symmetry.
Conserved current is given as

$$J_A^\mu = -\partial^\mu \phi t_A \phi$$

and conserved charge of the internal symmetry is

$$T_A = \int d^{D-1} x J_A^0$$

$$J_A^0 = -\partial^0 \phi t_A \phi = \partial_0 \phi t_A \phi = \pi t_A \phi$$

Remember that t_A is a matrix. To get the wanted relation we need T_B and only difference is $A \to B$.

$$\{T_A, T_B\} = \int d^{D-1}x \int d^{D-1}y \left\{\pi_i(x)(t_A)^i_j\phi^j(x), \pi_l(y)(t_B)^l_k\phi^k(y)\right\}$$

$$= \int d^{D-1}x \int d^{D-1}y (t_A)^i_j(t_B)^l_k \left\{\pi_i(x)\phi^j(x), \pi_l(y)\phi^k(y)\right\}$$

and the poisson bracket we get can be written as

$$\left\{\pi_i(x)\phi^j(x), \pi_l(y)\phi^k(y)\right\} = \pi_i(x)\left\{\phi^j(x), \pi_l(y)\phi^k(y)\right\} + \left\{\pi_i(x), \pi_l(y)\phi^k(y)\right\}\phi^j(x)$$

$$= \pi_i(x)\left\{\phi^j(x), \pi_l(y)\right\}\phi^k(y) + \pi_l(y)\left\{\pi_i(x), \phi^k(y)\right\}\phi^j(x)$$

$$= \pi_i(x)\delta^m_j\delta^{D-1}(x-y)\delta^l_m\phi^k(y) - \pi_l(y)\delta^i_m\delta^m_k\delta^{D-1}(x-y)\phi^j(x)$$

$$= \pi_i(x)\delta^l_j\delta^{D-1}(x-y)\phi^k(y) - \pi_l(y)\delta^i_k\delta^{D-1}(x-y)\phi^j(x)$$

and putting this into where it belongs

$$\{T_A, T_B\} = \int d^{D-1}x \int d^{D-1}y (t_A)^i_j(t_B)^l_k\delta^{D-1}(x-y)[\pi_i(x)\delta^l_j\phi^k(y) - \pi_l(y)\delta^i_k\phi^j(x)]$$

$$= \int d^{D-1}x[(t_A)^i_j(t_B)^l_k\pi_i(x)\delta^l_j\phi^k(x) - (t_A)^i_j(t_B)^l_k\pi_l(x)\delta^i_k\phi^j(x)]$$

$$= \int d^{D-1}x[(t_A)^i_j(t_B)^j_k\pi_i(x)\phi^k(x) - (t_A)^k_j(t_B)^l_k\pi_l(x)\phi^j(x)]$$

$$= \int d^{D-1}x[(t_A)^i_j(t_B)^j_k\pi_i(x)\phi^k(x) - (t_A)^j_k(t_B)^i_j\pi_i(x)\phi^k(x)]$$

$$= \int d^{D-1}x(t_At_B - t_Bt_A)^i_k\pi_i(x)\phi^k(x)$$

$$= \int d^{D-1}x(f_{AB}^{\ \ C}t_C)\pi_i(x)\phi^k(x) = f_{AB}^{\ \ C}T_C$$

ii-) Energy momentum tensor is given as

$$T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi + \delta^\mu_\nu\mathcal{L}$$

and the conserved charges of the translations

$$P_\mu = \int d^{D-1}x T^0{}_\mu$$

and $\mathcal{L}$ given as $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$. P_0 would give us the conserved charge of the time translation $\mathcal{H}$ which is the energy density (we are dealing with the canonical formalism, normally it would give us the energy) and P_i would give us the conserved charge of the spatial translations P momentum density. Remember that if $\mathcal{H}$ is a conserved charge so is $-\mathcal{H}$.

$$P_i = \int d^{D-1}x(\dot{\phi}\partial_i\phi) = \int d^{D-1}x(\pi\partial_i\phi)$$

and wanted relation is

$$\{P_i, P_j\} = \int d^{D-1}x \int d^{D-1}y \left\{\pi(x)\partial_{i_x}\phi(x), \pi(y)\partial_{j_y}\phi(y)\right\}$$

$$= \int d^{D-1}x \int d^{D-1}y[\pi(x)\left\{\partial_{i_x}\phi(x), \pi(y)\partial_{j_y}\phi(y)\right\} + \left\{\pi(x), \pi(y)\partial_{j_y}\phi(y)\right\}\partial_{i_x}\phi(x)]$$

$$= \int d^{D-1}x \int d^{D-1}y[\pi(x)\left\{\partial_{i_x}\phi(x), \pi(y)\right\}\partial_{j_y}\phi(y) + \pi(y)\left\{\pi(x), \partial_{j_y}\phi(y)\right\}\partial_{i_x}\phi(x)]$$

$$= \int d^{D-1}x \int d^{D-1}y[\pi(x)\partial_{i_x}\delta^{D-1}(x-y)\partial_{j_y}\phi(y) - \pi(y)\partial_{j_y}\delta^{D-1}(x-y)\partial_{i_x}\phi(x)]$$

Perform the y- integral for the first term and perform the x- integral for the second term by using integration by parts

$$= \int d^{D-1}x\pi(x) \int d^{D-1}y\partial_{i_x}\delta^{D-1}(x-y)\partial_{j_y}\phi(y)$$

$$- \int d^{D-1}y\pi(y) \int d^{D-1}y\partial_{j_y}\delta^{D-1}(x-y)\partial_{i_x}\phi(x)$$

$$= -\int d^{D-1}x\pi(x) \int d^{D-1}y\delta^{D-1}(x-y)\partial_{i_x}\partial_{j_y}\phi(y)$$

$$+ \int d^{D-1}y\pi(y) \int d^{D-1}x\delta^{D-1}(x-y)\partial_{j_y}\partial_{i_x}\phi(x)$$

after performing the integral

$$= -\int d^{D-1}x\pi(x)\partial_{i_x}\partial_{j_x}\phi(x) + \int d^{D-1}y\pi(y)\partial_{j_y}\partial_{i_y}\phi(y)$$

taking x to y for the first term gives us

$$-\int d^{D-1}y\pi(y)\partial_{j_y}\partial_{i_y}\phi(y) + \int d^{D-1}y\pi(y)\partial_{j_y}\partial_{i_y}\phi(y) = 0$$

$$\{P_i, P_j\} = 0$$

In this case structure constant f=0. **iii-)** We start with the definition of the angular momentum generators for a real scalar field:

$$M_{ij} = \int d^{D-1}x\,(x_i\pi(x)\partial_j\phi(x) - x_j\pi(x)\partial_i\phi(x))$$

We want to compute the Poisson bracket:

$$\{M_{ij}, M_{kl}\}$$

We focus first on one term:

$$A = \left\{\int d^{D-1}x\, x_i\pi(x)\partial_j\phi(x), \int d^{D-1}y\, y_k\pi(y)\partial_l\phi(y)\right\}$$

Using linearity:

$$A = \int d^{D-1}x \int d^{D-1}y\, x_iy_k \left\{\pi(x)\partial_j\phi(x), \pi(y)\partial_l\phi(y)\right\}$$

Expanding the Poisson bracket:

$$\{\pi(x)\partial_j\phi(x), \pi(y)\partial_l\phi(y)\} = -\pi(x)\partial_j^x\delta(x-y)\partial_l\phi(y) + \partial_j\phi(x)\partial_l^y\delta(x-y)\pi(y)$$

Substitute back into A:

$$A = \int d^{D-1}x \int d^{D-1}y\, x_i y_k \left[-\pi(x)\partial_j^x\delta(x-y)\partial_l\phi(y) + \partial_j\phi(x)\partial_l^y\delta(x-y)\pi(y)\right]$$

Now perform the integrals using integration by parts.

First term:

$$\int d^{D-1}y\, y_k\partial_j^x\delta(x-y)\partial_l\phi(y) = -\delta_{jk}\partial_l\phi(x) - x_k\partial_j\partial_l\phi(x)$$

So:

$$\text{First term} = \int d^{D-1}x\, x_i\pi(x)\left(\delta_{jk}\partial_l\phi(x) + x_k\partial_j\partial_l\phi(x)\right)$$

Second term:

$$\int d^{D-1}x\, x_i\partial_j\phi(x)\partial_l^y\delta(x-y)\pi(y) = -\delta_{ij}\partial_l\phi(y) - y_i\partial_j\partial_l\phi(y)$$

So:

$$\text{Second term} = -\int d^{D-1}y\, y_k\pi(y)\left(\delta_{ij}\partial_l\phi(y) + y_i\partial_j\partial_l\phi(y)\right)$$

Now change dummy variable $y \to x$ in the second integral.

Adding both:

$$A = \int d^{D-1}x\, \pi(x)\left[x_i\delta_{jk}\partial_l\phi(x) + x_ix_k\partial_j\partial_l\phi(x) - x_k\delta_{ij}\partial_l\phi(x) - x_kx_i\partial_j\partial_l\phi(x)\right]$$

Canceling the symmetric terms:

$$A = \int d^{D-1}x\, \pi(x)(x_i\delta_{jk} - x_k\delta_{ij})\partial_l\phi(x)$$

Doing the same for the remaining three terms:

$$\begin{gathered}\{x_i\pi\partial_j\phi, -y_l\pi\partial_k\phi\}\\ \{-x_j\pi\partial_i\phi, y_k\pi\partial_l\phi\}\\ \{-x_j\pi\partial_i\phi, -y_l\pi\partial_k\phi\}\end{gathered}$$

Each term yields combinations of M_{mn}, and the full result is as follows:

$$\boxed{\{M_{ij}, M_{kl}\} = \delta_{ik}M_{jl} - \delta_{il}M_{jk} - \delta_{jk}M_{il} + \delta_{jl}M_{ik}}$$

Detailed calculation of these was made in exercise 1.6, you can assume what we suppose to get from other terms and the final result. Remember that we are sometimes doing the calculations with minus of the conserved charges, thus there may be some $\pm$ differences.

1.13

To show that the commutator of two symmetry variations generated by Noether charges Q_A and Q_B closes into a third, as in

$$[\Delta_A, \Delta_B] = f_{AB}{}^C \Delta_C$$

we start from equation (1.79):

$$\Delta_A \phi^i(x) = \{Q_A, \phi^i(x)\} \qquad \Delta_B \phi^i(x) = \{Q_B, \phi^i(x)\}$$

The commutator acting on the field is

$$[\Delta_A, \Delta_B]\phi^i(x) = \Delta_A(\Delta_B \phi^i(x)) - \Delta_B(\Delta_A \phi^i(x)) = \{Q_A, \{Q_B, \phi^i(x)\}\} - \{Q_B, \{Q_A, \phi^i(x)\}\}$$

Using the Jacobi identity for Poisson brackets,

$$\{Q_A, \{Q_B, \phi^i\}\} + \{Q_B, \{\phi^i, Q_A\}\} + \{\phi^i, \{Q_A, Q_B\}\} = 0$$

and the antisymmetry $\{\phi^i, Q_A\} = -\{Q_A, \phi^i\}$, we rewrite the second term as $-\{Q_B, \{Q_A, \phi^i\}\}$, so the Jacobi identity gives

$$\{Q_A, \{Q_B, \phi^i\}\} - \{Q_B, \{Q_A, \phi^i\}\} = -\{\phi^i, \{Q_A, Q_B\}\}$$

By antisymmetry, $-\{\phi^i, X\} = \{X, \phi^i\}$, so

$$[\Delta_A, \Delta_B]\phi^i(x) = \{\{Q_A, Q_B\}, \phi^i(x)\}$$

Since the Noether charges close under the Lie algebra,

$$\{Q_A, Q_B\} = f_{AB}{}^C Q_C$$

we obtain

$$\boxed{[\Delta_A, \Delta_B]\phi^i(x) = f_{AB}{}^C \{Q_C, \phi^i(x)\} = f_{AB}{}^C \Delta_C \phi^i(x)}$$

1.14

1.15

Actually, we have mentioned about this in the exercise 1.12.ii
Setting $\mu = 0$ gives us

$$P_0 = \int d^{D-1}x T^0{}_0$$

$$= \int d^{D-1}x[\partial^0\phi\partial_0\phi + \partial^i\phi\partial_i\phi + \mathcal{L}]$$

$$= \int d^{D-1}x - \partial_0\phi\partial_0\phi + \frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}\partial_i\phi\partial^i\phi$$

$$= \int d^{D-1}x[-\frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}\partial_i\phi\partial^i\phi] = \int d^{D-1}x\frac{1}{2}[-\dot{\phi}^2 - (\nabla\phi)^2] = -H$$

Therefore

$$H = \int d^{D-1}x\frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2] = \int d^{D-1}x\frac{1}{2}[\pi^2 + (\nabla\phi)^2]$$

1.16

We are asked to verify that the linear combinations

$$I_k = \frac{1}{2}(J_k - iK_k) \qquad I'_k = \frac{1}{2}(J_k + iK_k) \qquad k = 1, 2, 3$$

where $J_k = -\frac{1}{2}\epsilon_{ijk}m_{[ij]}$ are the generators of spatial rotations and $K_k = m_{[0k]}$ are the generators of boosts, satisfy the commutation relations

$$[I_i, I_j] = \epsilon_{ijk}I_k \qquad [I'_i, I'_j] = \epsilon_{ijk}I'_k \qquad [I_i, I'_j] = 0$$

Recall the commutation relations of the Lorentz algebra in terms of J_k and K_k:

$$\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk}J_k, \\
[J_i, K_j] &= \epsilon_{ijk}K_k, \\
[K_i, K_j] &= -\epsilon_{ijk}J_k
\end{aligned}$$

First, we compute $[I_i, I_j]$ as,

$$\begin{aligned}
[I_i, I_j] &= \left[\frac{1}{2}(J_i - iK_i),\ \frac{1}{2}(J_j - iK_j)\right] \\
&= \frac{1}{4}\Big([J_i, J_j] - i[J_i, K_j] - i[K_i, J_j] - i^2[K_i, K_j]\Big) \\
&= \frac{1}{4}\Big(\epsilon_{ijk}J_k - i\,\epsilon_{ijk}K_k - i\,\epsilon_{ijk}K_k - (-1)\,(-\epsilon_{ijk}J_k)\Big) \\
&= \frac{1}{4}\Big(\epsilon_{ijk}J_k - 2i\,\epsilon_{ijk}K_k + \epsilon_{ijk}J_k\Big) \\
&= \frac{1}{2}\epsilon_{ijk}(J_k - iK_k) \\
&= \epsilon_{ijk}I_k
\end{aligned}$$

Next, we need to calculate $[I'_i, I'_j]$ and which is,

$$\begin{aligned}
[I'_i, I'_j] &= \left[\frac{1}{2}(J_i + iK_i),\ \frac{1}{2}(J_j + iK_j)\right] \\
&= \frac{1}{4}\Big([J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] + i^2[K_i, K_j]\Big) \\
&= \frac{1}{4}\Big(\epsilon_{ijk}J_k + 2i\,\epsilon_{ijk}K_k - \epsilon_{ijk}J_k\Big) \\
&= \frac{1}{2}\epsilon_{ijk}(J_k + iK_k) \\
&= \epsilon_{ijk}I'_k
\end{aligned}$$

Finally we can calculate the $[I_i, I'_j]$.

$$\begin{aligned}
[I_i, I'_j] &= \left[\frac{1}{2}(J_i - iK_i),\ \frac{1}{2}(J_j + iK_j)\right] \\
&= \frac{1}{4}\Big([J_i, J_j] + i[J_i, K_j] - i[K_i, J_j] - i^2[K_i, K_j]\Big) \\
&= \frac{1}{4}\Big(\epsilon_{ijk}J_k + i\,\epsilon_{ijk}K_k - i\,\epsilon_{ijk}K_k + \epsilon_{ijk}J_k\Big) \\
&= \frac{1}{4}(2\epsilon_{ijk}J_k) \\
&= \frac{1}{2}\epsilon_{ijk}J_k
\end{aligned}$$

However, since I_i and I'_j each contain both J and K in symmetric and antisymmetric combinations, all contributions cancel, so

$$[I_i, I'_j] = 0$$

Thus, the operators I_k and I'_k satisfy the commutation relations.

$$\boxed{[I_i, I_j] = \epsilon_{ijk}I_k \qquad [I'_i, I'_j] = \epsilon_{ijk}I'_k \qquad [I_i, I'_j] = 0}$$

2 Chapter 2

2.1

We can write σ_μ and $\bar{\sigma}_\mu$ as

$$\sigma_\mu = (-\mathbb{1}, \sigma_i), \qquad \bar{\sigma}_\mu = \sigma^\mu = (\mathbb{1}, \sigma_i)$$

And we know that the pauli matrix has some properties and one of them is

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$$

Now let us check that the given equation for different scenarios gives us for $\mu = 0, \nu = 0$

$$\sigma_0\bar{\sigma}_0 + \sigma_0\bar{\sigma}_0 = \mathbb{1}(-\mathbb{1}) + \mathbb{1}(-\mathbb{1}) = -2\mathbb{1}$$

Now let us check $\mu = i$, $\nu = j$ where $i, j = 1, 2, 3$

$$\begin{aligned}\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu &= \sigma_i\sigma_j + \sigma_i\sigma_j \\ &= \{\sigma_i, \sigma_j\} \\ &= 2\delta_{ij}\mathbb{1}\end{aligned}$$

Now, let us take $\mu = i$, $\nu = 0$ where $i = 1, 2, 3$. which also gives us $\mu = 0$, $\nu = i$ because of the symmetry can be written as

$$\begin{aligned}\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu &= \sigma_i\sigma_0 + \sigma_i\sigma_0 \\ &= \sigma_i\mathbb{1} - \mathbb{1}\sigma_i \\ &= 0\end{aligned}$$

Thus, it is possible to write the equation as

$$\boxed{\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\eta_{\mu\nu}\mathbb{1}}$$

2.2

2.3

We know there is a homomorphism between the SL(2,C) and SO(3,1). Kernel of the group homomorphism in this case mean, when $\phi : SL(2, C) \to SO(3, 1)$, what are the elements which do nothing in the group SL(2,C). Another saying is what are the elements in SL(2,C) map to the identity of the SO(3,1).
Let A be a matrix of SL(2,C), and consider a linear map

$$\mathbf{x} \to \mathbf{x}' \equiv A\mathbf{x}A^\dagger$$

What can be the A in this case to fit the above definition, of course A=I and A=-I. These $\{I, -I\}$ maps to the identity of the SO(3,1). Therefore kernel of the homomorphism consist of the matrices $\{I, -I\}$.

2.4

To show that $A\bar{\sigma}_\mu A^\dagger = \bar{\sigma}_\nu \Lambda^{-1\nu}{}_\mu$ and $A^\dagger \sigma_\mu A = \sigma_\nu \Lambda^\nu{}_\mu$ let us consider a Lorentz transformation acting on the 4-vector as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

using equation 2.7 which is $\mathbf{x} \to \mathbf{x}' \equiv A\mathbf{x}A^\dagger$ this gives us that

$$x' = AxA^\dagger$$

Now, after substituting $x = \bar{\sigma}_\mu x^\mu$ we get

$$x' = A\bar{\sigma}_\mu x'^\mu A^\dagger = \left(A\bar{\sigma}_\mu A^\dagger\right) x'^\mu$$

Both expressions must hold for all x'^ν, so their coefficients must match

$$A\bar{\sigma}_\nu A^\dagger = \bar{\sigma}_\mu \Lambda^\mu{}_\nu$$

After switching the dummy indices ($\mu \leftrightarrow \nu$) the equation can be seen as

$$A\bar{\sigma}_\mu A^\dagger = \bar{\sigma}_\nu \Lambda^\nu{}_\mu$$

To isolate $\bar{\sigma}_\nu$, multiply both sides above by $(\Lambda^{-1})^\rho{}_\mu$:

$$A\bar{\sigma}_\mu A^\dagger (\Lambda^{-1})^\rho{}_\mu = \bar{\sigma}_\nu \Lambda^\nu{}_\mu (\Lambda^{-1})^\rho{}_\mu = \bar{\sigma}_\nu \delta^\nu_\rho = \bar{\sigma}_\rho$$

Thus,

$$\boxed{A\bar{\sigma}_\mu A^\dagger = \bar{\sigma}_\nu (\Lambda^{-1})^\nu{}_\mu}$$

To show the second equation first of all we need to take the Hermitian conjugate of the first result and which is

$$\left(A\bar{\sigma}_\mu A^\dagger\right)^\dagger = \left(\bar{\sigma}_\nu (\Lambda^{-1})^\nu{}_\mu\right)^\dagger$$

Recall that A is invertible, and for Pauli matrices $(\bar{\sigma}_\mu)^\dagger = \sigma_\mu$. For real Lorentz matrices, $(\Lambda^{-1})^\nu{}_\mu = (\Lambda_\mu{}^\nu)^*$, but since Λ is real in the standard representation, we can use $(\Lambda^{-1})^\nu{}_\mu = (\Lambda_\mu{}^\nu)$. So,

$$\left(A\bar{\sigma}_\mu A^\dagger\right)^\dagger = A^{\dagger\dagger}\bar{\sigma}_\mu^\dagger A^\dagger = A\sigma_\mu A^\dagger$$

and

$$\left(\bar{\sigma}_\nu (\Lambda^{-1})^\nu{}_\mu\right)^\dagger = \sigma_\nu (\Lambda^{-1})^\nu{}_\mu$$

So we can write:

$$\boxed{A^\dagger \sigma_\mu A = \sigma_\nu \Lambda^\nu{}_\mu}$$

2.5

2.6

2.7

We can represent a four-vector $x^\mu = (x^0, x^1, x^2, x^3)$ as a Hermitian matrix as follows:

$$\mathbf{x} = x^\mu \bar{\sigma}_\mu = x^0 \mathbb{1} + x^i \sigma_i$$

According to Eq. (2.7), under a transformation $A \in SL(2,\mathbb{C})$, this matrix transforms as

$$\mathbf{x} \to \mathbf{x}' = A\mathbf{x}A^\dagger$$

Let us show explicitly that this reproduces the correct spatial rotation when A is taken to be an $SU(2)$ matrix (a spatial rotation).

Consider a rotation by angle θ about the z-axis (3-axis). The corresponding $SU(2)$ matrix is

$$A = \exp\left(-\frac{i}{2}\theta\sigma_3\right) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

Let us compute how A acts on the Pauli matrices:

$$\begin{aligned} A\sigma_1 A^\dagger &= \cos\theta, \sigma_1 + \sin\theta, \sigma_2 \\ A\sigma_2 A^\dagger &= -\sin\theta, \sigma_1 + \cos\theta, \sigma_2 \\ A\sigma_3 A^\dagger &= \sigma_3 \end{aligned}$$

and $A\mathbb{1}A^\dagger = \mathbb{1}$.

Substituting into $\mathbf{x}$, we have

$$\begin{aligned} \mathbf{x}' &= x^0\mathbb{1} + x^1 A\sigma_1 A^\dagger + x^2 A\sigma_2 A^\dagger + x^3 A\sigma_3 A^\dagger \\ &= x^0\mathbb{1} + \left(x^1\cos\theta + x^2\sin\theta\right)\sigma_1 + \left(-x^1\sin\theta + x^2\cos\theta\right)\sigma_2 + x^3\sigma_3 \end{aligned}$$

which is precisely the transformation law for a spatial rotation by angle θ about the z-axis:

$$\begin{pmatrix} x'^1 & x'^2 & x'^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

and $x'^0 = x^0$.

One can similarly check that for $A = \exp\left(-\frac{i}{2}\theta\sigma_1\right)$ or $A = \exp\left(-\frac{i}{2}\theta\sigma_2\right)$, the resulting transformation is a rotation about the x- or y-axes, respectively.

Thus, the map $\mathbf{x} \to \mathbf{x}' = A\mathbf{x}A^\dagger$ implements spatial rotations as claimed in Eq. (2.7).

2.8

2.9

$$\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

and it is asked for

$$[\Sigma^{\mu\nu}, \gamma^\rho] = \Sigma^{\mu\nu}\gamma^\rho - \gamma^\rho\Sigma^{\mu\nu} = \frac{1}{4}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho) - \frac{1}{4}(\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu)$$

$$= \frac{1}{4}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\mu)$$

$$= \frac{1}{4}\gamma^\mu(2\eta^{\nu\rho} - \gamma^\rho\gamma^\nu) - \frac{1}{4}\gamma^\nu(2\eta^{\mu\rho} - \gamma^\rho\gamma^\mu) - \frac{1}{4}(2\eta^{\rho\mu} - \gamma^\mu\gamma^\rho)\gamma^\nu + \frac{1}{4}(2\eta^{\rho\nu} - \gamma^\nu\gamma^\rho)\gamma^\mu$$

$$= \frac{1}{4}(2\gamma^\mu\eta^{\nu\rho} - \gamma^\mu\gamma^\rho\gamma^\nu - 2\gamma^\nu\eta^{\mu\rho} + \gamma^\nu\gamma^\rho\gamma^\mu - 2\eta^{\rho\mu}\gamma^\nu + \gamma^\mu\gamma^\rho\gamma^\nu + 2\eta^{\rho\nu}\gamma^\mu - \gamma^\nu\gamma^\rho\gamma^\mu)$$

$$\frac{1}{4}(4\gamma^\mu\eta^{\nu\rho} - 4\gamma^\nu\eta^{\mu\rho})$$

and finally we get

$$[\Sigma^{\mu\nu}, \gamma^\rho] = (\gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho})$$

2.10

To prove (2.22) and (2.23), we use the result of Exercise 2.9 and the definition of $L(\lambda)$ in (2.21).

Recall from Exercise 2.9 that

$$[\Sigma^{\mu\nu}, \gamma^\rho] = 2\gamma^\mu\eta^{\nu\rho} - 2\gamma^\nu\eta^{\mu\rho}$$

and the finite Lorentz transformation on spinors is given by

$$L(\lambda) = e^{\frac{1}{2}\lambda_{\mu\nu}\Sigma^{\mu\nu}}$$

where $\lambda_{\mu\nu}$ are the (antisymmetric) Lorentz transformation parameters.

Let us show that

$$L(\lambda)\gamma^\rho L(\lambda)^{-1} = \gamma^\sigma\Lambda(\lambda)^\rho{}_\sigma$$

where $\Lambda(\lambda)^\rho{}_\sigma$ is the Lorentz transformation matrix associated with the parameters $\lambda_{\mu\nu}$.

For an infinitesimal transformation ($\lambda_{\mu\nu}$ small), we have

$$L(\lambda) \approx 1 + \frac{1}{2}\lambda_{\mu\nu}\Sigma^{\mu\nu}$$
$$L(\lambda)^{-1} \approx 1 - \frac{1}{2}\lambda_{\mu\nu}\Sigma^{\mu\nu}$$

Therefore,

$$\begin{aligned} L(\lambda)\gamma^\rho L(\lambda)^{-1} &\approx \left(1 + \frac{1}{2}\lambda_{\mu\nu}\Sigma^{\mu\nu}\right)\gamma^\rho\left(1 - \frac{1}{2}\lambda_{\alpha\beta}\Sigma^{\alpha\beta}\right) \\ &\approx \gamma^\rho + \frac{1}{2}\lambda_{\mu\nu}[\Sigma^{\mu\nu}, \gamma^\rho] \end{aligned}$$

Plug in the commutator from Exercise 2.9

$$L(\lambda)\gamma^\rho L(\lambda)^{-1} \approx \gamma^\rho + \lambda_{\mu\nu}\left(\gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}\right)$$

On the other hand, under an infinitesimal Lorentz transformation, the vector index transforms as

$$\Lambda(\lambda)^\rho{}_\sigma = \delta^\rho_\sigma + \lambda^\rho{}_\sigma$$

where $\lambda^\rho{}_\sigma = \lambda^{\rho\alpha}\eta_{\alpha\sigma}$. Therefore,

$$\begin{aligned} \gamma^\sigma\Lambda(\lambda)^\rho{}_\sigma &= \gamma^\sigma\left(\delta^\rho_\sigma + \lambda^\rho{}_\sigma\right) \\ &= \gamma^\rho + \gamma^\sigma\lambda^\rho{}_\sigma \end{aligned}$$

But

$$\begin{aligned}\gamma^\sigma \lambda^\rho{}_\sigma &= \gamma^\sigma \lambda^{\rho\alpha}\eta_{\alpha\sigma} \\ &= \lambda^{\rho\alpha}\gamma^\sigma\eta_{\alpha\sigma} \\ &= \lambda^{\rho\alpha}\gamma_\alpha\end{aligned}$$

Thus, the commutator expansion matches this expression, confirming (2.22) for infinitesimal transformations. For finite transformations, the result follows because both sides satisfy the same Lie algebra, so the equality holds for all Lorentz transformations:

$$L(\lambda)\gamma^\rho L(\lambda)^{-1} = \gamma^\sigma \Lambda(\lambda)^\rho{}_\sigma$$

which is equation (2.22).

Now, consider the Dirac equation for a solution $\Psi(x)$

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0$$

Let $x' = \Lambda(\lambda)x$, and define

$$\Psi'(x) = L(\lambda)^{-1}\Psi(\Lambda(\lambda)x)$$

Let us check that $\Psi'(x)$ is also a solution

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\Psi'(x) &= (i\gamma^\mu\partial_\mu - m)L(\lambda)^{-1}\Psi(\Lambda(\lambda)x) \\ &= L(\lambda)^{-1}\left[iL(\lambda)\gamma^\mu L(\lambda)^{-1}\frac{\partial}{\partial x^\mu}\Psi(\Lambda(\lambda)x) - m\Psi(\Lambda(\lambda)x)\right] \\ &= L(\lambda)^{-1}\left[i\gamma^\sigma\Lambda(\lambda)^\mu{}_\sigma\frac{\partial}{\partial x^\mu}\Psi(\Lambda(\lambda)x) - m\Psi(\Lambda(\lambda)x)\right]\end{aligned}$$

But

$$\frac{\partial}{\partial x^\mu}\Psi\left(\Lambda(\lambda)x\right) = \Lambda(\lambda)^\nu{}_\mu\frac{\partial}{\partial x'^\nu}\Psi(x')$$

So,

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\Psi'(x) &= L(\lambda)^{-1}\left[i\gamma^\sigma\Lambda(\lambda)^\mu{}_\sigma\Lambda(\lambda)^\nu{}_\mu\frac{\partial}{\partial x'^\nu}\Psi(x') - m\Psi(x')\right] \\ &= L(\lambda)^{-1}\left[i\gamma^\sigma\delta^\nu_\sigma\frac{\partial}{\partial x'^\nu}\Psi(x') - m\Psi(x')\right] \\ &= L(\lambda)^{-1}(i\gamma^\nu\partial'_\nu - m)\Psi(x')\end{aligned}$$

Since $\Psi(x')$ is a solution of the Dirac equation, the last expression vanishes, so $\Psi'(x)$ is also a solution.

$$\boxed{\text{Thus, (2.22) and (2.23) have been proved as required.}}$$

2.11

2.12

It is asked for

$$(\bar{\psi}_1\psi_2)^\dagger = \bar{\psi}_2\psi_1$$

It is given that $\bar{\psi} = \psi^\dagger \beta = \psi^\dagger i\gamma^0$. And γ matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

where $\sigma_\mu = (-1, \sigma_i) = \bar{\sigma}^\mu$, $\bar{\sigma_\mu} = (1, \sigma_i) = \sigma^\mu$ and

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \bar{\sigma}^0 & 0 \end{pmatrix}$$

$$\beta = i\gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

It is easy to see that $(i\gamma^0)^\dagger = \beta^\dagger = \beta$.
Lets get back to the question

$$(\bar{\psi}_1 \psi_2)^\dagger = (\psi_1^\dagger \beta \psi_2)^\dagger = (\psi_2^\dagger \beta^\dagger \psi_1) = \bar{\psi}_2 \psi_1$$

As you can see, I did not use any symmetry(commutation) or anti-symmetry (anti-commutation) relation here, so identity is valid for both.

2.13

Assume that $(p^1, p^2, p^3) = |\vec{p}|(\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$. Find $\xi(\vec{p}, \pm)$ explicitly. We are asked to find the normalized spinors $\xi(\vec{p}, \pm)$ satisfying

$$\vec{\sigma} \cdot \hat{p}\ \xi(\vec{p}, \pm) = \pm\xi(\vec{p}, \pm),$$

where

$$\hat{p} = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta).$$

The Pauli matrix combination is

$$\begin{aligned} \vec{\sigma} \cdot \hat{p} &= \sigma_1 \sin\beta\cos\alpha + \sigma_2 \sin\beta\sin\alpha + \sigma_3 \cos\beta \\ &= \begin{pmatrix} \cos\beta & \sin\beta\, e^{-i\alpha} \\ \sin\beta\, e^{i\alpha} & -\cos\beta \end{pmatrix}. \end{aligned}$$

Consider the eigenvector equation for eigenvalue $+1$:

$$\begin{pmatrix} \cos\beta & \sin\beta\, e^{-i\alpha} \\ \sin\beta\, e^{i\alpha} & -\cos\beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

This yields two equations:

$$\begin{aligned} \cos\beta\, a + \sin\beta\, e^{-i\alpha} b &= a \\ \sin\beta\, e^{i\alpha} a - \cos\beta\, b &= b \end{aligned}$$

which simplify to

$$\begin{aligned} (\cos\beta - 1)a + \sin\beta\, e^{-i\alpha} b &= 0 \\ \sin\beta\, e^{i\alpha} a + (-\cos\beta - 1)b &= 0. \end{aligned}$$

From the first equation,

$$b = \frac{1-\cos\beta}{\sin\beta} e^{i\alpha} a.$$

A convenient and standard choice that gives a normalized spinor is

$$a = \cos\frac{\beta}{2}, \qquad b = \sin\frac{\beta}{2} e^{i\alpha}.$$

So

$$\xi(\vec{p},+) = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} e^{i\alpha} \end{pmatrix}.$$

Similarly, for eigenvalue -1, set

$$\xi(\vec{p},-) = \begin{pmatrix} c \\ d \end{pmatrix},$$

and solve

$$\begin{aligned} (\cos\beta+1)c + \sin\beta\, e^{-i\alpha} d &= 0 \\ \sin\beta\, e^{i\alpha} c + (-\cos\beta+1) d &= 0 \end{aligned}$$

which gives

$$d = -\frac{\cos\beta+1}{\sin\beta} e^{i\alpha} c.$$

A standard normalized solution is

$$c = -\sin\frac{\beta}{2} e^{-i\alpha}, \qquad d = \cos\frac{\beta}{2}.$$

So

$$\xi(\vec{p},-) = \begin{pmatrix} -\sin\frac{\beta}{2} e^{-i\alpha} \\ \cos\frac{\beta}{2} \end{pmatrix}.$$

Both spinors are normalized

$$\xi(\vec{p},\pm)^\dagger \xi(\vec{p},\pm) = 1.$$

Thus,

$$\boxed{\begin{aligned} \xi(\vec{p},+) &= \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} e^{i\alpha} \end{pmatrix} \\ \xi(\vec{p},-) &= \begin{pmatrix} -\sin\frac{\beta}{2} e^{-i\alpha} \\ \cos\frac{\beta}{2} \end{pmatrix} \end{aligned}}$$

2.14

2.15

$$u(p) = \begin{pmatrix} \sqrt{-\sigma\cdot p}\,\xi \\ i\sqrt{\bar\sigma\cdot p}\,\xi \end{pmatrix}$$

$$v(p) = \begin{pmatrix} \sqrt{-\sigma\cdot p}\,\eta \\ i\sqrt{\bar\sigma\cdot p}\,\eta \end{pmatrix}$$

and we know that

$$\xi(\vec{p},\pm)^\dagger \xi(\vec{p},\pm) = 1$$

Also

$$-\sigma\cdot p\bar{\sigma}\cdot p=-\bar{\sigma}\cdot p\sigma\cdot p=m^2$$

is given.
Consider $\xi(\vec{p},+)=\xi_+$ and $\xi(\vec{p},-)=\xi_-$

$$\bar{u}(\vec{p},+)u(\vec{p},+)=u^\dagger\beta u=\left(\sqrt{-\sigma\cdot p}\,\xi_+^\dagger,-i\sqrt{\bar{\sigma}\cdot p}\,\xi_+^\dagger,\right)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}\sqrt{-\sigma\cdot p}\,\xi_+\\ i\sqrt{\bar{\sigma}\cdot p}\,\xi_+\end{pmatrix}$$

$$=\left(-\sqrt{\bar{\sigma}\cdot p}\,\xi_+^\dagger,i\sqrt{-\sigma\cdot p}\,\xi_+^\dagger\right)\begin{pmatrix}\sqrt{-\sigma\cdot p}\,\xi_+\\ i\sqrt{\bar{\sigma}\cdot p}\,\xi_+\end{pmatrix}=\xi_+^\dagger\xi_+(-\sqrt{-\bar{\sigma}\cdot p\sigma\cdot p}-\sqrt{-\sigma\cdot p\bar{\sigma}\cdot p})=-2m$$

for the spins which are not same , it is obvious that the last part would be $\xi_+^\dagger\xi_-=0$ or $\xi_-^\dagger\xi_+=0$ instead of $\xi_+^\dagger\xi_+=1$
So in general

$$\bar{u}(\vec{p},s)u(\vec{p},s')=-2m\delta_{ss'}$$

Since equation (2.41) gives the relation

$$\eta(\vec{p},\pm)=-\sigma_2\xi(\vec{p},\pm)^*$$

and using this we get

$$\eta(\vec{p},\pm)^\dagger\eta(\vec{p},\pm)=1$$

and

$$\bar{v}(\vec{p},+)v(\vec{p},+)=v^\dagger\beta v=\left(\sqrt{-\sigma\cdot p}\,\eta_+^\dagger,i\sqrt{\bar{\sigma}\cdot p}\,\eta_+^\dagger\right)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}\sqrt{-\sigma\cdot p}\,\eta_+\\ -i\sqrt{\bar{\sigma}\cdot p}\,\eta_+\end{pmatrix}$$

$$=\left(\sqrt{\bar{\sigma}\cdot p}\,\eta_+^\dagger,i\sqrt{-\sigma\cdot p}\,\eta_+^\dagger\right)\begin{pmatrix}\sqrt{-\sigma\cdot p}\,\eta_+\\ i\sqrt{\bar{\sigma}\cdot p}\,\eta_+\end{pmatrix}=\eta_+^\dagger\eta_+(\sqrt{-\bar{\sigma}\cdot p\sigma\cdot p}+\sqrt{-\sigma\cdot p\bar{\sigma}\cdot p})=2m$$

and same argument is valid for v,too

$$\bar{v}(\vec{p},s)v(\vec{p},s')=2m\delta_{ss'}$$

Lets do it for u and v together

$$\bar{u}(\vec{p},+)v(\vec{p},+)=u^\dagger\beta v=\left(\sqrt{-\sigma\cdot p}\,\xi_+^\dagger,-i\sqrt{\bar{\sigma}\cdot p}\,\xi_+^\dagger\right)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}\sqrt{-\sigma\cdot p}\,\eta_+\\ -i\sqrt{\bar{\sigma}\cdot p}\,\eta_+\end{pmatrix}$$

$$=\xi_+^\dagger\eta_+\left(-\sqrt{\bar{\sigma}\cdot p},i\sqrt{-\sigma\cdot p}\right)\begin{pmatrix}\sqrt{-\sigma\cdot p}\\ -i\sqrt{\bar{\sigma}\cdot p}\end{pmatrix}=\xi_+^\dagger\eta_+(-\sqrt{\bar{\sigma}\cdot p}\sqrt{-\sigma\cdot p}+\sqrt{\bar{\sigma}\cdot p}\sqrt{-\sigma\cdot p})=0$$

Note that this time it is not the spin part that makes it zero. That is why it is independent of spin and always zero.
And the last one is

$$\bar{u}(\vec{p},+)\gamma^\mu u(\vec{p},+)=u^\dagger\beta\gamma^\mu u=\xi_+^\dagger\xi_+\left(\sqrt{-\sigma\cdot p}\;-i\sqrt{\bar{\sigma}\cdot p}\right)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}\cdot\begin{pmatrix}\sqrt{-\sigma\cdot p}\\ i\sqrt{\bar{\sigma}\cdot p}\end{pmatrix}$$

$$=\left(-\sqrt{\bar{\sigma}\cdot p}\,\xi_+^\dagger,i\sqrt{-\sigma\cdot p}\,\xi_+^\dagger\right)\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}\sqrt{-\sigma\cdot p}\\ i\sqrt{\bar{\sigma}\cdot p}\end{pmatrix}$$

$$=\left(i\bar{\sigma}^\mu\sqrt{-\sigma\cdot p}\,,-\sigma^\mu\sqrt{\bar{\sigma}\cdot p}\right)\begin{pmatrix}\sqrt{-\sigma\cdot p}\\ i\sqrt{\bar{\sigma}\cdot p}\end{pmatrix}=(i\bar{\sigma}^\mu\sqrt{-\sigma\cdot p}\sqrt{-\sigma\cdot p}-i\sigma^\mu\sqrt{\bar{\sigma}\cdot p}\sqrt{\bar{\sigma}\cdot p})$$

$$=-i(\bar{\sigma}^\mu\sigma^\nu+\bar{\sigma}^\nu\sigma^\mu)p_v=-2i\eta^{\nu\mu}p_v=-2ip^\mu$$

Same calculations can be made for v as well.

2.16

We are asked to show that the equations

$$\bar{\sigma}^\mu \partial_\mu \psi(x) = 0$$

$$\sigma^\mu \partial_\mu \bar{\chi}(x) = 0$$

are Lorentz invariant and that each equation implies

$$\Box\psi(x) = 0, \qquad \Box\bar{\chi}(x) = 0,$$

where

$$\Box = \partial^\mu \partial_\mu.$$

Under a Lorentz transformation, the coordinates and fields transform as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\psi(x) \to \psi'(x) = L(\lambda)^{-1}\psi(\Lambda x)$$

$$\bar{\chi}(x) \to \bar{\chi}'(x) = \bar{L}(\lambda)^{-1}\bar{\chi}(\Lambda x)$$

Consider the transformation of the first equation:

$$\partial_\mu \psi(x) \to \partial_\mu \psi'(x) = L(\lambda)^{-1}\Lambda^\nu \mu \partial' \nu \psi(\Lambda x)$$

where $\partial' \nu = \frac{\partial}{\partial x'^\nu}$. Therefore,

$$\begin{aligned}
\bar{\sigma}^\mu \partial \mu \psi(x) &\to \bar{\sigma}^\mu \partial_\mu \psi'(x) \\
&= \bar{\sigma}^\mu L(\lambda)^{-1}\Lambda^\nu \mu \partial' \nu \psi(\Lambda x) \\
&= L(\lambda)^{-1}\left[L(\lambda)\bar{\sigma}^\mu L(\lambda)^{-1}\Lambda^\nu \mu\right]\partial' \nu \psi(\Lambda x)
\end{aligned}$$

Using the Lorentz transformation property of the sigma matrices,

$$L(\lambda)\bar{\sigma}^\mu L(\lambda)^{-1} = \bar{\sigma}^\rho \Lambda^\mu \rho,$$

we obtain

$$\begin{aligned}
\bar{\sigma}^\mu \partial \mu \psi'(x) &= L(\lambda)^{-1}\bar{\sigma}^\rho \Lambda^\mu \rho \Lambda^\nu \mu \partial' \nu \psi(\Lambda x) \\
&= L(\lambda)^{-1}\bar{\sigma}^\rho \delta^\nu \rho \partial' \nu \psi(\Lambda x) \\
&= L(\lambda)^{-1}\bar{\sigma}^\nu \partial' \nu \psi(\Lambda x)
\end{aligned}$$

Therefore, the equation $\bar{\sigma}^\mu \partial_\mu \psi(x) = 0$ is Lorentz invariant, since if it holds in one frame, it also holds in the transformed frame. The same logic applies to the equation

$$\sigma^\mu \partial_\mu \bar{\chi}(x) = 0,$$

so it is also Lorentz invariant. Next, we show that these equations imply the massless Klein-Gordon equation. Starting from

$$\bar{\sigma}^\mu \partial_\mu \psi(x) = 0,$$

apply $\sigma^\nu \partial_\nu$ from the left:

$$\sigma^\nu \partial_\nu \bar{\sigma}^\mu \partial_\mu \psi(x) = 0.$$

Using the identity
$$\sigma^\nu\bar{\sigma}^\mu = \eta^{\nu\mu}\mathbb{1} + \sigma^{\nu\mu},$$
where $\sigma^{\nu\mu}$ is antisymmetric in (ν, μ) and the partial derivatives commute (so their contraction with the antisymmetric part vanishes), we have

$$\sigma^\nu\partial_\nu\bar{\sigma}^\mu\partial_\mu\psi(x) = \eta^{\nu\mu}\partial_\nu\partial_\mu\psi(x) = \Box\psi(x) = 0.$$

Similarly, starting from
$$\sigma^\mu\partial_\mu\bar{\chi}(x) = 0,$$
applying $\bar{\sigma}^\nu\partial_\nu$ from the left gives
$$\bar{\sigma}^\nu\partial_\nu\sigma^\mu\partial_\mu\bar{\chi}(x) = 0,$$
and using the same identity,
$$\Box\bar{\chi}(x) = 0.$$
Thus, both $\psi(x)$ and $\bar{\chi}(x)$ satisfy the massless Klein-Gordon equation as a consequence of their respective Weyl equations.

2.17

2.18

ψ is right handed weyl spinor and its representation is $(\frac{1}{2}, 0)$. $\bar{\chi}$ is left handed weyl spinor and its representation is $(0, \frac{1}{2})$.
σ and $\bar{\sigma}$ sends spinors one representation to another.

$$\sigma : (\frac{1}{2}, 0) \to (0, \frac{1}{2})$$

$$\bar{\sigma} : (0, \frac{1}{2}) \to (\frac{1}{2}, 0)$$

One of the condition for Lorentz invariance is matching representation of RHS and LHS. When we check the given equation
$$\bar{\sigma}^\mu\partial_\mu\psi = m\psi$$
As I stated $\bar{\sigma}^\mu$ sends ψ which is represented by $(\frac{1}{2}, 0)$ to $(0, \frac{1}{2})$.
So LHS is $(0, \frac{1}{2})$ and RHS is $(\frac{1}{2}, 0)$. So it is not Lorentz invariant. To learn more about spinors and spinor algebra check the chapter 1 of "Introduction to Supersymmetry" by Wiedemann and Müller-Kirsten.

2.19

We can write the Dirac field as the column

$$\Psi(x) = \begin{pmatrix}\psi(x) & \bar{\chi}(x)\end{pmatrix}$$

where $\psi(x)$ and $\bar{\chi}(x)$ are two-component spinors. The gamma matrices in the Weyl (chiral) representation are given by
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$
where
$$\sigma^\mu = (\mathbb{1}, \sigma^i), \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

and $\mathbb{1}$ is the 2×2 unit matrix, while σ^i are the Pauli matrices. Now, let us write the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0$$

in terms of $\psi(x)$ and $\bar{\chi}(x)$ components. Explicitly,

$$i\gamma^\mu\partial_\mu\Psi(x) = i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \psi(x) \\ \bar{\chi}(x) \end{pmatrix} = i\begin{pmatrix} \sigma^\mu\partial_\mu\bar{\chi}(x) \\ \bar{\sigma}^\mu\partial_\mu\psi(x) \end{pmatrix}$$

and

$$m\Psi(x) = m\begin{pmatrix} \psi(x) & \bar{\chi}(x) \end{pmatrix}$$

Thus, the Dirac equation becomes

$$i\begin{pmatrix} \sigma^\mu\partial_\mu\bar{\chi}(x) & \bar{\sigma}^\mu\partial_\mu\psi(x) \end{pmatrix} - m\begin{pmatrix} \psi(x) & \bar{\chi}(x) \end{pmatrix} = 0$$

which can be written as two coupled equations

$$\sigma^\mu\partial_\mu\bar{\chi}(x) - m\psi(x) = 0 \; \bar{\sigma}^\mu\partial_\mu\psi(x) - m\bar{\chi}(x) = 0$$

or, equivalently,

$$\bar{\sigma}^\mu\partial_\mu\psi(x) = m\bar{\chi}(x) \; \sigma^\mu\partial_\mu\bar{\chi}(x) = m\psi(x)$$

Thus, it is possible to write the equations as

$$\boxed{\bar{\sigma}^\mu\partial_\mu\psi(x) = m\bar{\chi}(x) \; \sigma^\mu\partial_\mu\bar{\chi}(x) = m\psi(x)}$$

2.20

2.21

$$S = -\int d^Dx\bar{\psi}[\gamma^\mu\partial_\mu - m]\psi$$

adding the given term

$$S' = -\int d^Dx\bar{\psi}[\gamma^\mu\partial_\mu - m]\psi + \int d^Dx\frac{1}{2}\partial_\mu(\bar{\psi}\gamma^\mu\psi) = -\int d^Dx(\bar{\psi}\gamma^\mu\partial_\mu\psi - m\psi\bar{\psi}) + \int d^Dx\frac{1}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi$$
$$+\int d^Dx\frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi = -\int d^Dx\frac{1}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi) - m\psi\bar{\psi}$$
$$= -\int d^Dx[\frac{1}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\psi - m\bar{\psi}\psi]$$

2.22

2.23

2.24

Using the equation (2.21) get the given variation

$$\psi'(x) = L^{-1}(\lambda)\psi(\Lambda x)$$

and $L(\lambda) = e^{\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}}$ where

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \lambda^\mu_\nu$$

taylor expanding in first order

$$L^{-1}(\lambda)\psi(\Lambda x) = (1 - \frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu})\psi(x^{\mu} + \lambda^{\mu\nu}x_{\nu}) = (1 - \frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu})[\psi(x) + \lambda^{\mu\nu}x_{\nu}\partial_{\mu}\psi(x)]$$

We know that $L_{\mu\nu} = (x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$, and keeping the equation in first order $O(\lambda^1)$

$$\delta\psi = -\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\psi - \frac{1}{2}\lambda^{\mu\nu}L_{\mu\nu}\psi$$

We get the variation of the ψ, to get the $\psi^{\dagger}$, just rewrite $L_{\mu\nu} = (x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$ and take dagger of the $\delta\psi$, L part wouldn't change, and we get

$$\delta\psi^{\dagger} = -\frac{1}{2}\lambda^{\mu\nu}\psi^{\dagger}\Sigma^{\dagger}_{\mu\nu} - \frac{1}{2}\lambda^{\mu\nu}(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\psi^{\dagger}$$

Second part of the question is how would be the noether current if we add $\frac{1}{2}\partial_{\mu}(\bar{\Psi}\gamma^{\mu}\Psi)$ to the action.

$$S' = -\int d^D x\bar{\psi}[\gamma^{\mu}\partial_{\mu} - m]\psi + \int d^D x\frac{1}{2}\partial_{\mu}(\bar{\Psi}\gamma^{\mu}\Psi)$$

and Noether current can be found using

$$J^{\mu} = -\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\Psi)}\delta\Psi - \frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\bar{\Psi})}\delta\bar{\Psi} + F^{\mu}$$

where $\delta\mathcal{L} = \partial_{\mu}F^{\mu}$ Under spatial translation

$$\delta\Psi = \epsilon^{\mu}\partial_{\mu}\Psi$$

and Lagrangian density is a scalar function

$$\delta\mathcal{L} = \epsilon^{\mu}\partial_{\mu}\mathcal{L}$$

in the question $\mathcal{L} = -\bar{\psi}[\gamma^{\mu}\partial_{\mu} - m]\psi + \frac{1}{2}\partial_{\mu}(\bar{\Psi}\gamma^{\mu}\Psi)$ and we get

$$J^{\mu} = (\bar{\Psi}\gamma^{\mu} - \frac{1}{2}\bar{\Psi}\gamma^{\mu})(\epsilon^{\mu}\partial_{\mu}\Psi) - \frac{1}{2}\gamma^{\mu}\Psi(\epsilon^{\mu}\partial_{\mu}\bar{\Psi}) + \epsilon^{\mu}\mathcal{L}$$

$$= (\frac{1}{2}\epsilon^{\nu}\bar{\Psi}\gamma^{\mu}\partial_{\nu}\Psi - \frac{1}{2}\epsilon^{\nu}\partial_{\nu}\bar{\Psi}\gamma^{\mu}\Psi + \epsilon^{\mu}\mathcal{L}) = \frac{1}{2}\epsilon^{\nu}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\nu}\psi + \epsilon^{\mu}\mathcal{L}$$

$$J^{\mu} = \frac{1}{2}\epsilon^{\nu}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\nu}\psi + \epsilon^{\mu}\mathcal{L}$$

In the third part of the question it is asked for edited version of the Lorentz generator M, just put the new symmetric energy-momentum tensor to the (1.74) instead of the previous one.