Schwarzschild solution

Schwarzschild solution is a matter free solution which means

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}\right) \quad \text{where } d=4 \quad \text{and} \quad T = g^{\mu\nu} T_{\mu\nu}$$

$$R - 2R = 8\pi G\, T$$

$$R = -8\pi G\, T$$

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(8\pi G T) = 8\pi G\, T_{\mu\nu} \Rightarrow R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

Matter free means $T_{\mu\nu}=0$ and when it is placed to the above equation we get $\underline{R_{\mu\nu}=0}$.

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2 \qquad \text{where } d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$\alpha = \alpha(r)$ and $\beta = \beta(r)$

non-zero components of Christoffels are

$$\Gamma^t_{tr} = \partial_r \alpha \qquad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\,\partial_r\alpha \qquad \Gamma^r_{rr} = \partial_r\beta$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta} \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}$$

$$\Gamma^r_{\varphi\varphi} = -re^{-2\beta}\sin^2\theta \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

Thus, components of the Ricci tensor are given as

$$R_{\mu\nu} = \partial_\nu \Gamma^\rho_{\mu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} = 0$$

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] = 0 \qquad (1)$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\beta = 0 \qquad (2)$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 = 0 \qquad (3)$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta} = 0 \qquad (4)$$

(1) includes the terms we are looking for

thus $\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\alpha = 0$

and $\quad -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\beta = 0$

$$\frac{2}{r}(\partial_r\alpha + \partial_r\beta) = 0 \quad \Rightarrow \alpha = -\beta$$

this leads (3) to be

$$e^{2\alpha}\left[r(2\partial_r\alpha) - 1\right] + 1 = 0$$

$$\frac{dt}{dr_*} = \pm 1$$

so

$$t = \pm r_* + \text{constant}$$

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr_*^2) + r^2 d\Omega^2$$

But still, $r \to 2M$ goes infinity

Lets define

$$u = t + r^* \qquad d(t \pm r^*) = 0 \text{ on radial null geodesics}$$

so that u is constant along ingoing radial null geodesics. Now lets use (u, r, θ, φ) as coordinates instead of (t, r, θ, φ). The new coordinates are called ingoing Eddington-Finkelstein coordinates. We eliminate t by $t = u - r_*(r)$ and hence

$$dt = du - \frac{dr}{1 - \frac{2M}{r}}$$

substituting this into metric gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 + 2du\,dr + r^2 d\Omega^2$$

written as a matrix we have, in these coordinates

$$g_{\mu\nu} = \begin{pmatrix} -(1-2M/r) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Unlike the metric components in Schwarzschild coordinates, the components of the above matrix are smooth for all $r > 0$, in particular they are smooth at $r = 2M$. Furthermore, this matrix has determinant $-r^4 \sin^2\theta$ and hence is non-degenerate for any $r > 0$ (except at $\theta = 0, \pi$ but this is just because the coordinates θ, φ are not defined at the poles of the sphere). This implies that its signature is Lorentzian for $r > 0$ since a change of signature would require an eigenvalue passing through zero.

Coordinate transformation ...

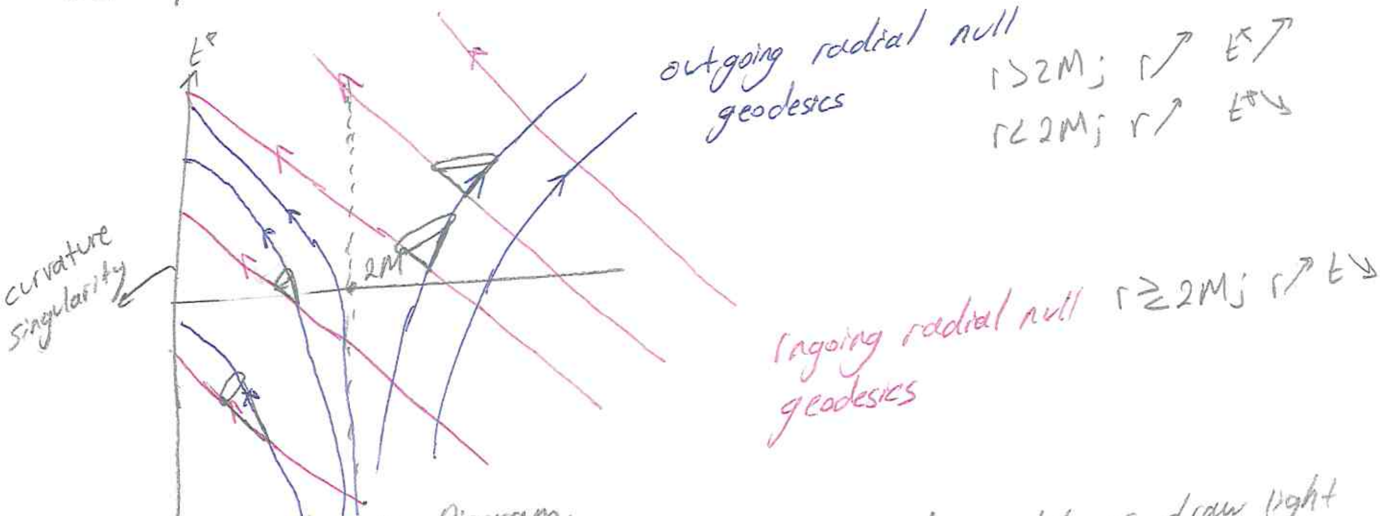
coordinates.

There are two ways for null geodesics, either $du = 0$ or

$$-\left(1-\frac{2M}{r}\right)du^2 + 2\,du\,dr = 0$$

$$\frac{du}{dr} = \begin{cases} 0 & \text{, ingoing} \\ \dfrac{2}{1-\frac{2M}{r}} & \text{, outgoing} \end{cases}$$

Lets plot the radial null geodesic by defining $t^* = u - r$



Finkelstein Diagram.

Knowing the ingoing and outgoing radial null geodesics let us draw light "cones" on the diagram. Radial timelike curves have tangent vectors that lie inside the light cone at any point.

The "outgoing" radial null geodesics have increasing r if $r>2M$. But if $r<2M$ then r decreases for both families of null geodesics. Both reach the curvature singularity at $r=0$ in finite affine parameter. Since nothing can travel faster than light, the same is true for radial timelike curves.

It will be shown that r decreases along any timelike or null curves (irrespective of whether or not it is radial or geodesics) in $r<2M$. Hence no signal can be sent from a point with $r<2M$ to a point with $r>2M$, in particular to a point with $r\to\infty$.

This is the defining property of a black hole: a region of an "asymptotically flat" spacetime from which it is impossible to send a signal to infinity

A way of checking if a singularity is coordinate singularity or not is looking for a scalar, for example the simplest non-trivial scalar constructed from the metric is Kretschmann scalar

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48 m^2}{r^6}$$

This object is singular at $r=0$ where $r=2m$ is finite. That is why $r=2m$ is a coordinate singularity, $r=0$ is an example of curvature singularity. Strictly speaking, $r=0$ is not part of the space time manifold because the metric is not defined there.

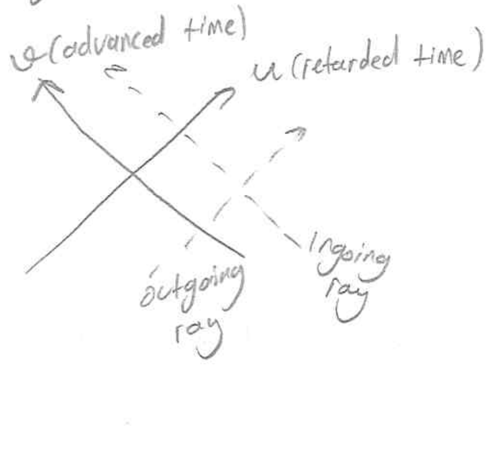
White Holes

In black holes nothing could escape once the point $r=2M$ is passed. We used the EF coordinates (v, r) to pass the horizon through the future, however we couldnt pass it towards the past. Now we can use u instead of v and metric becomes

$$ds^2 = -\left(1-\frac{2M}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2 \quad \text{where} \quad u = t - r^*$$

Now, we can pass the past horizon and we expanded the space-time in two ways past and future.

Lets use the v and u coordinates instead of t and r we get



$$ds^2 = -\frac{1}{2}\left(1-\frac{2M}{r}\right)(2\,du\,dv) + r^2 d\Omega^2$$

and r is a function of u and v

$$= \frac{1}{2}(u-v) = r + 2m \ln\left|\frac{r}{2m}-1\right|$$

u and v goes to $-\infty$ and $+\infty$ respectively when $r=2m$.

$$v - u = t + r^* - (t - r^*) = 2r^*\Big|_{r=2m} = \infty$$

and it is still a coordinate singularity.

To see how this coordinate singularity might be removed, we focus our attention on a small neighbourhood of the $r=2M$, in which the relation $r^*(r)$ can be approximated by $r^* \simeq 2M \ln\left|\frac{r}{2M}-1\right|$. This implies that

$$e^{r^*/2M} = \left|\frac{r}{2M}-1\right| \Rightarrow \frac{r}{2M}-1 = \pm e^{r^*/2M} \Rightarrow \frac{r}{2M} = 1 \pm e^{(v-u)/4M}$$

Here and below, the upper sign refers to the part of the neighbourhood corresponding to $r>2M$, while the lower sign refers to $r<2M$.

$$-\left(1-\frac{2M}{r}\right) = -\left(1-\left(1\pm e^{(v-u)/4M}\right)^{-1}\right) = -\left(1-\left(1\mp e^{(v-u)/4M}\right)\right)$$

$$= -\left(\cancel{1}-\cancel{1}\pm e^{(v-u)/4M}\right) = \mp e^{(v-u)/4M}$$

So the metric becomes

$$ds^2 \simeq \mp\left(e^{-u/4M}\,du\right)\left(e^{v/4M}\,dv\right) + r^2 d\Omega^2$$

This expression motivates the introduction of a new set of null coordinates, U and V, defined by

$$U = \mp e^{-u/4M}, \quad V = e^{v/4M}$$

In terms of these, the metric will be well behaved near $r=2M$. Going back to the exact expression for r^*, we have that $e^{r^*/2M} = e^{(v-u)/4M} = \mp UV$

or $$e^{r/2M}\left(\frac{r}{2M}-1\right) = -UV \quad \text{when } r^* = r + 2M\ln\left|\frac{r}{2M}-1\right|$$

which implicitly gives r as a function of U and V. Schwarzschild metric in new coordinates is

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}\,dU\,dV + r^2 d\Omega^2$$

This is obviously regular at $r=2M$. The coordinates U and V are called <u>null Kruskal coordinates</u>. In a Kruskal diagram (a map of the U-V plane) <u>outgoing light rays</u> move along curves $U=$ constant, while ingoing light rays move along curves $V=$ constant.

In the Kruskal coordinates, a surface of constant r is described by an equation of the form $UV=$ constant, which corresponds to a two branch hyperbola in the UV plane. (It looks like $y=\frac{c}{x}$ where c is a constant, the major difference is it is rotated)

For example, $r=2M$ becomes $UV=0$, while $r=0$ becomes $UV=1$. There are two copies of each surface $r=$ constant in a Kruskal diagram.

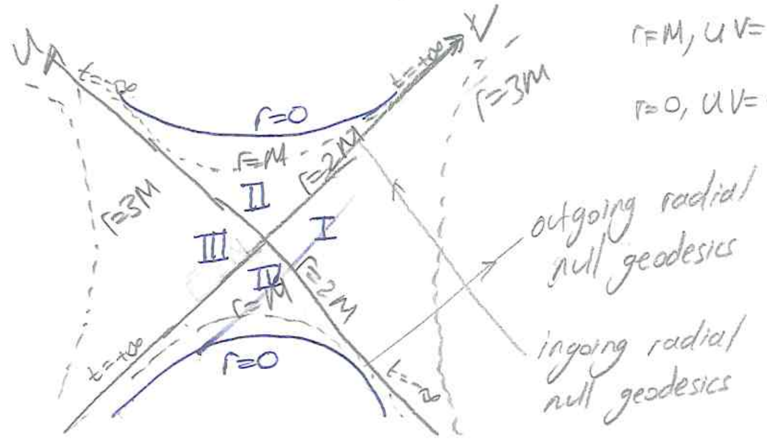
For example, $r=2M$ can be either $U=0$ or $V=0$. The Kruskal coordinates therefore reveal the existence of a much larger manifold than the portion covered by the original Schwarzschild coordinates. In a Kruskal diagram, this portion is labelled I.

The Kruskal coordinates do not only allow the continuation of the metric through $r=2M$ into region II, they also allow continuation into regions III and IV.

These additional regions, however, exist only in the maximal extension of the Schwarzschild spacetime. If the blackhole is the result of gravitational collapse, then the Kruskal diagram must be cut off at a timelike boundary representing the surface of the collapsing body. Regions III and IV then effectively disappear below the surface of the collapsing star.

$r=M$, $UV = e^{1/2}\frac{1}{2} = 0.82$

$r=0$, $UV=1$



Region I is the region we started in (the region $r>2M$ of the Schwarzschild solution). Region II is the black hole that we discovered using ingoing EF coordinates (note that these coordinates cover regions I and II of the Kruskal diagram). Region IV is the white hole that we will discover using the outgoing EF coordinates, and Region III is an entirely new region. In this region $r>2M$ and so this region is again described by the Schwarzschild solution with $r>2M$. This is a new asymptotically flat region. It is isometric to region I: the isometry is $(U,V)\to(-U,-V)$.

Note that it is impossible for an observer in Region I to send a signal to an observer in Region III. If they want to communicate then one or both of them will have to travel into Region II (and then hit the singularity)

Note that the singularity in region II appears to the future of any point. Therefore it is not appropriate to think of singularity as a "place" inside the black hole. It is more appropriate to think of it as a "time" at which tidal forces become infinite.
The black hole region is time-independent because, in Schwarzschild coordinates, it is r, not t, that plays the role of the time. The region can be thought as describing a homogeneous but anisotropic universe approaching a "big crunch". Conversly, the white hole singularity resembles a "big bang" singularity

White Holes revisited

We defined ingoing EF coordinates using ingoing radial null geodesics. Lets try the same procedure for outgoing radial null geodesics. Starting with the Schwarzschild solution in Schwarzschild coordinates with $r>2M$

$$u = t - r^*$$

so u = constant along outgoing radial null geodesics. Now introduce outgoing EF (u, r, θ, φ). The Schwarzschild metric becomes

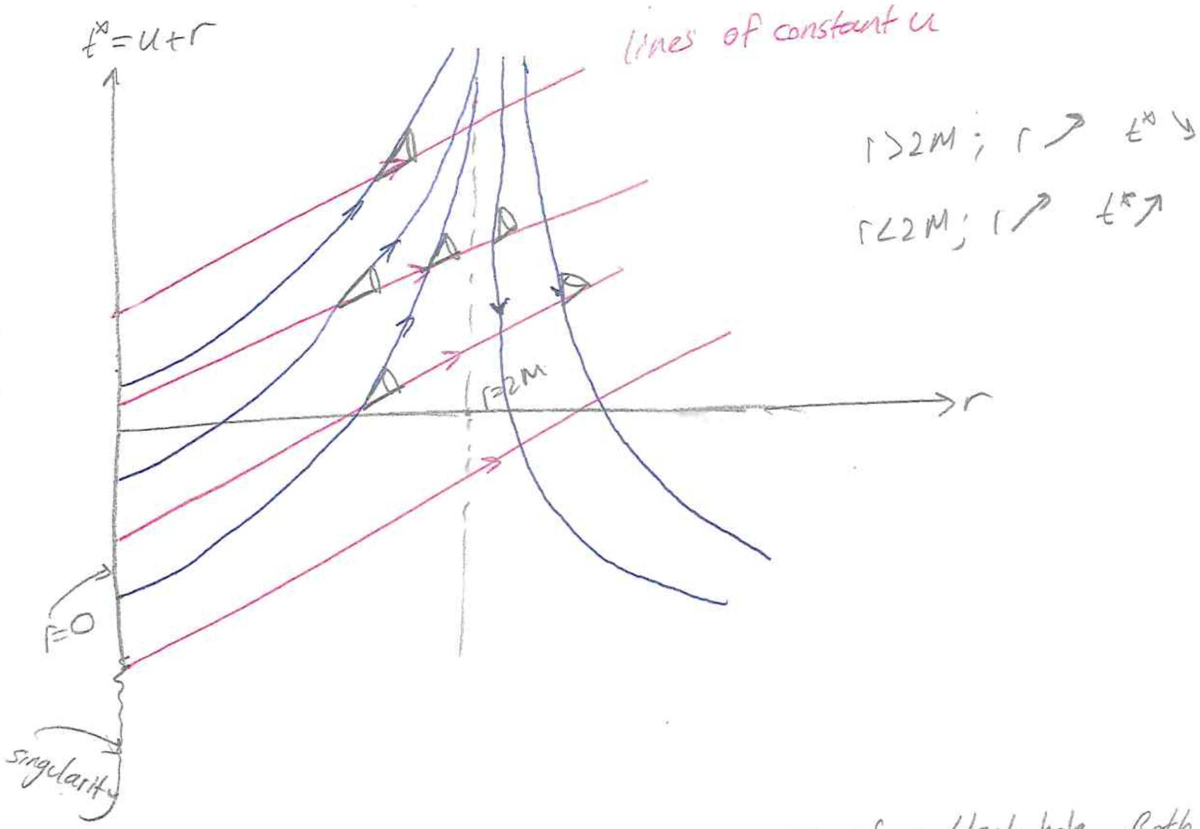
$$ds^2 = -\left(1-\frac{2M}{r}\right)du^2 - 2\,du\,dr + r^2 d\Omega^2$$

This metric is smooth with non-vanishing determinant for $r>0$ just as ingoing
EF coordinates, hence can be extended to a new region $r\leq 2M$. Once again
However, the $r<2M$ region in outgoing EF coordinates is not the same as the
$r<2M$ region in ingoing EF coordinates. To see this, note that for $r\leq 2M$

$$2\,dr\,du = -ds^2 + \left(\frac{2M}{r}-1\right)du^2 + r^2 d\Omega^2 \geq 0 \quad \text{when } ds^2\leq 0$$

$dr\,du\geq 0$ on timelike or null worldlines. But $du>0$ ($dt > dr^*$) when t and r
are moving forward) for future directed worldlines, so $dr\geq 0$ with equality when
$r=2M$, $d\Omega=0$ and $ds^2=0$. In this case a star with a surface at $r<2M$ must
expand and explode through $r=2M$, as illustrated in the following Finkelstein
diagram

$$\frac{du}{dr} = \begin{cases} 0, & \text{when } u=\text{constant} \\ \dfrac{2}{\frac{2M}{r}-1}, & \text{when } u=u(r) \end{cases}$$



This is a white hole, the time reverse of a black hole. Both black and white holes are allowed by GR because of the time reversebility of Einstein's equations but white holes require very special initial conditions near the singularity, whereas black holes do not, so only black holes can occur in practice (irreversibility in thermodynamics)

Penrose-Carter Diagram

Conformal Compactification

A black hole is a "region of spacetime from which no signal can escape to infinity" (Penrose). This is unsatisfactory because "infinity" is not part of the spacetime. However the "definition" concerns the causal structure of spacetime which is unchanged by conformal compactification.

$$ds^2 \to d\tilde{s}^2 = \Lambda^2(r,t)\, ds^2 \quad , \quad \Lambda \neq 0$$

we can choose Λ in such a way that all points at ∞ in the original metric are at finite affine parameter in the new metric. For this to happen we must choose Λ such that

$$\Lambda(\vec{r},t) \to 0 \quad \text{as } |\vec{r}| \to \infty \quad \text{and/or } |t| \to \infty$$

In this case "infinity" can be identified as those points $(\vec{r},t)$ for which $\Lambda(\vec{r},t) = 0$. These points <u>are not</u> part of the original spacetime but they can be added to it to yield a conformal compactification of the spacetime.

Example 1

Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

Let $\left.\begin{array}{l} u = t - r \\ v = t + r \end{array}\right\} \to ds^2 = -du\,dv + \dfrac{(u-v)^2}{4} d\Omega^2$

Now set

$$\left.\begin{array}{ll} u = \tan\tilde{u} & -\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2} \\ v = \tan\tilde{v} & -\frac{\pi}{2} < \tilde{v} < \frac{\pi}{2} \end{array}\right\} \quad \text{with } \tilde{v} \geq \tilde{u} \text{ since } r \geq 0$$

In these coordinates

$$ds^2 = (2\cos\tilde{u}\cos\tilde{v})^{-2}\left[-4\,d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\,d\Omega^2\right]$$

To approach infinity in this metric we must take $|\tilde{u}| \to \frac{\pi}{2}$ or $|\tilde{v}| \to \frac{\pi}{2}$ so by choosing $\Lambda = (2\cos\tilde{u}\cos\tilde{v})$

we bring these points to finite affine parameter in the new metric

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\,d\Omega^2$$

We can now add the "points at infinity". Taking the restriction $\tilde{v} \geq \tilde{u}$ into account

$$\left.\begin{matrix}\tilde{u} = -\frac{\pi}{2} \\ \tilde{v} = \frac{\pi}{2}\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} u \to -\infty \\ v \to +\infty\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} r \to \infty \\ t \to \text{finite}\end{matrix}\right\} \text{ spatial infinity, } i_0$$

$$\left.\begin{matrix}\tilde{u} = \pm\frac{\pi}{2} \\ \tilde{v} = \pm\frac{\pi}{2}\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} u \to \pm\infty \\ v \to \pm\infty\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} t \to \pm\infty \\ r \to \text{finite}\end{matrix}\right\} \text{ past and future temporal infinity, } i_\pm$$

$$\left.\begin{matrix}\tilde{u} = -\frac{\pi}{2} \\ |\tilde{v}| \neq \frac{\pi}{2}\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} u \to -\infty \\ v \to \text{finite}\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} r \to \infty \\ t \to -\infty \\ r+t \text{ finite}\end{matrix}\right\} \text{ past null infinity } \mathscr{I}^-$$

$$\left.\begin{matrix}|\tilde{u}| \neq \frac{\pi}{2} \\ \tilde{v} = \frac{\pi}{2}\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} u \to \text{finite} \\ v \to +\infty\end{matrix}\right\} \longleftrightarrow \left.\begin{matrix} r \to \infty \\ t \to \infty \\ t-r \text{ finite}\end{matrix}\right\} \text{ future null infinity } \mathscr{I}^+$$

Minkowski spacetime is conformally embedded in the new spacetime with metric $d\tilde{s}^2$ with boundary at $\Lambda = 0$

Introducing the new time and space coordinates τ, χ by

$$\tau = \tilde{v} + \tilde{u}, \quad \chi = \tilde{v} - \tilde{u}$$

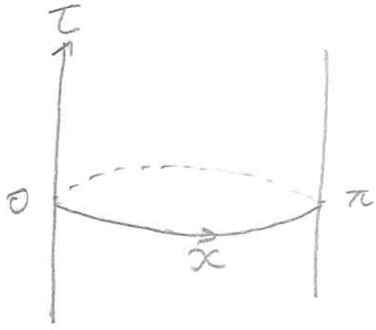
we have

$$d\tilde{s}^2 = \Lambda ds^2 = -d\tau^2 + d\chi^2 + \sin^2\chi\, d\Omega^2$$

$$\Lambda = (\cos\tau + \cos\chi)$$

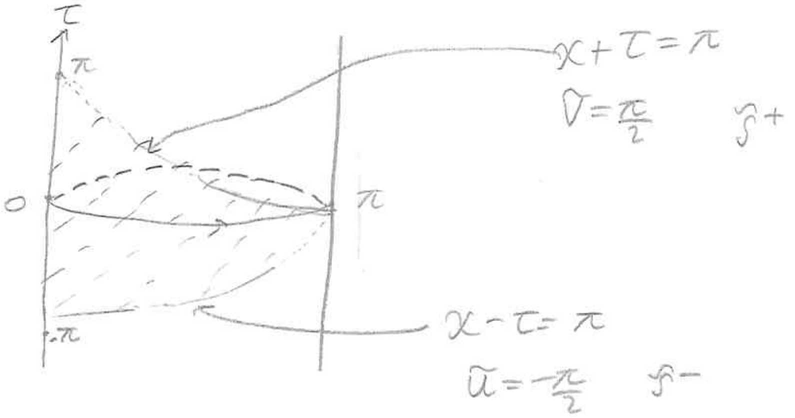
χ is an angular variable which must be identified modulo 2π, $\chi \sim \chi + 2\pi$. If no other restriction is placed on the ranges of τ and χ, then this metric $d\tilde{s}^2$ is that of the Einstein Static Universe, of topology $\mathbb{R}$ (time) $\times S^3$ (space).

The 2-spheres of constant $\chi \neq 0, \pi$ have a radius $|\sin\chi|$ (the points $\chi = 0, \pi$ are the poles of a 3-sphere). If we represent each 2-sphere of constant χ as a point E.S.U can be drawn as a cylinder.

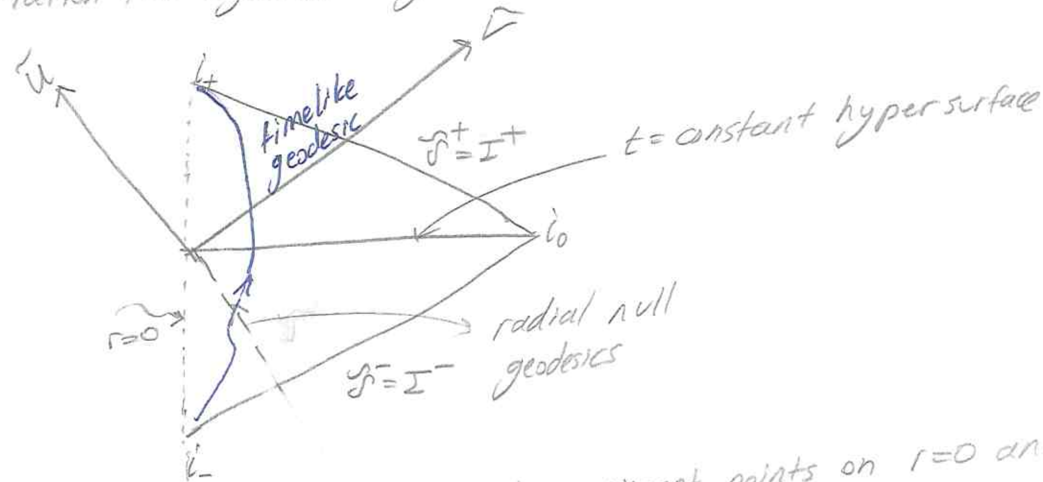


But compactified Minkowski spacetime is conformal to the triangular region

$-\pi \leq \tau \leq \pi, \quad 0 \leq \chi \leq \pi$



Flatten the cylinder to get the Carter-Penrose diagram of Minkowski spacetime



Each point represent a 2-sphere, except points on $r=0$ and i_0, $i_\pm$. Light rays travel at $45°$ from I^- through $r=0$ and then out to I^+ [$I^\pm$ are null hypersurfaces]

Spatial sections of the compactified spacetime are topologically S^3 because of the addition of the point i_0. Thus they are not only compact, but also have no boundary. This is not true for the whole spacetime. Asymptotically it is possible to identify points on the boundary of compactified spacetime to obtain a compact manifold without boundary (the group $U(2)$). More generally, this is not possible because $i_\pm$ are singular points that can not be added.

Example 2

Rindler space-time

$$ds^2 = -dU'dV'$$

Let

$$\left.\begin{array}{l} U' = \tan\tilde{U} \\ V' = \tan\tilde{V} \end{array}\right\} \begin{array}{l} -\frac{\pi}{2} < \tilde{U} < \frac{\pi}{2} \\ -\frac{\pi}{2} < \tilde{V} < \frac{\pi}{2} \end{array}$$

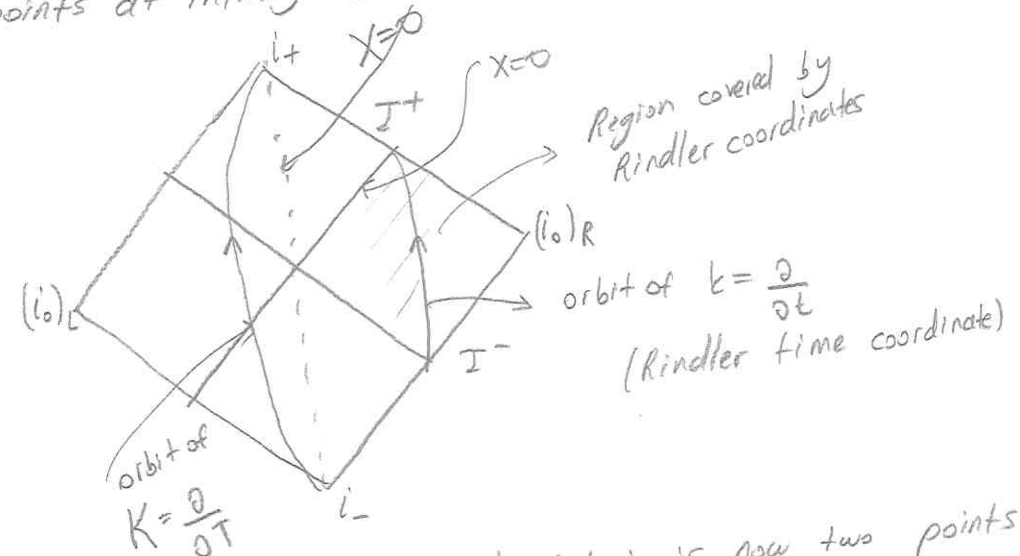
Then

$$ds^2 = -(\cos\tilde{U}\cos\tilde{V})^{-2}\, d\tilde{U}\, d\tilde{V}$$

$$= \Omega^{-2} d\tilde{s}^2 \qquad \text{where } \Omega = \cos\tilde{U}\cos\tilde{V}$$

i.e. conformally compactified space-time with metric $d\tilde{s}^2 = -d\tilde{U}\,d\tilde{V}$ is same as before but with the above <u>finite</u> ranges for coordinates $\tilde{U}, \tilde{V}$

The points at infinity are those for which $\Omega = 0$, $|\tilde{U}| = \frac{\pi}{2}$, $|\tilde{V}| = \frac{\pi}{2}$



Similar to 4-dimensional Minkowski but i_0 is now two points

Example 3

Kruskal space time

$$ds^2 = -\left(1-\frac{2M}{r}\right)du\,dv + r^2 d\Omega^2$$ in region I which black hole.

Let

$u = \tan\tilde{u}$, $-\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2}$

$v = \tan\tilde{v}$, $-\frac{\pi}{2} < \tilde{v} < \frac{\pi}{2}$

Then $$ds^2 = (2\cos\tilde{u}\cos\tilde{v})^{-2}\left[-4\left(1-\frac{2M}{r}\right)d\tilde{u}\,d\tilde{v} + r^2\cos^2\tilde{u}\cos^2\tilde{v}\,d\Omega^2\right]$$

Using the fact that

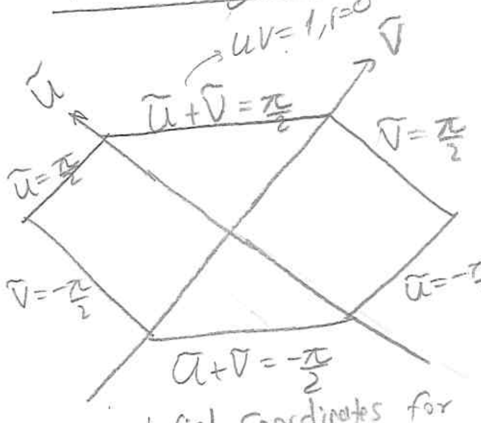
$$r^* = \frac{1}{2}(v-u) = \frac{\sin(\tilde{v}-\tilde{u})}{2\cos\tilde{u}\cos\tilde{v}}$$ $$r^* = r + 2M\ln\left|\frac{r}{2M}-1\right|$$

we have

$$d\tilde{s}^2 = \Omega^2 ds^2 = -4\left(1-\frac{2M}{r}\right)d\tilde{u}\,d\tilde{v} + \left(\frac{r}{r^*}\right)^2 \sin^2(\tilde{v}-\tilde{u})\,d\Omega^2$$

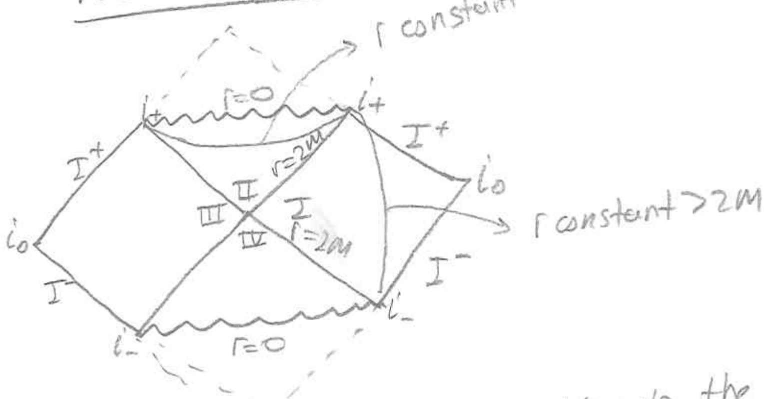
Kruskal is an example of an asymptotically flat space time. It approaches the metric of compactified Minkowski space time as $r \to \infty$ (with or without fixing t) so i_0, and $\mathcal{I}^\pm$ can be added as before. Near $r=2M$ we can introduce KS-type coordinates to pass through the horizon. In this way one can deduce that Carter-Penrose diagram for the Kruskal spacetime.

Kruskal Diagram



Compactified Coordinates for Schwarzschild space time

Penrose - Carter Diagram



The dotted lines gives the completion to the Penrose diagram of flat two dimensional minkowski space

Boundaries of the compactified Schwarzschild space time

$\mathcal{I}^+$	Future null infinity	$v=\infty$, $u=$ finite
$\mathcal{I}^-$	Past null infinity	$u=-\infty$, $v=$ finite
i_0	Spatial infinity	$r=\infty$, t finite
i_+	Future timelike infinity	$t=\infty$, $r=$ finite
i_-	Past timelike infinity	$t=-\infty$, $r=$ finite

Note

(i) All $r=$ constant hypersurfaces meet at i_+ including the $r=0$ hypersurface, which is singular, so i_+ is a singular point. Similarly for i_-, so these points can not be added

(ii) we can adjust Λ so that $r=0$ is represented by a straight line

Remember that while drawing the Penrose diagram of the Minkowski space

$$\tilde{V}=\frac{\tau+\chi}{2}, \quad \tilde{u}=\frac{\tau-\chi}{2}$$

so $$\tan\tilde{u}\tan\tilde{V}=1=\tan\left(\frac{\tau+\chi}{2}\right)\cdot\tan\left(\frac{\tau-\chi}{2}\right)=\frac{\sin\left(\frac{\tau+\chi}{2}\right)\sin\left(\frac{\tau-\chi}{2}\right)}{\cos\left(\frac{\tau+\chi}{2}\right)\cos\left(\frac{\tau-\chi}{2}\right)}$$

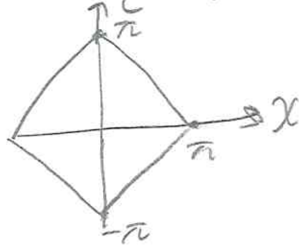
$$=\frac{\sin^2\left(\frac{\tau}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}{1-\sin^2\left(\frac{\tau}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)}$$

$$1-\sin^2\left(\frac{\tau}{2}\right)-\cancel{\sin^2\left(\frac{\theta}{2}\right)}=\sin^2\left(\frac{\tau}{2}\right)-\cancel{\sin^2\left(\frac{\theta}{2}\right)}$$

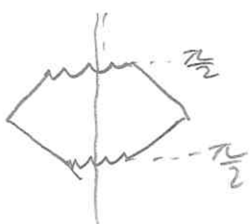
$$\sin^2\left(\frac{\tau}{2}\right)=\frac{1}{2}$$

$$\sin\left(\frac{\tau}{2}\right)=\pm\sqrt{\frac{1}{2}} \qquad \tau=\pm\frac{\pi}{2}$$

Penrose diagram of the minkowski is like



But τ was cut off at $\pm\frac{\pi}{2}$ which means



in $d>2$ dimension black hole it would be

Dotted line is $r=2M$ (Horizon)

Properties of null hypersurfaces

Let $\mathcal{N}$ be a null hypersurfaces with a normal l. A vector t, tangent to $\mathcal{N}$, is one for which $t.l=0$. But since, $\mathcal{N}$ is null, $l.l=0$, so l is itself a tangent vector: $l^\mu = \frac{dx^\mu}{d\lambda}$ for some null curve $x^\mu(\lambda)$ in $\mathcal{N}$.

Proposition: The curves $x^\mu(\lambda)$ are geodesics

Proof:

$s =$ const hypersurfaces

$n_\mu \propto -\partial_\mu s \qquad \underline{n_\mu = \tilde{f}\,\partial_\mu s}$

$n^\mu = \tilde{f} g^{\mu\nu}\partial_\nu s$

for this case $\underline{n^\mu = l^\mu}$

$l^\mu = \tilde{f} g^{\mu\nu}\partial_\nu s$, for $x^\mu(\lambda)$ to be geodesics

$l.Dl^\mu = 0$ must hold

$l^\rho D_\rho(\tilde{f} g^{\mu\nu}\partial_\nu s) = (l^\rho D_\rho \tilde{f}) g^{\mu\nu}\partial_\nu s +$

$= (l^\rho \partial_\rho \tilde{f}) g^{\mu\nu}\partial_\nu s + \tilde{f} g^{\mu\nu} l^\rho D_\nu \partial_\rho s$ (using $D_\mu D_\nu f = D_\nu D_\mu f$) symmetry of Γ

$= \left(l^\rho \frac{\partial_\rho \tilde{f}}{\tilde{f}}\right) \tilde{f} g^{\mu\nu}\partial_\nu s + \tilde{f} g^{\mu\nu} l^\rho D_\nu \partial_\rho s$

$= (l.\partial \ln\tilde{f})\, l^\mu + l^\rho \tilde{f} D^\mu (f^{-1} l_\rho)$

$= \left(\frac{dx^\rho}{d\lambda}\frac{\partial}{\partial x^\rho}\ln\tilde{f}\right) l^\mu - \frac{l^\rho l_\rho \tilde{f}\,\partial^\mu f}{\tilde{f}^2} + l^\rho D^\mu l_\rho$

$= \left(\frac{d}{d\lambda}\ln\tilde{f}\right) l^\mu - (\partial^\mu \ln\tilde{f})\, l^2 + l^\rho D^\mu l_\rho$

$= \left(\frac{d}{d\lambda}\ln\tilde{f}\right) l^\mu + \frac{1}{2} D^\mu l^2 - (\partial^\mu \ln\tilde{f})\, l^2$

$l^2\big|_{\mathcal{N}} = 0$, but it doesn't follow that $\partial^\mu l^2\big|_{\mathcal{N}} = 0$ unless the whole family of hypersurfaces $s =$ constant is null. However since l^2 is constant on $\mathcal{N}$ for any vector t tangent to $\mathcal{N}$.

$\underbrace{t^\mu \partial_\mu l^2 = 0}_{\text{change on the surface } \mathcal{N}}$ for any vector t tangent to $\mathcal{N}$.

Thus

$\partial_\mu l^2\big|_{\mathcal{N}} \propto l_\mu \propto -\partial_\mu s \quad : \quad l_\mu t^\mu = 0$

$l.Dl^\mu\big|_{\mathcal{N}} \propto l^\mu \quad : \quad l^\mu l_\mu = 0 \ : \text{RHS} = 0$ so $\quad l_\mu \underbrace{l^\alpha D_\alpha l^\mu}_{=0}$

Killing Horizons

Definition: A null hypersurface $\mathcal{N}$ is a Killing horizon of a Killing vector field ξ

If, on $\mathcal{N}$, ξ is normal to $\mathcal{N}$.

Let l be a normal to $\mathcal{N}$ such that $l^\alpha D_\alpha l^\mu = 0$ ($l \cdot D l^\mu$)

Then, since, on $\mathcal{N}$

$$\xi = f l$$

$$\xi^\alpha D_\alpha \xi^\mu = f l^\alpha D_\alpha (f l^\mu) = f^2 \underbrace{\cancel{l^\alpha D_\alpha l^\mu}}_{0} + f(l^\alpha D_\alpha f) l^\mu$$
$$= \underbrace{\frac{1}{f}(\xi^\alpha D_\alpha f)}_{\kappa} \xi^\mu$$

$$\kappa = \xi \cdot \partial \ln f$$

where κ is called surface gravity

Formula for surface gravity

Since ξ is normal to $\mathcal{N}$, "Frobenius theorem" (hypersurface orthogonality) implies that

$$\xi_{[\mu} D_\nu \xi_{\rho]}\Big|_{\mathcal{N}} = 0$$

for a killing vector field $D_\mu \xi_\nu = -D_\nu \xi_\mu = D_{[\mu}\xi_{\nu]} = C$

so symmetric part vanishes

$$(\xi_\mu D_\nu \xi_\rho - \xi_\nu D_\mu \xi_\rho)\Big|_{\mathcal{N}} + \xi_\rho D_\mu \xi_\nu\Big|_{\mathcal{N}} = 0$$

Multiplying by $D^\mu \xi^\nu$

$$(D^\mu \xi^\nu)(D_\mu \xi_\nu)\xi_\rho\Big|_{\mathcal{N}} = -(D^\mu \xi^\nu)(D_\nu \xi_\rho)\xi_\mu + \underbrace{(D^\mu \xi^\nu)(D_\mu \xi_\rho)\xi_\nu}_{= -(D^\mu \xi^\nu)(D_\nu \xi_\rho)\xi_\mu}$$

$$= -2(D^\mu \xi^\nu)(D_\nu \xi_\rho)\xi_\mu$$

$$= -2\underbrace{(\xi^\mu D_\mu \xi^\nu)}_{\kappa \xi^\nu}(D_\nu \xi_\rho)$$ (for killing horizon)

$$= -2\kappa \underbrace{\xi^\nu D_\nu \xi^\rho}_{\kappa \xi^\rho} = -2\kappa^2 \xi^\rho\Big|_{\mathcal{N}}$$

$$\kappa^2 = -\frac{1}{2}(D^\mu \xi^\nu)(D_\mu \xi_\nu)\Big|_{\mathcal{N}}$$ except $\xi = 0$ (2.89)

It will turn out that all points at which $\xi = 0$ are limit points of orbits of ξ for which $\xi \neq 0$, so continuity implies that this formula is valid even when $\xi = 0$ (Note that $\xi = 0 \not\Rightarrow D_\mu \xi_\nu = 0$)
$\Rightarrow \partial_\mu \xi_\nu = 0$

Killing Vector Lemma : For a Killing vector

- $D_\rho D_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma}\xi^\sigma$ where $R^\nu{}_{\mu\rho\sigma}$ is Riemann tensor

$[D_\rho, D_\mu]\xi^\nu = R^\nu{}_{\lambda\rho\mu}\xi^\lambda$

- $D_\rho D_\mu \xi_\nu - D_\mu D_\rho \xi_\nu = R_{\nu\lambda\rho\mu}\xi^\lambda \Rightarrow \underline{D_\rho D_\mu \xi_\nu} + D_\mu D_\nu \xi_\rho = R_{\nu\lambda\rho\mu}\xi^\lambda$
- $D_\mu D_\nu \xi_\rho - D_\nu D_\mu \xi_\rho = R_{\rho\lambda\mu\nu}\xi^\lambda \Rightarrow D_\mu D_\nu \xi_\rho + \underline{D_\nu D_\rho \xi_\mu} = R_{\rho\lambda\mu\nu}\xi^\lambda$
- $D_\nu D_\rho \xi_\mu - D_\rho D_\nu \xi_\mu = R_{\mu\lambda\nu\rho}\xi^\lambda \Rightarrow -\left(\underline{D_\nu D_\rho \xi_\mu} + D_\rho D_\mu \xi_\nu = R_{\mu\lambda\nu\rho}\xi^\lambda \right)$

$$2D_\mu D_\nu \xi_\rho = (R_{\nu\lambda\rho\mu} + R_{\rho\lambda\mu\nu} - R_{\mu\lambda\nu\rho})\xi^\lambda$$
$$= -\xi^\lambda(\underline{R_{\lambda\nu\rho\mu}} + \underline{R_{\lambda\rho\mu\nu}} - R_{\lambda\mu\nu\rho})$$

Symmetries of R

- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
- $R_{\nu\mu\rho\sigma} = -R_{\nu\mu\rho\sigma}$
- $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
- $R_{\mu[\nu\rho\sigma]} = 0$

$\rightarrow R_{\lambda\nu\rho\mu} + R_{\lambda\rho\mu\nu} + R_{\lambda\mu\nu\rho} = 0$

$$\cancel{2} D_\mu D_\nu \xi_\rho = \cancel{+2}\,\xi^\lambda(R_{\lambda\mu\nu\rho})$$
$$D_\mu D_\nu \xi_\rho = R_{\lambda\mu\nu\rho}\xi^\lambda$$
$$= R_{\nu\rho\lambda\mu}\xi^\lambda = R_{\rho\nu\mu\lambda}\xi^\lambda$$
$$\boxed{D_\mu D_\nu \xi^\rho = R^\rho{}_{\nu\mu\lambda}\xi^\lambda}$$

Proposition: K is constant on orbits of ξ

Proof: Let t be tangent to N. Then since (2.89) is valid everywhere on N

$$t^\rho \partial_\rho K^2 = -\frac{1}{2} t^\rho D_\rho (D^\mu \xi^\nu D_\mu \xi_\nu)$$
$$= -t^\rho D^\mu \xi^\nu D_\rho D_\mu \xi_\nu = -t^\rho D^\mu \xi^\nu R_{\nu\mu\rho\sigma}\xi^\sigma$$

Now ξ is tangent to N (also normal to N). Choosing $t = \xi$

$$\xi^\rho \partial_\rho K^2 = -\underbrace{\xi^\rho \xi^\sigma}_{\text{symmetric}} D^\mu \xi^\nu \underbrace{R_{\nu\mu\rho\sigma}}_{\text{anti symmetric}} = 0$$

: so K is constant on orbits of ξ
This means surface gravity does not change on the Killing Horizon.

Non-Degenerate Killing Horizons ($K \neq 0$)

Suppose $K \neq 0$ on one orbit of ξ in N. Then this orbit coincides with only part of a null generator of N. To see this choose coordinates on N such that

$\xi = \frac{\partial}{\partial \alpha}$ (except at points wher $\xi = 0$)

if $\alpha = \alpha(\lambda)$ on an orbit of ξ with affine parameter λ

$$\xi\Big|_{orbit} = \frac{d\lambda}{d\alpha}\frac{d}{d\lambda} = f\ell \qquad \begin{cases} f = \frac{d\lambda}{d\alpha} \\ \ell = \frac{d}{d\lambda} = \underbrace{\frac{dx^\mu(\lambda)}{d\lambda}}_{\ell^\mu} \partial_\mu = \ell^\mu \partial_\mu \end{cases}$$

It is shown previously when $\xi = f\ell$ $\qquad \xi = f\ell^\mu \partial_\mu = \xi^\mu \partial_\mu = \frac{\partial}{\partial \alpha}$

$$K = \xi \cdot \partial \ln f = \underline{\xi^\mu \partial_\mu} \ln f = \frac{\partial}{\partial \alpha} \ln f$$

where K is constant for orbit on N.

$f = f_0 e^{K\alpha}$. when α shifted by a constant $\frac{\partial}{\partial \alpha}$ and nothing else chang so that I can shift α to hold $f_0 = \pm K$ without loss of generality.

$$f = \frac{d\lambda}{d\alpha} = \pm K e^{K\alpha} \Rightarrow \lambda = \pm e^{K\alpha} + \text{const} \quad , \text{Let const} = 0$$

$$\lambda = \pm e^{K\alpha}$$

As α ranges from $-\infty$ to ∞ we cover the $\lambda > 0$ or the $\lambda < 0$ portion of the generator of N (geodesic in N with normal ℓ). The bifurcation point $\lambda = 0$ is a fixed point of ξ, which can be shown to be a 2-sphere, called the bifurcation 2-sphere

This is called a bifurcate Killing Horizon

<u>Proposition</u> : If $\mathcal{N}$ is a bifurcate Killing horizon of ξ, with bifurcation 2-sphe
B, then κ^2 is constant on $\mathcal{N}$.

<u>Proof</u>: κ^2 is constant on each orbit of ξ.

We know that

$$t^\mu \partial_\mu \kappa^2 = -(D^\mu \xi^\nu)\, t^\rho R_{\nu\mu\rho\sigma}\, \xi^\sigma \Big|_{\mathcal{N}}$$

$$= 0 \text{ on } B \text{ since } \xi^\sigma \Big|_B = 0$$

Since t can be any tangent to B, κ^2 is constant on B, and hence on $\mathcal{K}$
↳ Does not have to be $t = \xi$

<u>Example</u>

$\mathcal{N}$ is $\{U=0\} \cup \{V=0\}$ of Kruskal spacetime, and $\xi = k$, the time translation Killing vector field.

on $\mathcal{N}$

$$k = \left. \begin{cases} \underbrace{\frac{1}{4M} V}_{f} \overbrace{\frac{\partial}{\partial V}}^{l} & \text{on } \{U=0\} \\ \underbrace{-\frac{1}{4M} U}_{f} \underbrace{\frac{\partial}{\partial U}}_{l} & \text{on } \{V=0\} \end{cases} \right\} = fl$$

Since l is normal to $\mathcal{N}$, $\mathcal{N}$ is a Killing horizon of k. since $l \cdot D l = 0$

the surface gravity is

$$\kappa = k \cdot \partial \ln f = \begin{cases} \frac{1}{4M} V \partial_V \ln(V/4M) \text{ on } U=0 \\ -\frac{1}{4M} U \partial_U \ln(U/4M) \text{ on } V=0 \end{cases}$$

$$= \begin{cases} \frac{1}{4M} \text{ on } U=0 \\ -\frac{1}{4M} \text{ on } V=0 \end{cases}$$

So $\kappa^2 = 1/(4M)^2$ is indeed a constant on $\mathcal{N}$. Note that orbits of k lie either entirely in $U=0$ or in $V=0$ or are fixed points on B, which allows a difference of sign in κ on two branches of $\mathcal{N}$. $|\kappa| = \frac{c^4}{4GM}$

Normalization of κ

If $\mathcal{N}$ is a Killing horizon of ξ with surface gravity κ, then it is also Killing horizon of $c\xi$ with surface gravity $c^2\kappa$ for a constant c.

Thus surface gravity is not a property of $\mathcal{N}$ alone, it also depends on the normalization of ξ.

There is no natural normalization of ξ on $\mathcal{N}$ since $\xi^2=0$

But asymptotically flat spacetime, natural normalization at spatial infinity

for the time translation Killing vector field k we choose

$k^2 \to -1$ as $r\to\infty$, asymptotically $\underset{\substack{\text{future-directed}\\ \text{unit timelike vector}}}{k=\partial_t}$, $k^\mu=(1,0,0,0)$
$k_\mu = -\left(1-\frac{2M}{r}\right) = -1$

Now k is normalizable, hence κ : $\kappa = \lim_{r\to 2M} d(r) = \frac{1}{4M}$ $\quad k^2=-1$

Degenerate Killing Horizon ($\kappa=0$)

In this case group parameter $\alpha=\lambda=$ affine parameter, so there is no bifurcation 2-sphere

$$f=\frac{d\lambda}{d\alpha}=1$$

$$\kappa = \xi\cdot\partial\ln f = 0$$

#7

Return to Schwarzschild

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Horizon at $r = r_H = 2M$

Lets expand around the horizon

Let $r = r_H + \epsilon$, $\epsilon << r_H$

$$1-\frac{2M}{r} = \frac{r-2M}{r} = \frac{\epsilon}{r_H+\epsilon} \Rightarrow \frac{\epsilon}{r_H}\cdot\frac{1}{1+\frac{\epsilon}{r_H}} = \frac{\epsilon}{r_H}\left(1-\frac{\epsilon}{r_H}+O(\epsilon^2)\right)$$

$$= \frac{\epsilon}{r_H} - \cancel{\frac{\epsilon^2}{r_H^2}} = \frac{\epsilon}{r_H}$$

$$1-\frac{2M}{r} \simeq \frac{\epsilon}{r_H}$$

$$g_{tt} = -\frac{\epsilon}{r_H} , \; g_{rr} = \frac{r_H}{\epsilon} , \; d\rho^2 = g_{rr}\,dr^2$$

Proper radial distance : $\rho = \int_{r_H}^{r_H+\epsilon} \sqrt{g_{rr}}\,dr = \int_{r_H}^{r_H+\epsilon} \sqrt{\frac{r_H}{\epsilon}}\,dr \xrightarrow[d\epsilon' = dr]{\epsilon' = r - r_H} \int_0^{\epsilon} \sqrt{\frac{r_H}{\epsilon'}}\,d\epsilon'$

$$\rho = 2\sqrt{r_H\epsilon} , \; \rho^2 = 4r_H\epsilon$$

$$\epsilon = \frac{\rho^2}{4r_H} = \frac{\rho^2}{8M}$$

Now, near the horizon $(t, r$ part$)$

$$ds^2_{(2)} \simeq -\frac{\epsilon}{r_H}dt^2 + d\rho^2$$

$K = \frac{1}{4M}$ for Schwarzschild $= \frac{1}{2r_H}$

$$ds^2_{(2)} \simeq -\frac{\rho^2}{4r_H^2}dt^2 + d\rho^2$$

$$ds^2_{(2)} \simeq -K^2\rho^2 dt^2 + d\rho^2 \quad \text{and} \quad r^2 \simeq r_H^2$$

So, near the horizon

$$ds^2 = -K^2\rho^2 dt^2 + d\rho^2 + r_H^2 d\Omega^2$$

$$= \underbrace{-K^2\rho^2 dt^2 + d\rho^2}_{\text{2-dim Rindler spacetime}} + \underbrace{\frac{1}{4K^2}d\Omega^2}_{\text{2-sphere of radius } 1/2K}$$

So, we can expect to learn something about the spacetime near the Killing horizon at $r=2M$ by studying 2-dim Rindler spacetime. $(\rho = x)$

$$ds^2 = -(Kx)^2 dt^2 + dx^2 \quad (x>0)$$

$x=0$ is coordinate singularity, to handle that

$$u' = -x e^{-Kt}, \quad v' = x e^{Kt}$$

And Rindler metric becomes

$$ds^2 = -du'dv'$$

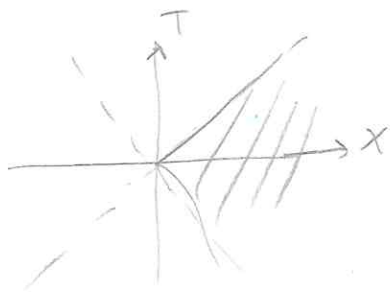
Now set

$$u' = T - X, \quad v' = T + X$$

to get

$$ds^2 = -dT^2 + dX^2, \quad T = x\sinh(Kt), \quad X = x\cosh(Kt)$$

$$X^2 - T^2 = x^2 > 0, \quad \text{so } X > |T|$$



So, Rindler coordinates with $x>0$ cover only the $u'<0$, $v'>0$ region of 2d Minkowski

From what we know about the surface $r=2M$ of Schwarzschild it follows that lines $u'=0, v'=0$ i.e. $x=0$ of Rindler is a Killing horizon of $k=\partial_t$ with surface gravity $\pm K$

i.) $n_\mu \propto \partial_\mu u' = (1,-1)$, $n_\mu = f(1,-1)$

$$n^\mu = g^{\mu\nu} n_\nu = g^{\mu T} n_T + g^{\mu X} n_X$$

$$n^T = -n_T, \; n^X = n_X, \quad n^\mu = f(-1,-1)$$

$$n^\mu n_\mu = f^2(1.(-1) + (-1)(-1)) = 0 \qquad \text{if its normal is null, itself is null}$$

similarly $l_\mu = f\partial_\mu v' = f(1,1)$

$$l^\mu = f(-1,1) \qquad l^\mu l_\mu = 0 \;//$$

ii-)

$k = \frac{1}{4M}(V\partial_V - U\partial_U)$ in Schwarzschild

which equals ∂_t in <u>region I</u>

Thus, using k being in Region I as Rindler spacetime

$$k = \underbrace{\frac{1}{4M}}_{K}(V'\partial_{V'} - U'\partial_{U'}) = K(V'\partial_{V'} - U'\partial_{U'})$$

$$k\Big|_{U'=0} = K\,V'\partial_{V'}$$

Normal to $U'=0$, $n_\mu = f\partial_\mu U' = f(1,0,0,0)$ in $(U', V', \theta, \varphi)$ coordinates

$n^\mu = g^{\mu\nu} n_\nu = g^{\mu U'} n_{U'}$

$n^\mu = (0,-2,0,0) \Rightarrow k^\mu \propto n^\mu$: killing vector is proportional to normal at $U'=0$.

So $U'=0$ is <u>Killing horizon</u>

iii-) $(k^\nu D_\nu k^\mu)\Big|_{U'=0} = k^\nu(\partial_\nu k^\mu + \Gamma^\mu_{\nu\lambda} k^\lambda)$

$= k^{U'}(\partial_{U'} k^\mu + \Gamma^\mu_{U'\lambda} k^\lambda) + k^{V'}(\partial_{V'} k^\mu + \Gamma^\mu_{V'\lambda} k^\lambda)$

$k^{U'} = -KU'$

$k^{V'} = KV'$

$= k^{U'}(\partial_{U'} k^\mu + \cancel{\Gamma^\mu_{U'U'} k^{U'}} + \cancel{\Gamma^\mu_{U'V'} k^{V'}}) + k^{V'}(\partial_{V'} k^\mu + \cancel{\Gamma^\mu_{V'U'} k^{U'}} + \cancel{\Gamma^\mu_{V'V'} k^{V'}})$

$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho})$; $g_{U'V'} = -\frac{1}{2}$

$g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2\sin^2\theta$

$\Gamma^\mu_{U'V'} = \frac{1}{2} g^{\mu\lambda}(\cancel{\partial_{U'} g_{\lambda V'}} + \cancel{\partial_{V'} g_{\lambda U'}} - \cancel{\partial_\lambda g_{U'V'}})$

$\Gamma^\mu_{U'U'} = \frac{1}{2} g^{\mu\lambda}(\cancel{2\partial_{U'} g_{U'\lambda}} - \cancel{\partial_\lambda g_{U'U'}})$

$(k^\nu D_\nu k^\mu)\Big|_{U'=0} = k^{U'}\partial_{U'} k^\mu + k^{V'}\partial_{V'} k^\mu$

$(k^\nu D_\nu k^{U'})\Big|_{U'=0} = (-KU')(-K)\Big|_{U'=0} + 0 = 0$

$(k^\nu D_\nu k^{V'})\Big|_{U'=0} = \underbrace{(KV')}_{k^{V'}}(K) = Kk^{V'}$: Due to k^μ having no other element at $U'=0$

$k^\mu\Big|_{U'=0} = k^{V'}$

$$\underline{(k^\nu D_\nu k^\mu)\Big|_{U'=0} = K k^\mu}$$

Note that $k^2 = -(Kx)^2 \to -\infty$ as $x \to \infty$, so there is natural normalization of k for Rindler.

Acceleration Horizons

Proposition: The proper acceleration of a particle at $x=a^{-1}$ in Rindler spacetime (i.e. an orbit of k) is constant and equal to a.

Proof: A particle on a timelike orbit $X^\mu(\tau)$ of a Killing vector field ξ (Particle moving along the way of the killing vector field) has 4-velocity.

$$\frac{dX^\mu}{d\tau} = u^\mu \propto \xi^\mu$$

$$u^\mu = \alpha \xi^\mu \qquad u^\mu u_\mu = -1 = \alpha^2 \xi^\mu \xi_\mu \;, \; \alpha^2 = \frac{-1}{\xi^2}$$

if ξ timelike $(\xi^2 < 0)$, $\alpha = \frac{1}{\sqrt{-\xi^2}}$

$$u^\mu = \frac{\xi^\mu}{(-\xi^2)^{1/2}}$$

Its proper 4-acceleration is

$$a^\mu = D_{(\tau)} u^\mu = u \cdot D u^\mu = \frac{\xi^\nu}{(-\xi^2)^{1/2}} D_\nu \left(\frac{\xi^\mu}{(-\xi^2)^{1/2}} \right)$$

$$= \frac{\xi^\nu}{(-\xi^2)^{1/2}} \cdot \frac{1}{(-\xi^2)^{1/2}} D_\nu \xi^\mu + \frac{\xi^\nu \xi^\mu}{(-\xi^2)^{1/2}} \left(\cancel{2}\left(-\frac{1}{\cancel{2}}\right)\right) (-\xi^2)^{-3/2} (\cancel{2}\xi^\sigma) D_\nu \xi_\sigma$$

$$\xi^\mu \underbrace{\xi^\nu \xi^\sigma}_{\text{sym}} \underbrace{D_\nu \xi_\sigma}_{\text{Antisy}} \to 0 \quad \text{for Killing vector field}$$

$$a^\mu = \frac{\xi^\nu}{-\xi^2} D_\nu \xi^\mu = \frac{\xi \cdot D \xi^\mu}{-\xi^2}$$

and "proper acceleration" is magnitude $|a|$ of a^μ

This is placed for square root not be minus

for $k^\mu = \xi^\mu = (-\kappa u', \kappa v', 0, 0)$ $\qquad k^\mu k_\mu = k^2 = \frac{\kappa^2}{2}(u'v' + v'u') = \kappa^2 u'v'$ no need a minus

$$k_\mu = \left(-\frac{\kappa}{2} v', \frac{\kappa u'}{2}, 0, 0\right)$$

$$a^\mu = \frac{k \cdot D \cdot k^\mu}{k^2} = \frac{k^{u'} \partial_{u'} k^\mu + k^{v'} \partial_{v'} k^\mu}{\kappa^2 u'v'} = \left(+\frac{1}{v'}, +\frac{1}{u'}, 0, 0\right)$$

$$a^\mu \partial_\mu = \frac{1}{v'} \partial_{u'} + \frac{1}{u'} \partial_{v'}$$

so $|a| = (a^\mu a^\nu g_{\mu\nu})^{1/2} = \left(a^{u'} a^{v'} \left(-\frac{1}{2}\right) + a^{v'} a^{u'} \left(-\frac{1}{2}\right)\right)^{1/2} = \left(-\frac{1}{u'v'}\right)^{1/2}$

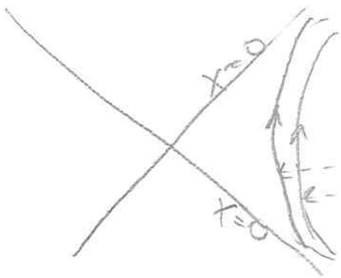
$$u'v' = T^2 - X^2 = X^2 \left(\underline{\sinh^2(\kappa\tau) - \cosh^2(\kappa\tau)}_{-1}\right) = -X^2$$

$$|a| = \left(-\frac{1}{-X^2}\right)^{1/2} = \frac{1}{X} \qquad |a| = \frac{1}{X}$$

So for $x = a^{-1}$ (constant) we have $|a| = a$, i.e. orbits of k in Rindler are worldlines of constant proper acceleration.

$x \to 0$, $a \nearrow$

so the Killing horizon at $x=0$ is called acceleration horizon



Although the proper acceleration of an $x =$ constant worldline diverges as $x \to 0$ its acceleration as measured by another $x =$ constant observer will remain finite.

<u>Since</u> $d\tau^2 = (\kappa x)^2 dt^2$ (for $x = a^{-1} = \text{const}$) $\quad ds^2 = -d\tau^2$

the acceleration as measured by an observer whose proper time is t is

$$\left(\frac{d\tau}{dt}\right) * \frac{1}{x} = \kappa x \cdot \frac{1}{x} = \kappa$$

, which has a finite limit, κ, as $x \to 0$

In Rindler spacetime such an observer is one with constant proper acceleration, κ, but these observers are in no way 'special' because the normalization of t was arbitrary

$$t \to \lambda t \Rightarrow \kappa \to \lambda^{-1} \kappa \quad (\lambda \in \mathbb{R})$$

For Schwarzschild

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 \to dt^2 \Rightarrow \begin{cases} r = \text{const} \to \infty \\ \theta, \varphi = \text{const} \end{cases}$$

$$d\tau^2 = dt^2$$

i.e. an observer whose proper time is t is one at spatial ∞. Thus

<u>Surface gravity</u> is the acceleration of a static particle near the horizon as measured at spatial infinity.

Surface Gravity and Hawking Temperature

Behavior of QFT in a BH using Euclidean path integrals.

In Minkowski spacetime this involves setting

$$t = i\tau$$

and continuing τ from imaginary to real values. Thus τ is "imaginary time" here (not proper time on some worldline)

In BH spacetime this leads to a continuation of the SM to the Euclidean SM

$$ds_E^2 = \left(1-\frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2$$

This is singular at $r=2M$. To examine the region near $r=2M$, we do the same thing as Rindler

$$r-2M = \frac{x^2}{8M} \quad \text{to get} \quad ds_E^2 = \underbrace{(Kx)^2 d\tau^2 + dx^2}_{\substack{\text{Euclidean} \\ \text{Rindler}}} + \frac{1}{4K^2}d\Omega^2$$

The metric near $r=2M$ is the product of the metric on S^2 and the Euclidean Rindler spacetime

$$ds_E^2 = dx^2 + x^2 d(K\tau)^2$$

This is just E^2 in plane polar coordinates if we make the periodic identification

$$K\tau \sim K\tau + 2\pi$$

$$\tau \sim \tau + \frac{2\pi}{K}, \ \beta$$

Path Integral and Statistical Mechanics

$$\langle q_f | e^{-iHt} | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} Dq \, e^{iS[q]}$$

$$Z = \operatorname{tr} e^{-\beta H} = \sum_n \langle n | e^{-\beta H} | n \rangle$$

$$U(t) = e^{-iHt} \xrightarrow[t=-i\tau]{\text{wick rotate}} U(-i\tau) = e^{-H\tau}, \quad U(-i\beta) = e^{-\beta H}$$

$$\langle q_f | e^{-iHt} | q_i \rangle \xrightarrow{t=-i\tau} \langle q_f | e^{-\tau H} | q_i \rangle = \int Dq \, e^{-S_E[q]}$$

$$Z = \int_{q(\tau=0)=q(\tau+\beta)} Dq \, e^{-S_E[q]} \qquad \beta = \frac{2\pi}{K} = \frac{1}{k_B T}$$

$$T_H = \frac{K \hbar}{2\pi k_B}$$

1

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$$T_H = \frac{K}{2\pi}$$

So we have deduce that a QFT can be in equilibrium with a black hole only at the Hawking temperature $T_H = \frac{\kappa}{2\pi}$

i-) At any other temperature, Euclidean Schwarzschild has a conical singularity (H $\theta \sim \theta + 2\pi\alpha$, $\alpha \neq 1$) $\longrightarrow$ no equilibrium

ii-) $T_H = \frac{1}{8\pi M}$, Equilibrium at Hawking temperature is unstable since if the BH absorbs radiation its <u>mass increases</u> and its <u>temperature decreases</u>

BH has negative specific heat, $E = M$, $T = \frac{1}{8\pi M}$ $\quad \frac{dE}{dT} = \frac{dM}{dT} = -\frac{1}{8\pi T^2} < 0$

Tolman Law - Unruh Temperature

$E(r) = -p_\mu u^\mu$, $u^\mu = \frac{\xi^\mu}{(-\xi^2)^{1/2}}$ $\qquad \xi^\mu \xi_\mu = g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt}(r)$

$\xi^\mu = (1,0,0,0)$

$E_\infty = -p_\mu \xi^\mu$: conserved energy $\qquad k \to \xi$

$$E(r) = \frac{E_\infty}{(-\xi^2)^{1/2}} = \frac{E_\infty}{\sqrt{-g_{tt}(r)}}$$

In statistical mechanics, equilibrium distribution is

$$f(E) \sim e^{-\beta E} \xrightarrow{\beta = 1/T} e^{-E/T}$$

f can not be different for the same assemble, so

$$\frac{E(r)}{T(r)} = \frac{E_\infty}{T_\infty} \Rightarrow \frac{\cancel{E_\infty}}{T(r)\sqrt{-g_{tt}(r)}} = \frac{\cancel{E_\infty}}{T_\infty} \Rightarrow T(r)\sqrt{-g_{tt}(r)} = T_\infty$$

<u>Lets give the definition</u>

<u>Tolman Law</u>: The local temperature T of a static self-gravitating system in thermal equilibrium satisfies

$$(-k^2)^{1/2} T = T_0$$

where T_0 is constant and k is timelike Killing vector field ∂_t. If $k^2 \to -1$ asymptotically we can identify T_0 as the temperature 'as seen from infinity'.

For a Schwarzschild BH

$$T_0 = T_H = \frac{\kappa}{2\pi} \qquad \text{why?}$$

Remember that K is arbitrary, but we dont like that, when $r \to \infty$

$$d\tau^2 = \left(1-\frac{2M}{r}\right) dt^2$$

$$= dt^2$$ we set t as proper time (τ) as $r \to \infty$
so we fixed K too.

<u>Surface gravity</u> is the acceleration of a static particle near the horizon as measured as <u>spatial infinity</u>

So, $(-k^2)^{1/2} T = \frac{K}{2\pi}$

Near $r = 2M$ we have, in Rindler coordinates

$k^2 = g_{tt} = -(Kx)^2$

$(\cancel{K}x) T = \frac{\cancel{K}}{2\pi} \Rightarrow T = \frac{x^{-1}}{2\pi}$ is the temperature measured by a static observer (on orbit of k) near the horizon.

But $x = a^{-1}$, constant, for such an observer, where a is proper acceleration. So

$T = \frac{a}{2\pi}$ is the local (Unruh) Temperature. It is a general feature of QM (Unruh effect) that an observer accelerating in Minkowski spacetime appears to be in a heat bath at Unruh temperature.

Since $T = \frac{x^{-1}}{2\pi}$ for x = constant, we deduce $T_0 = \frac{K}{2\pi}$, as in Schwarzschild, but this is now just the temperature of the observer with constant acceleration K, who is of no particular significance. Note that in Rindler spacetime

$T = \frac{x^{-1}}{2\pi} \to 0$ as $x \to \infty$

so the Hawking temperature (i.e. temperature as measured at spatial ∞) is actually zero.

This is expected because Rindler is just Minkowski in unusual coordinates, there is nothing inside which could radiate. But for a BH

$T_{local} \to T_H$ at infinity

The BH must be radiating at this temperature

Asymtopia

A spacetime (M, g) is asymptotically simple if there is a manifold $(\tilde{M}, \tilde{g})$ with boundary $\partial\tilde{M} = \bar{M}$ and a continious embedding $f(M): M \to \tilde{M}$ s.t.

i-) $f(M) = \tilde{M} - \partial\tilde{M}$: M is $\tilde{M}$ except one point (or a surface)

ii-) There is a smooth function Λ on $\tilde{M}$ with $\Lambda > 0$ on $f(M)$ and $\tilde{g} = \Lambda^2 f(g)$
(Embedded metric g has some proportional with the metric lives in $\tilde{M}$)

iii-) $\Lambda = 0$ but $d\Lambda \neq 0$ on $\tilde{M}$ which points boundary to interior points.
$\hookrightarrow$ it is a normal vector

iv-) Every null geodesic in M acquires 2 endpoints on ∂M, $\mathscr{I}^-$ $\mathscr{I}^+$
(The light rays can come from infinity and can escape to infinity)

Example : M = Minkowski, $\tilde{M}$ = compactified Minkowski

Due to a light ray might be falling singularity (not to $\mathscr{I}^+$) condition (iv)
excludes BH spacetime. This motivates following definition:

A weakly asymptotically simple space time (M, g) is one for which there exist an open set $U \subset M$ that is isometric to an open neighbourhood of $\partial\tilde{M}$, where $\tilde{M}$ is the "conformal compactification" of some asymptotically simple manifold

Example M = Kruskal, $\tilde{M}$ its conformal compactification

Note
i-) $\tilde{M}$ is not actually compact because $\partial\tilde{M}$ excludes $i_\pm$ (inside the horizon)
ii-) M is not asymptotically simple because geodesics that enter $r < 2M$ can not end on $\mathscr{I}^+$

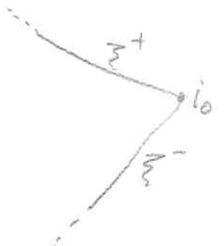
Asymptotic Flatness

An asymptotic flat spacetime is one that is both weakly asymptotically simple and asymptotically empty in the sense that

v) $R_{\mu\nu} = 0$ in an open neighbourhood of ∂M in $\bar{M}$

This excludes AdS space and space times with long range EM fields that we dont wish to exclude. So condition (V) requires modification to deal with EM fields

Asymptotically flat spacetimes have the same type of structure for $\mathscr{I}^\pm$ and i_0 as Minkowski space time.



In particular they admit vectors that are asymptotic to the Killing vectors of Minkowski spacetime near i_0, which allows a definition of total mass, momentum and angular momentum on spacelike hypersurfaces. The asymptotic symmetries on $\mathscr{I}^\pm$ are much more complicated (BMS group).

<u>The Event Horizon</u>

Assume spacetime M is weakly asymptotically flat. Define

$J^-(U)$