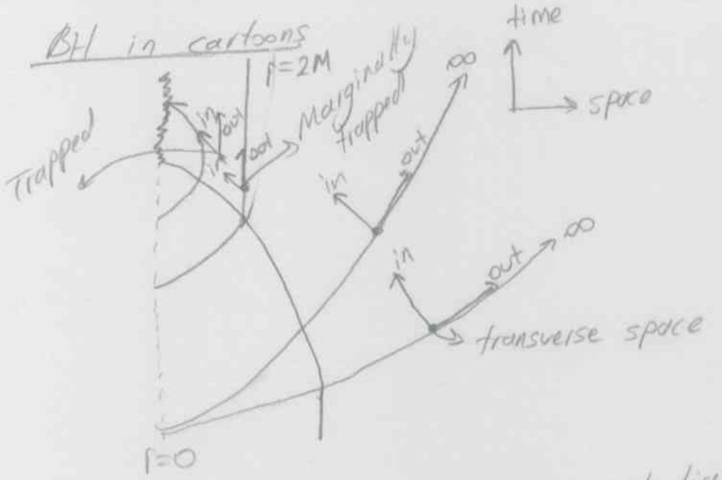
# Black Holes

## BH in cartoons




Transverse space - two null normal directions

$\Theta_{out} = \begin{cases} + & \text{outside BH} \\ - & \text{inside BH} \end{cases}$

$\Theta_{in} < 0$

**Trapped surface**: A 2D closed surface with converging ($\Theta < 0$) null geodesics in outgoing normal direction

**Marginally trapped surface**: A 2D closed surface with stationary ($\Theta = 0$) null geodesics in outgoing normal direction.

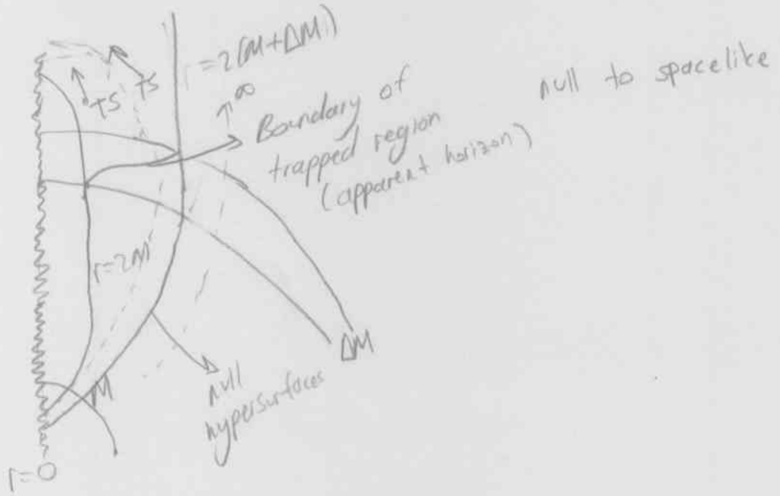
→ Boundary of trapped region

→ "apparent horizon"

**Event Horizon**:

Causal boundary of BH, events inside BH are causally disconnected from observers at infinity. Signals from inside event horizon can never reach outside.

Event horizon is null hypersurface



## Schwarzschild Spacetime

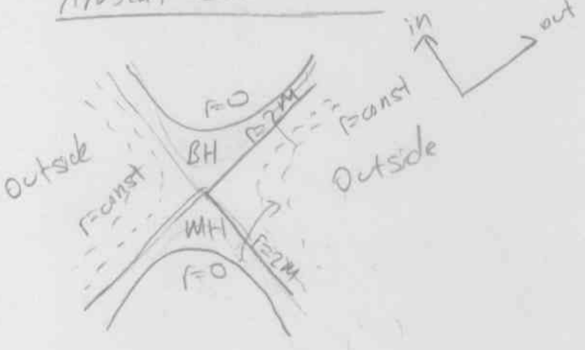
$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \qquad f = 1 - 2M/r$$

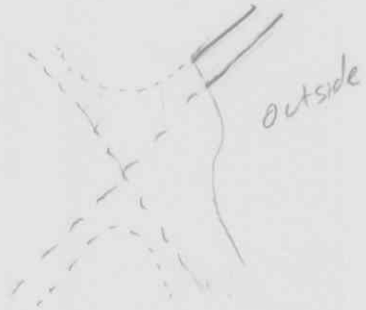
Birkhoff: Schwarzschild metric is the outside metric of any spherically symmetric body

Israel: " " " " unique solution of EFE in vacuum for a static asymptotically flat spacetime with a non-singular event horizon

(spherical symmetry is consequence, not assumption)

## Kruskal Coordinates



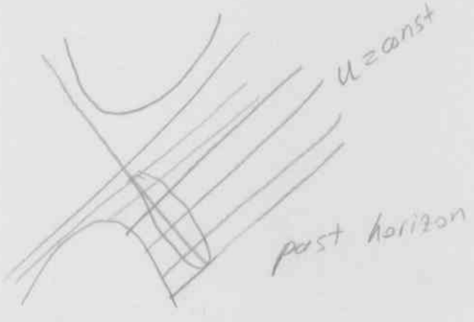




## Eddington-Finkelstein coordinates

$$ds^2 = -f(\underbrace{dt - f^{-1}dr}_{du})(\underbrace{dt + f^{-1}dr}_{dv}) + r^2 d\Omega^2$$

$u = t - \int \frac{dr}{f}$ - constant on outgoing light rays

$v = t + \int \frac{dr}{f}$ - " " ingoing " "





future horizon

Outside BH → inside BH

$$dt = dv - f^{-1} dr$$

$$ds^2 = -f(dv - f^{-1}dr)^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

$$= -f(dv^2 - 2f^{-1} dv\, dr + \cancel{f^{-2}dr^2}) + \cancel{f^{-1}dr^2} + r^2 d\Omega^2$$

$$ds^2 = -f dv^2 + 2 dv\, dr + r^2 d\Omega^2$$

outgoing light rays $\qquad (v, r, \theta, \varphi)$

$0 = du = dt - f^{-1}dr = dv - 2f^{-1}dr$

$u = \text{const}, \quad \bar{K}_\alpha = -\nabla_\alpha u = (-1, \frac{2}{f}, 0, 0)$ : normal to $u$ = const

$\bar{K}^\alpha = g^{\alpha\beta}\bar{K}_\beta = (\frac{2}{f}, 1, 0, 0)$ : normal to outgoing null rays

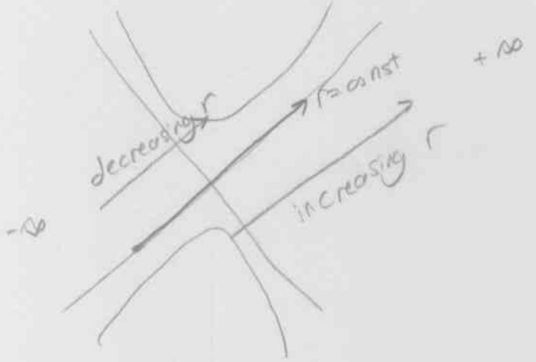
$\bar{K}_\alpha \bar{K}^\alpha = 0$

$\bar{K}^\beta \nabla_\beta \bar{K}^\alpha = 0$ : affine

$\frac{dv}{d\lambda} = \bar{K}^v = \frac{2}{f}$

$\frac{dr}{d\lambda} = \bar{K}^r = 1$

$\frac{d\theta}{d\lambda} = \frac{d\varphi}{d\lambda} = 0, \quad \theta, \varphi = \text{const}$ $\qquad$ set $r$ as a paramet[er] $r = \lambda$




We can not totally understand if the parametrization is $r$. At $r = 2M$ it is constant while outside it is increasing and inside it is decreasing.

We will change the $r$ parametrization.

$r$ X : affine
$\theta, \varphi$ > const X
$v$: ✓ : non affine

$\bar{K}^\alpha = \underbrace{\frac{2}{f}}_{1/\mu} K^\alpha, \quad K^\alpha = (1, \frac{f}{2}, 0, 0)$ , new parameter $v$

$K^\alpha = \mu \bar{K}^\alpha$

$\mu = \frac{f}{2} = \frac{dr}{dv}$

geodesic eqn: $K^\beta \nabla_\beta K^\alpha = \mu \bar{K}^\beta \nabla_\beta (\mu \bar{K}^\alpha) = \mu^2 \underbrace{\bar{K}^\beta \nabla_\beta \bar{K}^\alpha}_{0} + (\mu \bar{K}^\beta \nabla_\beta \mu) \bar{K}^\alpha$

$(K^\beta \nabla_\beta K^\alpha) = \underbrace{\frac{1}{\mu}(K^\beta \nabla_\beta \mu)}_{\kappa} K^\alpha$

$\kappa = \frac{1}{\mu}(K^\beta \nabla_\beta \mu)$

$\mu = \mu(r)$

$\nabla_r = \partial_r$ for scalar

$f = 1 - \frac{2M}{r} \quad \mu = \frac{f}{2}$

$\frac{df}{dr} = \frac{2M}{r^2}, \quad \frac{d\mu}{dr} = \frac{M}{r^2}$

$\kappa = \frac{2}{f} K^r \nabla_r \mu = \frac{2}{f}\frac{f}{2}\frac{d\mu}{dr} = \frac{M}{r^2}$

under the old parametrization

$\bar{\Theta} = \frac{1}{\delta A}\frac{d}{dr}\delta A = \nabla_\alpha \bar{K}^\alpha$

under the new parametrization

$\Theta = \frac{1}{\delta A}\frac{d}{dv}\delta A$ ?

$= \underbrace{\frac{dr}{dv}}_{\mu} \bar{\Theta} = \mu \bar{\Theta} = \mu \nabla_\alpha \bar{K}^\alpha = \mu \nabla_\alpha (\mu^{-1} K^\alpha)$

$= \nabla_\alpha K^\alpha - \underbrace{\frac{1}{\mu} K^\alpha \nabla_\alpha \mu}_{\kappa}$

$\Theta = \nabla_\alpha K^\alpha - \kappa$

$\nabla_\alpha K^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} K^\alpha) \quad ; \sqrt{-g} = r^2 \sin\theta$

$= \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{1}{2} f) = \frac{1}{2r^2}\frac{d}{dr}(r^2 - 2Mr) = \frac{1}{2r^2}(2r - 2M)$

$= \frac{r - M}{r^2} = \frac{1}{r} - \frac{M}{r^2}$

$$\Theta = \frac{1}{r} - \frac{M}{r^2} - \frac{M}{r^2} = \frac{r-2M}{r^2}$$

$$\Theta = \frac{r-2M}{r^2}$$


$r = 2M$: const $\quad \Theta = 0$, static, boundary of the trapped region (marginally trapped) = event horizon for SM

r increasing
$\Theta > 0$

<u>Vaidya space time</u> $\quad (v, r, \theta, \varphi)$

$$ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

$f = 1 - \frac{2M(v)}{r}$ . $M = M(v)$ means a BH with changing mass


Boundary of trapped region (apparent horizon)

only non vanishing $G$: $\quad G_{vv} = 2\dot{M}/r^2$

" " " $T$: $\quad T_{vv} = \frac{\dot{M}}{4\pi r^2}$

$T_{\alpha\beta} = \frac{\dot{M}}{4\pi r^2} l_\alpha l_\beta \quad ; l_\alpha = -\nabla_\alpha v$

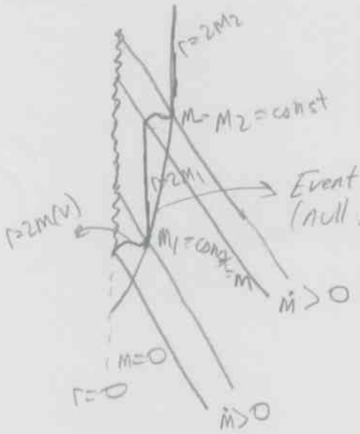
$\rightarrow$ null vector, tangent to ingoing light rays

$T_{\alpha\beta} = \mu u_\alpha u_\beta + \cancel{p(g_{\alpha\beta} + u_\alpha u_\beta)}$ : perfect fluid

no pressure $\rightarrow$ , not timelike : $u_\alpha \rightarrow l_\alpha (k_\alpha)$

$T_{\alpha\beta} = \frac{\dot{M}}{4\pi r^2} l_\alpha l_\beta$ : pressurless null fluid

Energy conditions $\Rightarrow \dot{M} > 0$ : BH's mass will increase obviously


- Boundary of trapped region ($\Theta = 0$) at $r = 2M(v)$
- spacelike when $\dot{M} \neq 0$
- null when $\dot{M} = 0$

Event horizon (null surface, asymptots to $r = 2M(\infty)$)

Outgoing null geodesics : $K^\alpha = (1, \frac{1}{2}f, 0, 0)$ $(v, r, \theta, \varphi)$

$$K_\alpha K^\alpha = 0$$

$$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha, \quad \kappa = m(v)/r^2 \text{ as before}$$

$$(14) = \nabla_\alpha K^\alpha - \kappa = \frac{r - 2m(v)}{r^2}$$

Surface $r = 2m(v) \rightarrow \Phi = r - 2m(v) = 0$

normal vector: $\nabla_\alpha \Phi = (-2\dot{m}, 1, 0, 0)$

$$g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi \quad \begin{matrix} > 0 & \text{spacelike} \\ < 0 & \text{timelike} \\ = 0 & \text{null} \end{matrix}$$

$$= -4\dot{m} = \begin{cases} < 0, \ \dot{m} > 0 \Rightarrow \text{normal is timelike, surface is spacelike} \\ = 0, \ \dot{m} = 0 \Rightarrow \text{" " null, surface is null} \end{cases}$$

## Kerr Black Hole

Schwarzschild : static space time → Killing vector $t^\alpha$ (hypersurface orthogonal) ↔ $\omega_{\alpha\beta} = 0$ : not rotati[ng]

Kerr : stationary space time → Killing vector $t^\alpha$ (not hypersurface orthogonal) ↔ $\omega_{\alpha\beta} \neq 0$ : rotatin[g]

$$\left.\begin{matrix} \hookrightarrow \text{mass } M \\ \text{angular momentum } J = aM \end{matrix}\right\} a \leq M$$

Kerr metric is a solution to EFE in vacuum

Uniqueness theorem : Kerr metric is unique solution to vacuum EFE for stationary, asymptotically flat space time with a non-singular event horizon

Schwarzschild :

spherical body — outside Schwarzschild ✓

Kerr :

rotating body — outside Kerr(?) X

Kerr metric is not the exterior metric of a rotating, unless rotating body is a BH

$\hookrightarrow \{M, M_{ab}^K, M_{abc}^K \ldots\}$

$\{J, J_{ab}^K, J_{abc}^K \ldots\}$

$\hookrightarrow \{M, M_{ab}, M_{abc} \ldots\}$ moments

$\{J, J_{ab}, J_{abc} \ldots\}$

Kerr metric  Boyer-Lindquist coordinates $(t, r, \theta, \varphi)$

Notation: $\rho^2 = r^2 + a^2\cos^2\theta$

$\Delta = r^2 - 2Mr + a^2$

$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$

$g_{tt} = -1 + 2Mr/\rho^2$

$g_{t\varphi} = -2aMr\sin^2\theta/\rho^2$

$g_{rr} = \rho^2/\Delta$

$g_{\theta\theta} = \rho^2$

$g_{\varphi\varphi} = \Sigma\, a \sin^2\theta/\rho^2$

$R_{\mu\nu} = 0$

$g^{tt} = -\Sigma/\rho^2\Delta$

$g^{t\varphi} = -2Mar/\rho^2\Delta$

$g^{rr} = \Delta/\rho^2$

$g^{\theta\theta} = 1/\rho^2$

$g^{\varphi\varphi} = \dfrac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}$

Killing vectors: $t^\alpha = (1,0,0,0)$ : (stationary)

$\varphi^\alpha = (0,0,0,1)$ : (axisymmetric)

Observers #1 - ZAMOs (zero angular momentum observers)

$\tilde{L} = u_\alpha \varphi^\alpha = u_\varphi$ = constant of motion $= 0$

$0 = u_\varphi = g_{\varphi\alpha}u^\alpha = g_{t\varphi}u^t + g_{\varphi\varphi}u^\varphi \Rightarrow \underbrace{\dfrac{u^\varphi}{u^t}}_{} = -\dfrac{g_{t\varphi}}{g_{\varphi\varphi}}$

$\dfrac{\dot\varphi}{\dot t} = \dfrac{d\varphi/\cancel{d\tau}}{dt/\cancel{d\tau}} = \dfrac{d\varphi}{dt}$

$\dfrac{d\varphi}{dt} = -\dfrac{\overset{<0}{g_{t\varphi}}}{g_{\varphi\varphi}} = \omega = \dfrac{2aMr\sin^2\theta\,\rho^2}{\rho^2\Sigma\sin^2\theta}$

$$\boxed{\dfrac{d\varphi}{dt} = \omega = \dfrac{2aMr}{\Sigma}}$$

asymptotically $\omega \simeq \dfrac{2J}{r^3}$

and it vanishes $r \to \infty$

"Lense-Thirring"

"Dragging of inertial frames"

Observers #2 - Static observers: $u^\alpha = \gamma(1,0,0,0) = \gamma t^\alpha$
(observers → timelike)

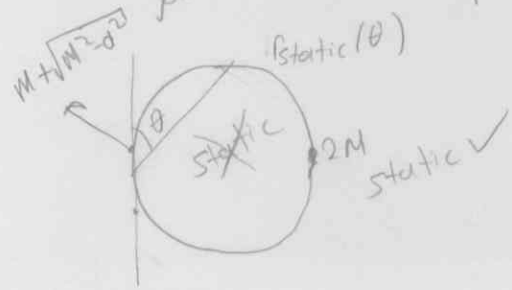
$\gamma$ must be finite to have a static observer

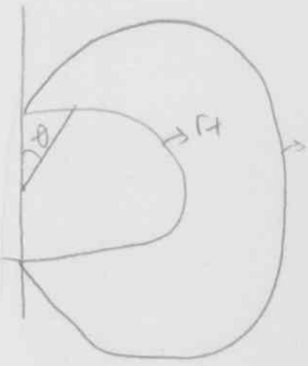
$g_{\alpha\beta}u^\alpha u^\beta = -1$, $\gamma^2 < \infty$, $(-g_{tt}) > 0$, unless it is null

$-1 = g_{\alpha\beta}u^\alpha u^\beta = \gamma^2 g_{tt} \Rightarrow \gamma^2 = -\dfrac{1}{g_{tt}}$

$1 - \dfrac{2Mr}{\rho^2} > 0 \Rightarrow \rho^2 - 2Mr > 0 \Rightarrow r^2 - 2Mr + a^2\cos^2\theta > 0$

$r > r_{static}(\theta) \equiv M + \sqrt{M^2 - a^2\cos^2\theta}$






$r_{static}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}$

$\rightarrow$ Timelike surface (not event horizon)

$r = \infty \rightarrow t^\alpha = $ null

Event horizon: $r = r_+ = M + \sqrt{M^2 - a^2}$
(null surface)

$\boxed{\Omega_H = \frac{a}{r_+^2 + a^2}}$ BH angular velocity

Stationary observers: we allow it to rotate

$$u^\alpha = \gamma(t^\alpha + \Omega\varphi^\alpha) = \gamma(1, 0, 0, \Omega)$$

$$-1 = g_{\alpha\beta}u^\alpha u^\beta = \gamma^2 g_{\alpha\beta}(t^\alpha + \Omega\varphi^\alpha)(t^\beta + \Omega\varphi^\beta) = \gamma^2(g_{tt} + 2g_{t\varphi}\Omega + g_{\varphi\varphi}\Omega^2)$$

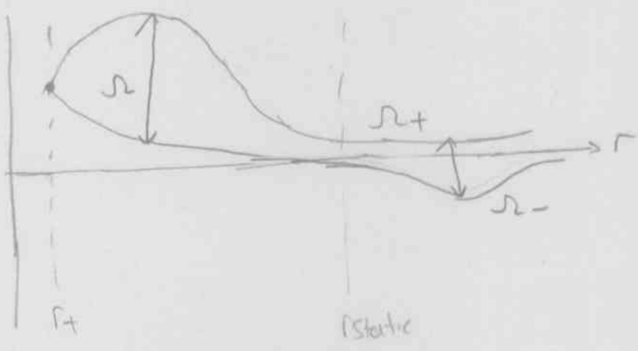
$$= -\gamma^2 g_{\varphi\varphi}\left(-\Omega^2 - 2\frac{g_{t\varphi}}{g_{\varphi\varphi}} - \frac{g_{tt}}{g_{\varphi\varphi}}\right) = -\gamma^2 g_{\varphi\varphi}\Big(\underbrace{-\Omega^2 + 2\omega\Omega - \frac{g_{tt}}{g_{\varphi\varphi}}}_{>0}\Big)$$

$\gamma < \infty$ when

$$-\Omega^2 + 2\omega\Omega - \frac{g_{tt}}{g_{\varphi\varphi}} > 0$$

$$(\Omega_+ - \Omega)(\Omega - \Omega_-) > 0 \Rightarrow \Omega_- < \Omega < \Omega_+$$

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma\sin\theta}$$




when $\Delta = 0 \rightarrow$ double root; $\Omega_- = \Omega_+$

$\Delta = r^2 - 2Mr + a^2$

$\Delta = 0 \Rightarrow r = r_+ = M + \sqrt{M^2 - a^2}$

$$\Omega_H = \Omega_- = \Omega_+ = \omega(r = r_+)$$

$$= \frac{2Mar_+}{(r_+^2 + a^2)^2} = \frac{2Mr_+ a}{(2Mr_+ - \cancel{a^2} + \underbrace{\Delta}_{0} + \cancel{a^2})^2} = \frac{2Mr_+ a}{(2Mr_+)^2}$$

$$\Omega_H = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}$$

Surface $r = r_+$ is null, Event horizon of the Kerr BH

$t^\alpha + \Omega_H\varphi^\alpha$ is null at $r = r_+$

$\frac{d\varphi}{dt} \rightarrow \Omega_H$ $\Omega_H$ is finite, $t \rightarrow \infty$, $\varphi \rightarrow \infty$ those are bad coordinates

## Regular coordinates

behaviour of incoming light rays (principal null congruence)

$V = t + \int \frac{r^2+a^2}{\Delta}\, dr$ (constant on incoming light rays)

$\Psi = \varphi + \int \frac{a}{\Delta}\, dr$ ( " " " " " )

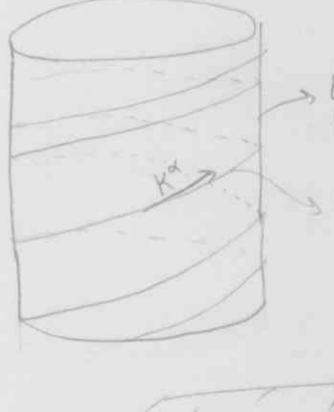
$$\left.\begin{aligned} dt &= dV - \frac{r^2+a^2}{\Delta}\, dr \\ d\varphi &= d\Psi - \frac{a}{\Delta}\, dr \end{aligned}\right.$$

$g^{Kerr}_{\alpha\beta}(V, r, \theta, \Psi)$
regular across the event horizon

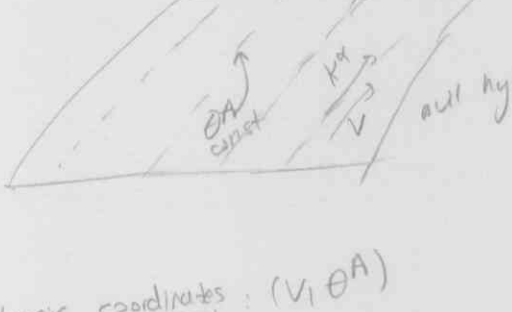
$(V, r, \theta, \Psi)$: $t^\alpha = (1,0,0,0)$
$\varphi^\alpha = (0,0,0,1)$

$K^\alpha = t^\alpha + \Omega_H \varphi^\alpha$
→ null on Event horizon (EH)
$r = r_+ = M + \sqrt{M^2 - a^2}$

## Kinematics of Kerr Horizon




$K^\alpha = t^\alpha + \Omega_H \varphi^\alpha = (1,0,0,\Omega_H)$




$K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$ : V is not affine

Calculation: $\kappa = \frac{r_+ - M}{r_+^2 + a^2}$ "surface gravity" constant across horizon

Schwarzschild → $\kappa = \frac{M}{(2M)^2} = \frac{M}{R^2}$

Intrinsic coordinates: $(V, \theta^A)$

generator labels: $\left.\begin{aligned}\theta^2 &= \theta \\ \theta^3 &= \chi\end{aligned}\right\}$ constant on generators

$0 = K^\theta = \frac{d\theta}{dV} \Rightarrow \theta = \underline{const}$

$\Omega_H = K^\Psi = \frac{d\Psi}{dV} \Rightarrow \Psi = \Omega_H V + \underbrace{const}_{\chi}$

Embedding relations: $X^\alpha = X^\alpha(V, \theta^A)$

$V = V$
$r = r_+$
$\theta = \theta$
$\Psi = \Omega_H V + \chi$

Tangent vectors to EH: $e^\alpha_a = \frac{\partial X^\alpha}{\partial y^a}$; $y^a = (V, \theta^A)$

$e^\alpha_V = \frac{\partial X^\alpha}{\partial V} = (1,0,0,\Omega_H) = K^\alpha$

$e^\alpha_\theta = \frac{\partial X^\alpha}{\partial \theta} = (0,0,1,0)$

$e^\alpha_\chi = \frac{\partial X^\alpha}{\partial \chi} = (0,0,0,1)$

So overall

Intrinsic coordinates : $y^a = (v, \theta, \chi)$

Embedding relations : $X^\alpha = X^\alpha(y^a)$ ;
$v = v$
$r = r_+$
$\theta = \theta$
$\varphi = \Omega_H v + \chi$

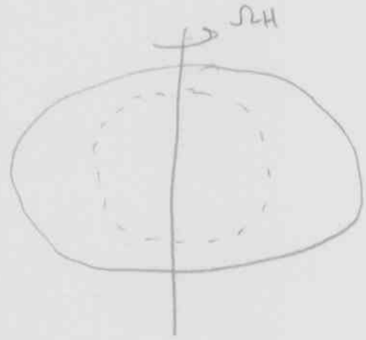
Tangent vectors

$e^\alpha_v = (1, 0, 0, \Omega_H)$

$e^\alpha_\theta = (0, 0, 1, 0)$

$e^\alpha_\chi = (0, 0, 0, 1)$

$\Omega_H = \dfrac{a}{r_+^2 + a^2}$ , $\mathcal{K} = \dfrac{r_+ - M}{r_+^2 + a^2}$ , $r_+ = M + \sqrt{M^2 - a^2}$



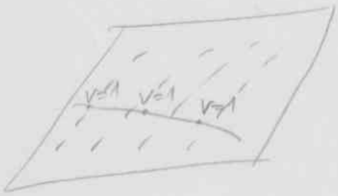

Induced metric : $\Omega_{AB} = g_{\alpha\beta}\, e^\alpha_A\, e^\beta_B$

$\Omega_{\theta\theta} = r_+^2 + a^2 \cos^2\theta$

$\Omega_{\chi\chi} = \dfrac{(r_+^2 + a^2)^2}{r_+^2 + a^2 \cos^2\theta} \sin^2\theta$

$\sqrt{\Omega} = (r_+^2 + a^2) \sin\theta$

Cross sectional area :



$A = \int \sqrt{\Omega}\, d^2\theta$

$= (r_+^2 + a^2) \underbrace{\int \sin\theta\, d\theta\, d\varphi}_{4\pi}$

$A = 4\pi \underbrace{(r_+^2 + a^2)}_{2Mr_+} = 8\pi M r_+$

$A = 8\pi M \left( M + \sqrt{M^2 - (J/M)^2} \right)$

Differential law : $A = A(J, M)$

BH #1 : $(M, J)$
BH #2 : $(M + dM, J + dJ)$

$dA = \left.\dfrac{\partial A}{\partial M}\right) dM + \left.\dfrac{\partial A}{\partial J}\right) dJ$

$\dfrac{\mathcal{K}}{8\pi} dA = dM - \Omega_H dJ$

$\dfrac{\mathcal{K}}{8\pi} = \left(\dfrac{\partial M}{\partial A}\right)_J$ , $\Omega_H = \left(\dfrac{\partial M}{\partial J}\right)_A$

$\rightarrow dM = \underbrace{\dfrac{\mathcal{K}}{8\pi} dA}_{\text{"TdS"}} + \Omega_H dJ$ ; $M = M(A, J)$

## Integral Law

Euler theorem on homogeneous functions : $f(x^\alpha)$ : $f(\lambda x^\alpha) = \lambda^k f(x^\alpha)$ : homogeneous function of degree $k$

$$kf = x^\alpha \partial_\alpha f$$

$[M] = L$

$[A] = L^2$

$[J] = L^2$

$M = M(A, J)$

$M(\lambda A, \lambda J) = \lambda^{1/2} M(A, J) \rightarrow M$ is homogeneous function of degree $1/2$

$$\text{Euler: } \frac{1}{2} M = A \frac{\partial M}{\partial A} + J \frac{\partial M}{\partial J}$$

$$= \frac{\kappa}{8\pi} A + \Omega_H J$$

$$M = \frac{\kappa}{4\pi} A + 2\Omega_H J \quad \text{(Smarr's formula)}$$

## Null hypersurfaces


$y^\alpha = (v, \theta^A)$

$k^\alpha = \dfrac{\partial x^\alpha}{\partial v}$, $e^\alpha_A = \dfrac{\partial x^\alpha}{\partial \theta^A}$, $N_\alpha$

$k_\alpha k^\alpha = 0$ $\quad N_\alpha k^\alpha = -1$

$k_\alpha e^\alpha_A = 0$ $\quad N_\alpha e^\alpha_A = 0$

$N_\alpha N^\alpha = 0$

$\Omega_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$

$k^\beta \nabla_\beta k^\alpha = \kappa k^\alpha$

$k^\beta \nabla_\beta e^\alpha_A = \omega_A k^\alpha + B_A{}^B e^\alpha_B$

$B_{AB} = e^\alpha_A e^\beta_B \nabla_\alpha k_\beta = \frac{1}{2} \Theta \Omega_{AB} + \sigma_{AB}$

$= \partial_v \Omega_{AB}$

$\Theta = \dfrac{1}{\sqrt{\Omega}} \partial_v \sqrt{\Omega}$, $\dfrac{dA}{dv} = \displaystyle\int_S \Theta \, dS = \int \Theta \sqrt{\Omega} \, d^2\theta$

## Integration :

$d\Sigma_\alpha = -k_\alpha \, dv \, dS$

$dS_{\alpha\beta} = 2 k_{[\alpha} N_{\beta]} dS$

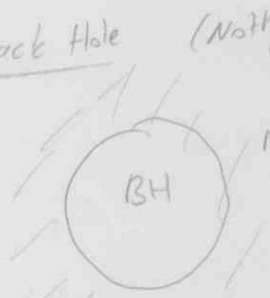
Gauss - Codazzi eqn's

$R_{\mu\nu} k^\mu k^\nu = -\partial_v \Theta + \kappa \Theta - \frac{1}{2} \Theta^2 - \sigma_{AB} \sigma^{AB}$ (Raychandini eqn)

$R_{\mu\nu} k^\mu e^\nu_A = \partial_v \omega_A - \partial_A \kappa - \frac{1}{2} \partial_A \Theta + D_B \sigma_A{}^B + \Theta \omega_A$

## Stationary Black Hole (Nothing depends on time)

Not Kerr




Hawking (1972)

- must be static ( $t^\alpha$ hypersurface orthogonal )
  $\rightarrow \omega_{\alpha\beta} = \nabla_{[\alpha} t_{\beta]} = 0$

or axisymmetric

$t^\alpha$ not hypersurface orthogonal)

$\varphi^\alpha$ axial killing vector

- Linear combination $t^\alpha + \underbrace{\Omega_H}_{\text{const}} \varphi^\alpha \equiv k^\alpha$

in null on the EH

Why stationary gives axisymmetry?


would be stationary if $\Omega$ = const

Tidal forces → $\Omega \downarrow$

↳ torquing


not evenly distributed matter

$\Omega_H \downarrow$ : stationary to static

axisymmetric → $\Omega_H$ = const : stationary

Stationary → axisymmetry → $K^\alpha = t^\alpha + \underbrace{\Omega_H}_{\text{const}} \varphi^\alpha$ = null on the EH

↳ tangent to null generator ; $K^\beta \nabla_\beta K^\alpha = \kappa K^\alpha$

$\Theta = \frac{1}{\delta A} \frac{d}{dv} \delta A$

$\partial_v$(anything) = 0

$\Theta = 0 , \quad \partial_v \Theta = 0$

Null energy condition: $T_{\alpha\beta} k^\alpha k^\beta \geq 0$

→ $R_{\alpha\beta} k^\alpha k^\beta \geq 0$

Raychaudri: $\underbrace{\sigma_{AB} \sigma^{AB}}_{\geq 0} + \underbrace{R_{\alpha\beta} k^\alpha k^\beta}_{\geq 0} = 0$

$\sigma_{AB} \sigma^{AB} = 0 \Rightarrow \sigma_{AB} = 0$

$R_{\alpha\beta} k^\alpha k^\beta = 0 = T_{\alpha\beta} k^\alpha k^\beta$

## Zeroth Law

$\kappa$ = constant on the horizon

$\partial_v \kappa = 0$ is obvious

$\partial_A \kappa = 0$

$\partial_A \kappa = -R_{\mu\alpha} K^\mu e^\alpha_A = -8\pi T_{\mu\alpha} k^\mu e^\alpha_A$

$J^\alpha \equiv -T_\mu{}^\alpha k^\mu \rightarrow \partial_A \kappa = 8\pi J_\alpha e^\alpha_A$

model: $T_{\alpha\beta} = \rho u_\alpha u_\beta$. (pressureless fluid = dust)

$0 = \nabla_\beta T^{\alpha\beta} = \nabla_\beta (\rho u^\alpha u^\beta) = \nabla_\beta (\rho u^\beta) u^\alpha + \rho \underbrace{u^\beta \nabla_\beta u^\alpha}_{a^\alpha}$

$0 = \nabla_\beta (\rho u^\beta)$

$0 = a^\alpha = u^\beta \nabla_\beta u^\alpha$

Conserved quantities

$$\tilde{E} = -u_\alpha t^\alpha = \text{energy per unit mass}$$
$$\tilde{L} = u_\alpha \psi^\alpha = \text{angular momentum}/m$$
$\Big\}$ constant of motion (motion is geodesic)

$$J^\alpha = -\rho\,(u_\mu k^\mu)\,u^\alpha = \rho\,(\underbrace{-u_\mu k^\mu}_{\geq 0})\,u^\alpha \equiv \text{energy flux vector}$$
"boost factor" $= \tilde{E} - \Omega_H \tilde{J}$

Dominant energy condition

$J^\alpha$ must be timelike or null "future directed"

$$J^\alpha = A k^\alpha + \cancel{B N^\alpha} + \cancel{C^A} e^\alpha_A$$

$$-T_{\mu\alpha} k^\mu k^\alpha = J_\alpha k^\alpha = 0 = -B \quad \Rightarrow B = 0$$

$$J_\alpha J^\alpha \leq 0 \Rightarrow (A k_\alpha + C^A e_{A\alpha})(A k^\alpha + C^B e_B{}^\alpha) \leq 0$$

$$C^A C^B e_{A\alpha} e_B{}^\alpha = \underbrace{C^A C^B \Omega_{AB}}_{\substack{\text{spatial} \\ \geq 0}} \leq 0$$

$$C^A C^B \Omega_{AB} = 0 \qquad C^A = 0$$

$$J^\alpha = A k^\alpha$$

$$\partial_A \kappa = 8\pi J_\alpha e^\alpha_A = 8\pi A k_\alpha e^\alpha_A = 0$$

$$\partial_A \kappa = 0 \checkmark$$

Mass law

$$M = \frac{\kappa}{4\pi} A + 2\Omega_H J_H - 2\int_\Sigma \left(T^\alpha_\beta - \frac{1}{2} T g^\alpha{}_\beta\right) t^\beta \, d\Sigma_\alpha$$

$\downarrow$ total mass measured at $\infty$

$H$

$\Sigma$ - stationary surface

What is total mass

coordinate answer: $g_{tt} = -1 + \frac{2M}{r} + O(r^{-2})$

geometrical answer: $\boxed{M = -\frac{1}{8\pi} \oint_\infty \nabla^\alpha t^\beta \, dS_{\alpha\beta}}$

$$dS_{\alpha\beta} = 2\, r_{[\alpha} n_{\beta]}\, dS$$

$$-\frac{1}{8\pi} \nabla_\alpha t_\beta \, (2 r^\alpha n^\beta)\, dS = -\frac{1}{4\pi} \underbrace{(\nabla_\alpha t^\beta) r^\alpha n_\beta}\, r^2 d\Omega$$
$$= \frac{1}{4\pi} M \, d\Omega$$

$$(\partial_\alpha t^\beta + \Gamma^\beta_{\alpha\gamma} t^\gamma)\, r^\alpha n_\beta = -\Gamma^t_{tr}$$

$$= \frac{1}{2}(-1)\left(\cancel{\partial_t g_{tr}} + \partial_r g_{tt} - \cancel{\partial_t g_{tr}}\right) = \frac{1}{2}\left(-\frac{2M}{r^2}\right) = -\frac{M}{r^2}$$

What is the angular momentum $J$?

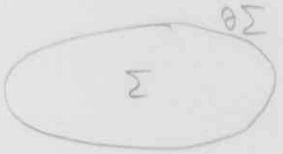
$$g_{t\varphi} = -\frac{2J}{r}\sin^2\theta + O(r^{-2})$$

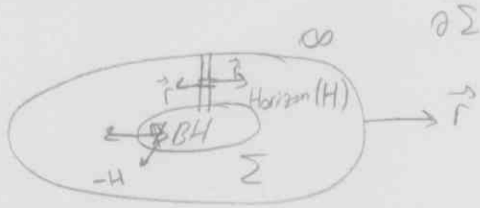
$$J = \frac{1}{16\pi}\oint_{\infty} \nabla^\alpha \varphi^\beta \, dS_{\alpha\beta}$$

Stokes Theorem

$$\frac{1}{2}\int_{\partial\Sigma} \underbrace{B^{\alpha\beta}}_{\text{anti-symmetric}} dS_{\alpha\beta} = \int_{\Sigma} \nabla_\beta B^{\alpha\beta} d\Sigma_\alpha$$


$\partial\Sigma = \infty \ominus H$




$dS_{\alpha\beta} = 2\, r_{[\alpha} n_{\beta]}\, dS$

$\rightarrow r^\alpha \equiv$ outward normal to $\partial\Sigma$

$$\oint_{\partial\Sigma} B^{\alpha\beta} dS_{\alpha\beta} = \int_{\infty} B^{\alpha\beta} dS_{\alpha\beta} - \oint_{H} B^{\alpha\beta} dS_{\alpha\beta}$$

$$\Rightarrow \oint_{\infty} B^{\alpha\beta} dS_{\alpha\beta} = \oint_{H} B^{\alpha\beta} dS_{\alpha\beta} + 2\int_{\Sigma} \nabla_\beta B^{\alpha\beta} d\Sigma_\alpha$$

$$B^{\alpha\beta} = -\frac{1}{8\pi}\nabla^\alpha t^\beta \longrightarrow M = M_H - \frac{1}{4\pi}\int_{\Sigma} \nabla_\beta (\nabla^\alpha t^\beta) d\Sigma_\alpha$$

$$M_H \equiv -\frac{1}{8\pi}\oint_{H} \nabla^\alpha t^\beta dS_{\alpha\beta}$$

$$J_H \equiv \frac{1}{16\pi}\oint_{H} \nabla^\alpha \varphi^\beta dS_{\alpha\beta}$$

$$M_H - 2\Omega_H J_H = -\frac{1}{8\pi}\oint_{H} \nabla^\alpha \underbrace{(t^\beta + \Omega_H \varphi^\beta)}_{K^\beta} dS_{\alpha\beta} = -\frac{1}{8\pi}\oint_{H} \nabla^\alpha K^\beta \underbrace{dS_{\alpha\beta}}_{2K_{[\alpha}N_{\beta]}dS}$$

$$= -\frac{1}{4\pi}\oint_{H} \nabla_\alpha K^\beta K^\alpha N_\beta \, dS = -\frac{1}{4\pi}\oint_{H} \kappa \underbrace{K^\beta N_\beta}_{-1} dS = \frac{1}{4\pi}\oint_{H} \kappa \, dS \overset{0\text{th law}}{=} \frac{\kappa}{4\pi}\oint_{H} dS = \frac{\kappa}{4\pi}A$$

$$\nabla_\beta(\nabla^\alpha t^\beta) = R^\alpha{}_\beta t^\beta = 8\pi\left(T^\alpha{}_\beta - \frac{1}{2}T g^\alpha{}_\beta\right)t^\beta$$

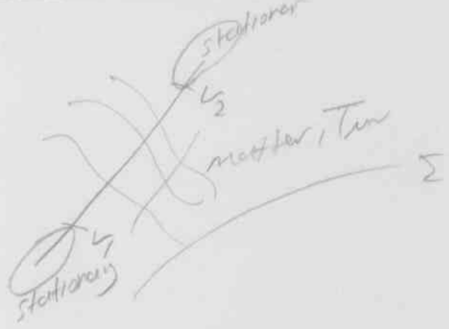
Generalized Smarr Formula

$$\boxed{M = \frac{\kappa}{4\pi}A + 2\Omega_H J_H - 2\int_{\Sigma}\left(T^\alpha{}_\beta - \frac{1}{2}T g^\alpha{}_\beta\right)t^\beta d\Sigma_\alpha}$$

First Law

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J_H + (\text{Matter terms})$$


<u>Physical process</u> - BH is stationary before $v_1$ and after $v_2$, changes $v_1 < v < v_2$




<u>Flux vectors</u>

$$\varepsilon^\alpha = -T^\alpha{}_\beta t^\beta$$

$$\ell^\alpha = T^\alpha_\beta \varphi^\beta$$

$$T^{\alpha\beta} = \rho u^\alpha u^\beta$$

$$\left.\begin{aligned} \varepsilon^\alpha &= \rho(-u_\beta t^\beta) u^\alpha = (\rho \tilde{E}) u^\alpha \\ \ell^\alpha &= \rho(u_\beta \varphi^\beta) u^\alpha = (\rho \tilde{L}) u^\alpha \end{aligned}\right\} \text{Divergence free}$$

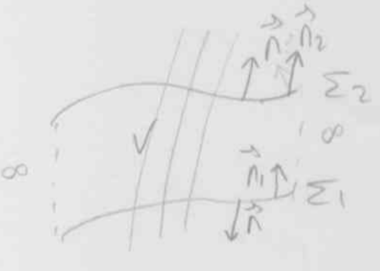
$$\nabla_\alpha \varepsilon^\alpha = -\nabla_\alpha (T^\alpha{}_\beta t^\beta) = -T^\alpha{}_\beta \nabla_\alpha t^\beta = -\underbrace{T^{\alpha\beta}}_{\text{symmetric}} \underbrace{\nabla_\alpha t_\beta}_{\text{antisymmetric}} = 0$$


Mass-energy crossing spacelike hypersurface $\Sigma \equiv \int_\Sigma \varepsilon^\alpha d\Sigma_\alpha = \delta M$

Angular momentum crossing spacelike hypersurface $\Sigma \equiv \int_\Sigma \ell^\alpha d\Sigma_\alpha = \delta J$

Conservation statements: $\delta M[\Sigma_1] = \delta M[\Sigma_2]$
$\delta J[\Sigma_1] = \delta J[\Sigma_2]$

<u>Gauss Theorem</u>: $\displaystyle\int_V \nabla_\alpha A^\alpha dV = \oint_{\partial V} A^\alpha \underbrace{d\Sigma_\alpha}_{-n_\alpha d\Sigma}$




$$0 = \int_V \nabla_\alpha \varepsilon^\alpha dV = \int_{\Sigma_2} (-\varepsilon^\alpha n^2_\alpha) d\Sigma - \int (-\varepsilon^\alpha n^1_\alpha) d\Sigma$$

$$\int_{\Sigma_1} = \int_{\Sigma_2}$$

Mass fluxing into horizon $\equiv \delta M_H = \int_H \varepsilon^\alpha d\Sigma_\alpha$ $\qquad d\Sigma_\alpha = -K_\alpha dV dS$

Angular momentum " " $= \delta J_H = \int_H \ell^\alpha d\Sigma_\alpha$


$$\delta M - \Omega_H \delta J = \int_H (\varepsilon^\alpha - \Omega_H \ell^\alpha) d\Sigma_\alpha = -\int_H T^\alpha_\beta \underbrace{(t^\beta + \Omega_H \varphi^\beta)}_{K^\beta} \underbrace{d\Sigma_\alpha}_{-K_\alpha dV dS}$$

$$= \int_H T_{\alpha\beta} K^\alpha K^\beta dV dS$$

## Raychaudri eqn

$$T_{\alpha\beta} k^\alpha k^\beta = \frac{1}{8\pi} R_{\alpha\beta} k^\alpha k^\beta = \frac{1}{8\pi}\left(-\partial_v \Theta + \kappa \Theta - \cancel{\frac{1}{2}\Theta^2} - \cancel{\sigma_{AB}\sigma^{AB}}\right)$$ , perturbative $\Theta$ ✓

$\Theta^2 \sim 0$

$\sigma_{AB}$ ✓

$\sigma_{AB}^2 \sim 0$

$$\delta M_H - \Omega_H \delta J_H = -\frac{1}{8\pi}\int_H \partial_v \Theta \, dv\, dS + \frac{1}{8\pi}\int_H \kappa \Theta \, dv\, dS$$

$$= -\frac{1}{8\pi}\cancel{\int_H \Theta \, dS \Big|_{v_1}^{v_2}} + \frac{\kappa}{8\pi}\int_H \Theta \, dv\, dS$$

$dS = \sqrt{\Omega}\, d^2\theta$

$\Theta = \frac{1}{\sqrt{\Omega}} \partial_v \sqrt{\Omega}$

stationary state
to stationary state
$\Theta(v_1) = \Theta(v_2) = 0$

$$= \frac{\kappa}{8\pi}\int_H \partial_v \sqrt{\Omega}\, dv\, d^2\theta = \frac{\kappa}{8\pi}\int_H \sqrt{\Omega}\, d^2\theta \Big|_{v_1}^{v_2} = \frac{\kappa}{8\pi}\delta A$$

$$\delta M_H = \Omega_H \delta J_H + \frac{\kappa}{8\pi}\delta A$$ : First law of BH thermodynamics

$+ \Phi_H \delta Q$ if there were charge

## Second Law

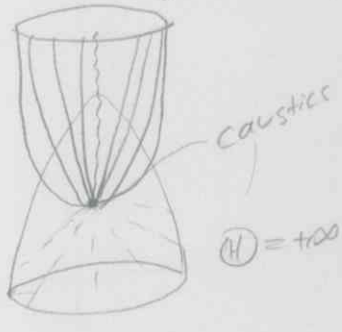
Null energy condition: $\delta A \geq 0$

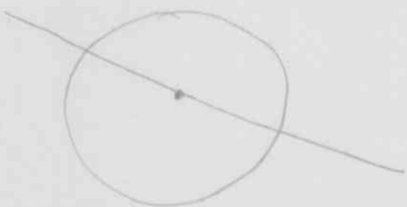
## Caustics

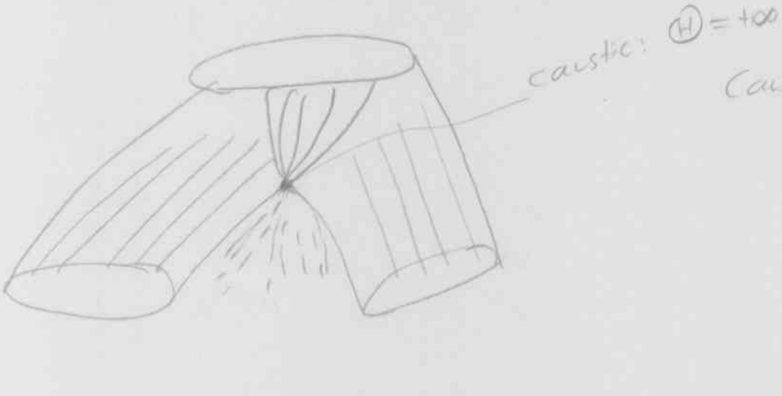
focusing theorem: null energy condition ; $\frac{d\Theta}{d\lambda} \leq 0$

$\Theta < 0 \longrightarrow \Theta \to -\infty$ (in finite $\lambda$) $\to$ unpredictable

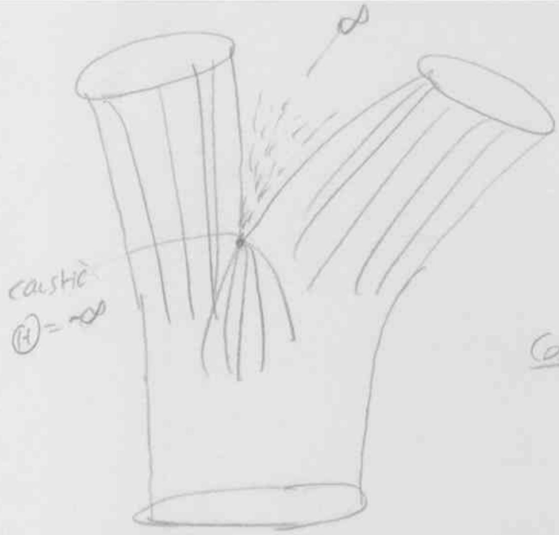
$\longrightarrow$ caustics






Caustics ( $\Theta = +\infty$ ) can occur on EH
in the past




future caustic
→ light rays reaching $\infty$
→ invalidates the definition of BH as causal boundary

Cosmic cencorship
→ past caustics are OK. but future caustics are ruled out

can ∧ seperate two BH

two statements:
1- $\Theta$ cannot go to $-\infty$ on EH (future caustics X) cosmic cencorship
2- $\Theta \to -\infty$ when at some initial time $\Theta < 0$ , so $\Theta \geq 0$

$$\frac{dA}{dV} = \int_H \overbrace{\Theta}^{\geq 0} ds \geq 0$$

$\boxed{\delta A \geq 0}$ : second law