Reissner-Nordström

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}\,(R-F_{\mu\nu}F^{\mu\nu})$$

$G_{\mu\nu}=8\pi T_{\mu\nu}$

Electromagnetic energy momentum tensor: $T_{\mu\nu}=(\underbrace{F_{\mu\lambda}F_{\nu}{}^{\lambda}}^{F_{\mu\lambda}F_{\nu\alpha}g^{\alpha\lambda}}-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma})$

$g^{\mu\nu}(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu})$

Trace free: $g^{\mu\nu}T_{\mu\nu}=0$

$R-2R=8\pi T=0$

$R=0$

$R_{\mu\nu}=8\pi T_{\mu\nu}$

- $F^{\mu\nu}{}_{;\nu}=0$
- $F_{\mu\nu;\lambda}+F_{\lambda\mu;\nu}+F_{\nu\lambda;\mu}=0$

Means $B=0=A_1=A_2=A_3=\underline{A_i=0}$

$A_0=-\phi(r)$

$F_{01}=\partial_0 A_1-\partial_1 A_0=-E_r$

Let metric to be

$ds^2=-A(t,r)\,dt^2+B(t,r)\,dr^2+\underline{r^2d\Omega^2}$ → spherically symmetric

Assumption is $F^{\mu\nu}$ has no components along the θ and ϕ direction; this ensures radial (E_r) that field is purely electric when measured by stationary observers

Non-vanishing christoffels are

$\Gamma^0_{00}=\frac{\dot{A}}{2A}$, $\quad\Gamma^0_{01}=\Gamma^0_{10}=\frac{A'}{2A}\quad\Gamma^0_{11}=\frac{\dot{B}}{2A}$

$\Gamma^1_{00}=\frac{A'}{2B}\quad\Gamma^1_{01}=\Gamma^1_{10}=\frac{\dot{B}}{2B}\quad\Gamma^1_{11}=\frac{B'}{2B}$

$\Gamma^1_{22}=-\frac{r}{B}\quad\Gamma^1_{33}=-\frac{r\sin^2\theta}{B}$

$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{r}\quad\Gamma^2_{33}=-\sin\theta\cos\theta$

$\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{r}\quad\Gamma^3_{23}=\Gamma^3_{32}=\cot\theta$

where dot is derivative wrt "t" and prime is derivative wrt "r".

$$R_{00}=\frac{A'}{4B}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{A''}{2B}-\frac{A'}{rB}-\frac{\ddot{B}}{2B}-\frac{\dot{B}}{4B}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)$$

$$R_{11}=-\frac{A'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{A''}{2A}-\frac{B'}{rB}+\frac{\ddot{B}}{2A}+\frac{\dot{B}}{4A}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)$$

$$R_{22}=\frac{r}{2B}\left(\frac{A'}{A}-\frac{B'}{B}\right)+\frac{1}{B}-1$$

$$R_{33}=R_{22}\sin^2\theta\ ,\quad R_{01}=R_{10}=-\frac{\dot{B}}{rB}$$

Assumption about the $F^{\mu\nu}$ leads us

$$F_{10}=-F_{01}=E_1=E_r$$

All non-diagonal components of $T_{\mu\nu}$ is zero

$A_0=-\phi(r)\ ,\ A_i=0$: spherically symmetric, θ and ϕ components does not exist.

$$F_{10}=\underline{\partial_1 A_0}-\cancel{\partial_0 A_0}=-\partial_r\phi=E_r=-\phi'(r)$$

$$\partial^\mu=g^{\mu\nu}\partial_\nu$$
$$\partial^0=g^{0\nu}\partial_\nu=g^{00}\partial_0=-A^{-1}.\partial_0$$
$$\partial^1=g^{11}\partial_1=B^{-1}\partial_1$$

$$A^\mu=g^{\mu\nu}\partial_\nu$$
$$A^0=g^{00}A_0=-A^{-1}A_0$$

$$F^{10}=\partial^1A^0-\cancel{\partial^0A^1}$$
$$=-A^{-1}B^{-1}\partial_1A_0=A^{-1}B^{-1}\phi'(r)$$

$$F^{10}=-A^{-1}B^{-1}F_{10}$$

$$F^{\mu\nu}F_{\mu\nu}=2F^{10}F_{10}=-2A^{-1}B^{-1}\phi'(r)^2$$

$$T_{\mu\nu}=(A,-B,r^2,r^2\sin^2\theta)\frac{\phi(r)^2}{8\pi A(r)B(r)}$$

- $T_{01}=0=R_{01}$

$$\dot{B}=0\qquad B=B(r)$$

- $$\frac{T_{00}}{A}+\frac{T_{11}}{B}=0=\frac{R_{00}}{A}+\frac{R_{11}}{B}=\frac{1}{rB}\underbrace{\left(\frac{A'}{A}+\frac{B'}{B}\right)}_{\frac{\partial}{\partial r}\left(\ln(AB)\right)=0}=0$$

$$AB=f(t)$$

$E_r = -\partial_r \phi$

$\phi = \frac{Q}{r} \Rightarrow E_r = \frac{Q}{r^2}$

$B(r) = \frac{f(t)}{A(r)}$

$B' = -\frac{f A'}{A^2}$

$\rightarrow 8\pi T_{22} = R_{22} = \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{B} - 1$

$= \frac{r}{2} \cdot \frac{A}{f}\left(\frac{A'}{A} + \frac{\cancel{f} A'}{A^2} \cdot \frac{A}{\cancel{f}}\right) + \frac{A}{f} - 1$

$= r \cdot \frac{\cancel{A}}{f}\left(\frac{A'}{\cancel{A}}\right) + \frac{A}{f} - 1 = \frac{1}{f}(rA' + A) - 1 = \underbrace{\frac{1}{f}\frac{\partial}{\partial r}(rA) - 1}_{RHS}$

$-\cancel{8\pi} \cdot r^2 \frac{\phi'^2}{\cancel{8\pi} AB f} = \frac{1}{f}\frac{\partial}{\partial r}(rA) - 1$

$-\frac{r^2 \phi'^2}{f} = \frac{1}{f}\frac{\partial}{\partial r}(rA) - 1 \Rightarrow -r^2\phi'^2 = \frac{\partial}{\partial r}(rA) - f$

$\cancel{r^2}\frac{Q^2}{\cancel{r^4}} = \frac{\partial}{\partial r}(rA) - f$

$\frac{\partial}{\partial r}(rA) = f - \frac{Q^2}{r^2}$

$Ar = fr + \frac{Q^2}{r} + C$

$A(r) = f(t) + \frac{C(t)}{r} + \frac{Q^2}{r^2}$

The constansts can be found taking the limit $Q \to 0$, it gives us schwartzschild metric which is $g_{00} = \cancel{-}\left(1 - \frac{2M}{r}\right) = \lim_{Q\to 0} A(r)$

$\lim_{Q\to 0} A(r) = f(t) + \frac{C(t)}{r} = 1 - 2M$

$f(t) = 1$

$C(t) = -2M$

So, $A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ and $A = B^{-1}$

The Reissner-Nordström metric is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2$$

The RN metric can be written as

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega^2$$

where

$$\Delta = r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-)$$

where $r_\pm$ are not necessary real

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

There are therefore 3 cases to consider

i-) $M < |Q|$

Δ has no real roots so there is no horizon and the singularity at $r=0$ is naked

This case is similar to $M<0$ Schwarzschild. This case could not occur in gravitational collapse. As confirmation, consider a shell of matter of charge Q and radius R in Newtonian gravity but incorporating

a-) Equivalence of inertial mass M with total energy, from SR

b-) " " and gravitational mass, from GR

$$\underbrace{M_{total}}_{\text{total energy}} = \underbrace{M_0}_{\text{Rest mass energy}} + \underbrace{\frac{GQ^2}{R}}_{\text{Culomb energy}} - \underbrace{\frac{GM^2}{R}}_{\text{Gravitational binding energy}}$$

(M = total mass)

This is quadratic equation for M. The solution with $M \to M_0$ as $R \to \infty$ is

$$M(R) = \frac{1}{2G}\left[(R^2 + 4GM_0R + 4G^2Q^2)^{1/2} - R\right] \quad \text{for } M>0$$

The shell will only undergo gravitational collapse if and only if M decreases with decreasing R (so allowing KE to increase). Now

$$M' = \frac{G(M^2 - Q^2)}{2MGR + R^2}$$

by taking the derivative of $M = M_0 + \frac{GQ^2}{R} - \frac{GM^2}{R}$ wrt R.

$$\frac{dM}{dR} = M' > 0$$

for gravitational collapse so it only occurs if $M > |Q|$ as expected

Now consider $M(R)$ as $R \to 0$

$M \to |Q|$ independent of M_0

So GR resolves the infinite self-energy problem of point particles in classical EM. A point particle becomes an extreme ($M = |Q|$) RN black hole (case (iii) below)

__Remark__: The electron has $M \ll |Q|$ (at least when probed at distances $\gg \frac{GM}{c^2}$) because the gravitational attraction is negligible compared to Coulomb repulsion. But the electron is intrinsically quantum mechanical, since its Compton wavelength $\gg$ Schwarzschild radius.

Clearly the applicability of GR requires

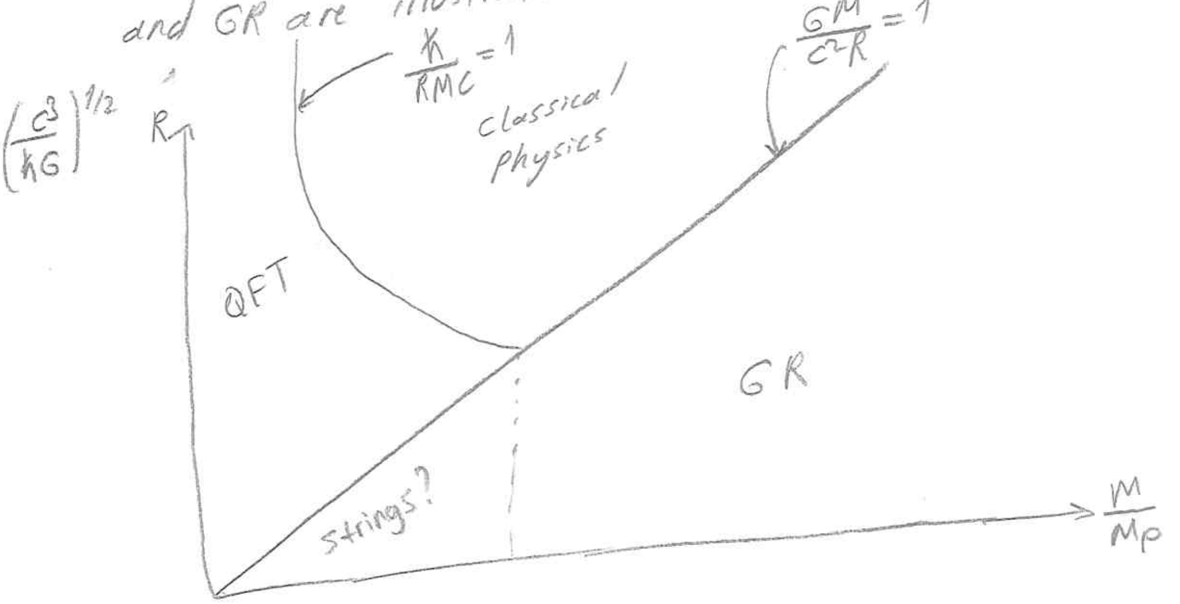
$$\frac{\text{Compton wavelength}}{\text{Schwarzschild radius}} = \frac{\hbar/Mc}{MG/c^2} = \frac{\hbar c}{M^2 G} \ll 1$$

i.e.

$$M \gg \left(\frac{\hbar c}{G}\right)^{1/2} \equiv M_P \text{ (Planck mass)}$$

This is satisfied by any macroscopic object but not by elementary particles.

More generally the domains of applicability of classical physics, QFT and GR are illustrated in the following diagram



ii-) $M > |Q|$

Δ vanishes at $r=r_+$ and $r=r_-$, so metric is singular there but these are coordinate singularities. To see this we proceed as for $r=2M$ in Schwarzschild. Define r^* by

$$dr^* = \frac{r^2}{\Delta}dr = \frac{dr}{\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)}$$

$$r^* = r + \frac{1}{2k_+}\ln\left(\frac{|r-r_+|}{r_+}\right) + \frac{1}{2k_-}\ln\left(\frac{|r-r_-|}{r_-}\right) + \text{const}$$

where $K_\pm = \frac{(r_\pm - r_\mp)}{2r_\pm^2}$

radial null coordinates

$v = t + r^*$

$u = t - r^*$

RN metric in EF coordinates (v, r, θ, φ)

$$ds^2 = -\frac{\Delta}{r^2}dv^2 + 2dv\,dr + r^2 d\Omega^2$$

, $r=0$ curvature singularities

$\Delta=0$ coordinate singularities

constant hypersurfaces of $r \Rightarrow r = \text{const}$

$\Phi = r - \text{const}$

$l_\alpha = -f\nabla_\alpha \Phi = -f(0,-1,0,0)$ → normalization func

$g^{\alpha\beta} l_\alpha l_\beta = g^{rr} \cdot 1 \cdot f^2$, $g^{rr} \to \frac{\Delta}{r^2}$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & & \\ 1 & \Delta/r^2 & & \\ & & 1/r^2 & \\ & & & 1/r^2\sin^2\theta \end{pmatrix}$$

Hypersurface is null when $\frac{\Delta}{r^2} = 0$

so when $\Delta = 0$, $r = r_+$ and $r = r_-$ are null hypersurfaces, $N_\pm$

<u>Proposition</u>: The null hypersurfaces $N_\pm$ of RN are Killing horizons of Killing vector field $k = \frac{\partial}{\partial v}$ (the extension of $\frac{\partial}{\partial t}$ in RN coordinates) with surface gravities $K_\pm$

<u>Proof</u>: <u>The normals</u> to $N_\pm$ are : $l^\alpha = g^{\alpha\beta} l_\beta = g^{\alpha r} l_r = f g^{\alpha r}$, $l = l^\alpha \partial_\alpha = l^r \partial_r + l^v \partial_v \ldots$

$$l_\pm = f_\pm (g^{rr}\partial_r + g^{vr}\partial_v)\Big|_{N_\pm} = f_\pm\,\partial_v$$

(note $g^{rr}=0$ on $N_\pm$ and $g^{vr}=1$) for some arbitrary $f_\pm$ which we can choose $l^\mu_\pm \partial_\mu l^\alpha_\pm = 0$ (tangent to affinely parametrized geodesic

so $l_\pm f_\pm^{-1} = \partial_v$

which shows that $N_\pm$ are Killing horizons of ∂_v (This is Killing because in EF coordinates the metric is v-independent.)

$$k^\mu D_\mu k^r = k^\mu(\partial_\mu k^r + \Gamma^r_{\mu\nu} k^\nu) = \underbrace{k^v}_{1}(\cancel{\partial_v k^r} + \Gamma^r_{v\nu} k^\nu) = \Gamma^r_{v\nu} k^\nu = \Gamma^r_{vv}$$

$$\left(k = (1,0,0,0) = k^\mu \partial_\mu = \underbrace{1}_{k^v} \cdot \partial_v\right)$$

$$= \frac{1}{2} g^{r\lambda}(\partial_v g_{\lambda v} + \partial_v g_{\lambda v} - \partial_\lambda g_{vv})$$

$$= \frac{1}{2} g^{rr}(2\partial_v \cancel{g_{rv}} - \partial_r \cancel{g_{vv}}) + \frac{1}{2} g^{rv}(\cancel{\partial_v g_{vv}})$$

$$= 0 \quad \text{on } \mathcal{N}_\pm$$

$$k^\mu D_\mu k^v\Big|_{r=r_\pm} = k^v(\cancel{\partial_v k^v} + \Gamma^v_{v\mu} k^\mu) = \Gamma^v_{vv} = \frac{1}{2} g^{v\lambda}(2\cancel{\partial_v g_{\lambda v}} - \partial_\lambda g_{vv}) = \frac{1}{2r^2}\partial_r \Delta + \cancel{\frac{1}{2}\Delta \partial_r\left(\frac{1}{r^2}\right)}\Big|_{r=r_\pm}$$

$$= -\frac{1}{2} g^{vr} g_{vv,r} = +\frac{1}{2}\, 1 \cdot \partial_r\left(\frac{\Delta}{r^2}\right)\Big|_{r=r_\pm}$$

$$= \frac{1}{2r^2}\partial_r\{(r-r_+)(r-r_-)\}\Big|_{r=r_\pm} = \frac{1}{2r^2}\{(r-r_-)+(r-r_+)\}\Big|_{r=r_\pm}$$

$$= \frac{1}{2r_\pm^2}\cdot\begin{cases} r=r_+ : (r_+-r_-) \\ r=r_- : (r_--r_+)\end{cases} = \frac{1}{2r_\pm^2}\{r_\pm - r_\mp\} = \boxed{K_\pm}$$

$k^\mu D_\mu k^\nu = K_\pm k^\nu$: due to other elements being zero.

Since $k = \partial_t$ in static coordinates we have asymptotically flat behaviour when $r \to \infty$ so k would be timelike killing vector: $k_\mu k^\mu = k^2 \to -1$

So we identifly $K_\pm$ as surface gravities of $\mathcal{N}_\pm$

Each of Killing Horizons $\mathcal{N}_\pm$ will have a bifurcation (distinction of two place) 2-sphere in the neighbourhood of which we can introduce the KS-type coordinates

$$U^\pm = -e^{K_\pm u}, \quad V^\pm = e^{K_\pm v}$$

For the $+$ sign $\xrightarrow{r > r_+}$

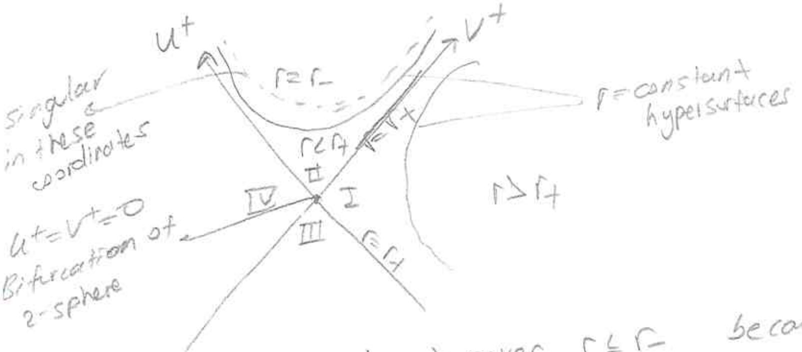
$$ds^2 = -\frac{r_+ r_-}{K_+^2}\,\frac{e^{-2K_+ r}}{r^2}\left(\frac{r_-}{r-r_-}\right)^{\left(\frac{K_+}{K_-}-1\right)} dU^+ dV^+ + r^2 d\Omega^2$$

where $r = r(U^+, V^+)$ is determined implicitly by

$$U^+V^+ = -e^{2K_+ r}\left(\frac{r-r_+}{r_+}\right)\left(\frac{r-r_-}{r_-}\right)^{K_+/K_-}$$

#

This metric covers four regions of the maximal analytic extension of RN

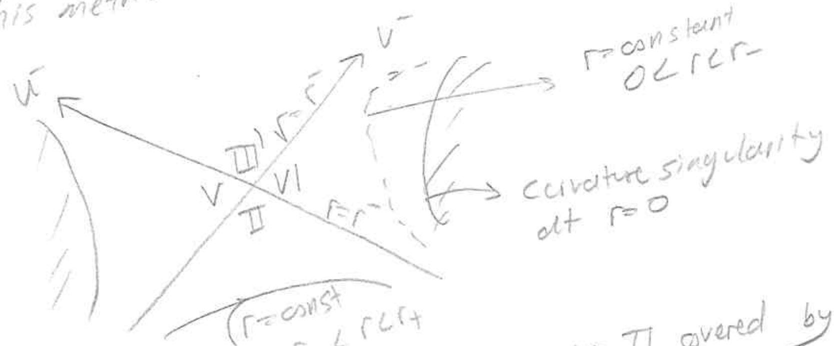


These coordinates do not cover $r\le r_-$ because $r=r_-$ is singularity but $r=r_-$ and a similar four regions covered by the (U_-, V_-) KS-type coordinates to this case

$$ds^2 = -\frac{r_+ r_-}{\kappa_-^2}\frac{e^{-2\kappa_- r}}{r^2}\left(\frac{r_+}{r_+ - r}\right)^{\frac{\kappa_-}{\kappa_+}-1} dU^- dV^- + r^2 d\Omega^2$$

$$U^- V^- = -e^{-2\kappa_- r}\left(\frac{r_- - r}{r_-}\right)\left(\frac{r_+ - r}{r_+}\right)^{\kappa_-/\kappa_+}$$

This metric covers 4 region around $U^- = V^- = 0$



Region II is the same as the region II covered by the (U^+, V^+) coordinates, which other regions are new. VI and V contain the curvature singularity at $r=0$, which is timelike because the normal to $r=$ const is spacelike for $\Delta>0$ $\left(\ell^M \ell_M = f^2 \frac{\Delta}{r^2}\right)$ e.g. in $r<r_-$.

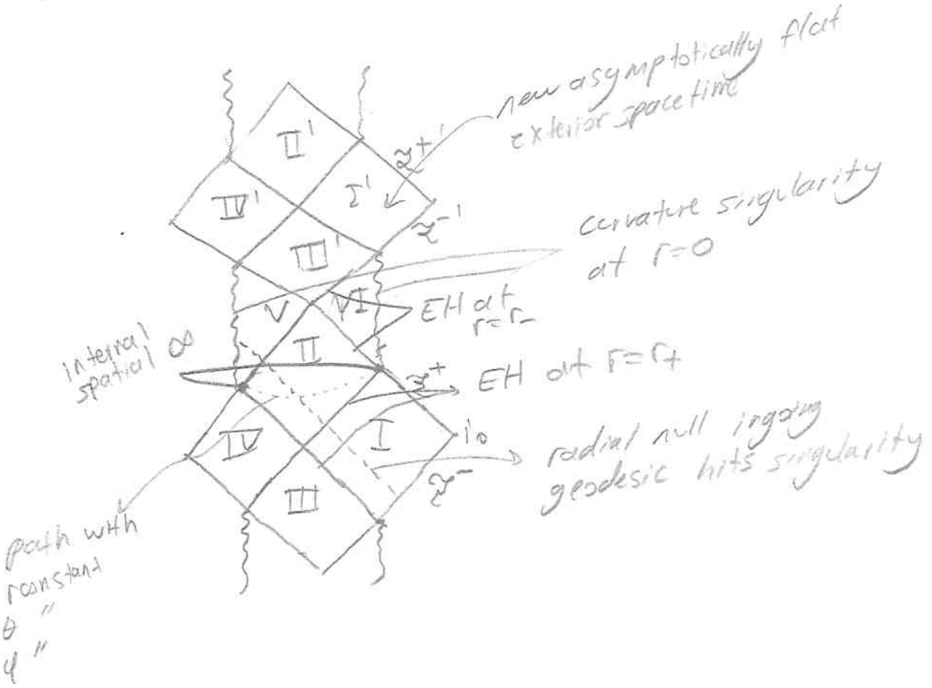
Region II is connected to an exterior spacetime in the past (I, III, IV)

Now (U^+, V^+) and (U^-, V^-) part  is like Region I and II of the Schwarzschild. If we apply the same thing we will get new asymptotically flat region as III and IV which I and III are isometric.

In RN BH, there must be continium as Schwarzschild, continium of the III' is new asymptotical region which we label I', II', IV'

And in new asymptotically flat region, observer might fell into RN' BH again so that diagram repeats.

And overall diagram can be shown as



Internal Infinities

Consider a path of constant r, θ, ϕ in any region for which $\underset{r_- < r < r_+}{\Delta < 0}$, e.g. region II.

In ingoing EF coordinates

$$ds^2 = -\frac{\Delta}{r^2}dv^2 = \frac{|\Delta|}{r^2}dv^2 > 0$$

Since $ds^2 > 0$ the path is spacelike. The distance along it from $v=0$ to $v=-\infty$ (i.e. $V^+=0$ or $V^-=0$) is

$$s = \int_{-\infty}^{0} \frac{|\Delta|^{1/2}}{r}dv = \frac{|\Delta|^{1/2}}{r}\int_{-\infty}^{0} dv = \infty$$

So there is an "internal" spatial infinity behind the $r=r_+$ horizon

(Note that one can still reach $V^\pm=0$ in finite proper time on timelike path so the null hypersurfaces $V^\pm=0$ are part of the spacetime)

If all points at ∞, external and internal, are brought to finite affine parameter by a conformal transformation one finds the above Penrose diagram.