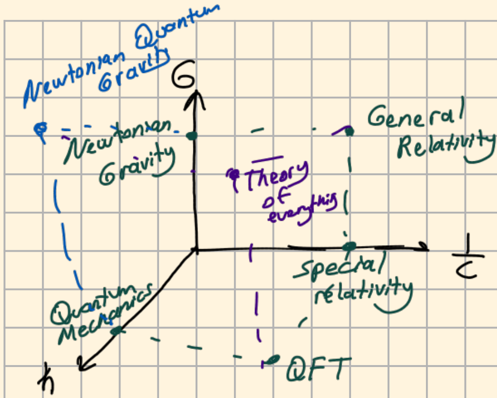


$$m\frac{v^2}{r} = \frac{GmM}{r^2} \Rightarrow \frac{v^2}{c^2} = \frac{GM}{rc^2} \quad R_s$$

$\frac{v}{c} \to 0$: non-relativistic limit

QM ⟶ QFT

Newton M. ⟶ Newton Quantum Gravity

GR ⟶ QM Gravity

Why did we need, why others werent enough

- $\nabla^2 \phi = 4\pi G \rho$ → This is not Lorentz invariant

$$\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$$

David Tong : Dynamics and Relativity

$$\begin{pmatrix} \overset{1\times1}{\rho} & \overset{1\times3}{p} \\ \underset{3\times1}{j} & \underset{3\times3}{T} \end{pmatrix}$$

Stress-Energy Tensor

$$T = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

$$\begin{tikzcd}
\text{Newtonian Gravity} \arrow[r, "\text{Geometrize}"] \arrow[d, "\text{Relativistic Model}"] & \text{Newton-Cartan Gravity} \arrow[d, "\text{Relativistic}"] \\
\text{Matrix Field Theory} \arrow[r, "\text{Geometize}"'] & \text{GR}
\end{tikzcd}$$

Principle of Equivalance

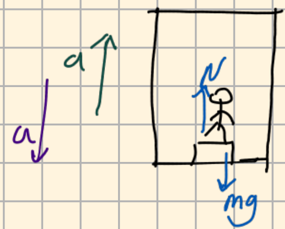
$\vec{F} = m_I \vec{a}$ ↳ inertial mass ; $\vec{F}_G = m\vec{g}$ ↳ passive gravitational mass

Strong Equivilance Principle

$$m_A = m_P \quad ; \quad m_{1A} m_{2P} = m_{1P} m_{2A} \Rightarrow \frac{m_{1A}}{m_{1P}} = \frac{m_{2A}}{m_{2P}} = 1$$

Weak Equivalance Principle V1.0

$$m_I = m_G \qquad \vec{a} = \left(\frac{m_G}{m_I}\right)\vec{g}$$



$N - mg = ma \implies N = m(g+a)$

$N + mg = ma \implies N = m(g-a)$; $g=a$ weightless

$g(\vec{x},t)$

$$x \to \vec{x}' = \vec{x}' + \vec{b}(t)$$

$$\frac{d^2\vec{x}}{dt^2} = \vec{g}(x,t)$$

$$\frac{d^2\vec{x}'}{dt^2} = g(\vec{x},t) + \ddot{\vec{b}}(t) = \tilde{g}(\vec{x},t)$$

if $\vec{g}$ is uniform (g is x independent)

$$\tilde{g} = 0$$

$$x \to x' = x - \tfrac{1}{2} g t^2$$

if $\vec{g}$ is non-uniform; then one can always find a "local inertial frame"; $\tilde{g} = 0$

In the presence of a gravitational field, we define a local inertial frame to be the set of coordinates $(t, \vec{x})$ that a freely falling observer would define in the same way as coordinates defined in Minkowski spacetime (relativistic)

Galileon space-time for non-relativistic

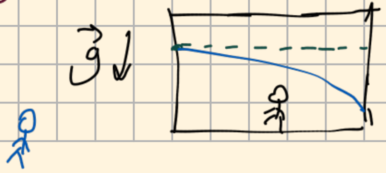
The Einstein Equivalance Principle

1-) Weak EP is valid

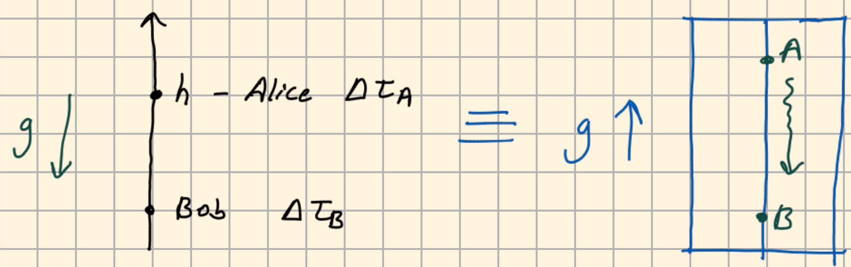
2-) In a local inertial-frame the results of all non-gravitational experiment are indistinguishable form the results of the same experiments performs in an inertial frame in Minkowski space-time

This implies that gravity is the curvature of space time

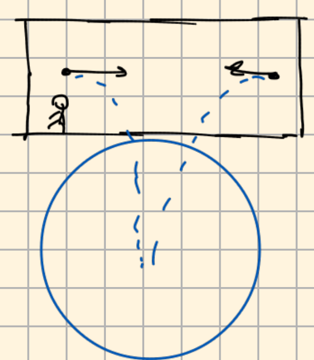
<u>Bending of Light</u>



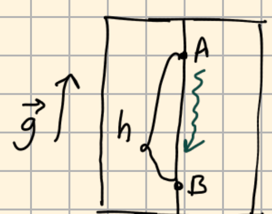
Gravitational Redshift



Global



Tidal Force (Gelgit)



$\frac{gt}{c}$ (small parameter)

Pound - Rebka Experiment

- Alice send the first light signal at $t = t_1$

$z_A(t) = h + \frac{1}{2} g t^2$

$z_B(t) = \frac{1}{2} g t^2$

$z = h + \frac{1}{2} g t^2 - c(t - t_1)$

which reaches to Bob at $t = T_1$

$$h + \frac{1}{2} g t_1^2 - c(T_1 - t_1) = \frac{1}{2} g T_1^2 \quad (1)$$

The second light is send at $t = t_1 + \Delta\tau_A$ ⟶ proper time

let it reach to Bob at $t = T_1 + \Delta\tau_B$

$$h + \frac{1}{2} g (t_1 + \Delta\tau_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \frac{1}{2} g (T_1 + \Delta\tau_B)^2 \quad (2)$$

which leads to

$$g t_1 \Delta\tau_A + \cancel{\frac{1}{2} g \Delta\tau_A^2} - c(\Delta\tau_B - \Delta\tau_A) = g T_1 \Delta\tau_B + \cancel{\frac{1}{2} g \Delta\tau_B^2}$$

$$(c + g t_1)\Delta\tau_A = (c + g T_1)\Delta\tau_B \Rightarrow \frac{\left(1 + \frac{g t_1}{c}\right)}{\left(1 + \frac{g T_1}{c}\right)} \Delta\tau_A = \Delta\tau_B$$

expansion

$$\left(1 + \frac{g t_1}{c}\right)\left(1 - \frac{g T_1}{c} + \ldots\right)\Delta\tau_A = \Delta\tau_B$$

$$\boxed{\Delta\tau_B = \left(1-\frac{g}{c}(T_1-t_1)\right)\Delta\tau_A}$$

If relativitistic effects included other terms in the expansion will be included

Use

$$\frac{g}{c^2}\left(h+\frac{1}{2}gt_1^2 - c(T_1-t_1) = \frac{1}{2}gt_1^2\right)$$

$$\frac{gh}{c^2} + \cancel{\frac{1}{2}\frac{g^2t^2}{c^2}} - \frac{g}{c}(T_1-t_1) = \cancel{\frac{1}{2}\frac{g^2t_1^2}{c^2}}$$

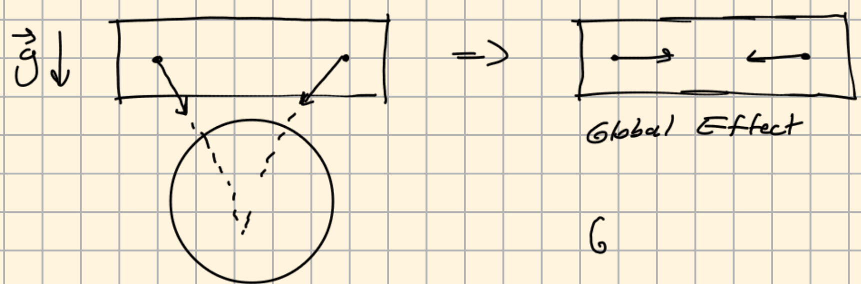
$$\Delta\tau_B = \left(1-\frac{gh}{c^2}\right)\Delta\tau_A \qquad \Delta\tau_A > \Delta\tau_B$$

Thus

$$\lambda_B = \left(1-\frac{gh}{c^2}\right)\lambda_A$$

Tidal Force

g is non-uniform



Geodesic Deviation

$$\vec{N}$$

$g(\vec{x},t) \qquad g(\vec{x}+N,t)$

$$\frac{d^2\vec{x}}{dt^2} = g(\vec{x},t) \qquad \frac{d^2x'}{dt^2} = g(\vec{x}+\vec{N},t)$$

$$\underbrace{\frac{d^2x}{dt^2}}_{g(x,t)} + \frac{d^2N}{dt^2} = g(x+N,t)$$

Thus

$$\frac{d^2N}{dt^2} = g(x+N,t) - g(x,t)$$
$$= (\vec{N}\cdot\vec{\nabla})\vec{g} + \cancel{\ldots} \text{ ignore}$$

$$\frac{d^2N^i}{dt^2} = (\vec{N}\cdot\vec{\nabla})g^i$$

$$g(t+\delta t) - g(t) = \left(\cancel{g(t)} + \delta t\,\partial_t g + \frac{1}{2}\delta t^2 \partial_t^2 g + \ldots\right) - \cancel{g(t)}$$

$$\boxed{\frac{d^2N^i}{dt^2} = (N_j\,\partial_j)g^i}$$

$$\vec{g} = -\vec{\nabla}\phi \Rightarrow g^i = -\partial^i\phi$$

$E_{ij} = -\partial_j g_i = \partial_i \partial_j \phi$

$$\boxed{\frac{d^2 N^i}{dt^2} = E_{ij} N_j}$$ ↝ Geodesic Derivation

Galilean Physics

Time Translation $\quad t \to t+c$: Energy

Spatial Translation $\quad x^i \to x^i + \xi^i = x^{i'}$: Linear momentum

Spatial Rotations $\quad x^i \to R^i_j x^j = x^{i'}$: Angular Momentum

R^i_j ↳ Rotation Matrix

Galilei Transformation $\quad x^i \to x^i + V^i \longrightarrow$ conserved charge $\{G\}$

conserved ↓ mass

Parity: Det=1 olmalı ama Det(parity) = −1 //

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

$O(3) \longrightarrow SO(3)$

↳ parity excluded

Review of Special Relativity

i-) Notation and Metric

- $v_i \neq v^i$

$$\delta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \underbrace{g_{ij}}_{\text{metric}} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$$

$dx^i = (dx, dy, dz) \qquad dx^i = (dr, d\theta, d\phi)$

$ds^2 = \delta_{ij}\, dx^i dx^j \qquad ds^2 = g_{ij}\, dx^i dx^j$

ii-) Galilei Transformation

- Time translation $t \to t' = t + c$
- Spatial translation $x^i \to x^{i'} = x^i + \xi^i$
- Galilean Boosts $x^i \to x^{i'} = x^i + v^i t$
- Rotations $x^i \to x^{i'} = R^i_j x^j \Rightarrow x^{1'} = R^1_1 x^1 + R^1_2 x^2 + R^1_3 x^3 = \sum_{i=1}^{3} R^i_j x^j$

$$\delta_{ij}\, x^i x^j \longrightarrow x'^i x'^j \delta_{ij} = \delta_{ij} \underbrace{R^i{}_k x^k}_{x'^i} \underbrace{R^j{}_m x^m}_{x'^j}$$

$$= \underbrace{\delta_{ij} R^i{}_k R^j{}_m}_{\delta_{km} \text{ olmalı}} x^m x^k$$

$$\delta_{km} = \delta_{ij} R^i{}_k R^j{}_m$$

Matrix product : $A \cdot B = A_{ij} B_{jk}$

$$1\!\!1_{km} = (R^T)_k{}^i \, 1\!\!1_{ij} R^j{}_m$$

$$(R^T)_k{}^j R^j{}_m$$

$$R^T R = 1 \qquad R^T = R^{-1} \qquad R \in O(3)$$

$$(R^{-1})^k{}_i \, (x'^i = R^i{}_j \, x^j)$$

$$(R^{-1})^k{}_i \, x'^i = \underbrace{\delta^k_j \, x^j}_{x^k} \quad \rightarrow \quad x^k = (R^{-1})^k{}_i \, x'^i$$

$$\boxed{x^k = R_i{}^k \, x'^i}$$

- $$\frac{\partial}{\partial x'^i} = \underbrace{\frac{dx^k}{\partial x'^i}}_{R_i{}^k} \cdot \frac{\partial}{\partial x^k} \quad \Rightarrow \quad \partial_i' = R_i{}^j \partial_j$$

$$R_i{}^k \cdot \partial_k$$

Scalar:

$$\phi(x) = \phi'(x')$$

Covariant Vector

$$W(x) = W_i(x)\, e^i \longrightarrow W'(x') = W_i'(x')\, e'^i$$

$$= \underbrace{W_i'(x')\, R^i{}_j}_{W_j(x)} \, e^j$$

$$(W_j(x) = W_i'(x')\, R^i{}_j)(R^{-1})^j{}_k$$

$$W_j(x)(R^{-1})^j{}_k = W_i'(x')\, \delta^i{}_k$$

$$\Rightarrow W'_k(x) = R_k{}^j \, W_j(x) \,//$$

Contravariant Vector

$$V^i(x)\, W_i(x) \longrightarrow V'^i(x')\, W_i'(x') = \underbrace{V'^i(x')\, R_i{}^j}_{V^j(x)} \, W_j(x)$$

$$[\, V^j(x) = V'^i(x)\, R_i{}^j \,](R^{-1})_j{}^k$$

$$\rightarrow V'^i(x) = V^j(x) \underbrace{(R^{-1})_j{}^k}_{R^k{}_j} = V^j(x)\, R^k{}_j \,//$$

$\underline{p,q \text{ tensor}}$ $\quad T^{i_1 i_2 \cdots i_p}{}_{j_1 j_2 \cdots j_q}$

$$T'^{i_1 i_2 \cdots i_p}{}_{j_1 j_2 \cdots j_q} = R^{i_1}{}_{m_1} \cdots R^{i_p}{}_{m_p} R_{j_1}{}^{n_1} \cdots R_{j_q}{}^{n_q} T^{m_1 \cdots m_p}{}_{n_1 \cdots n_q}$$

Levi Civita Tensor

$$\epsilon^{ijk} \longrightarrow \epsilon'^{ijk} = R^i{}_m R^j{}_n R^k{}_l \epsilon^{mnl} = \det R \ \epsilon^{ijk}$$

$$R = \text{parity ise} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hugh Osborn
"Group Theory"

Lorentz Transformation

$$x^\mu = (\underbrace{x^0}_{ct}, x^i)$$

set $c=1 \Rightarrow x^0 = t$

$$\mu, \nu, \rho \ldots = 0,1,2,3$$

$$ds^2 = -dt^2 + d\vec{x}\, d\vec{x}$$

$$\Big\downarrow \ x^\mu \longrightarrow x'^\mu$$

$$ds^2 = -c^2 dt'^2 + d\vec{x}'\, d\vec{x}'$$

Lorentz leaves Minkowski invariant

Galileo " rotation invariant

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \underbrace{(-,+,+,+)}$$

Minkowski space-time

$$\eta_{\mu\nu} (\eta^{-1})^{\nu\rho} = \delta_\mu{}^\rho$$

$$(\eta^{-1})^{\mu\nu} = \eta^{\mu\nu}$$

$g_{\mu\nu}$: metric

$g^{\mu\nu}$: inverse metric

$$g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho$$

$$\eta_{\mu\nu} \begin{cases} \longrightarrow \eta_{00} = -1 \\ \longrightarrow \eta_{ij} = \delta_{ij} \end{cases}$$

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad x^\mu \to x'^\mu = x'^\mu(x^\nu)$$

$$dx^\mu \to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = \underbrace{\eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\lambda}}_{\eta_{\rho\lambda}} dx^\rho\, dx^\lambda$$

$$\eta_{\rho\lambda} = \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\lambda} \quad //$$

- $0 = \partial_\sigma \eta_{\rho\lambda} = \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\lambda} \eta_{\mu\nu} + \boxed{\frac{\partial^2 x'^\nu}{\partial x^\lambda \partial x^\sigma} \frac{\partial x'^\mu}{\partial x^\rho} \eta_{\mu\nu}}$

$\sigma \to \lambda$, $\lambda \to \sigma$ — $\cancel{}$

- $0 = \partial_\lambda \eta_{\rho\sigma} = \boxed{\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\lambda} \frac{\partial x'^\nu}{\partial x^\sigma} \eta_{\mu\nu}} + \boxed{\frac{\partial^2 x'^\nu}{\partial x^\lambda \partial x^\sigma} \frac{\partial x'^\mu}{\partial x^\rho} \eta_{\mu\nu}}$

(1) $\sigma \to \rho$, $\rho \to \sigma$

- $0 = \partial_\rho \eta_{\sigma\lambda} = \frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho} \frac{\partial x'^\nu}{\partial x^\lambda} \eta_{\mu\nu} + \boxed{\frac{\partial^2 x'^\nu}{\partial x^\lambda \partial x^\rho} \frac{\partial x'^\mu}{\partial x^\sigma} \eta_{\mu\nu}}$

metric is symmetric so $\mu \to \nu$, $\nu \to \mu$ doesn't change anything

$$0 = \cancel{2}\, \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma}} \underbrace{\frac{\partial x'^\nu}{\partial x^\lambda}}_{\text{invertable}} \overbrace{\eta_{\mu\nu}}^{\text{Invertable}}$$

$$\frac{\partial^2 x^\mu}{\partial x^\rho \partial x^\sigma} = 0 \Rightarrow \underbrace{x'^\mu = \underbrace{a^\mu}_{\text{translation}} + \overbrace{\Lambda^\mu{}_\nu x^\nu}^{\text{Lorentz transformation}}}_{\text{Poincare' Transformation}}$$

$$\boxed{\eta_{\rho\lambda} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\lambda} \eta_{\mu\nu}}$$

$$\to \eta_{\rho\lambda} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta_{\mu\nu}$$

$$\eta_{\rho\lambda} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta_{\mu\nu} = (\Lambda^T)_\rho{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\lambda$$

$$\eta = \Lambda^T \eta \Lambda$$

Lorentz Group

Let Λ_1 and Λ_2 be two lorentz transformation

And let $\Lambda_3 = \Lambda_1 \Lambda_2$

$$(\Lambda_3)^T \eta \Lambda_3 = (\Lambda_2^T \underbrace{\Lambda_1^T \eta \Lambda_1}_{\eta} \Lambda_2) = \Lambda_2^T \eta \Lambda_2 = \eta$$

$$\det(\eta) = \det(\Lambda^T \eta \Lambda) \Rightarrow 1 = \det(\Lambda)^2 \qquad \det(\Lambda) = \pm 1$$

$$\eta^{-1}(\eta = \Lambda^T \eta \Lambda)\Lambda^{-1} \Rightarrow \Lambda^{-1} = \eta^{-1}\Lambda^T\eta$$

$$(\Lambda^{-1})^\mu{}_\nu = (\eta^{-1}\Lambda^T\eta)^\mu{}_\nu$$

$$= (\eta^{-1})^{\mu\rho}(\Lambda^T)_\rho{}^\sigma \eta_{\sigma\nu} = (\Lambda^T)^\mu{}_\nu$$

- $$(\Lambda^{-1})^T\eta\Lambda^{-1} = (\eta\Lambda\underbrace{\eta^{-1}})\eta\Lambda^{-1} = \eta \quad \text{yani} \quad (\Lambda^{-1})^T\eta\,\Lambda^{-1} = \eta \checkmark$$

Structure of Lorentz Transformation

$$\eta_{\rho\lambda} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta_{\mu\nu}$$

$$\eta_{00} = \Lambda^\mu{}_0\Lambda^\nu{}_0\eta_{\mu\nu}$$

$$= \Lambda^0_0\Lambda^0_0\eta_{00} + \Lambda^i_0\Lambda^j_0\underbrace{\eta_{ij}}_{\delta_{ij}}$$

$$-1 = -(\Lambda^0_0)^2 + (\Lambda^i_0)^2$$

$$\Rightarrow (\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2$$

$$\Lambda^0_0 \geq 1 \quad \text{or} \quad \Lambda^0_0 \leq -1$$

	$\det\Lambda$	Λ^0_0		Name	Contains
Proper	+1	$\geq +1$	1 1 1 1	Proper Ortchronous	$\Rightarrow \mathbb{1}_4$ → Lorentz invariant da genelde bu anlaşılır.
	+1	≤ -1	-1 -1 -1 -1	Proper non-Ortchronous	$\Rightarrow$ Parity Time reversal (PT)
Improper	-1	$\geq +1$	1 -1 -1 -1	Improper Ortchronous	$\Rightarrow$ Parity (P)
	-1	≤ -1	-1 1 1 1	Improper non-ortchronous	$\Rightarrow$ Time reversal (T)

<u>Example</u>: Spatial Rotations by an angle θ

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

- $\Lambda^T = \Lambda$

$$\Lambda = \begin{pmatrix} \lambda & \alpha n^T \\ \alpha n & B \end{pmatrix} \qquad B^T = B$$

$n^T n = 1$
$n n^T =$ 3x3 bir matrice

$$\Lambda^T\eta\Lambda = \eta$$

$$\underbrace{\begin{pmatrix} \lambda & \alpha n^T \\ \alpha n & B \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_3 \end{pmatrix}}_{\begin{pmatrix} -\lambda & \alpha n^T \\ -\alpha n & B \end{pmatrix}}\begin{pmatrix} \lambda & \alpha n^T \\ \alpha n & B \end{pmatrix} = $$

$$= \begin{pmatrix} -\lambda^2 + \alpha^2, & -\lambda\alpha n^T + \alpha n^T B \\ -\lambda\alpha n + \alpha B n, & -\alpha^2 n n^T + B^2 \end{pmatrix}$$

Thus ; $-\lambda^2+\alpha^2=-1 \Rightarrow \lambda^2=1+\alpha^2$

$\alpha=\sinh\theta$, $\lambda=\cosh\theta$

$Bn=\lambda n \Rightarrow Bn=\cosh\theta\, n$

$B^2=\mathbb{1}_3+\alpha^2 nn^T$

$\tanh\theta=\frac{v}{c}$

$1-\tanh^2\theta=\mathrm{sech}^2\theta=\frac{1}{\cosh^2\theta}$

$1-\frac{v^2}{c^2}=\frac{1}{\cosh^2\theta}$

$\rightarrow \cosh\theta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\gamma$

$\sinh\theta=\beta\gamma$

Thus

$Bn=\cosh\theta\, n$

$B^2=1+\sin^2\theta\, nn^T$

$(\cosh^2\theta-1)=(\cosh\theta-1)^2 nn^T+2\cosh\theta\, nn^T-2nn^T$

$B=\mathbb{1}_3+(\cosh\theta-1)nn^T \quad \Rightarrow \Lambda=\begin{pmatrix}\cosh\theta & \sinh\theta\, n^T\\ \sinh\theta\, n & B\end{pmatrix}$

$V'^i=R^i{}_j V^j$

$W_{j'}=R_i{}^j W_j$ $\qquad SO(3) \Rightarrow \delta_{ij}=R_i{}^k R_j{}^l \delta_{kl}$

$SO(3,1) \Rightarrow \eta_{\mu\nu}=\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} \qquad \eta=\Lambda^T\eta\Lambda$

$V'^\mu=\Lambda^\mu{}_\nu V^\nu$

$W'_\mu=\Lambda_\mu{}^\nu W_\nu$

<u>Spatial Rotations</u>

$\Lambda=\begin{pmatrix}1&0\\0&R\end{pmatrix}$ $\hookrightarrow R^TR=\mathbb{1}_3$

$\Lambda^T=\Lambda$: Lorentz boostları verir.

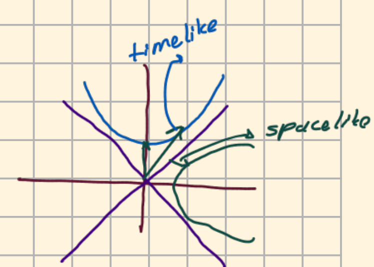
$x'^\mu=\Lambda^\mu{}_\nu x^\nu \qquad \begin{pmatrix}x'^0\\ \vec{x}'\end{pmatrix}=\Lambda\begin{pmatrix}x^0\\ \vec{x}\end{pmatrix}$

$t'=\cosh\theta\, t+\sinh\theta\,\hat{n}\cdot\vec{x}$

$\vec{v}=c\tanh\theta\,\hat{n}\cdot j \quad \mathrm{sech}^2\theta=1-\tanh^2\theta \Rightarrow \mathrm{sech}\,\theta=\left(1-\frac{v^2}{c^2}\right)^{1/2}$

$\cosh\theta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\gamma$; $\sinh\theta=\tanh\theta\cosh\theta=\gamma\frac{v}{c}$

$t'=\gamma\left(t+\frac{v}{c}\right)$



Lorentz boost = hyperbolic rotation

$\vec{x}'=t\sinh\theta\, n+(\mathbb{1}_3+(\cosh\theta-1)nn^T)\vec{x}$

$=t\,\gamma\frac{v}{c}n+\vec{x}+(\cosh\theta-1)\vec{v}\frac{(\vec{v}\cdot\vec{x})}{|v|^2}$

c=1 alabiliriz

$=t\,\gamma\vec{v}+\vec{x}+(\cosh\theta-1)\vec{v}\frac{(\vec{v}\cdot\vec{x})}{|v|^2}$

Worldlines and 4-velocity

$ds^2=-dt^2+\delta_{ij}dx^i dx^j$

$=0$ (null interval)

<0 (timelike)

>0 (spacelike)

$ds^2=-dt^2+\delta_{ij}dx^i dx^j=\eta_{\mu\nu}dx^\mu dx^\nu$

$ds^2=-d\tau^2$

$d\tau^2=-ds^2=dt^2-\delta_{ij}dx^i dx^j$

$\hookrightarrow$ proper time

$$\tau = \int_{\tau_1}^{\tau_2} \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu}$$
$$= \int_{\tau_1}^{\tau_2} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

$x^\mu(\lambda)$

$\eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \le 0$ Timelike or null

If we use τ as parameter

$$d\tau = \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau$$

$$\eta_{\mu\nu} u^\mu u^\nu = -1$$

$u^\mu = \frac{dx^\mu}{d\tau}$ (proper velocity)

Lorentz Transformation

$v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \to v' = \begin{pmatrix} v_0' \\ v_1' \end{pmatrix}$

$\Lambda = \begin{pmatrix} \gamma & +\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$

$v_0' = 1$

$v_1' = 0$

Maxwell Equations

- $\vec{\nabla}\cdot\vec{B} = 0$; $\partial^i B_i = 0$
- $\left(\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t}\right)^i = 0$; $\epsilon^{ijk}\partial_j E_k + \frac{1}{c}\partial_t B^i$
- $\left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi \vec{J}}{c}\right)^i$; $\epsilon^{ijk}\partial_j B_k - \frac{1}{c}\partial_t E^i = \frac{4\pi}{c} J^i$
- $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$; $\partial^i E_i$

Let $F_{\mu\nu}$ be an anti-symmetric $(0,2)$-tensor

- $F_{\mu\nu} = -F_{\nu\mu}$ (anti-symmetric)
- F_{0i} (3 components)
- $F_{ij} = -F_{ji}$ (3 components)

Let

$$E_i = F_{i0} = -F_{0i}$$

$$B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$$

$$B_i\, \epsilon^{ilm} = \frac{1}{2}\underbrace{\epsilon_{ijk}\epsilon^{ilm}}_{\delta_j^l \delta_k^m - \delta_j^m \delta_k^l} F^{jk}$$

$$\frac{1}{2}(\delta_j^l\delta_k^m - \delta_j^m\delta_k^l)F^{jk} = B_i\epsilon^{ilm}$$

$$\frac{1}{2}(F^{lm} - F^{ml}) = B_i\epsilon^{ilm} \Rightarrow \boxed{F^{lm} = B_i\epsilon^{ilm}}$$

- $\partial^i B_i = \frac{1}{2}\epsilon^{ijk}\partial_i F_{jk} = 0$
- $\epsilon^{ijk}\partial_j F_{k0} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon^{ijk}F_{jk}\right) = 0$

$$\epsilon^{ijk}\left(\partial_j F_{k0} + \frac{1}{2}\partial_0 F_{jk}\right) = 0$$

$$\epsilon^{ijk}\partial_j \underset{\substack{\downarrow \\ F_{k0}}}{E_k}$$

$x = (ct, x^i)$

$$\frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$$

These two equations can be written as a single equation

$$0 = \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}$$

$\epsilon^{0123} = 1$

$\epsilon_{0123} = -1$

$\mu = 0\,;\quad 0 = \underbrace{\epsilon^{0ijk}}_{\epsilon^{ijk}}\partial_i F_{jk}$

$$= \epsilon^{ijk}\partial_i F_{jk} = \partial^i B_i$$

$\mu = i\,;\quad 0 = \epsilon^{i\nu\rho\sigma}\partial_\nu F_{\rho\sigma}$

$$= \underbrace{\epsilon^{i0jk}}_{-\epsilon^{0ijk}}\partial_0 F_{jk} + \epsilon^{ij0k}\partial_j F_{0k} + \underbrace{\epsilon^{ijk0}\partial_j F_{k0}}_{\epsilon^{ij0k}\partial_j F_{0k}}$$

$$0 = -\epsilon^{0ijk}\partial_0 F_{jk} + 2\underbrace{\epsilon^{ij0k}}_{\epsilon^{0ijk}}\partial_j F_{0k}$$

$$0 = -\epsilon^{0ijk}\partial_0 F_{jk} + 2\epsilon^{0ijk}\partial_j F_{0k}$$

$$0 = 2\epsilon^{0ijk}\left(\partial_j F_{0k} - \frac{1}{2}\partial_0 F_{jk}\right)$$

$$0 = -2\epsilon^{ijk}\left(\partial_j F_{k0} + \frac{1}{2}\partial_0 F_{jk}\right)$$

$$j^\mu = \begin{pmatrix} j^0 \\ j^i \end{pmatrix} = \begin{pmatrix} \rho c \\ \vec{J} \end{pmatrix}$$

$$j_\mu = \begin{pmatrix} j_0 \\ j_i \end{pmatrix} = \begin{pmatrix} -\rho c \\ \vec{J} \end{pmatrix}$$

since $j^\mu = \eta^{\mu\nu} j_\nu$

$$\partial_\mu j^\mu = \partial_0 j^0 + \partial_i j^i$$

$$= \frac{1}{c}\frac{\partial}{\partial t}(\rho c) + \partial_i j^i$$

$$= \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}$$

$\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$

Let $\mu = 0$

$$\partial^\nu F_{0\nu} = \frac{4\pi}{c} j_0$$

$$\partial^i F_{0i} = \frac{4\pi}{c} j_0$$

$$-\partial^i E_i = \frac{4\pi}{c}(-c\rho) \Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

Let $\mu = i$

$$\partial^\nu F_{i\nu} = \frac{4\pi}{c} j_i$$

$$\partial^0 F_{i0} + \partial^j F_{ij} = \frac{4\pi}{c} j_i \Rightarrow -\frac{1}{c}\frac{\partial E_i}{\partial t} + \epsilon_{ijk}\partial^j B^k = \frac{4\pi}{c} j_i$$

Thus

$$\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0$$

$$\partial^\nu F_{\mu\nu} = \frac{4\pi}{c} j_\mu$$

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \partial^i B_i = 0$ Thus $\vec{B} = \vec{\nabla}\times\vec{A}$; $B^i = \epsilon^{ijk}\partial_j A_k$

$$0 = \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$= \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$= \vec{\nabla}\times\underbrace{\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)}_{-\vec{\nabla}\phi} \quad \Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

$A^\mu = (\phi, \vec{A})$ or $A_\mu = (-\phi, \vec{A})$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $\quad F_{i0} = E_i$; $F_{ij} = \epsilon_{ijk} B^k$

then

$$F_{i0} = \partial_i A_0 - \partial_0 A_i$$

$$= -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A} \,//$$

$$\epsilon^{kij}\left(F_{ij} = \partial_i A_j - \partial_j A_i\right)$$

$$\epsilon^{kij} F_{ij} = \epsilon^{ijk}\partial_i A_j - \underbrace{\epsilon^{kij}}_{-\epsilon^{kji}}\partial_j A_i$$

$$\epsilon^{kij} F_{ij} = \underbrace{2\epsilon^{ijk}\partial_i A_j}_{2B^k} \quad \Rightarrow \quad B^k = \frac{1}{2}\epsilon^{kij} F_{ij}$$

$$F'_{\mu\nu} = \partial'_\mu A'_\nu - \partial'_\nu A'_\mu$$

$$= \Lambda^\rho_\mu \Lambda^\sigma_\nu\, \partial_\rho A_\sigma - \Lambda^\rho_\mu \Lambda^\sigma_\nu\, \partial_\sigma A_\rho$$

$$= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \underbrace{\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right)}_{F_{\rho\sigma}} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma} \,//$$

For a single charge

$\rho = q\,\delta^3(\vec{r} - \vec{r}\,')$ $\quad$ Let $x^0 = ct$ and $\vec{r}\,' = (x'_0, x'_1, x'_2, x'_3)$

$\vec{j} = q\dfrac{d\vec{r}\,'}{dt}\delta^3(\vec{r} - \vec{r}\,')$ $\quad j^0 = \rho c$

$$\delta^{(4)}(x^\nu - x'^\nu(t')) = \delta(t - t')\,\delta^{(3)}(\vec{r} - \vec{r}\,')$$

then

$$j^\mu = q\,\delta^3(\vec{r} - \vec{r}\,')\frac{dx^\mu}{dt} = q\int dt'\,\delta^{(4)}(x^\nu - (x'^\nu(t')))\frac{dx'^\mu}{dt'}$$

$t' \to t'(\lambda)$ leaves invariant too

$dt' = \frac{dt'}{d\lambda} d\lambda \quad ; \quad \frac{d}{dt'} = \frac{d\lambda}{dt'} \frac{d}{d\lambda}$

then

$$dt' = \frac{dt'}{d\tau} d\tau$$

$$j^\mu = q \int d\tau \, \delta^{(4)}(x^\nu - x'^\nu(\tau)) \frac{dx^\mu}{d\tau}$$

$$j^\mu = \sum_n q_n \int d\tau \, \delta^{(4)}(x_n^\nu - x_n^\nu(\tau)) \frac{dx^\mu}{d\tau}$$

$$0 = \epsilon^{\mu\nu\rho\sigma} \partial_\nu (\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

↳ Bianchi identity

$$L = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{q} \dot{q}$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\vec{\nabla}\phi)^2$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{A}' \longrightarrow \vec{A} + \vec{\nabla}\lambda$$

$$\partial_i A'_j \to \partial_i A_j + \partial_i \partial_j \lambda$$

$$\partial_i A'_j - \partial_j A'_i = \partial_i A_j + \cancel{\partial_i \partial_j \lambda} - \partial_j A_i - \cancel{\partial_j \partial_i \lambda}$$

$$= \partial_i A_j - \partial_j A_i$$

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ would be easiest Lagrangian we can choose

Euler-Lagrange Equation

$$S[x(t)] = \int dt \, L(x^i, \dot{x}^i)$$

$$\delta S = \int dt \left(\frac{\partial L}{\partial x^i} \delta x^i + \frac{\partial L}{\partial \dot{x}^i} \delta \dot{x}^i \right) \quad \delta\dot{x}^i \to \frac{d}{dt} \delta x^i$$

$$= \int_{x_1}^{x_2} dt \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \delta x^i \right) + \underbrace{\left(\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) \right)}_{\text{Euler-Lagrange}} \delta x^i \right]$$

$$\cancel{\frac{\partial L}{\partial \dot{x}} \delta x^i \Big|_{x_1}^{x_2}} \, .$$

Flat Space

$$L = \frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j \quad ; \quad \frac{\partial}{\partial \dot{x}^m} \left(\frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j \right) = \frac{1}{2} m \, \delta_{ij} \delta_m^i \dot{x}^j + \frac{1}{2} m \delta_{ij} \delta_m^j \dot{x}^i$$

$$\frac{\partial L}{\partial x^i} = 0 \quad ; \quad \frac{\partial L}{\partial \dot{x}^i} = m \dot{x}^j \delta_{ij} = m \dot{x}^i \qquad = m \dot{x}^i$$

Generalize

$$L = \frac{1}{2} m\, g_{ij}(x)\, \dot{x}^i \dot{x}^j$$

$$\frac{\partial L}{\partial x^i} = \frac{1}{2} m\, \partial_i g_{jk}(x)\, \dot{x}^j \dot{x}^k$$

$$\frac{\partial L}{\partial \dot{x}^i} = m g_{ik} \dot{x}^k$$

$$\frac{1}{2} m g_{jk}(x) \left[\dot{x}^j \delta_i^k + \dot{x}^k \delta_i^j \right] =$$

$$= \frac{1}{2} m \left\{ g_{ji} \dot{x}^j + \dot{x}^k g_{ik} \right\} = m g_{ik} \dot{x}^k //$$

Euler-Lagrange

$$0 = \frac{1}{2} m\, \partial_i g_{jk} \dot{x}^j \dot{x}^k - \frac{d}{dt}\left(m g_{ik} \dot{x}^k \right)$$

$$= \frac{1}{2} \partial_i g_{jk} \dot{x}^j \dot{x}^k - \partial_j g_{ik} \dot{x}^j \dot{x}^k - g_{ik} \ddot{x}^k$$

$$\partial_j g_{ik} \dot{x}^j \dot{x}^k = \frac{1}{2} \partial_j g_{ik} \dot{x}^j \dot{x}^k + \frac{1}{2} \partial_k g_{ij} \dot{x}^j \dot{x}^k$$

$$= \frac{1}{2}\left(\partial_i g_{jk} - \partial_j g_{ik} - \partial_k g_{ij} \right) \dot{x}^j \dot{x}^k - g_{ik} \ddot{x}^k$$

$$g^{il}\left[g_{ik} \ddot{x}^k - \frac{1}{2}\left(\partial_i g_{jk} - \partial_j g_{ik} - \partial_k g_{ij} \right) \dot{x}^j \dot{x}^k \right] = 0$$

$$\ddot{x}^l + \underbrace{\frac{1}{2} g^{il}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk} \right)} \dot{x}^j \dot{x}^k = 0$$

$\Gamma^l_{jk} \equiv$ Christoffel symbol //

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 \rightarrow \text{Geodesic equation}$$

Lectures on QM Dirac //

Electron radiates in gravity when consider equivalence principle (??)

Geodesics in Equation

$$S = \int_{t_i}^{t_f} dt\, L(x, \dot{x})$$

$$\delta S = \int dt \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right)$$

$$= \int_{t_1}^{t_2} dt \left\{ \underbrace{\left[\frac{\partial L}{\partial x} - \partial_t \left(\frac{\partial L}{\partial \dot{x}} \right) \right]}_{\text{Euler Lagrange}} \delta t + \partial_t \left(\frac{\partial L}{\partial \dot{x}} \delta x \right) \right\} \qquad \delta x \Big|_{t_i} = \delta x \Big|_{t_f} = 0$$

Why Lagrangian is not $L = L(q, \dot{q}, \ddot{q}, \dddot{q} \ldots)$

For a well defined vacum we need positive definite Hamiltonian ???
If $L = L(\ldots, \ddot{q}, \dddot{q} \ldots)$ Hamiltonian wouldnt be positive definite

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2} m\, \delta_{ij} \dot{x}^i \dot{x}^j$$

$\delta_{ij} \rightarrow g_{ij}$

$$L = \frac{1}{2} m\, g_{ij}(x)\, \dot{x}^i \dot{x}^j$$

- $\dfrac{\partial L}{\partial x^i} = \dfrac{1}{2} m\, \partial_i g_{jk} \dot{x}^j \dot{x}^k$

- $\dfrac{\partial L}{\partial \dot{x}^i} = \dfrac{1}{2} m\, g_{jk} \left(\delta^{ij} \dot{x}^k + \delta^{ik} \dot{x}^j \right) = \dfrac{1}{2} m\, g_{ik}(x) \dot{x}^k + \dfrac{1}{2} m g_{kj} \dot{x}^k \delta_i{}^j = m g_{ik}(x) \dot{x}^k //$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = m\, \partial_j g_{ik}\, \dot{x}^j \dot{x}^k + m\, g_{ik}\, \ddot{x}^k$ //

$$0 = \frac{1}{2} m\, \partial_i g_{jk}\, \dot{x}^j \dot{x}^k - m\, \partial_j g_{ik}\, \dot{x}^j \dot{x}^k - m\, g_{ik}\, \ddot{x}^k$$

$$0 = g_{ik}\, \ddot{x}^k + \partial_j g_{ik}\, \dot{x}^j \dot{x}^k - \frac{1}{2} \partial_i g_{jk}\, \dot{x}^j \dot{x}^k$$

$$g^{im}\left[0 = g_{ik}\, \ddot{x}^k + \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right) \dot{x}^j \dot{x}^k\right] \qquad g_{ij} = g_{ji}$$

$$0 = \ddot{x}^m + \underbrace{\frac{1}{2} g^{im}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right)}_{\Gamma^m{}_{jk} = \Gamma^m{}_{kj}} \dot{x}^j \dot{x}^k$$

$$\boxed{\ddot{x}^m + \Gamma^m{}_{jk}\, \dot{x}^j \dot{x}^k = 0} \rightarrow \text{Geodesic Equation}$$

Flat Space in Spherical Coordinates

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/r^2 \sin^2\theta \end{pmatrix}$$

$$\Gamma^r{}_{ij} = \frac{1}{2} g^{rr}\Big(\underbrace{\partial_i g_{jr} + \partial_j g_{ir}}_{\text{vanishes for all choices of } (i,j)} - \partial_r g_{ij}\Big) = -\frac{1}{2} g^{rr} \partial_r g_{ij}$$

$$\Gamma^r{}_{\theta\theta} = -r$$

$$\Gamma^r{}_{\phi\phi} = -r \sin^2\theta$$

$$\vdots$$

$$\Gamma^\theta{}_{ij} = \frac{1}{2} g^{\theta l}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

$$= \frac{1}{2r^2}\left(\partial_i g_{j\theta} + \partial_j g_{i\theta} - \partial_\theta g_{ij}\right)$$

$\underline{i = j}$

$$\Gamma^\theta{}_{\phi\phi} = -\frac{1}{2r^2} \partial_\theta g_{\phi\phi} = -\sin\theta \cos\theta$$

$\underline{i \neq j}$

$$\Gamma^\theta{}_{\theta r} = \Gamma^\theta{}_{r\theta} = \frac{1}{2r^2} \partial_r g_{\theta\theta} = \frac{1}{r}$$

$$\Gamma^\phi{}_{ij} = \frac{1}{2} g^{\phi l}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

$$= \frac{1}{2r^2\sin^2\theta}\left(\partial_i g_{j\phi} + \partial_j g_{i\phi} - \cancel{\partial_\phi g_{ij}}\right)$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{2r^2\sin^2\theta}\,\partial_r g_{\phi\phi} = \frac{1}{r}$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \frac{1}{2r^2\sin^2\theta}\,\partial_\theta g_{\phi\phi} = \cot\theta$$

$$L = \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$0 = -\frac{\partial L}{\partial r} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$= \ddot{r} \underbrace{- r}_{\Gamma^r_{\theta\theta}}\dot{\theta}^2 \underbrace{- r\sin^2\theta}_{\Gamma^r_{\phi\phi}}\dot{\phi}^2$$

$$0 = -\frac{\partial L}{\partial \phi} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$= \frac{d}{dt}\left(r^2\sin^2\theta\,\dot{\phi}\right)$$

$\vdots$

$$\partial^\nu F_{\mu\nu} = 4\pi j_\mu$$

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu$$

$$S = \int d^4x \left(-\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}\right) \qquad -\frac{1}{16\pi}\left(F^{\mu\nu}\delta F_{\mu\nu} + F_{\mu\nu}\delta F^{\mu\nu}\right) = 2 F_{\mu\nu}\delta F^{\mu\nu}$$

$$\delta S = \int d^4x\left(-\frac{1}{8\pi} F_{\mu\nu}\delta F^{\mu\nu}\right)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F_{\mu\nu}\left(\delta F^{\mu\nu} = \partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu\right)$$

$$= F_{\mu\nu}\partial^\mu\delta A^\nu - F_{\mu\nu}\partial^\nu\delta A^\mu$$

$$= F_{\mu\nu}\partial^\mu\delta A^\nu - F_{\nu\mu}\partial^\mu\delta A^\nu$$

$$= 2F_{\mu\nu}\partial^\mu\delta A^\nu$$

$$= \int d^4x \underbrace{\left(-\frac{1}{8\pi} F_{\mu\nu}\partial^\mu\delta A^\nu + \frac{1}{8\pi}F_{\mu\nu}\partial^\nu\delta A^\mu\right)}_{-\frac{1}{4\pi}F_{\mu\nu}\underbrace{\partial^\mu\delta A^\nu}_{F_{\mu\nu}}}$$

$F^{\mu\nu}$ yazılıydı bence yanlış

$$= \int\left(-\frac{1}{4\pi}F_{\mu\nu}\partial^\mu\right)\delta A^\nu \quad F^{\mu\nu} \qquad = \int \frac{1}{4\pi}F_{\mu\nu}\partial^\mu + J_\nu)\,\delta A^\nu$$

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\Lambda \qquad S = \int d^4x\,\mathcal{L}' = \int d^4x\,\mathcal{L} - \int d^4x\,\underbrace{\partial_\mu j^\mu}_{0}\Lambda$$

$$J^\mu A'_\mu = J^\mu A_\mu + J^\mu\partial_\mu\Lambda \qquad \vec{A} \to \vec{A}' = \vec{A}' + \vec{\nabla}\Lambda$$

$$\phi \to \phi' = \phi - \frac{\partial\Lambda}{\partial t}$$

Relativistic Particles

A particle in Minkowski Space

$$S = -mc \int \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} \qquad [Js]$$

→ Timelike trajectory

$$= -mc \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}$$

Let $x'^\mu = \dfrac{dx^\mu}{d\sigma}$

$$p_\mu = \frac{\partial L}{\partial x'^\mu} = \frac{m\, \eta_{\mu\nu} x'^\nu}{\sqrt{-\eta_{\mu\nu} x'^\mu x'^\nu}}$$

$$p_\mu p^\mu = -m^2 \quad ; \quad (p^0)^2 = m^2 + \vec{p}^{\,2}$$

$$H = x'^\mu p_\mu - L = 0$$

→ Need to be studied with constraint

$\boxed{\sigma' = \sigma'(\sigma)}$ → constrained (Parametrizasyon Invariance is needed)

$$S = -mc \int c\,dt \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d(ct)} \frac{dx^\nu}{d(ct)}} = -mc \int c\,dt \underbrace{\sqrt{1 - \frac{v^2}{c^2}}}_{\sim 1 - \frac{1}{2}\frac{v^2}{c^2}} \quad ; \quad v \ll c$$

$$S = \int dt \left(-mc^2 + \frac{1}{2} m v^2 \right)$$

Gravitational : $S = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2} + \frac{2\phi}{c^2}} \rightarrow \eta_{\mu\nu} \rightarrow g_{\mu\nu}$

EMT : $S = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} - \int V\, dt$

Lorentz Force

$$L = \frac{1}{2} m \dot{x}^2 + A_\mu J^\mu \quad ; \quad J^\mu = q \dot{x}^\mu \ , \quad A^\mu = (\phi, \vec{A}), \quad A_\mu = (-\phi, \vec{A})$$

Then

$$L = \frac{1}{2} m \dot{x}^2 - q\phi + q \vec{A}\cdot\dot{\vec{x}}$$

$\phi = \phi(t, \vec{x})$; $\vec{A}\cdot\dot{\vec{x}} = A_i \dot{x}^i$

$$\frac{\partial L}{\partial x^i} = -q\, \partial_i \phi + q (\partial_i A_j) \dot{x}^j$$

$$\frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + qA^i \qquad A = A(t, x) \ ; \ \frac{dA^i}{dt} = \frac{\partial A^i}{\partial t} + \frac{\partial A^i}{\partial x^j}\frac{dx^j}{dt} = \partial_t A^i + \partial_j A^i \dot{x}^j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} \right) = m\ddot{x}^i + q\, \partial_t A^i + q\, \partial_j A^i \dot{x}^j$$

$$0 = m\ddot{x}^i + q\,\partial_t A_i + q\,\partial_j A_i\,\dot{x}^j + q\,\partial_i \phi - q(\partial_i A_j)\dot{x}^j$$

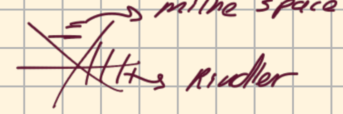
$$= m\ddot{x}^i + q\underbrace{(\partial_t A_i + \partial_i \phi)}_{-E_i} - q\underbrace{(\partial_i A_j - \partial_j A_i)}_{\epsilon_{ijk}B^k}\dot{x}^j$$

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

$$F^{il} = \epsilon^{ilm} B_m$$
$$F_{il} = \epsilon_{ilm} B^m$$

$$m\ddot{x}^i = qE_i + q(\vec{v}\times\vec{B})_i \,//$$

Rindler Space



$$\frac{\frac{v_1+v_2}{c}}{1+\frac{v_1 v_2}{c^2}}$$

$$\frac{v}{c} = \tanh\theta$$

$$v' = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}} \qquad \tanh(\theta_1+\theta_2) = \frac{\tanh\theta_1+\tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2}$$

Observer with constant acceleration ??

(Tolman Law, Unruh Temp.).

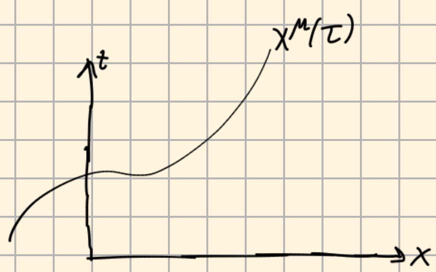
$$S = -m\int d\tau\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + A_\mu J^\mu \quad (J^\mu \to q\dot{x}^\mu,\ \eta_{\mu\nu}\to g_{\mu\nu})$$

$$-m\int d\tau\sqrt{1-\dot{x}^2+2\phi}$$

Rindler Space

Constant accelaration

$$x' = x + \frac{1}{2}at^2$$
$$t' = t$$



$$u^\mu u_\mu = \frac{dx^\mu dx_\mu}{(d\tau)^2} = \frac{\eta_{\mu\nu}dx^\mu dx^\nu}{d\tau^2} = -1 \qquad (\eta_{\mu\nu}dx^\mu dx^\nu = -d\tau^2)$$

- $u^\mu = \frac{dx^\mu}{d\tau}$ • $u^\mu u_\mu = -1$; $\frac{d}{d\tau}(u^\mu u_\mu) = 0$
- $a^\mu = \frac{du^\mu}{d\tau}$ (4-accelaration) $\qquad u^\mu a_\mu = 0$

- constant accelaration

$$a^\mu a_\mu = +\alpha^2 \qquad \text{purely spacelike}$$

Then we can reduce this problem to two dimensions (t, x)

Then
- $-(u^0)^2 + (u^x)^2 = -1$
- $-u^0 a_0 + u^x a_x = 0 \Rightarrow u^0 a_0 = u^x a_x \Rightarrow (\cancel{u^0 a_0})^2 = (u^x a_x)^2$
- $-(a^0)^2 + (a^x)^2 = \alpha^2$

$$= (-1+(u^0)^2)(\alpha^2+(a^0)^2)$$
$$= -\alpha^2 - (a^0)^2 + \alpha^2(u^0)^2 + \cancel{(u^0)^2} a_0^2$$
$$0 = \alpha^2(-1+(u^0)^2) - a_0^2$$
$$0 = \alpha^2(u^x)^2 - (a^0)^2$$

$$a_0 = \alpha u^x$$

On the other hand

$$\cancel{(u^x)^2}(a^x)^2 = (u^0)^2 (a^0)^2$$
$$= (1+(u^x)^2)\left((a^x)^2-\alpha^2\right)$$
$$= (a^x)^2 - \alpha^2 + \cancel{(u^x)^2}(a^x)^2 - \alpha^2 (u^x)^2$$
$$= (a^x)^2 - \alpha^2 (1+(u^x)^2)$$
$$= (a^x)^2 - \alpha^2 (u^0)^2$$
$$a^x = \alpha u^0$$

Since

$$a^0 = \frac{du^0}{d\tau} = \alpha u^x$$
$$a^x = \frac{du^x}{d\tau} = \alpha u^0 \qquad \text{we have} \qquad \frac{d^2 u^0}{d\tau^2} = \alpha \frac{du^x}{d\tau} = \alpha^2 u^0$$
$$\frac{d^2 u^x}{d\tau^2} = \alpha \frac{du^0}{d\tau} = \alpha^2 u^x$$

Let $u^x(\tau=0)=0$ $\curvearrowright$

$$u^x = \sinh(\alpha\tau)$$
$$u^0 = \cosh(\alpha\tau) \quad \Rightarrow \quad \frac{dx^x}{d\tau} = \sinh \alpha\tau$$

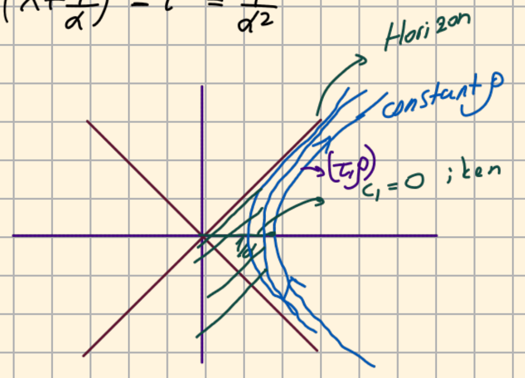
Thus $a^x = \alpha \cosh(\alpha\tau)$ $\qquad \frac{dx^0}{d\tau} = \cosh(\alpha\tau)$

$a^0 = \alpha \sinh(\alpha\tau)$

Thus

$$x = \int \sinh(\alpha\tau)\,d\tau \;:\; x = \frac{1}{\alpha}\cosh(\alpha\tau) + c_1 \quad \left(c_1 = -\frac{1}{\alpha}\right)$$
$$t = \int \cosh(\alpha\tau)\,d\tau \;:\; t = \frac{1}{\alpha}\sinh(\alpha\tau) + \cancel{c_2}$$

$$\left(x+\frac{1}{\alpha}\right)^2 - t^2 = \frac{1}{\alpha^2}$$



asimptot "c" olacak şekilde farklı seçimler olabilir.

genelleştirmek için

$$t = \left(\rho + \frac{1}{\alpha}\right)\sinh(\alpha\tau)$$

$$x = \left(\rho + \frac{1}{\alpha}\right)\cosh(\alpha\tau) - \frac{1}{\alpha}$$

$$dt = d\rho \sinh(\alpha\tau) + \alpha\rho \cosh(\alpha\tau) d\tau + \cosh(\alpha\tau) d\tau$$

$$= d\rho \sinh(\alpha\tau) + (1+\alpha\rho)\cosh(\alpha\tau))\, d\tau$$

$$dx = d\rho \cosh(\alpha\tau) + (1+\alpha\rho)\sinh(\alpha\tau) d\tau$$

Then

$$ds^2 = -dt^2 + dx^2$$

$$= -(1+\alpha\rho)^2 d\tau^2 + d\rho^2 + dy^2 + dz^2$$

pure state $S=0$

$$S = -m \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

$$= -m \int \sqrt{(1+\alpha\rho)^2 - \left(\frac{d\rho}{d\tau}\right)^2}\, d\tau \checkmark$$

$$= -m \int \sqrt{\left(1 - \frac{d\rho}{d\tau}^2 + 2\alpha\rho + \alpha^2\rho^2\right)}\, d\tau \checkmark$$

$$= \int d\tau \left(-m + \frac{1}{2}m\left(\frac{d\rho}{d\tau}\right)^2 - m\alpha\rho + \underbrace{\dots}_{\text{ignore higher order}}\right)$$

- Path Integral ↓
- Hugh Osborn Ch-1 Advanced QFT
- Paul Townsend "Black Hole" Rindler space

$$L = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j \longrightarrow \ddot{x}^i + \Gamma^i{}_{jk}\dot{x}^j \dot{x}^k = 0$$

$$L = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \longrightarrow S = -mc \int d\sigma\, L$$

σ → some arbitrary parameter

$g_{\mu\nu} = g_{\nu\mu}$
and
$g_{\mu\nu} = g_{\mu\nu}(x)$
and
$g_{\mu\nu} g^{\nu\rho} = \delta_\mu{}^\rho$

$$= -mc \int d\sigma \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\sigma}$$

$$0 = \frac{\partial L}{\partial x^\rho} - \frac{\partial}{\partial \sigma}\left(\frac{\partial L}{\partial \dot{x}^\rho}\right)$$

- $\frac{\partial L}{\partial x^\rho} = -\frac{1}{2L}\partial_\rho g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$
- $\frac{\partial L}{\partial \dot{x}^\rho} = -\frac{1}{L} g_{\rho\nu}\dot{x}^\nu$

Then

$$\frac{\partial}{\partial\sigma}\left(-\frac{1}{L} g_{\rho\nu}\dot{x}^\nu\right)$$

Thus

$$0 = -\frac{1}{2L}\,\partial_\rho g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu + \frac{d}{d\sigma}\left(\frac{1}{L} g_{\rho\nu}\,\dot{x}^\nu\right)$$

$$\partial_\mu g_{\rho\nu}\,\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\,\partial_\mu g_{\rho\nu}\,\dot{x}^\mu \dot{x}^\nu + \frac{1}{2}\,\partial_\nu g_{\rho\mu}\,\dot{x}^\mu \dot{x}^\nu$$

$$= \frac{1}{L}\left(-\frac{1}{2}\,\partial_\rho g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu + \underbrace{\partial_\mu g_{\rho\nu}\,\dot{x}^\mu \dot{x}^\nu} + g_{\rho\nu}\,\ddot{x}^\nu\right) + \frac{d}{d\sigma}\left(\frac{1}{L}\right) g_{\rho\nu}\,\dot{x}^\nu$$

$$A_{\mu\nu} B^{\mu\nu} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) B^{\mu\nu} \; ; \; B \text{ symmetric}$$

in other works

$$g^{\rho\lambda}\left(g_{\rho\nu}\,\ddot{x}^\nu + \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)\dot{x}^\mu \dot{x}^\nu = \frac{1}{L}\frac{dL}{d\sigma} g_{\mu\nu}\,\dot{x}^\nu\right)$$

if $\frac{dL}{d\sigma} = 0$? if $\sigma = a\tau + b$

$$\underbrace{\ddot{x}^\lambda + \Gamma^\lambda{}_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0}_{\text{Affinely parametrized geodesic}}$$

$$S_{useful} = \int d\tau\, L$$

$L = g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu$: its equations of motion is identical to an affinely parametrize geodesic

if time like < 0

$$g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = -1$$

if null $\Rightarrow$ imposed by hand

$$g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu = 0$$

What is the compton wave length we can not ignore the quantum effects

De Broglie

A First Look at the

Schwarzschild Metric

Kinesel simetrikler statiktir.

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$g_{00} = 1 - \frac{2\phi}{c^2} \longrightarrow \phi = \frac{GM}{r}$$

$r > R_s$

$$R_s = \frac{2GM}{c^2} \Rightarrow ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$= -A(r)\,dt^2 + \frac{1}{A(r)}\,dr^2 + r^2\, d\Omega_s^2$$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$
metric of a unit 2-sphere

$$S_{useful} = \int d\tau\, L$$

$$= \int d\tau\, g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu$$

$$= \int d\tau\left(-A(r)c^2\dot{t}^2 + A^{-1}(r)\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right)$$

$$0 = \frac{\partial L}{\partial \theta} - \partial_\tau \left(\frac{\partial L}{\partial \dot\theta} \right) \quad (?)$$

initial conditions

$\theta = \frac{\pi}{2}$, $\dot\theta = 0$

$$\frac{\partial L}{\partial \theta} = \cancel{2}\cancel{r^2}\sin\theta\cos\theta\,\dot\phi^2 \;;\quad \partial_\tau\left(\cancel{2}\cancel{r^2}\dot\theta\right)$$

$$\frac{d}{d\tau}(\dot\theta) = \underbrace{\sin\theta\cos\theta\,\dot\phi^2}_{=0}$$

$\dot\theta$ = cst

$$0 = \partial_t\left(\frac{\partial L}{\partial \dot t}\right)$$

$$\frac{\partial L}{\partial x^0} = p_0 = -p^0$$

$$-2E = -2A(r)c^2\dot t$$

Thus

$$\boxed{E = A(r)c^2\dot t}$$

$$2\ell = \frac{\partial L}{\partial \dot\phi} = 2r^2\dot\phi$$

$$\ell = r^2\dot\phi$$

Then we have the additional constraind

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2 \;(m\neq0) \qquad g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \;(m=0)$$

$$c^2 g_{00}\dot t^2 + g_{rr}\dot r^2 + g_{\phi\phi}\dot\phi^2 = -c^2$$

$$\underbrace{-c^2A(r)\dot t^2}_{\frac{E^2}{A(r)c^2}} + \frac{1}{A(r)}\dot r^2 + \underbrace{r^2\dot\phi^2}_{\frac{\ell^2}{r^2}} = -c^2 \Rightarrow \frac{1}{2}\dot r^2 + \frac{1}{2}A(r)\left(\frac{\ell^2}{r^2} + c^2\right) = \frac{1}{2}\frac{E^2}{c^2}$$

$$\frac{1}{2}\dot r^2 + \frac{1}{2}\left(c^2 + \frac{\ell^2}{r^2}\right)\underbrace{\left(1 - \frac{2GM}{rc^2}\right)}_{A(r)} = \frac{1}{2}\frac{E^2}{c^2}$$

Thus

$$\frac{1}{2}\dot r^2 + V_{eff} = \frac{E^2}{c^2}$$

$$V_{eff} = \frac{1}{2}\left(c^2 + \frac{\ell^2}{r^2}\right)\left(1 - 2\frac{GM}{rc^2}\right)$$

$$= \frac{1}{2}c^2 \underbrace{- \frac{GM}{r} + \frac{\ell^2}{2r^2}}_{\substack{\text{newtonian} \\ V_N(r)}} \underbrace{- \frac{GM\ell^2}{r^3c^2}}_{\text{General Relativity}}$$

When r is smaller this effects would be obvious.

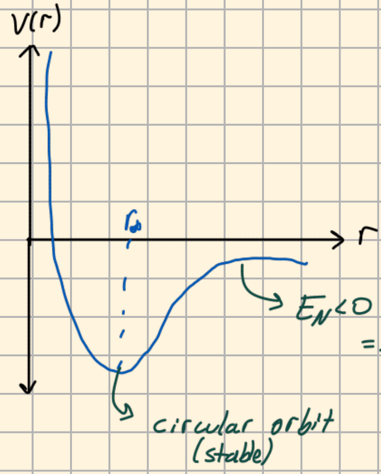
Planetory Orbits in Newtonian Mechanics

$$V_N(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2}$$

$$V_N'(r) = \frac{GM}{r^2} - \frac{\ell^2}{r^3} = 0 \qquad \boxed{r_0 = \frac{\ell^2}{GM}} \to \text{minimum}$$

$$V''_N(r) = -\frac{2GM}{r^3} + \frac{3\ell^2}{r^4}$$

$$= -\frac{2(GM)^4}{\ell^6} + \frac{3(GM)^4}{\ell^6} = \frac{GM}{\ell^6} > 0$$



$\Rightarrow$ particle cannot escape to ∞

$$E_N = \frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2}$$

Let $r = r(\phi)$

$u = \frac{1}{r}$ leading us to

$$\dot{u} = \frac{du}{d\phi}\dot{\phi} = \ell u^2 \frac{du}{d\phi}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\ell\frac{du}{d\phi}$$

Then

$$\frac{1}{2}\ell^2\left(\frac{du}{d\phi}\right)^2 - GMu + \frac{\ell^2}{2}u^2 = E_N$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{\ell^2}u = \frac{2E_N}{\ell^2}$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{GM}{\ell^2}\right)^2 = \frac{2E_N}{\ell^2} + \frac{G^2M^2}{\ell^4}$$

The solution is given by

$$u(\phi) = \frac{GM}{\ell^2}(1 + e\cos\phi)$$

$$r(\phi) = \frac{\ell^2}{GM}\frac{1}{(1+e\cos\phi)} \qquad e \equiv \left(1 + \frac{2E_N\ell^2}{G^2M^2}\right)^{1/2}$$

Planetary Orbits in General Relativity

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(c^2 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2GM}{rc^2}\right) = \frac{1}{2}\frac{E^2}{c^2}$$

and

$$V_{eff}(r) = \frac{1}{2}c^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2} \underbrace{- \frac{GM\ell^2}{r^3c^2}}_{GR}$$

We have

$$V'(r) = \frac{GM}{r^2} - \frac{\ell^2}{r^3} + \frac{3GM\ell^2}{r^4c^2}$$

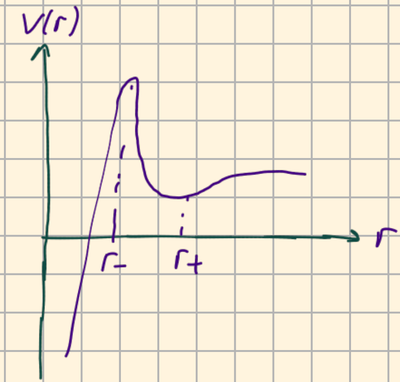
$$GMr^2 - \ell^2 r + \frac{3GM\ell^2}{c^2} = 0$$

$$\ell^4 - \frac{4\cdot 3G^2M^2\ell^2}{c^2} > 0$$

$$\ell^2 - 12\frac{G^2M^2}{c^2} > 0 \rightarrow$$ two solutions r_+ and r_- with $r_+ > r_-$

r_+ = minima (stable)

r_- = maxima (unstable)



if $\ell^2 = 12\frac{G^2M^2}{c^2}$

$r_{isco} = 6\frac{GM}{c^2}$ (Inner stable circular orbit)

Perihelion Precession

$$\frac{1}{2}\ell\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}c^2 - Gmu + \frac{\ell^2}{2}u^2 - \frac{GM\ell^2u^3}{c^2} = \frac{1}{2}\frac{E^2}{c^2}$$

$$\frac{d}{d\phi}\left[\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{\ell^2}u - \frac{2GM}{c^2}u^3 = \frac{E^2}{\ell^2c^2} - \frac{c^2}{\ell^2}\right]$$

$$2\frac{du}{d\phi}\left(\frac{d^2u}{d\phi^2}\right) + 2u\frac{du}{d\phi} - \frac{2GM}{\ell^2}\frac{du}{d\phi} - \frac{6GM}{c^2}u^2\frac{du}{d\phi} = 0$$

Lets ignore the circular orbits: $\frac{d^2u}{d\phi^2} + u - \frac{GM}{\ell^2} - \frac{3GM}{c^2}u^2 = 0$

$\beta = \frac{3G^2M^2}{\ell^2c^2}$ → perturbation

$$u(\phi) = u_0 + \beta u_1 + \beta^2 u_2 \ldots$$

we have

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{\ell^2} = \beta \frac{\ell^2}{GM} u^2$$

$$u_0 = \frac{GM}{\ell^2}(1 + e\cos\phi)$$

$$\frac{d^2(u_0+\beta u_1)}{d\phi^2} + (u_0 + \beta u_1) - \frac{GM}{\ell^2} = \frac{\beta \ell^2}{GM}(u_0 + \beta u_1)^2$$

$$\Downarrow$$

$$\underbrace{\left(\frac{d^2u_0}{d\phi^2} + u_0 - \frac{GM}{\ell^2}\right)}_{0} + \beta\frac{d^2u_1}{d\phi^2} + \beta u_1 = \frac{\beta \ell^2}{GM}(u_0^2 + \cancel{2\beta u_0 u_1} + \cancel{\beta^2 u_1^2})$$

$$\Rightarrow \frac{d^2u_1}{d\phi^2} + u_1 = \frac{\ell^2}{GM}\frac{G^2M^2}{\ell^4}(1 + e\cos\phi)^2$$

$$u_1(\phi) = \frac{GM}{\ell^2}\left[\left(1 + \frac{e^2}{2}\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right]$$

$$\frac{du}{d\phi} \approx \frac{d}{d\phi}(u_0 + \beta u_1) = 0$$

$$\hookrightarrow -e\sin\phi + \beta\left(e\sin\phi + e\phi\cos\phi + \frac{e^2}{3}\sin 2\phi\right) = 0$$

- One solution is $\phi = 0$
- Next solution is $\phi = 2\pi + \delta$

$\delta \ll$

$$\sin(2\pi + \delta) = \sin\delta = \delta$$

$\beta \cdot \delta \ll$

$$-e\sin\phi + \beta\left(e\sin\phi + e\phi\cos\phi + \frac{e^2}{3}\sin 2\phi\right) = 0$$

$$-e\delta + \beta(e 2\pi) = 0$$

$$\therefore \delta = 2\pi\beta = \frac{6\pi G^2M^2}{\ell^2c^2}$$