Chapter: 06

Linear programming:

Definition:

Linear programming is the method of solving the problem by optimizing the given objective-linear function by satisfying the given constraints / conditions.

The objective of the linear programming is just to find the maximum ~~and~~ or minimum (or optimum) solution of the linear objective function by satisfying the given conditions / in-equalities. Then such type of problem is known as the LP. problem.

mathematically a L.P.
problem can be defined as;

Objective,
(optimize) $Z = ax+by$
subject to,
$$\left.\begin{aligned} a_1x+by &\leq c_1 \\ a_2x+b_2y &\leq c_2 \end{aligned}\right\} \text{— (1)}$$
where, $x \geq 0$, $y \geq 0$.

Here, $ax+by = z$ is the objective function which is to be ~~maxi~~ optimized.

and the conditions (1) are the constraints of the objective function and the variables x & y are the decision variable.

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objective function:

~~The~~ A function of the form $f(x,y) = ax + by$ is called objective function which is to be optimized.

Decision variables:

The non-negative variables which are used in the LP problem ~~constraints~~ to get the optimal solution are known as the decision variables.

constraints:

The condition or inequalities in which the decision variables are used under certain restriction ~~are~~ known as the constraints.

Feasible region:

Feasible region is an optimal solution which is obtained by the intersection of the finite no. of closed half planes.

Standard maximization problem:

A LP problem of the form;

objective,

$$\max \quad Z = ax + by$$

$$\left.\begin{aligned} a_1x + b_1y &\le c_1 \\ a_2x + b_2y &\le c_2 \\ a_3x + b_3y &\le c_3. \end{aligned}\right\} \quad \text{— (1)}$$

Where, $x \geq 0, y \geq 0, c_i \geq 0.$

$i = 1, 2, 3.$

x, y are the decision variables, and the given inequations are the constraints and function $z = ax + by$ is the objective function which is to be maximized under the conditions (1).

standard minimization problem:

A LP problem of the form:

$\min z = ax + by$

subject to the constraints;

$a_1x + b_1y \geq c_1$

$a_2x + b_2y \geq c_2$

$a_3x + b_3y \geq c_3$.

$x \geq 0,\ y \geq 0,\ c_i \geq 0$

$i = 1,2,3$.

feasible region:

The region which is obtained by the finite intersection of the planes determined by a set of constraints, is known as the feasible region. And the corner points of the feasible region are known as the vertices of the feasible region.

Linear programming problem.

Exercise : 5.1

1. Soln Given LPP;

$$\min Z = 45x_1 + 22.5x_2$$

s.t. $-x_1 + x_2 \geq -5$

$2x_1 + x_2 \geq 10,$

$x_2 \geq 4,$

$10x_1 + 15x_2 \leq 150$

The boundary lines of the given inequalities are

$-x_1 + x_2 = -5$ — (1)

$2x_1 + x_2 = 10$ — (2)

$x_2 = 4$ — (3)

$10x_1 + 15x_2 = 150$ — (4)

From eqn (1)

If.

x_1	0	5
x_2	-5	0

We get line (1) passes through the points (0,-5) and (5,0). Let us take (0,0) as test point, then, $0 \geq -5$ (~~False~~). So, ~~line~~ Inequality $-x_1 + x_2 \geq -5$ [redacted] contain the origin.

Again, From (2)

x_1	0	5
x_2	10	0

Line (2) passes through the points (0,10) & (5,0) and inequality $2x_1 + x_2 \geq 10$ doesnot contain the origin since $0 \geq 10$ (False).

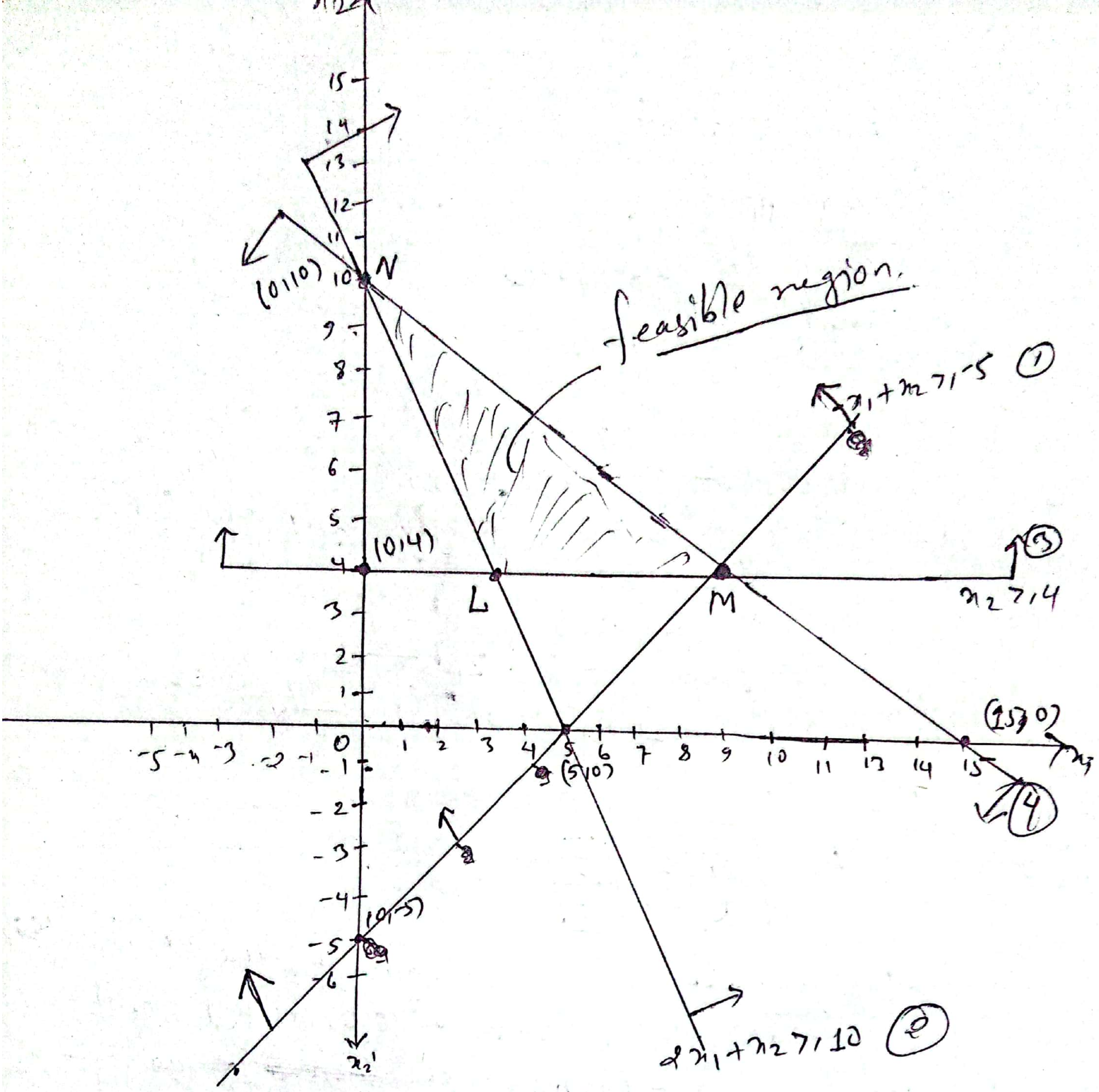
Again, from (3)

$x_2 = 4$ & $x_1 = 0$. This line also doesnot ~~the~~ contain the origin. since $0 \geq 4$ (False).

lastly, From eqn (4)

x_1	0	15
x_2	10	0

$\therefore$ line (4) passes through the points (0,10) & (15,0) and contains the origin since, $0 \leq 150$ (True).



solving the line (2) & (3) we get

$x_2 = 4$ & $x_1 = 3$.

Again, from the lines (1), (3) & (4) we get

$x_2 = 4$, and $x_1 = 9$.

From figure, LMN is the feasible region and L(3,4), M(9,4) & N(0,10) are the coordinate points of feasible region.

∴ The required maximum value of Z is;

Vertices	Value of $z = 45x_1 + 22.5x_2$
(3,4)	$z = 45\times3 + 22.5\times4 = 225$
(9,4)	$z = 45\times9 + 22.5\times4 = 495$ (maximum value)
(0,10)	$z = 45\times0 + 22.5\times10 = 225$

Q.no: 11. optimize $z = 2x_1 + x_2$,

subject to $x_1 + 2x_2 \le 10$

$x_1 + x_2 \ge 1$

$0 \le x_2 \le 4$,

$x_1 \ge 0$.

Soln: The boundary lines of the given inequality are:

$x_1 + 2x_2 = 10$ — (1)

$x_1 + x_2 = 1$ — (2)

$x_2 = 4$ — (3)

From (1):

If $x_1 = 0$, then, $x_2 = 5$

If $x_2 = 0$, then, $x_1 = 10$

$\therefore$ line (1) passes through the points (0,5) & (10,0) and $0 \le 10$ (true). So, the inequality $x_1 + 2x_2 \le 10$ contains the origin.

From eqn (2)

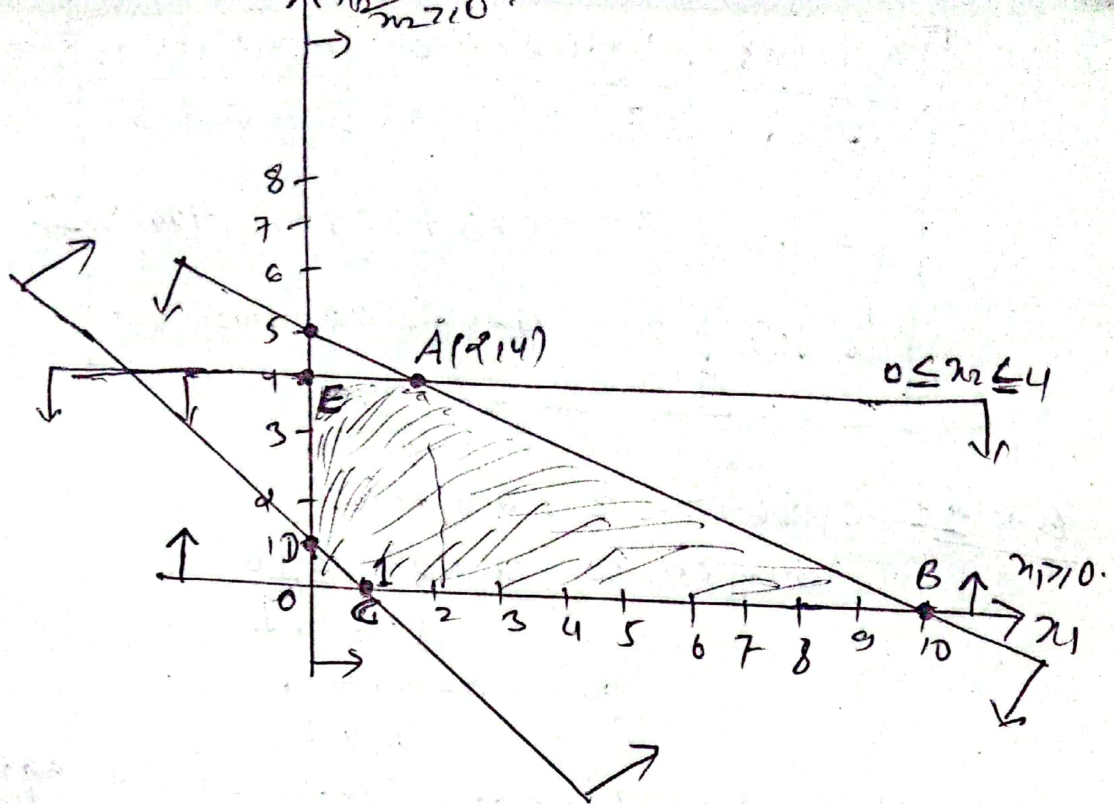
x_1	0	1
x_2	1	0

$\therefore$ line (2) passes through the points (0,1) & (1,0). Let us take (0,0) as the test point then inequality $0 \ge 1$ (false). So, $x_1 + x_2 \ge 1$ doesnot contain the origin.

From (3)

$x_2 = 4$ & $x_1 = 0$; is passes through the point (0,4) parallel to the x-axis.

and Lastly, $x_1 \ge 0$, and $x_2 \ge 0$.



From figure ABCDE is our feasible region and our optimum solution is;

vertices (x_1, x_2)	value of $z = 2x_1 + x_2$
$A(2,4)$	$z = 2\times2+4 = 8$
$B(10,0)$	$z = 2\times10+0 = 20 \rightarrow$ (maximum)
$C(1,0)$	$z = 2\times1+0 = 2$
$D(0,1)$	$z = 2\times0+1 = 1 \rightarrow$ (minimum)
$E(0,4)$	$z = 2\times0+4 = 4$

Hence, the given objective function gives maximum value at $B(10,0)$ and minimum value at $D(0,1)$.

#

Simplex Method

Exercise 5.2

1. Solve the Linear programming problem using the simplex method.

maximize $Z = 30x_1 + 20x_2$

subject to $-x_1 + x_2 \le 5$

$2x_1 + x_2 \le 10$

Solution:

Introducing non-negative slack variables x_3 and x_4 we have

$-x_1 + x_2 + x_3 + 0 \cdot x_4 = 5$

$2x_1 + x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 10$

$-30x_1 - 20x_2 + 0x_3 + 0 \cdot x_4 + Z = 0$.

The initial tableau is;

Basic variables	x_1	x_2	x_3	x_4	Z	R.H.S.
x_3	-1	1	1	0	0	5
x_4	$\boxed{2}$	1	0	1	0	10
	-30	-20	0	0	1	0

From this initial tableau -30 is the most negative entry. so, first column is pivote column and [redacted] element is 2 since $\frac{5}{-1} = -5$ and $\frac{10}{2} = 5$ where 5 is the smallest non-negative [redacted] value.

Applying $R_2 \to \frac{R_2}{2}$.

B.V.	x_1	x_2	x_3	x_4	Z	R.H.S.
x_3	-1	1	1	0	0	5
x_1	1	1/2	0	1/2	0	~~10~~ 5
	-30	-20	0	0	1	0

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 30R_2$

B.V.	x_1	x_2	x_3	x_4	Z	R.H.S.
x_3	0	$\boxed{3/2}$	1	1/2	0	10
x_1	1	1/2	0	1/2	0	5
	0	-5	0	15	1	150

Again, In the last row of this tableau we get negative entry i.e. -5. So the 2nd column is pivot column and $\frac{3}{2}$ is pivot element since the ratio $\frac{10}{3/2} = \frac{20}{3}$ (minimum) and $\frac{5}{1/2} = 10$

i.e. $\frac{20}{3} < 10$.

B.V.	x_1	x_2	x_3	x_4	z	R.H.S.
x_2	0	1	2/3	1/3	0	20/3
x_1	1	1/2	0	1/2	0	5
	0	-5	0	15	1	150

Applying $R_1 \to R_1 \times 2/3$

Again, Applying $R_2 \to R_2 - \frac{1}{2}R_1$ & $R_3 \to R_3 + 5R_1$

B.V.	x_1	x_2	x_3	x_4	z	R.H.S.
x_2	0	1	2/3	1/3	0	20/3
x_1	1	0	-1/3	1/6	0	$\frac{5}{3}$
	0	0	10/3	$\frac{50}{3}$	1	$\frac{550}{3}$

∴ This is the optimal basic solution of the given LPP because the last row has positive entry.

∴ Max $z = \frac{550}{3}$ when $x_1 = \frac{5}{3}$ and $x_2 = \frac{20}{3}$.

Also, max. $z = 30 \times x_1 + 20 x_2$

$= 30 \times 5/3 + 20 \times \frac{20}{3}$

$= \frac{150 + 400}{3}$

$= \frac{550}{3}$, which is true. #

In short way.

1. maximize $z = 30x_1 + 20x_2$.

s.t. $-x_1 + x_2 \leq 5$

$2x_1 + x_2 \leq 10$.

Soln: Introducing non-negative slack variables x_3 & x_4.

$\therefore -x_1 + x_2 + x_3 + 0 \cdot x_4 = 5$

$2x_1 + x_2 + 0 \cdot x_3 + x_4 = 10$

$-30x_1 + 20x_2 + 0 \cdot x_3 + 0 \cdot x_4 + z = 0$

$\therefore$ Initial tableaue is;

Row.	B.V.	x_1	x_2	x_3	x_4	z	R.H.S.	Ratio
R_1	x_3	-1	1	1	0	0	5	$\frac{5}{-1} = -5$
R_2	x_4	$\boxed{2}$	1	0	1	0	10	$10/2 = 5$
R_3		-30	-20	0	0	1	0	

(-ve values)

Here, -30 is the smallest negative entry. so, 1st column is the pivote column and 2 is the pivote element.

Now, Applying, $R_2 \to \frac{R_2}{2}$, $R_1 \to R_1 + R_2$, $R_3 \to R_3 + 30R_2$

Row	B.V.	x_1	x_2	x_3	x_4	z	R.H.S.	Ratio
R_1	x_3	0	$\boxed{3/2}$	1	$1/2$	0	10	$\frac{10}{3/2} = \frac{20}{3}$ (min)
R_2	x_1	1	$1/2$	0	$1/2$	0	5	$\frac{5}{1/2} = 10$
R_3		0	-5	0	15	1	150	

Again we get here, -5 is the smallest negative entry in the last row. so, 2nd column is the pivote column and 3/2 is pivote element since $\frac{20}{3} < 10$.

Now, Applying $R_1 \to R_1 \times 2/3$, $R_2 \to R_2 - \frac{1}{2}R_1$ & $R_3 \to R_3 + 5R_1$

Row	B.V.	x_1	x_2	x_3	x_4	z	R.H.S.	Ratio
R_1	x_2	0	1	$2/3$	$1/3$	0	$20/3$	
R_2	x_1	1	0	$-1/3$	$\frac{1}{3}$	0	$5/3$	
R_3		0	0	$\frac{10}{3}$	$\frac{50}{3}$	1	$150 + 5 \times \frac{20}{3} = \frac{550}{3}$	

$\therefore z = \frac{550}{3}$

when $x_1 = 5/3$ & $x_2 = \frac{20}{3}$

Q no: 3. Sol^n: Given Linear programming problem;

$\min Z = 5x_1 - 20x_2$
s.t. $-2x_1 + 10x_2 \le 5$
$2x_1 + 5x_2 \le 10$

OR: $\min Z = 5x_1 - 20x_2$
s.t. $2x_1 - 10x_2 \ge -5$
$-2x_1 + 5x_2 \ge -10$
$x_1, x_2 \ge 0$

$\max Z^* = -\min Z$

$\therefore \max Z^* = -5x_1 + 20x_2$
s.t. $-2x_1 + 10x_2 \le 5$
$2x_1 + 5x_2 \le 10$

Now, Introducing the non-negative slack variables; x_3 & x_4.

$\therefore$ ~~max~~ $-2x_1 + 10x_2 + x_3 = 5$
$2x_1 + 5x_2 + x_4 = 10$
$Z^* + 5x_1 + 20x_2 = 0.$

$2R_1$
-4 20 2 0 0 10
5 -20 0 0 1 0
1 0 2 0 1 10

simplex table: 1

Row	B.V.	x_1	x_2	x_3	x_4	Z^*	R.H.S	Ratio
R_1	x_3	-2	$\boxed{10}$	1	0	0	5	$\frac{5}{10} = \frac{1}{2}$ → (min)
R_2	x_4	2	5	0	1	0	10	$\frac{10}{5} = 2$
R_3	Z^*	5	$\boxed{-20}$	0	0	1	0	

$\therefore$ 10 is the pivote element & 2nd column is the pivote column. Since from ratio column $\frac{1}{2} < 2$.

$R_1 \to \frac{R_1}{10}$, $R_2 \to R_2 - 5R_1$, $R_3 \to R_3 + 20R_1$

$(-1, 5, \frac{1}{2}, 0, 0, \frac{5}{2})R_1$

Row	B.V.	x_1	x_2	x_3	x_4	Z^*	R.H.S	Ratio
R_1	x_2	$-\frac{1}{5}$	1	$\frac{1}{10}$	0	0	$\frac{1}{2}$	
R_2	x_4	3	0	$-\frac{1}{2}$	1	0	$\frac{15}{2}$	
R_3	Z^*	1	0	2	0	1	10	

Here, All entries in the last row are non-negative. so the process is terminates.

$\therefore \max Z^* = 10$ when, $x_2 = \frac{1}{2}$ & $x_1 = 0$.

But $\max Z^* = -\min Z$

i.e. $\min Z = -\max Z^*$
$= -10$ is our required solution.

The minimum value of the given O.F. is -10 at $(0, \frac{1}{2})$.