cs109a_hw3

October 4, 2017

1 CS 109A/STAT 121A/AC 209A/CSCI E-109A: Homework 3

2 Multiple Linear Regression, Subset Selection, Cross Validation

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2.0.1 INSTRUCTIONS

- To submit your assignment follow the instructions given in canvas.
- Restart the kernel and run the whole notebook again before you submit.
- Do not include your name(s) in the notebook if you are submitting as a group.
- If you submit individually and you have worked with someone, please include the name of your [one] partner below.

Your partner's name (if you submit separately): Everett Sussman, Jan Geffert
Enrollment Status (109A, 121A, 209A, or E109A): 109A
Import libraries:

In [2]:
```
import sys
import numpy as np
import pandas as pd
import matplotlib
import matplotlib.pyplot as plt
import seaborn as sns
from sklearn.metrics import r2_score
from scipy import stats
import statsmodels.api as sm
from statsmodels.api import OLS
from sklearn.preprocessing import PolynomialFeatures
from sklearn.model_selection import KFold
from sklearn.linear_model import Ridge
from sklearn.linear_model import Lasso
from sklearn.linear_model import RidgeCV
from sklearn.linear_model import LassoCV
%matplotlib inline
sns.set_context("poster")
```

```
/Applications/anaconda/lib/python3.6/site-packages/statsmodels/compat/pandas.py:56:
FutureWarning: The pandas.core.datetools module is deprecated and will be removed in a
future version. Please use the pandas.tseries module instead.
  from pandas.core import datetools
```

3 Forecasting Bike Sharing Usage

In this homework, we will focus on multiple linear regression and will explore techniques for subset selection. The specific task is to build a regression model for a bike share system that can predict the total number of bike rentals in a given day, based on attributes about the day. Such a demand forecasting model would be useful in planning the number of bikes that need to be available in the system on any given day, and also in monitoring traffic in the city. The data for this problem was collected from the Capital Bikeshare program in Washington D.C. over two years.

The data set is provided in the files `Bikeshare_train.csv` and `Bikeshare_test.csv`, as separate training and test sets. Each row in these files contains 10 attributes describing a day and its weather: - season (1 = spring, 2 = summer, 3 = fall, 4 = winter) - month (1 through 12, with 1 denoting Jan) - holiday (1 = the day is a holiday, 0 = otherwise) - day_of_week (0 through 6, with 0 denoting Sunday) - workingday (1 = the day is neither a holiday or weekend, 0 = otherwise) - weather - 1: Clear, Few clouds, Partly cloudy, Partly cloudy - 2: Mist + Cloudy, Mist + Broken clouds, Mist + Few clouds, Mist - 3: Light Snow, Light Rain + Thunderstorm + Scattered clouds, Light Rain + Scattered clouds - 4: Heavy Rain + Ice Pallets + Thunderstorm + Mist, Snow + Fog - temp (temperature in Celsius) - atemp (apparent temperature, or relative outdoor temperature, in Celsius) - humidity (relative humidity) - windspeed (wind speed)

and the last column 'count' contains the response variable, i.e. total number of bike rentals on the day.

3.1 Part (a): Data Exploration & Preprocessing

As a first step, identify important characteristics of the data using suitable visualizations when necessary. Some of the questions you may ask include (but are not limited to):

- How does the number of bike rentals vary between weekdays and weekends?
- How about bike rentals on holidays?
- What effect does the season have on the bike rentals on a given day?
- Is the number of bike rentals lower than average when there is rain or snow?
- How does temperature effect bike rentals?
- Do any of the numeric attributes have a clear non-linear dependence with number of the bike rentals?

In [3]:
```
# Create Dataframes
trainDf = pd.read_csv("Bikeshare_train.csv")
testDf = pd.read_csv("Bikeshare_test.csv")
```

In [4]:
```
trainDf.head()
```

Out[4]:

	Unnamed: 0	season	month	holiday	day_of_week	workingday	weather	temp
0	0	2	5	0	2	1	2	24
1	1	4	12	0	2	1	1	15
2	2	2	6	0	4	1	1	26
3	3	4	12	0	0	0	1	0
4	4	3	9	0	3	1	3	23

	atemp	humidity	windspeed	count
0	26	76.5833	0.118167	6073

```
1      19    73.3750    0.174129    6606
2      28    56.9583    0.253733    7363
3       4    58.6250    0.169779    2431
4      23    91.7083    0.097021    1996
```

```
In [5]: # trainDf summary statistics
        trainDf.shape
```

```
Out[5]: (331, 12)
```

```
In [6]: # testDf summary statistics
        testDf.shape
```

```
Out[6]: (400, 12)
```

3.1.1 Importing the Data

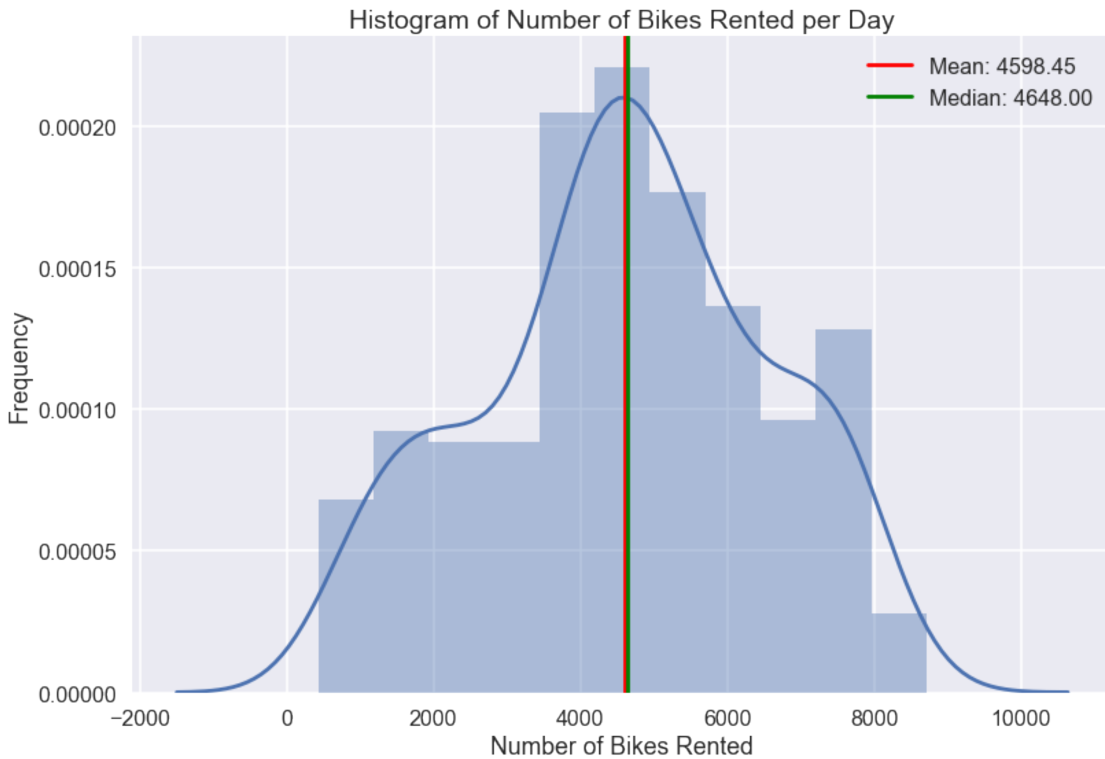
We noticed that our data set appears to be very clean - all columns are filled with values, each data point is atomic, etc. We also noticed that our testing data table contains more data points than our training data table.

3.1.2 Number of Rentals per Day

The first question our group wanted to answer was what the overall distribution of our y variable, the number of bikes rented per day, looked like. Thus, we created the below histogram.

```
In [7]: # General histogram of count data
        meanBikesRented = np.mean(trainDf["count"])
        medianBikesRented = np.median(trainDf["count"])


        fig, ax = plt.subplots(1)
        ax = sns.distplot(trainDf["count"])
        ax.set_title("Histogram of Number of Bikes Rented per Day")
        ax.set_xlabel("Number of Bikes Rented")
        ax.set_ylabel("Frequency")

        ax.axvline(x = meanBikesRented, color="r", label = "Mean:
        {0:.2f}".format(meanBikesRented))
        ax.axvline(x = medianBikesRented, color="g", label = "Median:
        {0:.2f}".format(medianBikesRented))
        ax.legend()
```

```
Out[7]: <matplotlib.legend.Legend at 0x11f21e0f0>
```



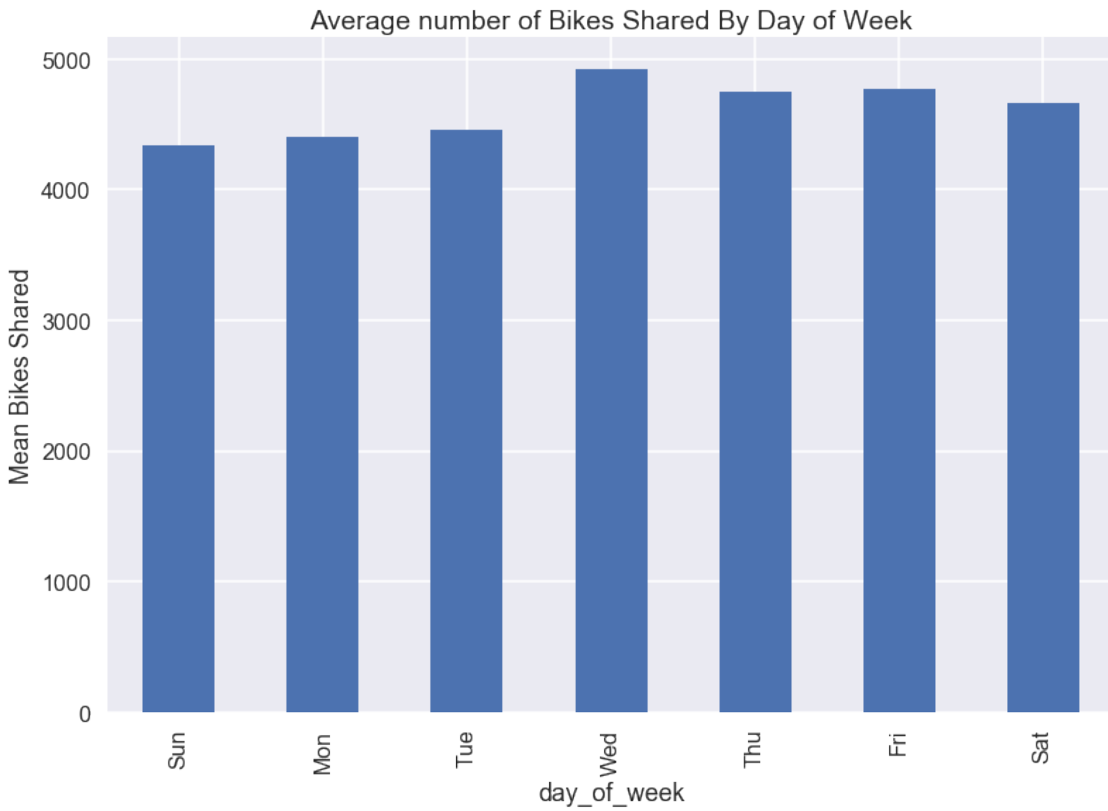
The histogram above shows a fairly symmetrical distribution of values, as the mean and median are very similar. On an average day, 4598 bikes are rented. We also note that there tend to be small *sub-peaks* in our histogram at around 1500 bikes rented and 7500 bikes rented. There are no nonsensical values, such as negative bikes rented in a day.

3.1.3 Weekdays vs. Weekends

Next, we investigated whether bikes, on average, were more likely to be used during the weekend versus weekdays.

```
In [33]: dayGroup = trainDf.groupby("day_of_week", as_index=True)
         ax = dayGroup["count"].mean().plot(kind="bar", title="Average number of Bikes Shared By
         Day of Week")
         plt.xticks(range(7), ["Sun", "Mon", "Tue", "Wed", "Thu", "Fri", "Sat"])
         ax.set_ylabel("Mean Bikes Shared")
```

```
Out[33]: <matplotlib.text.Text at 0x122dc51d0>
```




```
In [9]: # Find mean for weekdays, mean for weekends
        weekMean = np.mean(dayGroup["count"].mean()[1:-2])
        weekendMean = np.mean([dayGroup["count"].mean()[0],dayGroup["count"].mean()[6]])
        print("Mean bikes rented during the week: {}, Mean bikes rented during the weekend:
        {}".format(weekMean, weekendMean))
```

```
Mean bikes rented during the week: 4634.760765349032, Mean bikes rented during the
weekend: 4503.6314465408805
```

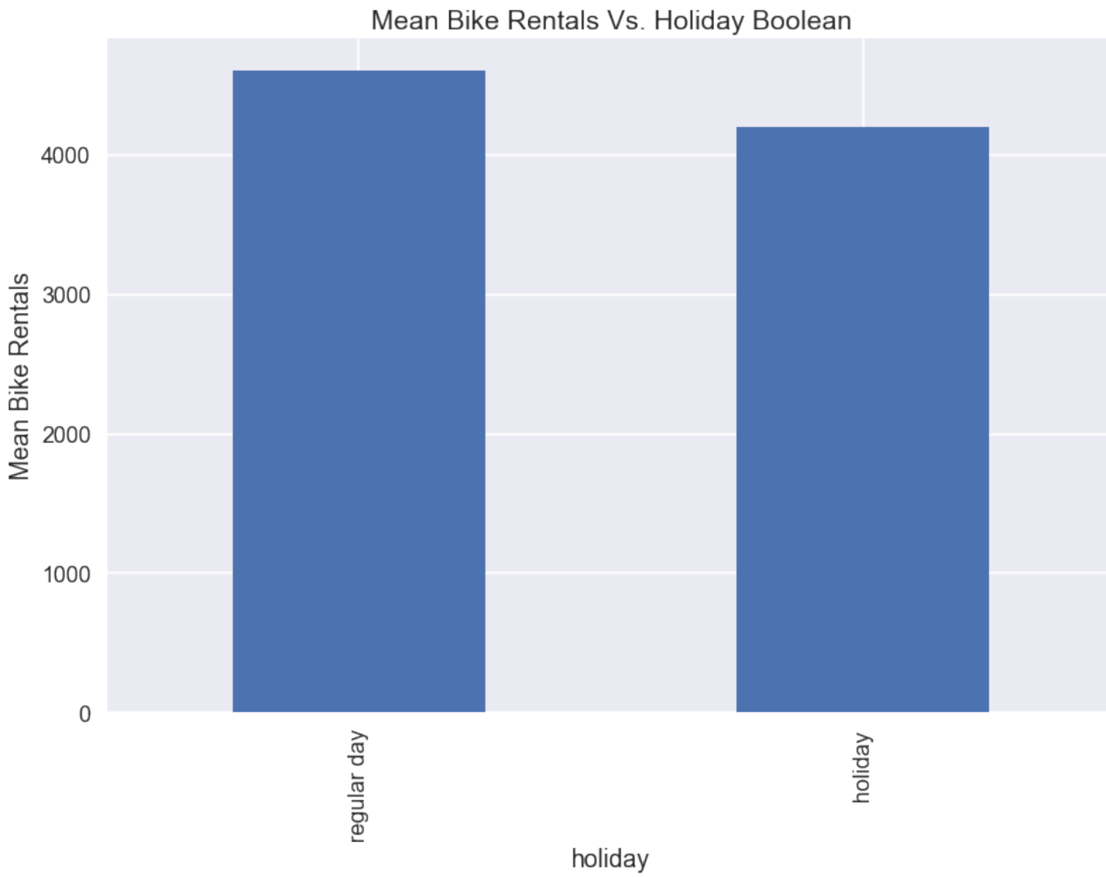
We find that, on average, slightly more bikes are rented during the week (4635 > 4504). This suggests the usage of rental bikes by commuters, as opposed to leisure.

3.1.4 Holidays

Next, we investigated whether the indicator of a day being a holiday impacted overal bike rentals.

```
In [10]: # Holiday Analysis
         holidayGroup = trainDf.groupby("holiday")
         ax = holidayGroup["count"].mean().plot(kind="bar", title="Mean Bike Rentals Vs. Holiday
         Boolean")
         ax.set_ylabel("Mean Bike Rentals")
         plt.xticks([0,1], ["regular day", "holiday"])
```

```
Out[10]: ([<matplotlib.axis.XTick at 0x11f517358>,
           <matplotlib.axis.XTick at 0x11f517b00>],
          <a list of 2 Text xticklabel objects>)
```




```
In [11]: hMeans = holidayGroup["count"].mean()
         print("Mean Bike Rentals during a non-Holiday: {}, Mean Bike Rentals during a Holiday:
         {}.".format(hMeans[0], hMeans[1]))
```

```
Mean Bike Rentals during a non-Holiday: 4612.171875, Mean Bike Rentals during a
Holiday: 4199.181818181818.
```

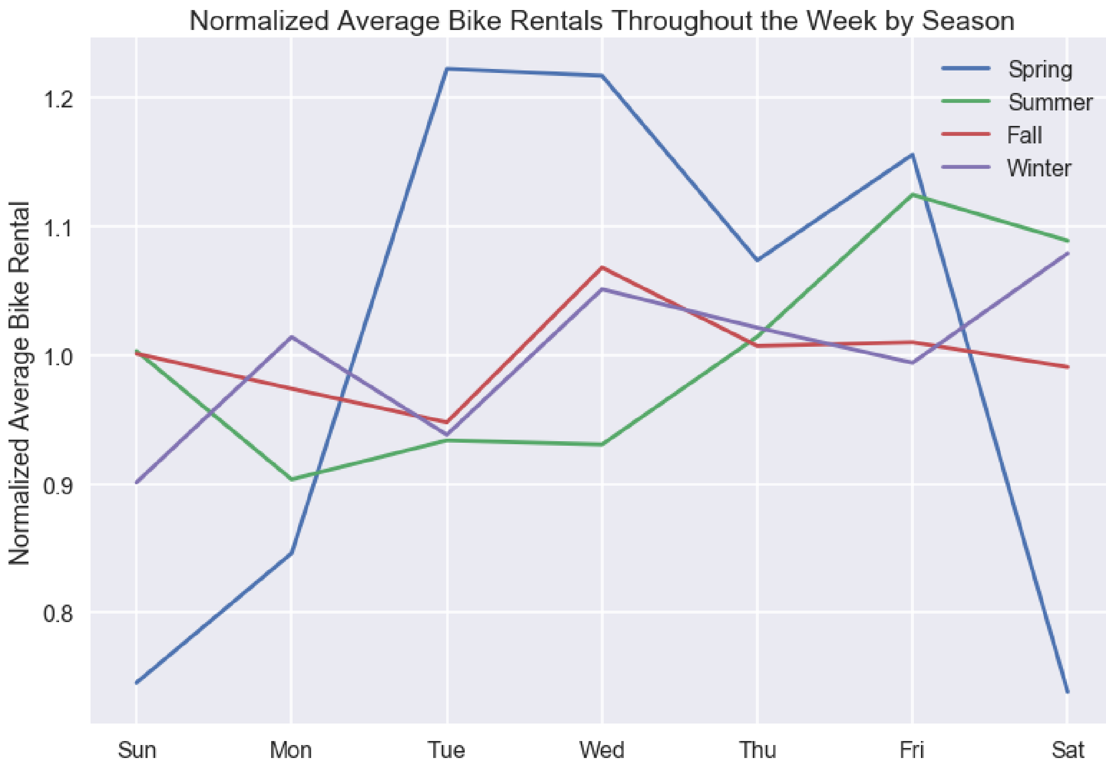
We find that, on average, more bikes are rented during a non-holiday. This further hints at the commuter-hypothesis.

3.1.5 Seasonal effects

Now, our group wanted to determine whether the season impacted the relative popularity of bike rentals throughout the week. For example, in the summer, we would imagine that more people would rent a bike during the weekend than in the winter time, when the only people biking would be those that relied on bikes to commute.

To visualize this, in the below graph, we plotted 4 trend lines throughout the week detailing the normalized average bike rentals per day. To normalize each trend line, we divided the daily average by the seasonal average.

```
In [12]: seasonDayGroup = trainDf.groupby(["season", "day_of_week"])["count"].mean()
         fig, ax = plt.subplots(1)
         seasonNames = ["Spring", "Summer", "Fall", "Winter"]
         for season in range(1,5):
             mean = np.mean(seasonDayGroup[season])
             normalizedRentals = list(map(lambda x: x / mean, seasonDayGroup[season]))
             ax.plot(normalizedRentals, label=seasonNames[season-1])
             ax.legend()
             ax.set_title("Normalized Average Bike Rentals Throughout the Week by Season")
             plt.xticks(range(7), ["Sun", "Mon", "Tue", "Wed", "Thu", "Fri", "Sat"])
             ax.set_ylabel("Normalized Average Bike Rental")
```



We find that there is considerably more variation in the spring time than in the other seasons. Also, in the spring, weekends are surprisingly unpopular in terms of average bike rentals compared to the weekdays. However, in contrast, in the summer, Fridays are very popular. Lastly, in the winter time, Saturday is the most popular day to rent a bike, relative to other days in the week. This was incredibly surprising, and throws a wrench into our initial hypothesis that most bike rentals were for commuting purposes only.

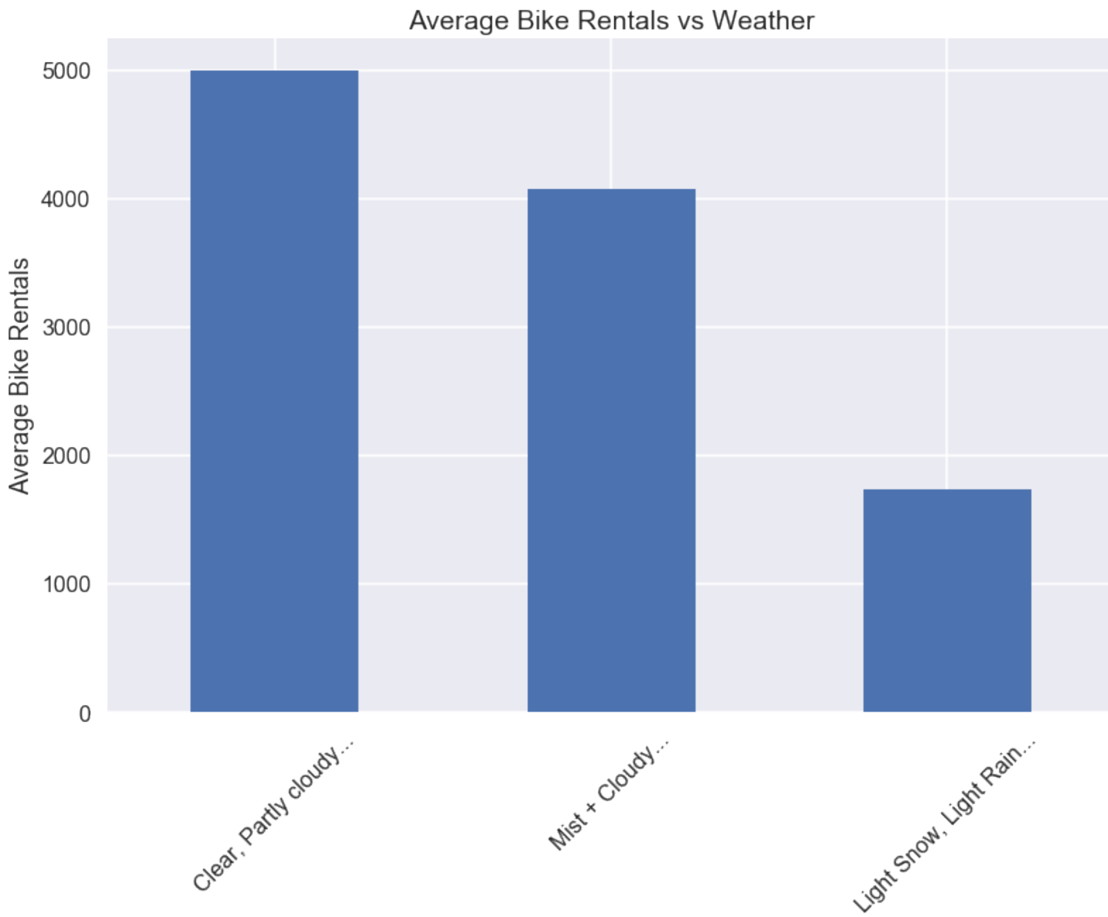
3.1.6 Weather Effects

Now, our group investigated whether the severity of weather impacted bike rentals in any given day.

```
In [35]: weatherGroup = trainDf.groupby("weather")["count"].mean()

         ax = weatherGroup.plot(kind="bar", title="Average Bike Rentals vs Weather")
         plt.xticks(range(0,3), ["Clear, Partly cloudy...", "Mist + Cloudy...", "Light Snow,
```

```
Light Rain..."], rotation=45)
ax.set_xlabel("")
ax.set_ylabel("Average Bike Rentals")
```

Out[35]: <matplotlib.text.Text at 0x1210110b8>



In the above graph, we see that, as expected, when the weather worsens, people rent fewer bikes.

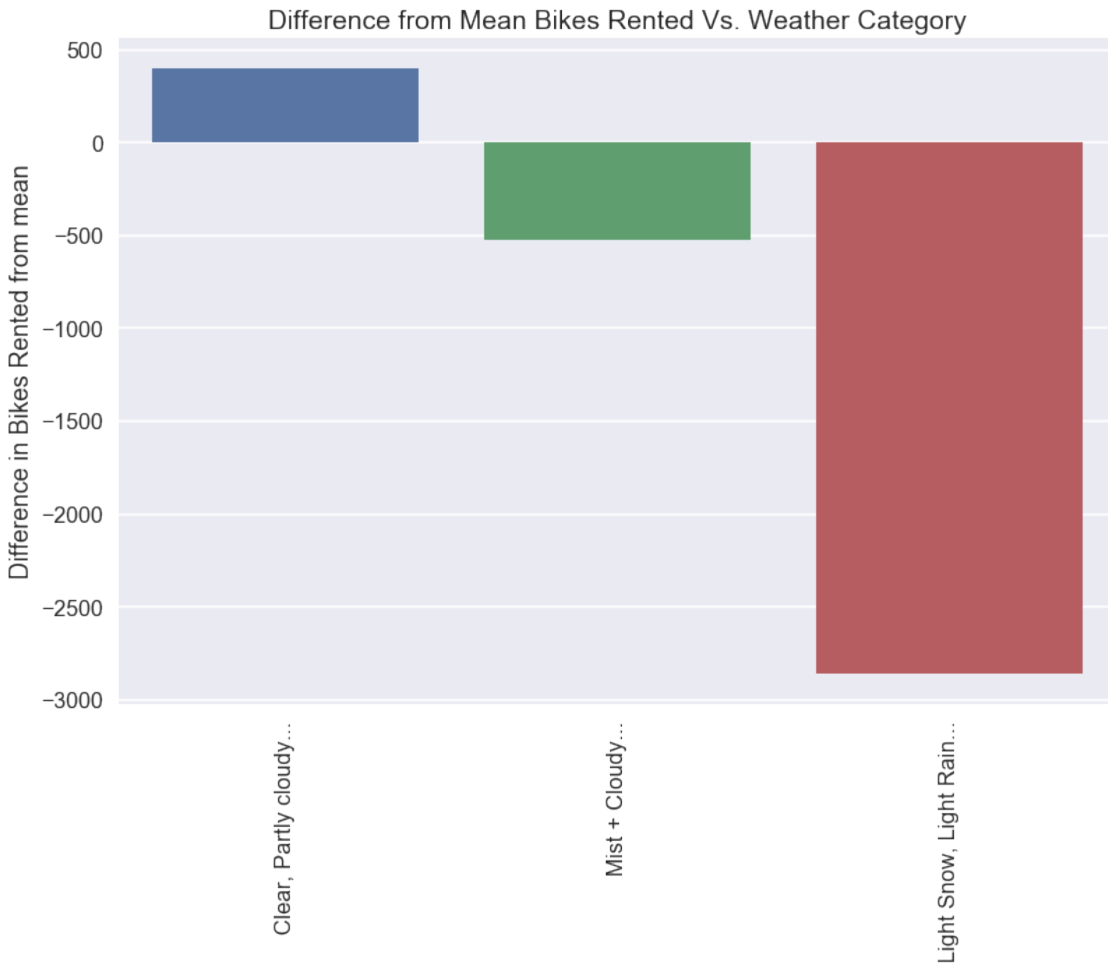
In [14]:
```
# Print out means:
weatherDiff = []

for weather in range(1,4):
    weatherMean = np.mean(weatherGroup[weather])
    weatherDiff.append(weatherMean - meanBikesRented)
    if weatherDiff[-1] > 0:
        print("Weather {} has a mean of: {}, which is {} more than the annual
mean.".format(weather, weatherMean, np.abs(weatherDiff[-1])))
    else:
        print("Weather {} has a mean of: {}, which is {} less than the annual
mean.".format(weather, weatherMean, np.abs(weatherDiff[-1])))
```

```
Weather 1 has a mean of: 5001.476415094339, which is 403.02928518497356 more than the
annual mean.
Weather 2 has a mean of: 4077.1651376146788, which is 521.2819922946869 less than the
```

```
annual mean.
Weather 3 has a mean of: 1736.2, which is 2862.247129909366 less than the annual mean.
```

```
In [15]: fig, ax = plt.subplots(1)
         ax = sns.barplot(list(range(3)), weatherDiff)
         ax.set_title("Difference from Mean Bikes Rented Vs. Weather Category")
         plt.xticks(range(0,3), ["Clear, Partly cloudy...", "Mist + Cloudy...", "Light Snow,
         Light Rain..."], rotation=90)
         ax.set_ylabel("Difference in Bikes Rented from mean")
```

```
Out[15]: <matplotlib.text.Text at 0x11fec8198>
```



The above visualization takes this notion one step further, by revealing how far fewer bikes are rented in bad weather than the total average of bikes rented.

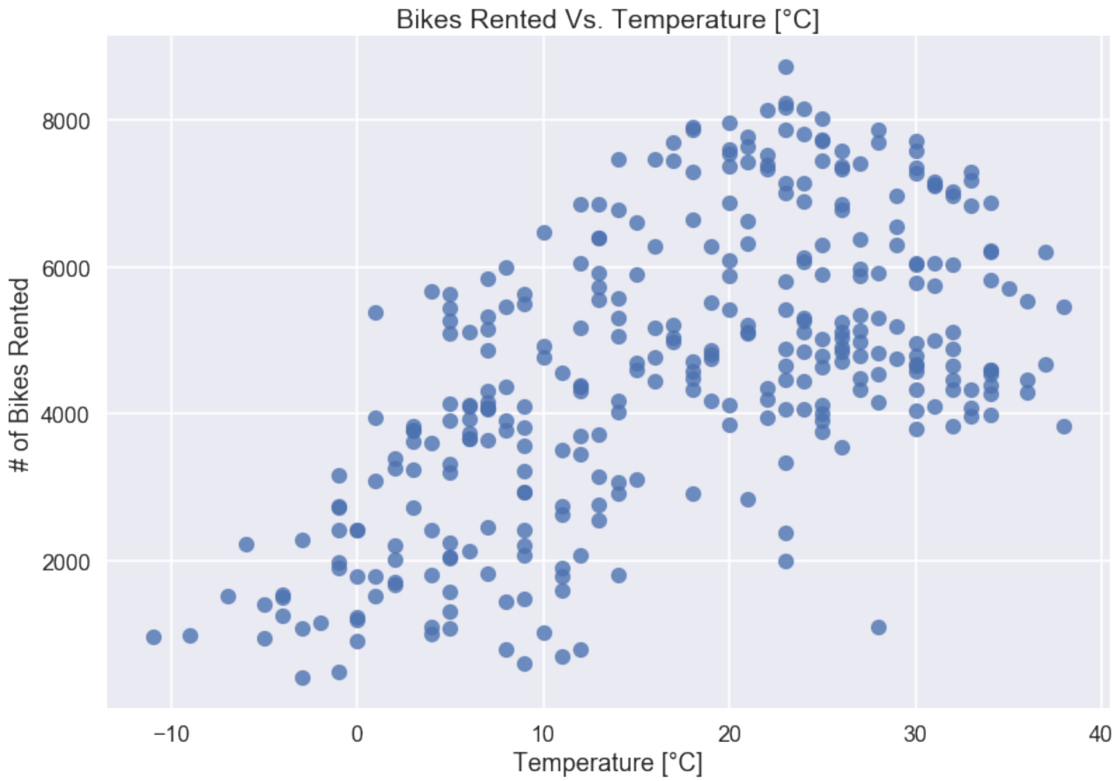
3.1.7 Temperature

Here, our group investigated the impact of temperature on bike rentals.

```
In [16]: fig, ax = plt.subplots(1)
         ax = sns.regplot(trainDf["temp"], trainDf["count"], fit_reg=False)
```

```
ax.set_title("Bikes Rented Vs. Temperature [řC]")
ax.set_xlabel("Temperature [řC]")
ax.set_ylabel("# of Bikes Rented")
```

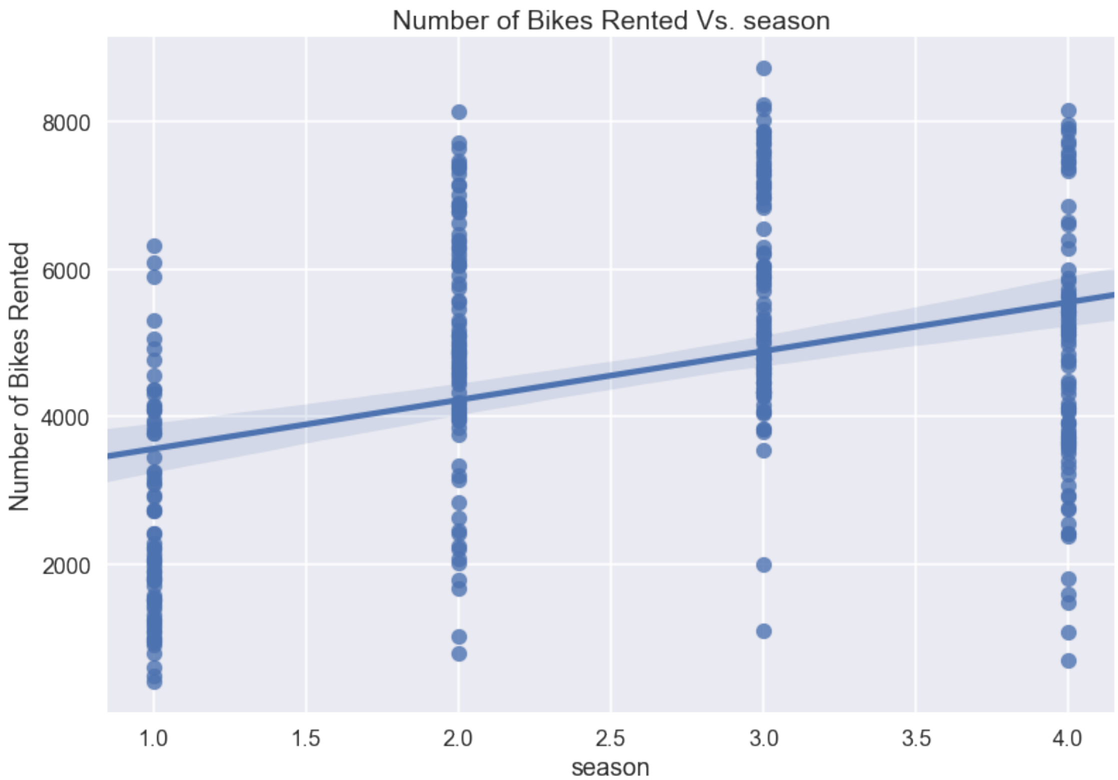
Out[16]: <matplotlib.text.Text at 0x12019d550>

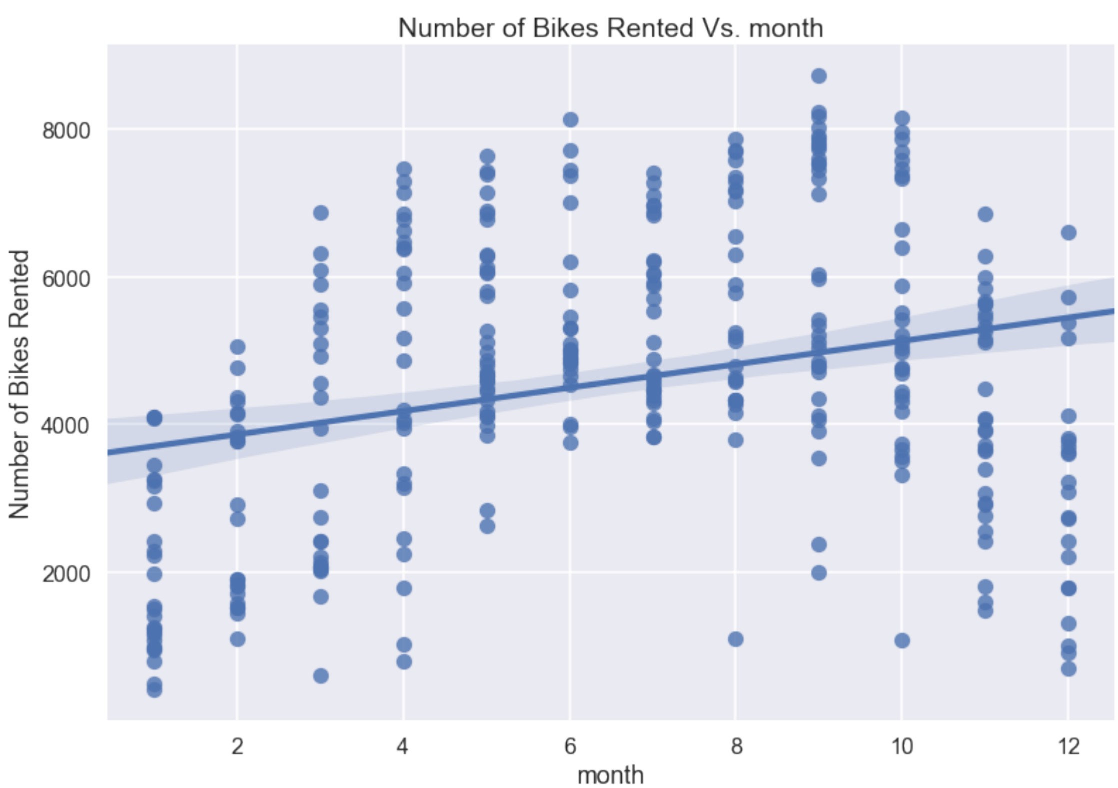


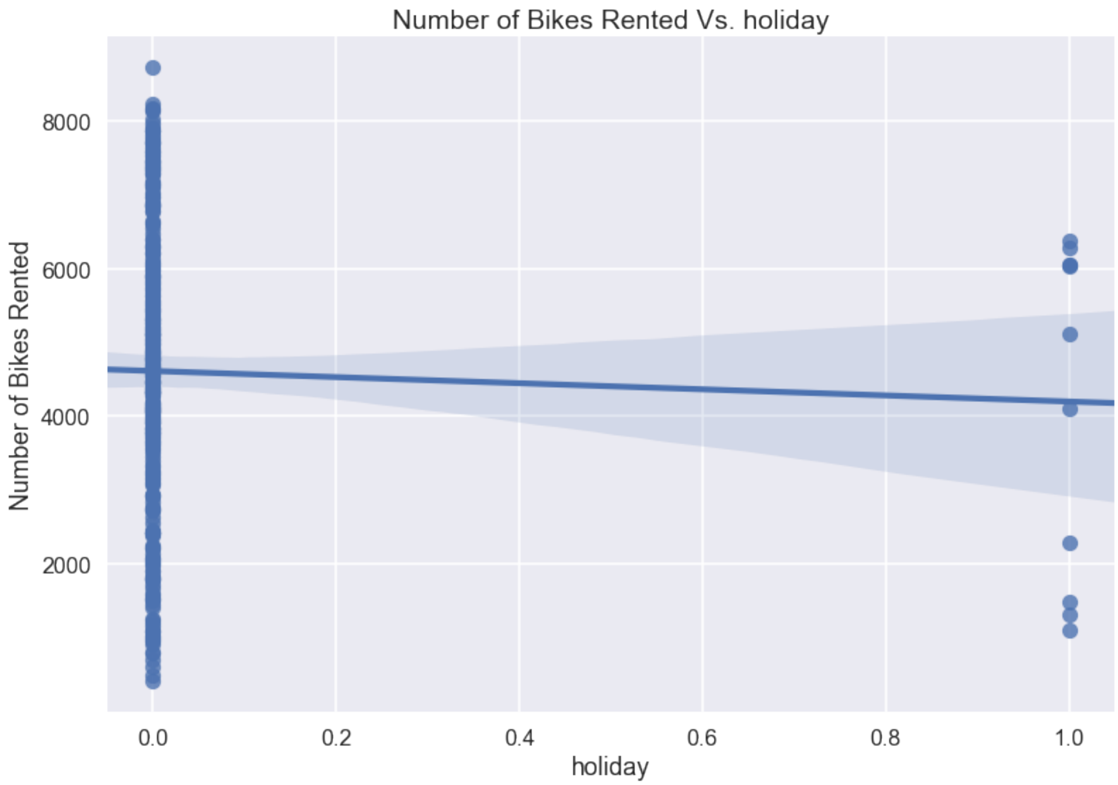
There seems to be a generally positive correlation of *number of bikes rented* and *temperature*, which makes sense, given our prior findings. At temperatures of 25 degrees or more, however, higher temperatures are associated with a reduction in ridership, presumably because riding becomes more cumbersome (and sweaty).

3.1.8 Checking for Non-linear Dependencies

In [17]:
```
initPredictors = ["season", "month", "holiday", "day_of_week", "workingday", "weather",
"temp", "atemp", "humidity", "windspeed"]

for i, predictor in enumerate(initPredictors):
    fig, ax = plt.subplots()
    ax = sns.regplot(trainDf[predictor], trainDf["count"])
    ax.set_title("Number of Bikes Rented Vs. {}".format(predictor))
    ax.set_xlabel("{}".format(predictor))
    ax.set_ylabel("Number of Bikes Rented")
```

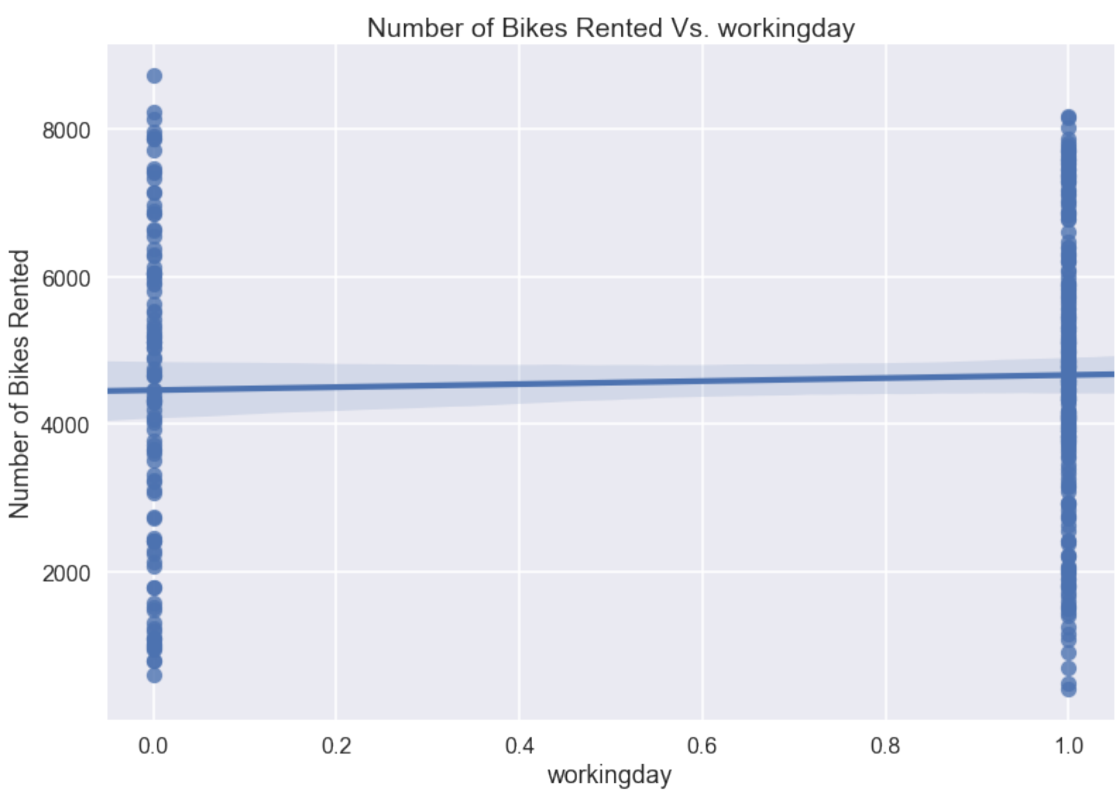

Number of Bikes Rented Vs. season
Number of Bikes Rented
2000
4000
6000
8000
1.0
1.5
2.0
2.5
3.0
3.5
4.0
season

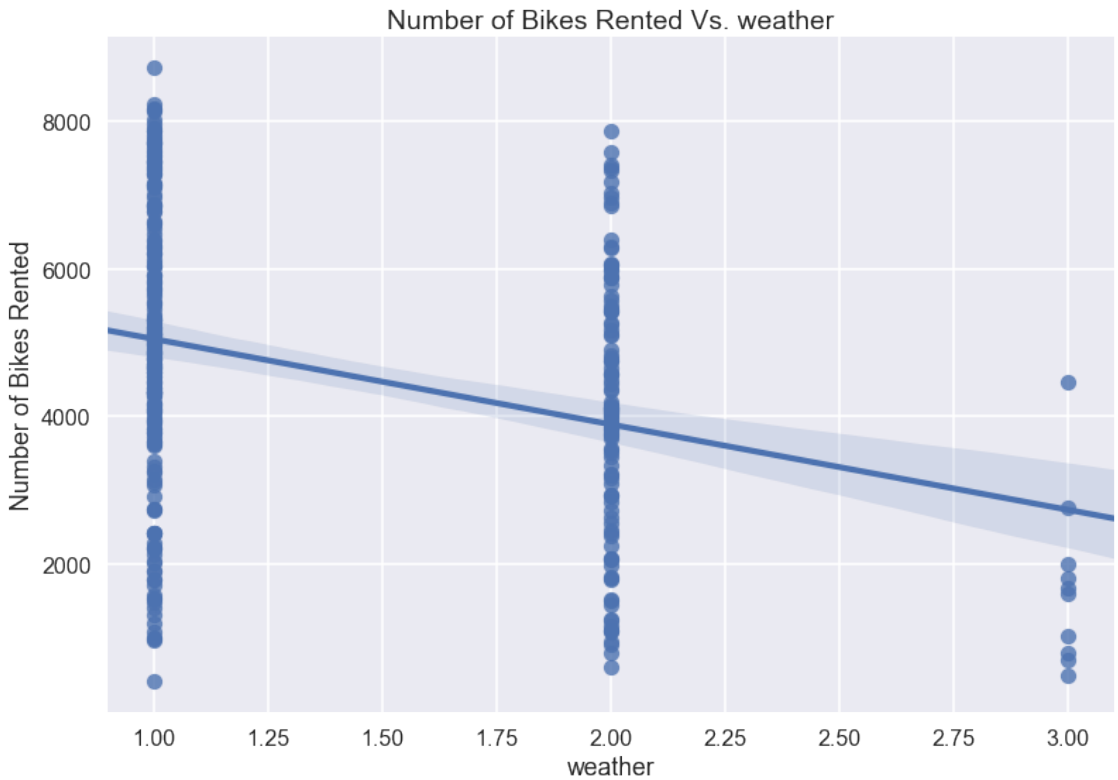

Number of Bikes Rented Vs. month
Number of Bikes Rented
2000
4000
6000
8000
2
4
6
8
10
12
month

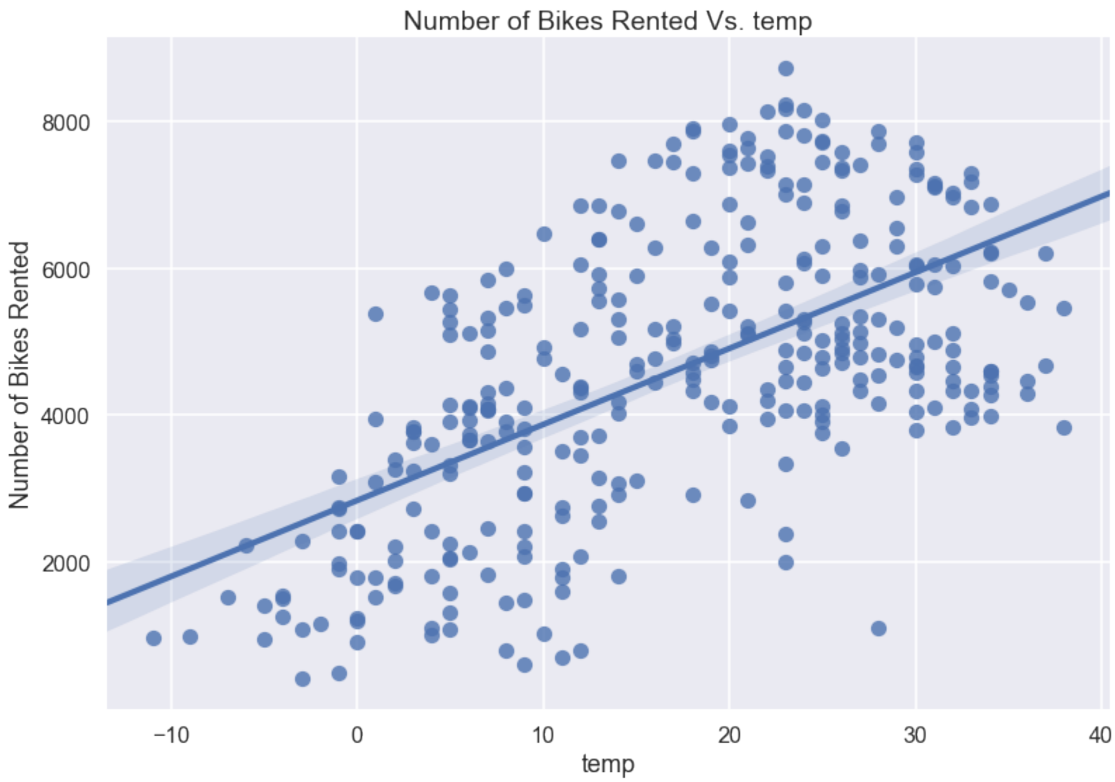


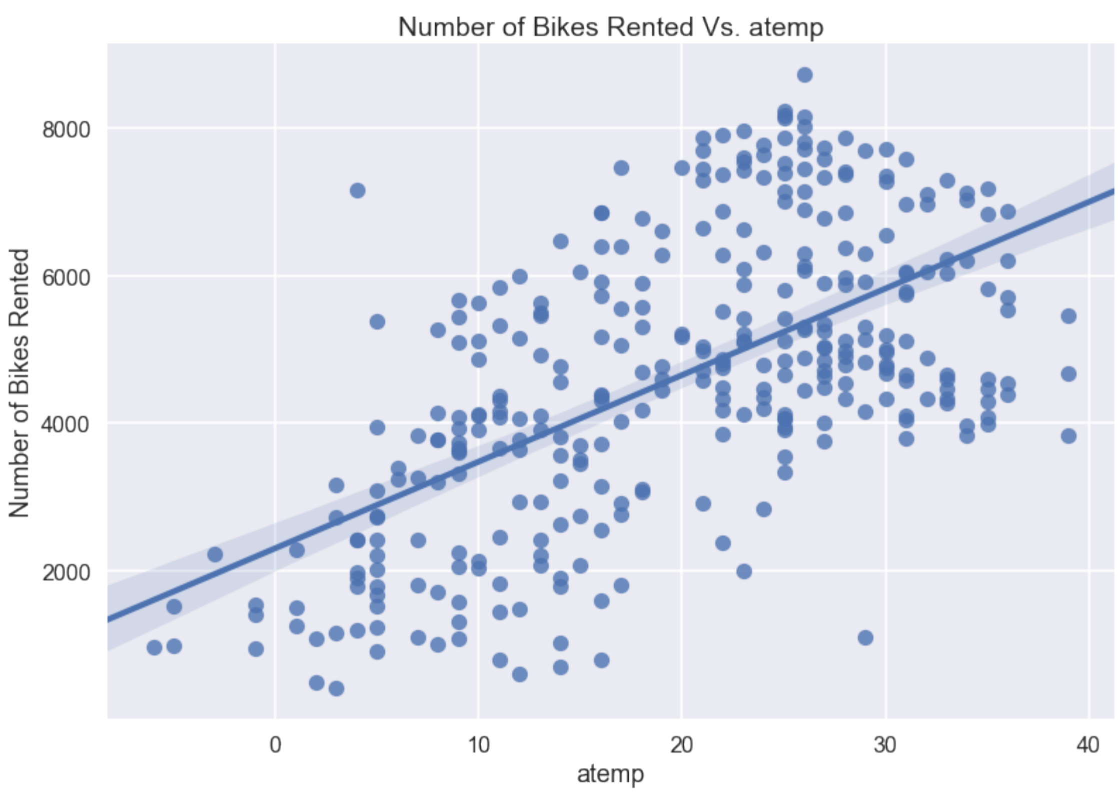
Number of Bikes Rented Vs. holiday
8000
6000
4000
2000
Number of Bikes Rented
0.0
0.2
0.4
0.6
0.8
1.0
holiday

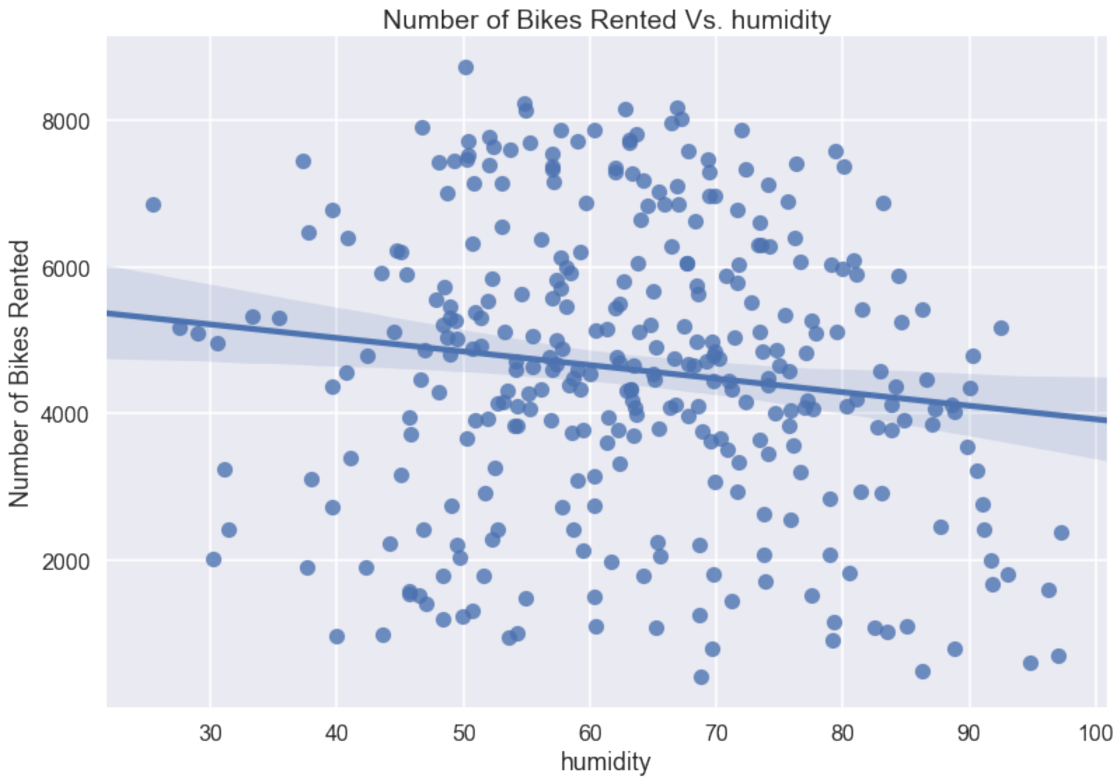

Number of Bikes Rented Vs. day_of_week
Number of Bikes Rented
2000
4000
6000
8000
0
1
2
3
4
5
6
day_of_week

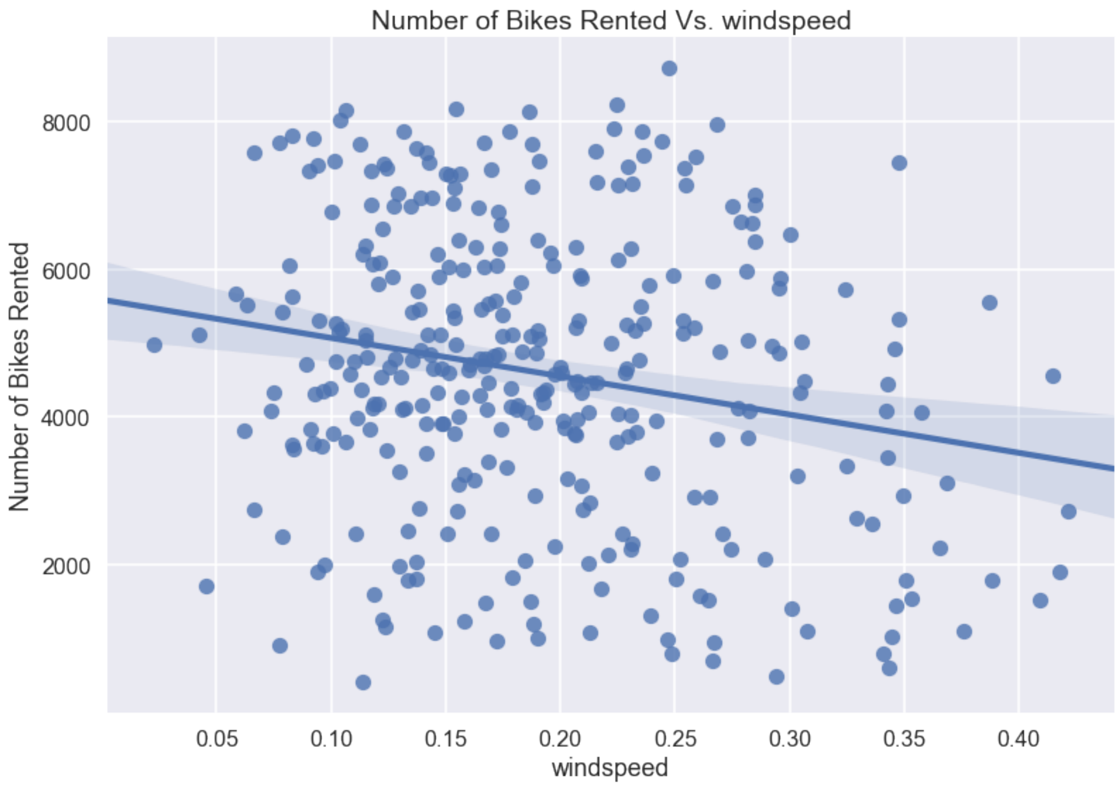

Number of Bikes Rented Vs. workingday
Number of Bikes Rented
2000
4000
6000
8000
0.0
0.2
0.4
0.6
0.8
1.0
workingday


Number of Bikes Rented Vs. weather
Number of Bikes Rented
8000
6000
4000
2000
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
weather


Number of Bikes Rented Vs. temp
Number of Bikes Rented
8000
6000
4000
2000
−10
0
10
20
30
40
temp


Number of Bikes Rented Vs. atemp
Number of Bikes Rented
8000
6000
4000
2000
0
10
20
30
40
atemp


Number of Bikes Rented Vs. humidity
Number of Bikes Rented
2000
4000
6000
8000
humidity
30
40
50
60
70
80
90
100



We observe that `season`, `month`, `temp`, and `atemp` seem to have a non-linear relationship with `number of bikes rented`. The former is probably best described by a sinusoidal function due to the periodicity of seasons, whereas the latter two would fit a negative quadratic function.

We next require you to pre-process the categorical and numerical attributes in the data set:

- Notice that this data set contains categorical attributes with two or more categories. **Why can't they be directly used as predictors?** Convert these categorical attributes into multiple binary attributes using one-hot encoding: in the place of every categorical attribute x_j that has categories $1, \ldots, K_j$, introduce $K_j - 1$ binary predictors $x_{j1}, \ldots, x_{j,K_j-1}$ where x_{jk} is 1 whenever $x_j = k$ and 0 otherwise. ** Why is it okay to not have a binary column for the K_j-th category? **

- Since the attributes are in different scales, it is a good practice to standardize the continuous predictors, i.e. to scale each continuous predictor to have zero mean and a standard deviation of 1. This can be done by applying the following transform to each continuous-valued predictor j: $\hat{x}_{ij} = (x_{ij} - \bar{x}_j)/s_j$, where $\bar{x}_j$ and s_j are the sample mean and sample standard deviation (SD) of predictor j in the training set. We emphasize that the mean and SD values used for standardization must be estimated using only the training set observations, while the transform is applied to both the training and test sets. ** Why shouldn't we include the test set observations in computing the mean and SD? **

- Provide a table of the summary statistics of the new attributes ('pd.describe()' function will help).

Hint: You may use the `pd.get_dummies` function to convert a categorical attribute in a data frame to one-hot encoding. This function creates K binary columns for an attribute with K categories. We suggest that you delete the last (or first) binary column generated by this function.

Note: We shall use the term "attribute" to refer to a categorical column in the data set, and the term "predictor" to refer to the individual binary columns resulting out of one-hot encoding.

3.1.9 Answers to Questions above

- We cannot use categorical variables directly as predictors since non-integer values have no meaning.
- The K_j-th category does not need to have a binary column as it is implied as the baseline, i.e. the 0-vector.
- As a general principle, we do not want to train on the test data. This way, we avoid fitting the training data which would lead to a less general model.

In [18]:
```
# Create dummy variables
trainBinaryDf = pd.get_dummies(trainDf, columns=["season", "month", "day_of_week",
"weather"])
testBinaryDf = pd.get_dummies(testDf, columns=["season", "month", "day_of_week",
"weather"])

trainBinaryDf.head()
```

Out[18]:

	Unnamed: 0	holiday	workingday	temp	atemp	humidity	windspeed	count	\
0	0	0	1	24	26	76.5833	0.118167	6073	
1	1	0	1	15	19	73.3750	0.174129	6606	
2	2	0	1	26	28	56.9583	0.253733	7363	
3	3	0	0	0	4	58.6250	0.169779	2431	
4	4	0	1	23	23	91.7083	0.097021	1996	

	season_1	season_2	...	day_of_week_0	day_of_week_1	day_of_week_2	\
0	0	1	...	0	0	1	
1	0	0	...	0	0	1	
2	0	1	...	0	0	0	
3	0	0	...	1	0	0	
4	0	0	...	0	0	0	

	day_of_week_3	day_of_week_4	day_of_week_5	day_of_week_6	weather_1	\
0	0	0	0	0	0	
1	0	0	0	0	1	
2	0	1	0	0	1	
3	0	0	0	0	1	
4	1	0	0	0	0	

	weather_2	weather_3
0	1	0
1	0	0
2	0	0
3	0	0
4	0	1

```
[5 rows x 34 columns]
```

In [19]:
```python
# Normalize predictors
for column in ["temp", "atemp", "humidity", "windspeed"]:
    colMean = np.mean(trainBinaryDf[column])
    colSD = np.std(trainBinaryDf[column])
    trainBinaryDf[column + "_norm"] = [(x - colMean)/float(colSD) for x in
trainBinaryDf[column]]
    testBinaryDf[column + "_norm"] = [(x - colMean)/float(colSD) for x in
testBinaryDf[column]]
trainBinaryDf.head()
```

Out[19]:
```
   Unnamed: 0  holiday  workingday  temp  atemp  humidity  windspeed  count  \
0           0        0           1    24     26   76.5833   0.118167   6073
1           1        0           1    15     19   73.3750   0.174129   6606
2           2        0           1    26     28   56.9583   0.253733   7363
3           3        0           0     0      4   58.6250   0.169779   2431
4           4        0           1    23     23   91.7083   0.097021   1996

   season_1  season_2      ...        day_of_week_4  day_of_week_5  \
0         0         1      ...                    0              0
1         0         0      ...                    0              0
2         0         1      ...                    1              0
3         0         0      ...                    0              0
4         0         0      ...                    0              0

   day_of_week_6  weather_1  weather_2  weather_3  temp_norm  atemp_norm  \
0              0          0          1          0   0.624743    0.651090
1              0          1          0          0  -0.180583   -0.054841
2              0          1          0          0   0.803704    0.852785
3              0          1          0          0  -1.522794   -1.567551
4              0          0          0          1   0.535262    0.348548

   humidity_norm  windspeed_norm
0       0.922058       -0.930164
1       0.697907       -0.213825
2      -0.449062        0.805143
3      -0.332616       -0.269507
4       1.978781       -1.200843

[5 rows x 38 columns]
```

In [20]:
```python
trainBinaryDf.describe()
```

Out[20]:
```
       Unnamed: 0     holiday  workingday        temp       atemp    humidity  \
count  331.000000  331.000000  331.000000  331.000000  331.000000  331.000000
mean   165.000000    0.033233    0.670695   17.018127   19.543807   63.385776
std     95.695698    0.179515    0.470672   11.192515    9.930991   14.334789
min      0.000000    0.000000    0.000000  -11.000000   -6.000000   25.416700
25%     82.500000    0.000000    0.000000    7.500000   11.000000   52.702900
50%    165.000000    0.000000    1.000000   18.000000   21.000000   63.291700
```

```
75%    247.500000    0.000000    1.000000   26.000000   27.000000   73.500000
max    330.000000    1.000000    1.000000   38.000000   39.000000   97.250000

        windspeed        count    season_1    season_2        ...         \
count  331.000000   331.000000  331.000000  331.000000        ...
mean     0.190833  4598.447130    0.217523    0.259819        ...
std      0.078240  1935.319338    0.413186    0.439199        ...
min      0.022392   431.000000    0.000000    0.000000        ...
25%      0.133083  3370.000000    0.000000    0.000000        ...
50%      0.178479  4648.000000    0.000000    0.000000        ...
75%      0.235380  5981.000000    0.000000    1.000000        ...
max      0.421642  8714.000000    1.000000    1.000000        ...

       day_of_week_4  day_of_week_5  day_of_week_6   weather_1   weather_2  \
count     331.000000     331.000000     331.000000  331.000000  331.000000
mean        0.123867       0.145015       0.135952    0.640483    0.329305
std         0.329929       0.352649       0.343256    0.480585    0.470672
min         0.000000       0.000000       0.000000    0.000000    0.000000
25%         0.000000       0.000000       0.000000    0.000000    0.000000
50%         0.000000       0.000000       0.000000    1.000000    0.000000
75%         0.000000       0.000000       0.000000    1.000000    1.000000
max         1.000000       1.000000       1.000000    1.000000    1.000000

        weather_3     temp_norm    atemp_norm  humidity_norm  windspeed_norm
count  331.000000  3.310000e+02  3.310000e+02   3.310000e+02    3.310000e+02
mean     0.030211 -3.823729e-17 -1.214202e-16  -8.439037e-16    1.549616e-15
std      0.171428  1.001514e+00  1.001514e+00   1.001514e+00    1.001514e+00
min      0.000000 -2.507081e+00 -2.576025e+00  -2.652747e+00   -2.156128e+00
25%      0.000000 -8.516886e-01 -8.616201e-01  -7.463695e-01   -7.392325e-01
50%      0.000000  8.785869e-02  1.468532e-01  -6.572679e-03   -1.581428e-01
75%      0.000000  8.037042e-01  7.519372e-01   7.066402e-01    5.702098e-01
max      1.000000  1.877473e+00  1.962105e+00   2.365957e+00    2.954455e+00

[8 rows x 38 columns]
```

In [21]:
```python
# Drop k-th dummies
predictors = ["holiday", "workingday",
"temp_norm","atemp_norm","humidity_norm","windspeed_norm",
              "weather_1", "weather_2", "day_of_week_0", "day_of_week_1",
"day_of_week_2",
              "day_of_week_3", "day_of_week_4", "day_of_week_5", "season_1",
              "season_2", "season_3", "month_1", "month_2", "month_3", "month_4",
"month_5",
              "month_6", "month_7", "month_8", "month_9", "month_10", "month_11",
"const"]

# Add a constant term to our predictors
trainBinaryDf = sm.add_constant(trainBinaryDf)
testBinaryDf = sm.add_constant(testBinaryDf)

trainPredict = trainBinaryDf[predictors]
testPredict = testBinaryDf[predictors]
```

3.2 Part (b): Multiple Linear Regression

We are now ready to fit a linear regression model and analyze its coefficients and residuals.

- Fit a multiple linear regression model to the training set, and report its R^2 score on the test set.
- *Statistical significance*: Using a t-test, find out which of estimated coefficients are statistically significant at a significance level of 5% (p-value<0.05). Based on the results of the test, answer the following questions:
  - Which among the predictors have a positive correlation with the number of bike rentals?
  - Does the day of a week have a relationship with bike rentals?
  - Does the month influence the bike rentals?
  - What effect does a holiday have on bike rentals?
  - Is there a difference in the coefficients assigned to `temp` and `atemp`? Give an explanation for your observation.
- *Residual plot:* Make a plot of residuals of the fitted model $e = y - \hat{y}$ as a function of the predicted value $\hat{y}$. Note that this is different from the residual plot for simple linear regression. Draw a horizontal line denoting the zero residual value on the Y-axis. Does the plot reveal a non-linear relationship between the predictors and response? What does the plot convey about the variance of the error terms?

In [22]:
```python
# Fit OLS multilinear regression on test set
X = trainPredict
Y = trainDf["count"]

allKModel = sm.OLS(Y, X).fit()
predictions = allKModel.predict(X)
allKModel.summary()
```

Out[22]:
```
<class 'statsmodels.iolib.summary.Summary'>
"""
                            OLS Regression Results
==============================================================================
Dep. Variable:                  count   R-squared:                       0.576
Model:                            OLS   Adj. R-squared:                  0.538
Method:                 Least Squares   F-statistic:                     15.25
Date:                Wed, 04 Oct 2017   Prob (F-statistic):           6.56e-42
Time:                        19:31:50   Log-Likelihood:                -2832.1
No. Observations:                 331   AIC:                             5720.
Df Residuals:                     303   BIC:                             5827.
Df Model:                          27
Covariance Type:            nonrobust
==================================================================================
                     coef    std err          t      P>|t|      [0.025      0.975]
----------------------------------------------------------------------------------
holiday         -616.6027    405.068     -1.522      0.129   -1413.706     180.500
workingday       -24.0933    174.143     -0.138      0.890    -366.777     318.590
temp_norm        924.3344    473.819      1.951      0.052      -8.058    1856.727
atemp_norm       311.9618    429.337      0.727      0.468    -532.898    1156.822
```

```
humidity_norm   -547.6638    113.029     -4.845      0.000    -770.085    -325.243
windspeed_norm  -254.7369     80.644     -3.159      0.002    -413.431     -96.043
weather_1       1581.9783    529.223      2.989      0.003     540.560    2623.396
weather_2       1565.4117    478.500      3.271      0.001     623.807    2507.016
day_of_week_0   -465.1450    269.154     -1.728      0.085    -994.794      64.504
day_of_week_1   -256.6501    172.675     -1.486      0.138    -596.445      83.145
day_of_week_2   -328.1845    204.810     -1.602      0.110    -731.214      74.845
day_of_week_3     37.6128    214.916      0.175      0.861    -385.304     460.530
day_of_week_4    -71.6425    208.337     -0.344      0.731    -481.612     338.327
day_of_week_5    -21.8317    200.742     -0.109      0.913    -416.858     373.194
season_1       -1226.1865    506.763     -2.420      0.016   -2223.407    -228.966
season_2        -327.3575    573.373     -0.571      0.568   -1455.654     800.939
season_3        -193.3050    449.171     -0.430      0.667   -1077.194     690.584
month_1          118.8358    505.353      0.235      0.814    -875.611    1113.282
month_2          207.7759    516.216      0.402      0.688    -808.047    1223.599
month_3          358.0167    511.391      0.700      0.484    -648.310    1364.344
month_4          452.1849    657.792      0.687      0.492    -842.234    1746.604
month_5           53.0233    700.991      0.076      0.940   -1326.403    1432.450
month_6         -673.4271    696.142     -0.967      0.334   -2043.313     696.458
month_7        -1161.1512    701.261     -1.656      0.099   -2541.109     218.806
month_8         -657.6397    684.628     -0.961      0.338   -2004.868     689.588
month_9          523.9804    548.284      0.956      0.340    -554.945    1602.906
month_10         605.0867    439.844      1.376      0.170    -260.449    1470.623
month_11         231.5175    413.966      0.559      0.576    -583.094    1046.129
const           3672.2940    664.433      5.527      0.000    2364.807    4979.781
==============================================================================
Omnibus:                       28.947   Durbin-Watson:                   1.912
Prob(Omnibus):                  0.000   Jarque-Bera (JB):                9.753
Skew:                           0.054   Prob(JB):                      0.00762
Kurtosis:                       2.166   Cond. No.                     1.05e+16
==============================================================================

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specif
[2] The smallest eigenvalue is 7.45e-30. This might indicate that there are
strong multicollinearity problems or that the design matrix is singular.
"""
```

3.2.1 Assesement of The Regression Model Above

- The following predictors have a statistically significant correlation with the number of bike rentals:
  - *weather_1*, *weather_2*, *humidity_norm*, *windspeed_norm*
  - *month_4* April, *month_9* September, *month_10* October, *month_11* November, *season_1* Spring
  - *const*
- We determined these statistically significant predictors by looking at the `P>|t|` values

above. If the value is less than .05, then that predictor's coefficient is significant.

- The day of a week does not have a statistically significant relationship with bike rentals.
- The month certainly influences the number of bike rentals (as evidenced by the significant predictors above)
- Ceteris paribus, a holiday does not have a significant relationship with bike rentals.
- There is no statistical significant difference in the coefficients assigned to *temp* and *atemp*, as both 95%-CIs have similar bounds, including 0. However, *temp* is almost significant, while *atemp* is nowhere close to being significant. This difference is surprising, but might be explained by the fact that apparent temperatures can vary significantly depending on other weather factors, while temperature is a baseline measure.

In [23]:
```
testY = testDf["count"]
testYHat = allKModel.predict(testPredict)
print("Test R-Squared: {}".format(r2_score(testY, testYHat)))
```

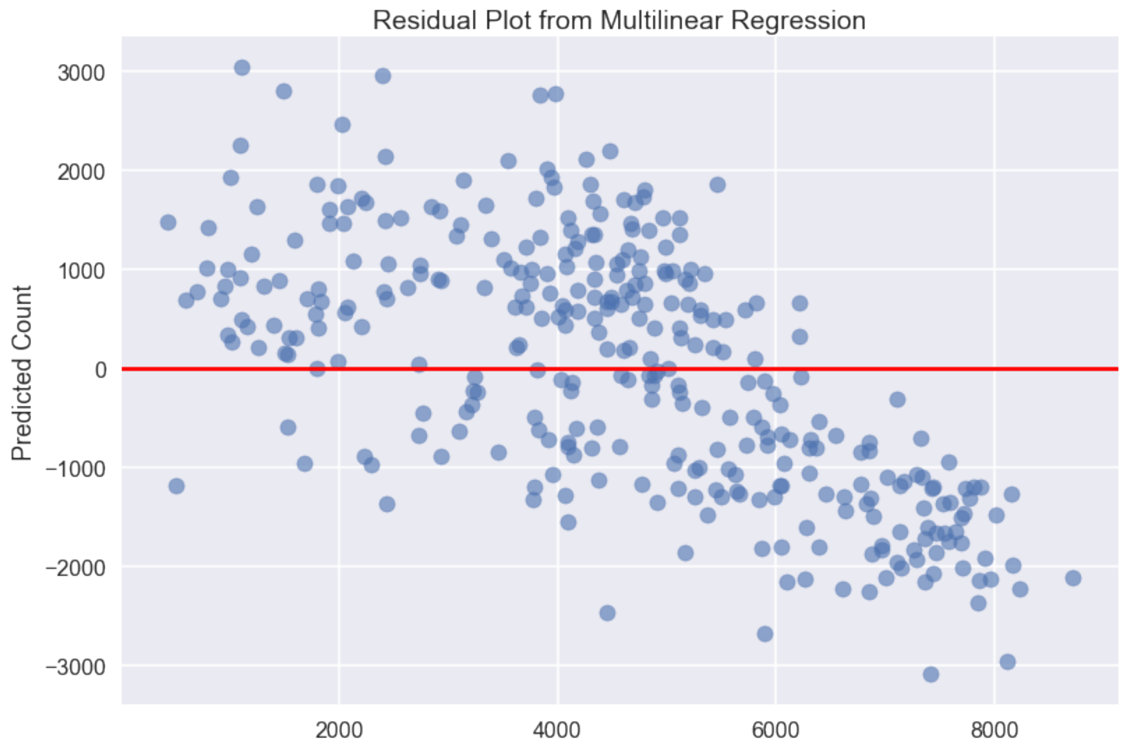
```
Test R-Squared: 0.24934211146527574
```

3.3 Low R^2 Value Interpretation

The comparatively low R^2 value indicates that our model is overfitting the training data, especially given the larger R^2 value for the training data.

In [24]:
```
residuals = list(map(lambda yhat, y: yhat - y, predictions, Y)) #YNorm instead of Y
fig, ax = plt.subplots()

ax.scatter(Y, residuals, alpha=.6)
ax.set_title("Residual Plot from Multilinear Regression")
ax.axhline(y=0, color="r")
# ax.set_xlabel("Normalized True Count") True?
ax.set_ylabel("Predicted Count")
```

Out[24]: <matplotlib.text.Text at 0x1229aa160>



3.3.1 Residuals analysis

The plot reveals a non-linear relationship between the predictors and response, since the error terms are not evenly distributed around the mean, and their variance depends on y, decreasing as y grows. This violates our regression assumption of normal, homoskedastic noise.

3.4 Part (c): Checking Collinearity

Does the data suffer from multi-collinearity? To answer this question, let us first analyze the correlation matrix for the data. Compute the (Pearson product-moment) correlation matrix for the predictor variables in the training set, and visualize the matrix using a heatmap. For categorical attributes, you should use each binary predictor resulting from one-hot encoding to compute their correlations. Are there predictors that fall into natural groups based on the correlation values?

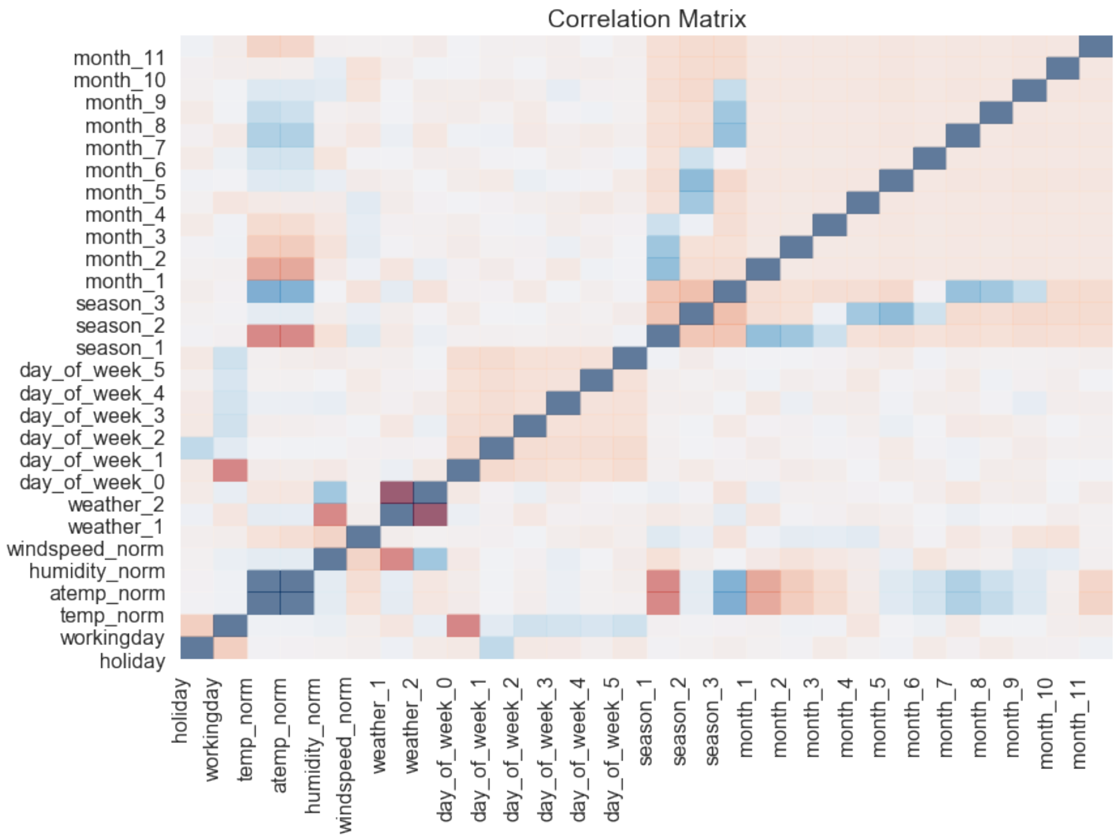
Hint: You may use the `np.corrcoef` function to compute the correlation matrix for a data set (do not forget to transpose the data matrix). You may use `plt.pcolor` function to visualize the correlation matrix.

```
In [25]: dpal = sns.choose_colorbrewer_palette('diverging', as_cmap=True)
```

```
In [26]: plt.pcolor(X.ix[:, X.columns != "const"].corr(), cmap=dpal, alpha=.6)
         ax = plt.gca()
         ax.set_title("Correlation Matrix")
         labels = predictors[:-1]
         plt.xticks(range(len(labels)), labels, rotation=90)
         plt.yticks(range(len(labels)), labels)
```

```
/Applications/anaconda/lib/python3.6/site-packages/ipykernel_launcher.py:1:
DeprecationWarning:
.ix is deprecated. Please use
.loc for label based indexing or
.iloc for positional indexing

See the documentation here:
http://pandas.pydata.org/pandas-docs/stable/indexing.html#deprecate_ix
  """Entry point for launching an IPython kernel.
```

```
Out[26]: ([<matplotlib.axis.YTick at 0x122ce8b38>,
           <matplotlib.axis.YTick at 0x122cd4eb8>,
           <matplotlib.axis.YTick at 0x122df7e48>,
           <matplotlib.axis.YTick at 0x122de99e8>,
           <matplotlib.axis.YTick at 0x122dd9c88>,
           <matplotlib.axis.YTick at 0x122dc5cc0>,
           <matplotlib.axis.YTick at 0x122dc08d0>,
           <matplotlib.axis.YTick at 0x122dadcc0>,
           <matplotlib.axis.YTick at 0x122cefbe0>,
           <matplotlib.axis.YTick at 0x122e03550>,
           <matplotlib.axis.YTick at 0x122e18240>,
           <matplotlib.axis.YTick at 0x122e18cc0>,
           <matplotlib.axis.YTick at 0x122e1e780>,
           <matplotlib.axis.YTick at 0x122e27240>,
           <matplotlib.axis.YTick at 0x122e27cc0>,
           <matplotlib.axis.YTick at 0x122e2d780>,
           <matplotlib.axis.YTick at 0x122e33240>,
           <matplotlib.axis.YTick at 0x122e33cc0>,
           <matplotlib.axis.YTick at 0x122e38780>,
           <matplotlib.axis.YTick at 0x122e3f240>,
           <matplotlib.axis.YTick at 0x122e3fcc0>,
           <matplotlib.axis.YTick at 0x122e45780>,
           <matplotlib.axis.YTick at 0x122e4c240>,
           <matplotlib.axis.YTick at 0x122e4ccc0>,
           <matplotlib.axis.YTick at 0x122e51780>,
           <matplotlib.axis.YTick at 0x122e56240>,
           <matplotlib.axis.YTick at 0x122e56cc0>,
           <matplotlib.axis.YTick at 0x122e5f780>],
          <a list of 28 Text yticklabel objects>)
```



We observe the following relationships: * *temp_norm* and *atemp_norm* are unsurprisingly very much positively correlated. * *weather_1* and *weather_2* are negatively correlated, which makes sense since they derive from categorical data. * We can see "boxes" for *day_of_week_j* and *month_k* which is intuitive. * *temp_norm* and *atemp_norm* are correlated with the months. * Naturally, the seasons are correlated with the months.

Note that we excluded the constant term from our correlation matrix, as the variance of a constant is 0, so calculating the correlation of a variable with a constant would yield an undefined value.

3.5 Part (d): Subset Selection

Apply either one of the following subset selection methods discussed in class to choose a minimal subset of predictors that are related to the response variable: - Step-wise forward selection - Step-wise backward selection

We require you to implement both these methods *from scratch*. You may use the Bayesian Information Criterion (BIC) to choose the subset size in each method. Do these methods eliminate one or more of the redundant predictors (if any) identified in Part (c)? In each case, fit linear regression models using the identified subset of predictors to the training set. How do the test R^2 scores for the fitted models compare with the model fitted in Part (b) using all predictors?

In [27]:
```
# Implement step-wise forward selection

# The set of predictors
```

```
predictors = ["const", "holiday", "workingday", "temp_norm", "atemp_norm",
"humidity_norm", "windspeed_norm",
              "weather_1", "weather_2", "day_of_week_0", "day_of_week_1",
"day_of_week_2",
              "day_of_week_3", "day_of_week_4", "day_of_week_5", "season_1",
              "season_2", "season_3", "month_1", "month_2", "month_3", "month_4",
"month_5",
              "month_6", "month_7", "month_8", "month_9", "month_10", "month_11"]


forwardPredictors = []
forwardModels = []

for _ in range(len(predictors)):

    betterModel, betterPredictor, betterBIC = None, None, sys.maxsize

    for addedPredictor in predictors:
        X = trainBinaryDf[forwardPredictors + [addedPredictor]]
        newModel = sm.OLS(Y, X).fit()

        bestBIC = min([model.bic for model in forwardModels] + [betterBIC])
        if newModel.bic < bestBIC:
            betterModel = newModel
            betterPredictor = addedPredictor
            betterBIC = newModel.bic

    if betterModel:
        forwardModels.append(betterModel)
        forwardPredictors.append(betterPredictor)
    else:
        break

forwardBICs = [model.bic for model in forwardModels]
forwardRSquareds = [model.rsquared for model in forwardModels]
print("Predictors: {}".format(forwardPredictors))
print("Rsquareds: {}".format(forwardRSquareds))
print("BICs: {}".format(forwardBICs))
```

```
Predictors: ['const', 'atemp_norm', 'humidity_norm', 'season_1', 'month_9',
'month_10', 'windspeed_norm', 'month_7']
Rsquareds: [0.0, 0.36094811139379512, 0.41237745384733182, 0.46809945208979198,
0.48910179719539404, 0.5104124449264057, 0.5237587713442825, 0.53432886723575779]
BICs: [5954.1722009863015, 5811.7625734619323, 5789.7935136501037, 5762.6186708109581,
5755.0860658498423, 5746.7852518967693, 5743.438926291893, 5741.8118001623388]
```

3.6 Results

The step-wise forward regression yields a 6-parameter model. *temp_norm* and some *season/month* redundancies (by collinearity) have been resolved. We correctly see that the BIC of each iteration of this algorithm decreases, signifying an improvement overall in our model. Further, our R^2 values increase to just below the original model with all predictors. This is encouraging, since our new model is significantly smaller but about as powerful as the first model.

3.7 Testing

```
In [28]: testY = testDf["count"]
         testPredict = testBinaryDf[forwardPredictors]
         testYHat = forwardModels[-1].predict(testPredict)
         print("Test R-Squared: {}".format(r2_score(testY, testYHat)))
```

```
Test R-Squared: 0.23390728606336764
```

Testing the resulting model on the test dataset yields a slightly lower R^2 value than the one in Part (b).

```
In [29]: # Implement step-wise backward selection

         # The set of predictors
         predictors = set(["const", "holiday", "workingday", "temp_norm", "atemp_norm",
         "humidity_norm", "windspeed_norm",
                           "weather_1", "weather_2", "day_of_week_0", "day_of_week_1",
         "day_of_week_2",
                           "day_of_week_3", "day_of_week_4", "day_of_week_5", "season_1",
                           "season_2", "season_3", "month_1", "month_2", "month_3", "month_4",
         "month_5",
                           "month_6", "month_7", "month_8", "month_9", "month_10", "month_11"])


         backwardPredictors = set(predictors)
         backwardModels = []

         for _ in range(len(predictors)):

             betterModel, betterPredictor, betterBIC = None, None, sys.maxsize

             for removedPredictor in backwardPredictors:
                 X = trainBinaryDf[list(backwardPredictors - set([removedPredictor]))]
                 newModel = sm.OLS(Y, X).fit()

                 bestBIC = min([model.bic for model in backwardModels] + [betterBIC])
                 if newModel.bic < bestBIC:
                     betterModel = newModel
                     betterPredictor = removedPredictor
                     betterBIC = newModel.bic

             if betterModel:
                 backwardModels.append(betterModel)
                 backwardPredictors.remove(betterPredictor)
             else:
                 break

         backwardBICs = [model.bic for model in backwardModels]
         backwardRSquareds = [model.rsquared for model in backwardModels]
         print("Predictors: {}".format(backwardPredictors))
         print("Rsquareds: {}".format(backwardRSquareds))
         print("BICs: {}".format(backwardBICs))
```

```
Predictors: {'weather_2', 'day_of_week_5', 'month_8', 'humidity_norm', 'month_7',
'const', 'windspeed_norm', 'weather_1', 'workingday', 'month_6', 'day_of_week_2',
'season_1', 'day_of_week_3', 'day_of_week_4', 'holiday', 'temp_norm', 'day_of_week_1'}
Rsquareds: [0.57612017442132424, 0.57605080910137485, 0.57588543174157469,
0.5756513106142167, 0.57513980584466751, 0.57432856240861385, 0.57347456671855168,
0.57236259109017618, 0.57102272820283972, 0.56788327070409084, 0.56358808011277584,
0.5584752558959003]
BICs: [5820.9282263781215, 5815.1802696804134, 5809.507245152513, 5803.8877960781911,
5798.484420821208, 5793.3137231919627, 5788.175002389833, 5783.2346965933684,
5778.4680378881812, 5775.0795120710272, 5772.5512515638839, 5770.6044544609886]
```

3.7.1 Results of Backwards Selection

Here, we recognize several problems with the result of this backwards selection. First, the number of predictors in this model is still very large, and there are still many predictors that are colinear with one another. Secondly, the R^2 values are decreasing, as well as our BIC values, which means that while our scoring metric (BIC) improves, we lose explanatory power of the data as well.

3.7.2 Testing

```
In [30]: testY = testDf["count"]
         testPredict = testBinaryDf[list(backwardPredictors)]
         testYHat = backwardModels[-1].predict(testPredict)
         print("Test R-Squared: {}".format(r2_score(testY, testYHat)))
```

```
Test R-Squared: 0.25606288780692477
```

Testing the resulting model on the test dataset yields a marginally higher R^2 value than the one in Part (b).

3.8 Part (e): Cross Validation

- Perform a 10-fold cross-validation procedure to select between the 3 competing models you have so far: the model with the best BIC from Step-wise forward selection, the model with the best BIC from Step-wise backward selection (if it is different), and the model with all possible predictors. Report the average R^2 across all 10 validation sets for each model and compare the results. Why do you think this is the case?
- Evaluate each of the 3 models on the provided left out test set by calculating R^2. Do the results agree with the cross-validation? Why or why not?

```
In [31]: forwardModel = forwardModels[-1]
         backwardModel = backwardModels[-1]

         kf = KFold(n_splits=10)
         X = trainPredict
         Y = trainDf["count"]

         allKRSquares = []
         forwardRSquares = []
         backwardRSquares = []

         for trainIndices, validationIndices in kf.split(X):
             trainX = X.iloc[trainIndices]
             trainY = Y.iloc[trainIndices]

             validationX = X.iloc[validationIndices]
             validationY = Y.iloc[validationIndices]

             allKX = trainX
             allK = sm.OLS(trainY, allKX).fit()

             forwardX = trainX[list(forwardPredictors)]
             forward = sm.OLS(trainY, forwardX).fit()

             backwardX = trainX[list(backwardPredictors)]
             backward = sm.OLS(trainY, backwardX).fit()

             allKYHat = allK.predict(validationX)
             forwardYHat = forward.predict(validationX[list(forwardPredictors)])
             backwardYHat = backward.predict(validationX[list(backwardPredictors)])

             allKRSquares.append(r2_score(validationY, allKYHat))
             forwardRSquares.append(r2_score(validationY, forwardYHat))
             backwardRSquares.append(r2_score(validationY, backwardYHat))
```

```
print("Original model mean: {}, Forward model mean: {}, Backwards model mean:
{}.".format(np.mean(allKRSquares), np.mean(forwardRSquares), np.mean(backwardRSquares)))
```

```
Original model mean: 0.34257855093958417, Forward model mean: 0.44036505151424993,
Backwards model mean: 0.4499215974237565.
```

3.8.1 Summary of 10-fold cross-validation

We see that our average R^2 score for 10-fold cross-validation from our original model was just .34, which is significantly smaller than both the forward and backwards models. This is likely due to the overfitting of our original model with the training dataset that it is given for each data split. As we have seen earlier, our step-wise backward model actually scored the best on our test data our of any other model. This is surprising for numerous reasons, but mostly because there still are a large amount of parameters in that model that might indicate overfitting.

```
In [ ]:
```