HW1

HW 1

Let us choose a different representation for the qubit, say,
$|0\rangle \doteq \begin{pmatrix} e^{i\varphi/2}\cos\theta/2 \\ e^{-i\varphi/2}\sin\theta/2 \end{pmatrix}$, $|1\rangle \doteq \begin{pmatrix} -e^{i\varphi/2}\sin\theta/2 \\ e^{-i\varphi/2}\cos\theta/2 \end{pmatrix}$,
where θ and φ are arbitrary real angles. Show that $|0\rangle$ and $|1\rangle$ form an orthonormal basis (for any choices of θ and φ).

We can check that

$$\begin{aligned}
\langle 0 \mid 0\rangle &\doteq \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 & e^{i\varphi/2}\sin\theta/2 \end{pmatrix}\begin{pmatrix} e^{i\varphi/2}\cos\theta/2 \\ e^{-i\varphi/2}\sin\theta/2 \end{pmatrix} \\
&= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1, \\
\langle 0 \mid 1\rangle &\doteq \begin{pmatrix} e^{-i\varphi/2}\cos\theta/2 & e^{i\varphi/2}\sin\theta/2 \end{pmatrix}\begin{pmatrix} -e^{i\varphi/2}\sin\theta/2 \\ e^{-i\varphi/2}\cos\theta/2 \end{pmatrix} \\
&= -\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0, \\
\langle 1 \mid 0\rangle &\doteq \begin{pmatrix} -e^{-i\varphi/2}\sin\theta/2 & e^{i\varphi/2}\cos\theta/2 \end{pmatrix}\begin{pmatrix} e^{i\varphi/2}\cos\theta/2 \\ e^{-i\varphi/2}\sin\theta/2 \end{pmatrix} \\
&= -\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0, \\
\langle 1 \mid 1\rangle &\doteq \begin{pmatrix} -e^{-i\varphi/2}\sin\theta/2 & e^{i\varphi/2}\cos\theta/2 \end{pmatrix}\begin{pmatrix} -e^{i\varphi/2}\sin\theta/2 \\ e^{-i\varphi/2}\cos\theta/2 \end{pmatrix} \\
&= \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1.
\end{aligned} \tag{1}$$

HW2

HW 2

Let $\boldsymbol{m}$ and $\boldsymbol{n}$ be three-component real unit vectors. Define the operator $\boldsymbol{m}\cdot\boldsymbol{\sigma} = m_x\,\sigma^x + m_y\,\sigma^y + m_z\,\sigma^z$ for the vector $\boldsymbol{m} = (m_x, m_y, m_z)$, similarly for $\boldsymbol{n}$.
(i) Write down the matrix representation of $\boldsymbol{m}\cdot\boldsymbol{\sigma}$ in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis.
(ii) If we measure the observable $\boldsymbol{m}\cdot\boldsymbol{\sigma}$, what are the possible measurement outcomes?
(iii) Let $\boldsymbol{m} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Calculate eigenvalues and eigenvectors (in terms of θ and φ) of $\boldsymbol{m}\cdot\boldsymbol{\sigma}$.
(iv) What is the probability of observing $\boldsymbol{n}\cdot\boldsymbol{\sigma} = +1$ when measuring the observable $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ on the state $|\boldsymbol{m}\cdot\boldsymbol{\sigma} = +1\rangle$? (in terms of $\boldsymbol{m}$ and $\boldsymbol{n}$)
(v) What is the expectation value of the operator $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ on the state $|\boldsymbol{m}\cdot\boldsymbol{\sigma} = +1\rangle$? (in terms of $\boldsymbol{m}$ and $\boldsymbol{n}$)

(i) Matrix representation

$$\boldsymbol{m}\cdot\boldsymbol{\sigma} \doteq \begin{pmatrix} m_3 & m_1 - i\,m_2 \\ m_1 + i\,m_2 & -m_3 \end{pmatrix}. \tag{2}$$

(ii) The possible outcomes are $\pm\sqrt{\boldsymbol{m}\cdot\boldsymbol{m}} = \pm 1$.

(iii) Eigenvalues are ± 1. Corresponding eigenvectors are

$$|\boldsymbol{m}\cdot\boldsymbol{\sigma} = +1\rangle \doteq \begin{pmatrix} e^{i\,\varphi/2}\cos\theta/2 \\ e^{-i\,\varphi/2}\sin\theta/2 \end{pmatrix},\ |\boldsymbol{m}\cdot\boldsymbol{\sigma} = -1\rangle \doteq \begin{pmatrix} -e^{i\,\varphi/2}\sin\theta/2 \\ e^{-i\,\varphi/2}\cos\theta/2 \end{pmatrix}. \tag{3}$$

(iv) The probability is $\frac{1}{2}(\boldsymbol{m}\cdot\boldsymbol{n}+1)$.

(v) The expectation value is $\boldsymbol{m}\cdot\boldsymbol{n}$.

▪ HW3

$$\begin{aligned} H &= h_0\,\mathbb{1} + h_x\,\sigma^x + h_y\,\sigma^y + h_z\,\sigma^z \\ &= h_0\,\mathbb{1} + \boldsymbol{h}\cdot\boldsymbol{\sigma}, \end{aligned} \tag{4}$$

$$\begin{aligned} U(t) &= e^{-i\,H\,t} \\ &= e^{-i\,h_0\,t}\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} - i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big), \end{aligned} \tag{5}$$

$$\begin{aligned} |\psi(t)\rangle &= U(t)\,|\psi(0)\rangle \\ &= e^{-i\,h_0\,t}\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} - i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big)\,|\psi(0)\rangle. \end{aligned} \tag{6}$$

$$\begin{aligned} \langle\boldsymbol{\sigma}\rangle_t &= \langle\psi(t)|\,\boldsymbol{\sigma}\,|\psi(t)\rangle \\ &= \cos(2\,|\boldsymbol{h}|\,t)\,\langle\boldsymbol{\sigma}\rangle_0 + \sin(2\,|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\times\langle\boldsymbol{\sigma}\rangle_0 + (1-\cos(2\,|\boldsymbol{h}|\,t))\,\hat{\boldsymbol{h}}\Big(\hat{\boldsymbol{h}}\cdot\langle\boldsymbol{\sigma}\rangle_0\Big). \end{aligned} \tag{7}$$

HW 3

(i) Derive Eq. (5) from Eq. (4).
(ii) Derive Eq. (7) from Eq. (6).

Solution (HW 3)

(i) Given $H = h_0\,\mathbb{1} + \boldsymbol{h}\cdot\boldsymbol{\sigma}$,

$$\begin{aligned} U(t) &= e^{-i\,H\,t} \\ &= e^{-i\,(h_0\,\mathbb{1}+\boldsymbol{h}\cdot\boldsymbol{\sigma})\,t} \\ &= e^{-i\,h_0\,t}\exp\Big(-i\,|\boldsymbol{h}|\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\,t\Big) \\ &= e^{-i\,h_0\,t}\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} - i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big). \end{aligned} \tag{8}$$

The last step is by Taylor expansion and using the fact that $\left(\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\right)^2 = \mathbb{1}$.

(ii) Instead of evaluating $\langle\boldsymbol{\sigma}\rangle_t$, we first consider $\boldsymbol{m}\cdot\langle\boldsymbol{\sigma}\rangle_t$

$$\begin{aligned} &\boldsymbol{m}\cdot\langle\boldsymbol{\sigma}\rangle_t = \\ &\quad\langle\psi(t)|\,\boldsymbol{m}\cdot\boldsymbol{\sigma}\,|\psi(t)\rangle = \langle\psi(0)|\,\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} + i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big)\,\boldsymbol{m}\cdot\boldsymbol{\sigma}\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} - i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big)\,|\psi(0)\rangle. \end{aligned} \tag{9}$$

The operator inside the bracket reads

$$\begin{aligned} &\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} + i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big)\,\boldsymbol{m}\cdot\boldsymbol{\sigma}\Big(\cos(|\boldsymbol{h}|\,t)\,\mathbb{1} - i\sin(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big) \\ &= \cos^2(|\boldsymbol{h}|\,t)\,\boldsymbol{m}\cdot\boldsymbol{\sigma} + i\sin(|\boldsymbol{h}|\,t)\cos(|\boldsymbol{h}|\,t)\,\Big(\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\,\boldsymbol{m}\cdot\boldsymbol{\sigma} - \boldsymbol{m}\cdot\boldsymbol{\sigma}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\Big) + \sin^2(|\boldsymbol{h}|\,t)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\,\boldsymbol{m}\cdot\boldsymbol{\sigma}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}. \end{aligned} \tag{10}$$

Given $\boldsymbol{a}\cdot\boldsymbol{\sigma}\,\boldsymbol{b}\cdot\boldsymbol{\sigma} = \boldsymbol{a}\cdot\boldsymbol{b}\,\mathbb{1} + i\,(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{\sigma}$, we have

$$\begin{aligned}
&\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\,\boldsymbol{m}\cdot\boldsymbol{\sigma} = \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\mathbb{1} + i\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\cdot\boldsymbol{\sigma},\\
&\boldsymbol{m}\cdot\boldsymbol{\sigma}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} = \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\mathbb{1} - i\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\cdot\boldsymbol{\sigma},\\
&\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\,\boldsymbol{m}\cdot\boldsymbol{\sigma}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} = \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} + i\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\cdot\boldsymbol{\sigma}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\\
&= \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} + i\left(\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\cdot\hat{\boldsymbol{h}}\,\mathbb{1} + i\left(\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\times\hat{\boldsymbol{h}}\right)\cdot\boldsymbol{\sigma}\right)\\
&= \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} - \left(\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\times\hat{\boldsymbol{h}}\right)\cdot\boldsymbol{\sigma}\\
&= \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} - \left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{h}}\,\boldsymbol{m}\cdot\boldsymbol{\sigma} - \hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}\right)\\
&= 2\,\hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} - \boldsymbol{m}\cdot\boldsymbol{\sigma}.
\end{aligned} \tag{11}$$

Eq. (10) becomes

$$\begin{aligned}
&\cos^2(|\boldsymbol{h}|\,t)\,\boldsymbol{m}\cdot\boldsymbol{\sigma} - 2\sin(|\boldsymbol{h}|\,t)\cos(|\boldsymbol{h}|\,t)\left(\hat{\boldsymbol{h}}\times\boldsymbol{m}\right)\cdot\boldsymbol{\sigma} + \sin^2(|\boldsymbol{h}|\,t)\left(2\,\hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma} - \boldsymbol{m}\cdot\boldsymbol{\sigma}\right)\\
&= \cos(2\,|\boldsymbol{h}|\,t)\,\boldsymbol{m}\cdot\boldsymbol{\sigma} + \sin(2\,|\boldsymbol{h}|\,t)\,\boldsymbol{m}\cdot\left(\hat{\boldsymbol{h}}\times\boldsymbol{\sigma}\right) + (1-\cos(2\,|\boldsymbol{h}|\,t))\,\hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{\sigma}.
\end{aligned} \tag{12}$$

Plugging back to Eq. (9)

$$\boldsymbol{m}\cdot\langle\boldsymbol{\sigma}\rangle_t = \cos(2\,|\boldsymbol{h}|\,t)\,\boldsymbol{m}\cdot\langle\boldsymbol{\sigma}\rangle_0 + \sin(2\,|\boldsymbol{h}|\,t)\,\boldsymbol{m}\cdot\left(\hat{\boldsymbol{h}}\times\langle\boldsymbol{\sigma}\rangle_0\right) + (1-\cos(2\,|\boldsymbol{h}|\,t))\,\hat{\boldsymbol{h}}\cdot\boldsymbol{m}\,\hat{\boldsymbol{h}}\cdot\langle\boldsymbol{\sigma}\rangle_0. \tag{13}$$

Now we take derivatives with respect to $\boldsymbol{m}$ on both sides to get Eq. (7).

HW4

HW 4 Show that the fact that two operator commute (or not commute) is independent of the choice of basis, i.e. suppose $A' = U\,A\,U^\dagger$ and $B' = U\,B\,U^\dagger$, then $[A, B] = 0 \Leftrightarrow [A', B'] = 0$.

Solution (HW 4)

Given $A' = U\,A\,U^\dagger$ and $B' = U\,B\,U^\dagger$,

$$\begin{aligned}
&[A', B'] = A'\,B' - B'\,A'\\
&= U\,A\,U^\dagger\,U\,B\,U^\dagger - U\,B\,U^\dagger\,U\,A\,U^\dagger\\
&= U\,A\,B\,U^\dagger - U\,B\,A\,U^\dagger\\
&= U\left(A\,B - B\,A\right)U^\dagger\\
&= U\,[A, B]\,U^\dagger.
\end{aligned} \tag{14}$$

So if $[A, B] = 0$, then $[A', B'] = U\,0\,U^\dagger = 0$.

HW5

HW 5 Suppose A and B are Hermitian operators.
(i) Show that $\langle A^2\rangle$, $\langle B^2\rangle$ and $i\,\langle[A, B]\rangle$ are real.
(ii) Show that $[\Delta A, \Delta B] = [A, B]$.

Solution (HW 5)

(i) One can check

$$\left(A^2\right)^\dagger = \left(A\, A\right)^\dagger = A^\dagger\, A^\dagger = A\, A = A^2, \tag{15}$$

similarly $\left(B^2\right)^\dagger = B^2$, also

$$\begin{aligned}&\left(i\left[A,\, B\right]\right)^\dagger = -i\left[A,\, B\right]^\dagger = -i\left(A\,B - B\,A\right)^\dagger\\ &= -i\left(B^\dagger\, A^\dagger - A^\dagger\, B^\dagger\right) = -i\left(B\,A - A\,B\right)\\ &= -i\left[B,\, A\right] = i\left[A,\, B\right].\end{aligned} \tag{16}$$

Therefore A^2, B^2 and $i\left[A,\, B\right]$ are all Hermitian, and their expectation values are all real.

(ii) Given $\Delta A = A - \langle A\rangle\, \mathbb{1}$ and $\Delta B = B - \langle B\rangle\, \mathbb{1}$,

$$\begin{aligned}&\left[\Delta A,\, \Delta B\right] = \left[A - \langle A\rangle\, \mathbb{1},\, B - \langle B\rangle\, \mathbb{1}\right]\\ &= \left[A,\, B\right] - \langle A\rangle[\mathbb{1},\, B] - \langle B\rangle\left[A,\, \mathbb{1}\right] + \langle A\rangle\, \langle B\rangle[\mathbb{1},\, \mathbb{1}]\\ &= \left[A,\, B\right],\end{aligned} \tag{17}$$

because the identity operator $\mathbb{1}$ commutes with any operator.

HW6

HW 6

> Consider a single-qubit Hamiltonian $H = \boldsymbol{h}\cdot\boldsymbol{S}$, where $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is the spin operator.
> (i) Show that the expectation values of the spin operator evolves as $\partial_t\langle\boldsymbol{S}\rangle = \boldsymbol{h}\times\langle\boldsymbol{S}\rangle$.
> (ii) Show that
> $\langle\boldsymbol{S}(t)\rangle = \cos(|\boldsymbol{h}|\, t)\,\langle\boldsymbol{S}(0)\rangle + \sin(|\boldsymbol{h}|\, t)\,\hat{\boldsymbol{h}}\times\langle\boldsymbol{S}(0)\rangle + (1-\cos(|\boldsymbol{h}|\, t))\,\hat{\boldsymbol{h}}\left(\hat{\boldsymbol{h}}\cdot\langle\boldsymbol{S}(0)\rangle\right)$
> is a solution of $\partial_t\langle\boldsymbol{S}\rangle = \boldsymbol{h}\times\langle\boldsymbol{S}\rangle$, where $\hat{\boldsymbol{h}} = \boldsymbol{h}/|\boldsymbol{h}|$.
> This describes the dynamics of a spin in a magnetic field $\boldsymbol{h}$.
> (iii) Show that the spin component along the magnetic field $\hat{\boldsymbol{h}}\cdot\boldsymbol{S}$ is a conserved quantity, that generates the SO(2) symmetry of the Hamiltonian.

Solution (HW 6)

We will use

$$i\,\hbar\,\partial_t\langle L(t)\rangle = \langle[L(t),\, H]\rangle. \tag{18}$$

(i) According to Eq. (18),

$$\begin{aligned}&\partial_t\langle\boldsymbol{S}\rangle = -\frac{i}{\hbar}\,\langle[\boldsymbol{S},\, \boldsymbol{h}\cdot\boldsymbol{S}]\rangle\\ &= -\frac{i\,\hbar}{4}\,\langle[\boldsymbol{\sigma},\, \boldsymbol{h}\cdot\boldsymbol{\sigma}]\rangle\\ &= -\frac{i\,\hbar}{2}\, i\,\boldsymbol{h}\times\langle\boldsymbol{\sigma}\rangle\\ &= \boldsymbol{h}\times\langle\boldsymbol{S}\rangle.\end{aligned} \tag{19}$$

(ii) Left hand side

$$\begin{aligned}\partial_t\langle \boldsymbol{S}\rangle &= -\sin(|\boldsymbol{h}|\ t)\,|\boldsymbol{h}|\left(\langle \boldsymbol{S}(0)\rangle - \hat{\boldsymbol{h}}\left(\hat{\boldsymbol{h}}\cdot\langle \boldsymbol{S}(0)\rangle\right)\right) + \cos(|\boldsymbol{h}|\ t)\,|\boldsymbol{h}|\,\hat{\boldsymbol{h}}\times\langle \boldsymbol{S}(0)\rangle \\ &= \cos(|\boldsymbol{h}|\ t)\,\boldsymbol{h}\times\langle \boldsymbol{S}(0)\rangle - \sin(|\boldsymbol{h}|\ t)\,|\boldsymbol{h}|\left(\langle \boldsymbol{S}(0)\rangle - \hat{\boldsymbol{h}}\left(\hat{\boldsymbol{h}}\cdot\langle \boldsymbol{S}(0)\rangle\right)\right) \\ &= \cos(|\boldsymbol{h}|\ t)\,\boldsymbol{h}\times\langle \boldsymbol{S}(0)\rangle - \sin(|\boldsymbol{h}|\ t)\left(|\boldsymbol{h}|\,\langle \boldsymbol{S}(0)\rangle - \hat{\boldsymbol{h}}(\boldsymbol{h}\cdot\langle \boldsymbol{S}(0)\rangle)\right),\end{aligned} \tag{20}$$

Right hand side

$$\begin{aligned}\boldsymbol{h}\times\langle \boldsymbol{S}\rangle &= \cos(|\boldsymbol{h}|\ t)\,\boldsymbol{h}\times\langle \boldsymbol{S}(0)\rangle + \sin(|\boldsymbol{h}|\ t)\,\boldsymbol{h}\times\left(\hat{\boldsymbol{h}}\times\langle \boldsymbol{S}(0)\rangle\right) \\ &= \cos(|\boldsymbol{h}|\ t)\,\boldsymbol{h}\times\langle \boldsymbol{S}(0)\rangle + \sin(|\boldsymbol{h}|\ t)\left(\hat{\boldsymbol{h}}(\boldsymbol{h}\cdot\langle \boldsymbol{S}(0)\rangle) - |\boldsymbol{h}|\,\langle \boldsymbol{S}(0)\rangle\right).\end{aligned} \tag{21}$$

The two sides match.

(iii) $\hat{\boldsymbol{h}}\cdot\boldsymbol{S}$ is conserved, because it commutes with the Hamiltonian

$$\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{S},\ H\right] = |\boldsymbol{h}|\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{S},\ \hat{\boldsymbol{h}}\cdot\boldsymbol{S}\right] = 0. \tag{22}$$

■ HW7

Quantum Tomography: reconstruction of the *density matrix* from (repeated) *measurements* on the systems taken from the *ensemble*. For a single qubit, by measuring $\langle\boldsymbol{\sigma}\rangle$, the density matrix can be reconstructed as

$$\boxed{\rho = \frac{1}{2}(\mathbb{1} + \langle\boldsymbol{\sigma}\rangle\cdot\boldsymbol{\sigma}).} \tag{23}$$

HW 7

> Check that the density matrix ρ in Eq. (23) is normalized $\operatorname{Tr}\rho = 1$ and reproduces all measurement expectation values $\operatorname{Tr}\rho\,\boldsymbol{\sigma} = \langle\boldsymbol{\sigma}\rangle$.

Solution (HW 7)

In the qubit Hilbert space,

$$\operatorname{Tr}\mathbb{1} = 2,\ \ \operatorname{Tr}\sigma^a = 0, \tag{24}$$

therefore

$$\operatorname{Tr}\sigma^a\,\sigma^b = \operatorname{Tr}\left(\delta_{ab}\,\mathbb{1} + i\,\epsilon^{abc}\,\sigma^c\right) = \delta_{ab}\operatorname{Tr}\mathbb{1} + i\,\epsilon^{abc}\operatorname{Tr}\sigma^c = 2\,\delta_{ab}. \tag{25}$$

With these it is straight forward to check $\operatorname{Tr}\rho = 1$ and $\operatorname{Tr}\rho\,\boldsymbol{\sigma} = \langle\boldsymbol{\sigma}\rangle$.

■ HW8

$$\boxed{i\,\hbar\,\partial_t\rho(t) = [H,\,\rho(t)].} \tag{26}$$

$$\boxed{i\,\hbar\,\partial_t|\psi(t)\rangle = H\,|\psi(t)\rangle.} \tag{27}$$

HW 8

> In the case of $\rho(t) = |\psi(t)\rangle\,\langle\psi(t)|$, derive the von Neumann equation Eq. (26) from the Schrödinger equation Eq. (27).

Solution (HW 8)

Starting from the Schrödinger equation

$$i\,\hbar\,\partial_t|\psi(t)\rangle = H\,|\psi(t)\rangle, \tag{28}$$

take Hermitian conjugate on both sides,

$$-i\,\hbar\,\partial_t\langle\psi(t)| = \langle\psi(t)|\,H^\dagger = \langle\psi(t)|\,H. \tag{29}$$

Given $\rho(t) = |\psi(t)\rangle\,\langle\psi(t)|$,

$$\begin{aligned} i\,\hbar\,\partial_t\rho(t) &= (i\,\hbar\,\partial_t|\psi(t)\rangle)\,\langle\psi(t)| + |\psi(t)\rangle(i\,\hbar\,\partial_t\langle\psi(t)|) \\ &= H\,|\psi(t)\rangle\,\langle\psi(t)| - |\psi(t)\rangle\,\langle\psi(t)|\,H \\ &= [H, \rho(t)]. \end{aligned} \tag{30}$$