Consider a spin - 1/2 particle in SHO w/ magnetic field

$$H_0 = \left(\frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 \right)\mathbb{1} + \frac{\omega}{2}\sigma^z = \omega(a^\dagger a + \tfrac{1}{2}) + \frac{\omega}{2}\sigma^z$$

1) What are energies and eigenstates? List gs & first excited states!

$|n\rangle \otimes |\uparrow\rangle$ and $|m\rangle \otimes |\downarrow\rangle$ w/ $E_n = \omega(n+\tfrac{1}{2}) + \frac{\omega}{2}$

$$= \omega(n+1)$$

$$\& \; E_m = \omega(m+\tfrac{1}{2}) - \frac{\omega}{2}$$

$$= \omega m$$

$|0,\downarrow\rangle$ is ground-state w/ $E_0 = 0$

$|1,\downarrow\rangle$ is excited w/ $E_1 = \omega$

$|0,\uparrow\rangle$ is deg. excited w/ $E_1 = \omega$

2) Consider the interaction $H_1 = \frac{\Omega}{2}(a\sigma^+ + a^\dagger\sigma^-)$
where $\sigma^\pm = \sigma^x \pm i\sigma^y$

Write the 3x3 matrix rep of H_1 is the space above

$H_1|0,\downarrow\rangle = 0$ $\quad H_1|1,\downarrow\rangle = \frac{\Omega}{2} a\sigma^+|1,\downarrow\rangle = \frac{\Omega}{2}|0,\uparrow\rangle$

$H_1|0,\uparrow\rangle = \frac{\Omega}{2}a^\dagger\sigma^-|0,\uparrow\rangle = \frac{\Omega}{2}|1,\downarrow\rangle$

$$\hookrightarrow H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega/2 \\ 0 & \Omega/2 & 0 \end{pmatrix}$$

3) Compute the first order correction in Ω to both the ground state energy and excited states. Does it split them?

$$E_0 = 0 + \langle 0 \downarrow | H_1 | 0 \downarrow \rangle = 0$$

Excited states are degenerate and to first order the effective hamiltonian is

$$H_{eff} = \begin{pmatrix} 0 & \Omega/2 \\ \Omega/2 & 0 \end{pmatrix} = \frac{\Omega}{2} \sigma^x$$

$\hookrightarrow$ $E_{\pm} = \omega \pm \frac{\Omega}{2}$ w/ $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,\uparrow\rangle \pm |1,\downarrow\rangle\right)$