Quantum Mechanics A (Physics 212A) Fall 2018
Worksheet 1 – Solutions

Problems

1. **Normal matrices.**

   An operator (or matrix) $\hat{A}$ is *normal* if it satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

   (a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal.

   Real symmetric is a special case of hermitian.

   Let H be hermitian. $[H, H^\dagger] = [H, H] = 0$

   Real orthogonal is a special case of unitary.

   Let U be unitary. $[U, U^\dagger] = UU^\dagger - U^\dagger U = \mathbb{1} - \mathbb{1} = 0$

   (b) Show that any operator can be written as $\hat{A} = \hat{H} + \mathbf{i}\hat{G}$ where $\hat{H}, \hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger, \hat{A} - \hat{A}^\dagger$.] Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

   Let $H = \frac{1}{2}(A + A^\dagger)$ and $G = \frac{1}{2i}(A - A^\dagger)$. By inspection H and G are hermitian.

   The combination $H + iG = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger) = A$

   $[A, A^\dagger] = [H + iG, H - iG] = [H, -iG] + [iG, H] = 2i[G, H]$ which is 0 iff $[H, G] = 0$

   (c) Show that a normal operator $\hat{A}$ admits a spectral representation

   $$\hat{A} = \sum_{i=1}^{N} \lambda_i \hat{P}_i$$

   for a set of projectors $\hat{P}_i$, and complex numbers λ_i.

   By the above if A is normal then $[H, G] = 0$ which allows us to simultaneously diagonalize them with the same set of projectors $\{P_j\}$. Denote their respective eigenvalues h_j and g_j.

   $A = \sum_j (h_j + ig_j) P_j$

2. **Gone with a Trace**

   Recall the trace of an operator $\text{Tr}\,[A] = \sum_m \langle m|A|m\rangle$ for the some basis set $\{|m\rangle\}$

   (a) Prove that this definition is independent of basis.

   This implies if A is diagonalizable with eigenvalues λ_i that $\text{Tr}\,[A] = \sum_i \lambda_i$

   Consider a second basis $\{|n\rangle\}$ for which we compute $\text{Tr}\,[A] = \sum_n \langle n|A|n\rangle$

   Insert $\mathbb{1} = \sum_m |m\rangle\langle m| \to \text{Tr}\,[A] = \sum_n \sum_m \langle n|m\rangle\langle m|A|n\rangle = \sum_m \sum_n \langle m|A|n\rangle\langle n|m\rangle$

   Now remove an identity $\mathbb{1} = \sum_n |n\rangle\langle n|$ to give $\text{Tr}\,[A] = \sum_m \langle m|A|m\rangle$

(b) Prove the cycle property: Tr $[ABC]$ = Tr $[BCA]$ = Tr $[CAB]$

Tr $[ABC] = \sum_m \langle m|ABC|m\rangle = \sum_{m,n,k} \langle m|A|n\rangle\langle n|B|k\rangle\langle k|C|m\rangle$

The above product can be cyclically rearranged and returns the appropriate traces after removing the two insertions of identity.

(c) Consider an operator A. Show the following identity

$$\det e^A = e^{\text{Tr }[A]} \tag{1}$$

Hint: Recall that the determinant is the product of eigenvalues

We can diagonalize A with the unitary transformation $A = U^\dagger \Lambda U$ where Λ is the matrix of eigenvalues.

The determinant of a unitary is a phase $e^{\mathbf{i}\phi}$ and the determinant satisfies $\det(AB) = \det(A)\det(B)$. If the eigenvalues of A are λ_i then the eigenvalues of e^A are e^{λ_i}

These facts allow us to write $\det(e^A) = \prod_i e^{\lambda_i} = e^{\sum_i \lambda_i} = e^{\text{Tr } A}$

3. **Clock and shift operators.**

Consider an N-dimensional Hilbert space, with orthonormal basis $\{|n\rangle, n = 0, \ldots, N-1\}$. Consider operators $\mathbf{T}$ and $\mathbf{U}$ which act on this N-state system by

$$\mathbf{T}|n\rangle = |n+1\rangle, \qquad \mathbf{U}|n\rangle = e^{\frac{2\pi \mathbf{i} n}{N}}|n\rangle \ .$$

In the definition of $\mathbf{T}$, the label on the ket should be understood as its value modulo N, so $N + n \equiv n$ (like a clock).

(a) Find the matrix representations of $\mathbf{T}$ and $\mathbf{U}$ in the basis $\{|n\rangle\}$.

Define $\omega = e^{\frac{2\pi i}{N}}$. $\mathbf{T}^\dagger = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix}$

(b) What are the eigenvalues of $\mathbf{U}$? What are the eigenvalues of its adjoint, $\mathbf{U}^\dagger$?

$e^{\frac{2\pi \mathbf{i} n}{N}}$ and $e^{\frac{-2\pi \mathbf{i} n}{N}}$ respectively for $n \in \{0, \cdots, N-1\}$

(c) Show that

$$\mathbf{UT} = e^{\frac{2\pi \mathbf{i}}{N}}\mathbf{TU}.$$

$\mathbf{UT}|n\rangle = \mathbf{U}|n+1\rangle = e^{\frac{2\pi i(n+1)}{N}}|n+1\rangle$

$\mathbf{TU}|n\rangle = \mathbf{T}e^{\frac{2\pi i n}{N}}|n\rangle = e^{\frac{2\pi i n}{N}}|n+1\rangle$

Comparing the coefficients yields the result above.

(d) From the definition of adjoint, how does $\mathbf{T}^\dagger$ act?

$$\mathbf{T}^\dagger|n\rangle = ?$$

$T^\dagger|n\rangle = |n-1\rangle$

(e) Show that the 'clock operator' $\mathbf{T}$ is normal – that is, commutes with its adjoint – and therefore can be diagonalized by a unitary basis rotation.

Consider $[\mathbf{T}, \mathbf{T}^\dagger]|n\rangle = \mathbf{T}\mathbf{T}^\dagger|n\rangle - \mathbf{T}^\dagger\mathbf{T}|n\rangle = \mathbf{T}|n-1\rangle - \mathbf{T}^\dagger|n+1\rangle = 0$

(f) Find the eigenvalues and eigenvectors of $\mathbf{T}$.

[Hint: consider states of the form $|\theta\rangle \equiv \sum_n e^{\mathbf{i}n\theta}|n\rangle$.]

Consider $\mathbf{T}|\theta\rangle = \mathbf{T}|0\rangle + \mathbf{T}e^{i\theta}|1\rangle + \cdots + \mathbf{T}e^{i(N-1)\theta}|N-1\rangle$

$= |1\rangle + e^{i\theta}|2\rangle + \cdots + e^{i(N-1)\theta}|0\rangle = e^{-i\theta}|\theta\rangle$ where θ must be such that $e^{iN\theta} = 1$

The most general solution to $e^{iN\theta} = 1$ is for $\theta = \frac{2\pi j}{N}$ for $j \in \{0, \cdots, N-1\}$

This defines a basis of $|\omega^j\rangle \equiv \sum_n \omega^{j*n}|n\rangle$ where j runs from 0 to $N-1$.