University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics B (Physics 212B) Winter 2019
Worksheet 4 – Solutions

Problems

1. **Shape Invariance**

   So far you've learned how to solve the SHO via an algebraic method, and you may be aware that there exist other exactly solvable potentials (hydrogen atom, etc.)

   The following machinery massively generalizes the algebraic "ladder operator" method, gives an explanation for the exact solvability of many potentials in terms of a hidden supersymmetry, and is a machine for making new exactly solvable models.

   Suppose you have a Hamiltonian:

$$H_1 = -\frac{1}{2}\partial_x^2 + V_1(x) \tag{1}$$

   with a known groundstate $\psi_0^{(1)}(x)$ of energy $E_0 = 0$.

   (a) Express $V_1(x)$ in terms of $\psi_0^{(1)}$

   $H_1\psi_0^{(1)} = 0 \implies V_1 = \frac{1}{2}\frac{\partial_x^2\psi_0^{(1)}}{\psi_0^{(1)}}$

   (b) Assume an ansatz $H_1 = A^\dagger A$ where $A^{(\dagger)} = \pm\frac{1}{\sqrt{2}}\partial_x + W(x)$. Relate $W(x)$ to $V_1(x)$ and therefore solve for $W(x)$ in terms of $\psi_0^{(1)}$. Show this implies $\psi_0^{(1)}$ is the ground-state of energy $E_0 = 0$

   $V_1(x) = W^2 - \frac{1}{\sqrt{2}}W'$ and therefore $W(x) = -\frac{1}{\sqrt{2}}\frac{\partial_x\psi_0^{(1)}}{\psi_0^{(1)}}$

   This relation implies $A\psi_0^{(1)} = 0$ explicitly and therefore $H_1\psi_0^{(1)} = A^\dagger A\psi_0^{(1)} = 0$

   (c) Consider the Hamiltonian $H_2 = AA^\dagger$. Write this in terms of a potential $V_2(x)$ using the function $W(x)$.

   $V_2(x) = W^2 + \frac{1}{\sqrt{2}}W'$

   (d) Show that the spectrum of H_2, denoted as $E_n^{(2)}$, is related to the spectrum of H_1 by $E_n^{(2)} = E_{n+1}^{(1)}$. Give the relation between their eigenfunctions.

   Suppose $H_1\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)}$

   Then $H_2(A\psi_n^{(1)}) = AA^\dagger A\psi_n^{(1)} = E_n^{(1)}(A\psi_n^{(1)})$

   Similarly if $H_2\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)}$ then $H_1(A^\dagger\psi_n^{(2)}) = A^\dagger AA^\dagger\psi_n^{(2)} = E_n^{(2)}(A^\dagger\psi_n^{(2)})$

   This gives the spectral relation with $\psi_n^{(2)} \propto A\psi_{n+1}^{(1)}$ and $\psi_{n+1}^{(1)} \propto A^\dagger\psi_n^{(2)}$

   The operators $A^{(\dagger)}$ and function W should remind you of the SUSY QM problem from before. This ultimately explains the degeneracy between H_1 and H_2.

(a) Show for $Q = A\sigma^-$ and $Q^\dagger = A^\dagger\sigma^+$ that there's a SUSY Hamiltonian given by $H = \{Q, Q^\dagger\} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$.

This is identical algebra to the SUSY problem before

(b) For $H = (\frac{P^2}{2} + V)\mathbb{1} + B\sigma^z$ express $V(x)$ and $B(x)$ in terms of V_1 and V_2.

$V = \frac{1}{2}(V_1 + V_2) = W^2$ and $B = \frac{1}{2}(V_1 - V_2) = -\frac{1}{\sqrt{2}}W'$

Many properties of the Hamiltonians H_1 and H_2 can then be reasoned about using the technology of SUSY introduced before. Let's consider a new ingredient though.

Let's make a further restriction on $V_1(x; a_1)$ and $V_2(x; a_1)$.

Suppose they satisfy the condition:

$$V_2(x; a_1) - V_1(x; a_2) = R(a_1) \tag{2}$$

where a_1 is a set of parameters and $a_2 = f(a_2)$ is a function of those parameters. The important thing is that $R(a_1)$ is independent of position.

Another way of saying this: The potential V_2 is equal to the potential V_1 if you "shifted" the parameters and added the constant $R(a_1)$.

This is an "integrability condition" known as "shape invariance" and it gives nice expressions for the spectrum of the Hamiltonians H_1 and H_2.[1]

(a) Using the degeneracy relation of $H_2(x; a_1)$ and $H_1(x; a_1)$ and shape invariance (SI) together imply: $E_1^{(1)}(a_1) = E_0^{(2)}(a_1) = R(a_1)$

SI gives: $H_2(x; a_1) = \frac{P^2}{2} + V_2(x; a_1) = \frac{P^2}{2} + V_1(x; a_2) + R(a_1) = H_1(x; a_2) + R(a_1)$

Consider the groundstate of the shifted $H_1(x; a_2)$ which I'll denote $\psi_0^{(1)}(x; a_2)$

Act: $H_2(x; a_1)\psi_0^{(1)}(x; a_2) = [H_1(x; a_2) + R(a_1)]\psi_0^{(1)}(x; a_2) = R(a_1)\psi_0^{(1)}(x; a_2)$

In the last step I used $H_1(x; a_2)\psi_0^{(1)}(x; a_2) = 0$

This proves that the shifted wavefunction $\psi_0^{(1)}(x; a_2)$ is an eigenfunction of the unshifted $H_2(x; a_1)$. In fact it must be the ground state of $H_2(x; a_1)$ by the positivity of the spectrum of $H_1(x; a_2)$

(b) What does this imply for the excited state wavefunction $\psi_1^{(1)}(x; a_1)$

This also means: $\psi_0^{(2)}(x; a_1) \propto A(a_1)\psi_1^{(1)}(x; a_1) = \psi_0^{(1)}(x; a_2)$. The action of the lowering operator gives the groundstate wavefunction but with shifted parameters!

Similarly we know $\psi_1^{(1)}(x; a_1) \propto A^\dagger(a_1)\psi_0^{(2)}(x; a_1) \propto A^\dagger(a_1)\psi_0^{(1)}(x; a_2)$

(c) We can iterate this process to produce the entire spectrum and excited state wavefunctions from our function R. Define the Hamiltonians

$$H_s = -\frac{1}{2}\partial_x^2 + V_1(x; a_s) + \sum_{k=1}^{s-1} R(a_k) \tag{3}$$

[1] I will implicitly assume SUSY is unbroken. This means we can find a groundstate with vanishing energy.

where $a_s = f^{s-1}(a_1)$; apply the shift map $s-1$ times.

Show that H_s and H_{s+1} are SUSY partners with $E_0^{(s)} = \sum_{k=1}^{s-1} R(a_k)$

This is an identical argument to the proof for $H_2(x; a_1)$

(d) Show this implies the relations:

$E_n^{(1)}(a_1) = \sum_{k=1}^{n} R(a_k)$ and $\psi_n^{(1)}(a_1) \propto A^\dagger(x; a_1)A^\dagger(x; a_2)\cdots A^\dagger(x; a_n)\psi_0^{(1)}(x; a_{n+1})$

This is induction and the argument about the spectrum and excited states before as a base case.

2. **Shape-invariant potentials applied**

A particle of unit mass moves in a certain one-dimensional potential $V_-(x)$. The boundstates $u_0, u_1 ... u_n$ have energies $E_0, E_1 ... E_n$ respectively, in order of increasing energy. The groundstate wavefunction is

$$u_0 \propto (\mathrm{sech}\beta x)^p$$

where $\beta, p > 0$ are real parameters.

(a) Find the superpotential $W(x)$.

$W(x) = -\frac{1}{\sqrt{2m}} \frac{\partial_x u_0}{u_0} = \frac{1}{\sqrt{2m}} p\beta \tanh \beta x$

(b) Find V_- and its partner potential V_+.

$V_- \equiv V_1 = W^2 - \frac{1}{\sqrt{2m}}\partial_x W = \frac{\beta^2 p\left(p-(p+1)\mathrm{sech}^2(\beta x)\right)}{2m}$

$V_+ \equiv V_2 = W^2 + \frac{1}{\sqrt{2m}}\partial_x W = \frac{\beta^2 p\left(p\tanh^2(\beta x)+\mathrm{sech}^2(\beta x)\right)}{2m}$

(c) Obtain a general formula for the E_n. What restriction applies to p so that there exist n boundstates?

We want to relate V_2 and V_1 by shape invariance:

$V_2(x; p, \beta) = V_1(x; q, \alpha) + R(p, \beta)$ where α and q are functions of β and q.

With some forsight let $\alpha = \beta$. So $R = V_2[p] - V_1[q] = \frac{\beta^2(p+q)\mathrm{sech}^2(\beta x)((p-q)\cosh(2\beta x)-p+q+2)}{4m}$

So if $q = p-1$ then $R = \frac{\beta^2(2p-1)}{2m}$ which has no x dependence as desired.

What was the point of doing this? We're guaranteed that $H_2 = \frac{P^2}{2m} + V_2(x; p, \beta)$ has a ground state with energy equal to the energy of the first excited state of H_1

But now $H_2 = \frac{P^2}{2m} + V_1(x; q, \alpha) + R(p, \beta)$

Consider $H_2|\psi_0^{(1)}(q)\rangle$ where $H_1[q]|\psi_0^{(1)}(q)\rangle = 0$ so $H_2|\psi_0^{(1)}(q)\rangle = R(p, \beta)|\psi_0^{(1)}(q)\rangle$

Thus the ground-state wavefunction of H_1 (with shifted parameters) is the also the ground-state of H_2 which has energy $R(p, \beta)$

This implies by above that $E_1^{(1)} = R(p, \beta)$

Notice we can repeat this process creating new super potentials and a whole list of potentials V_n of the same form.

Shape invariance gives $V_n(x; p, \beta) = V_1(x; p-n, \beta) + \sum_{i=0}^{n-1} R(p-n, \beta)$ so $E_n^{(1)} = \sum_{i=0}^{n-1} R(p-n, \beta) = -\frac{\beta^2 n(n-2p)}{2m}$

Notice that the energies start decreasing for $n > p$. This doesn't make sense as the excited state energies should be monotically increasing. So in order to have found n bound states we must have $p > n$. Another way to see this is to look at V_n and see that unless $p > n$ there's always a convex potential for some n which wouldn't have a boundstate. For the special case of $p = n$ the last potential vanishes!

(d) Find u_1, up to normalization.

From the shape invariance relation we have $\psi_1^{(1)}(x;p) \propto A^\dagger(p)\psi_0^{(1)}(x;p-1)$
Where $A^\dagger = -\frac{\mathbf{i}}{\sqrt{2m}}(-\mathbf{i}\partial_x) + W(x)$ so $\psi_1^{(1)}(x;p) \propto \beta(2p-1)\sinh(\beta x)\mathrm{sech}^p(\beta x)$

(e) Use the result of part 2c to discuss the potential $V = -V_0 \mathrm{sech}^2 \beta x$ where $V_0 > 0$, and show the boundstate eigenvalues for $m = \hbar = 1$ are given by

$$E_n = -\frac{\beta^2}{8}\left(-(1+2n) + \sqrt{1 + 8V_0/\beta^2}\right)^2.$$

Notice that $V_1(x;p) = -\frac{\beta^2}{2}(p+p^2)\mathrm{sech}^2\beta x + \frac{p^2\beta^2}{2}$ so we can match $V_0 = \frac{\beta^2}{2}(p + p^2) \implies p = \frac{1}{2}(\sqrt{1+\frac{8V_0}{\beta^2}} - 1)$
Therefore $V_1 - C = -V_0\mathrm{sech}^2(\beta x)$ for $C = \frac{1}{4}\left(\beta^2 - \sqrt{\beta^4 + 8\beta^2 V_0}\right) + V_0$
Then the spectrum of 2c is the spectrum of this potential with additional constant shift. The condition $p > n$ imposes a condition on V_0