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Quantum Mechanics B (Physics 212B) Winter 2019
Worksheet 2 – Solutions

Problems

1. **Give it a Kick**

   Consider the $D = 1$ simple harmonic oscillator in its groundstate. Suppose something kicks the system imparting an additional momentum p_0. What's the probability the system remains in the ground state?

   (a) What's the new Hamiltonian for the system? Express this in terms of the usual ladder operators $\hat{a}$ and $\hat{a}^\dagger$

   $H_{new} = \frac{(p+p_0)^2}{2m} + \frac{1}{2}m\omega^2 x^2 = H_{old} + \frac{p\ p_0}{m} + \frac{p_0^2}{2m} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \mathbf{i}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a}) + \frac{p_0^2}{2m}$

   (b) Define a new operator $\hat{A} \equiv \hat{a} - \beta$ where $\beta \equiv \frac{1}{\mathbf{i}\omega}\frac{p_0}{m}\sqrt{\frac{m\omega}{2}}$.

   Show that the $\hat{A}$ are ladder operators: $[\hat{A}, \hat{A}^\dagger] = 1$

   This follows immediately from $[\hat{a}, \hat{a}^\dagger] = 1$ and that β is a constant.

   (c) Rewrite the new Hamiltonian in terms of these operators, what do you find?

   $H_{new} = \omega(\hat{A}^\dagger\hat{A} + \frac{1}{2})$

   (d) Relate the original groundstate $|0\rangle$ to the new groundstate $|\beta\rangle$

   Since the new Hamiltonian is another harmonic oscillator it must be that:

   $\hat{A}|\beta\rangle = 0 = (\hat{a} - \beta)|\beta\rangle$ or in other words $\hat{a}|\beta\rangle = \beta|\beta\rangle$ this is a *coherent* state.

   (e) Using $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$ compute $P = |\langle 0|\beta\rangle|^2$

   Hint: Insert identity and use the relation above.

   $|\beta\rangle = \mathbb{1}|\beta\rangle = \sum_n |n\rangle\langle n|\beta\rangle = \sum_n |n\rangle\langle 0|\frac{(\hat{a})^n}{\sqrt{n!}}|\beta\rangle = (\sum_n \frac{\beta^n}{\sqrt{n!}}|n\rangle)\langle 0|\beta\rangle$

   Knowing this consider $\langle\beta|\beta\rangle = 1 = (\sum_n \frac{(|\beta|^2)^n}{n!}\langle n|n\rangle)|\langle 0|\beta\rangle|^2 = e^{|\beta|^2}|\langle 0|\beta\rangle|^2$

   Therefore: $|\langle 0|\beta\rangle|^2 = e^{-|\beta|^2}$

2. **Creation and Annihilation**

   Consider a two-level fermion system with basis states $|0\rangle, |u_1\rangle, |u_2\rangle, |u_1, u_2\rangle$

   Construct the matrix representations of the operators $\{a(u_1), a^\dagger(u_1), a(u_2), a^\dagger(u_2)\}$ and verify they satisfy the fermionic algebra:

   $\{a(u_1), a^\dagger(u_1)\} = \mathbb{1} = \{a(u_2), a^\dagger(u_2)\}$

   $(a^\dagger(u_1))^2 = 0 = (a^\dagger(u_2))^2$

   $\{a^\dagger(u_1), a^\dagger(u_2)\} = 0 = \{a(u_1), a(u_2)\}$

   For reference, in this basis:

$$|0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|u_1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|u_2\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|u_1, u_2\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

As was constructed: $a^\dagger(u_1) = \boldsymbol{\sigma}^+ \otimes \mathbb{1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Similarly it must be that:

$a(u_1)|u_1\rangle = |0\rangle$

$a(u_1)|0\rangle = 0 = a(u_1)|u_2\rangle$

$a(u_1)|u_1, u_2\rangle = |u_2\rangle$

This yields: $a(u_1) = \boldsymbol{\sigma}^- \otimes \mathbb{1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Which one expects from the action of dagger.

For $a^\dagger(u_2)$ however we have this algebra:

$a^\dagger(u_2)|u_1\rangle = -|u_1, u_2\rangle$

$a^\dagger(u_2)|0\rangle = |u_2\rangle$

$a^\dagger(u_2)|u_1, u_2\rangle = 0 = a^\dagger(u_2)|u_2\rangle$

This gives: $a^\dagger(u_2) = -\boldsymbol{\sigma}^z \otimes \boldsymbol{\sigma}^+ = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Similarly: $a(u_2) = -\boldsymbol{\sigma}^z \otimes \boldsymbol{\sigma}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Then it is easy to check with Pauli algebra or direct matrix multiplcation that the algebra is satisfied.