1) Prove that the entropy $S = -tr\{\rho \log \rho]$ is time independent for any density matrix

$\rho(t) = U(t)\, \rho_0\, U^{+}(t) \qquad U = e^{-iHt}$

$\frac{\partial}{\partial t} \log \rho = \frac{\rho'(t)}{\rho(t)}$

$\frac{\partial S}{\partial t} = -tr\left[\frac{\partial}{\partial t}\left[\rho \log \rho\right]\right] = -tr\left[\frac{\partial \rho}{\partial t} \log \rho(t) + \frac{\partial \rho}{\partial t}\right]$

$\frac{\partial \rho}{\partial t} = -i[H, \rho] \qquad \stackrel{?}{=} tr\left[[H,\rho] \log \rho + [H,\rho]\right]$

$tr[A + B] = tr[A] + tr[B]$ (Linearity)

$tr[AB] = tr[BA]$ (cyclic)

$tr\left[[H,\rho]\right] = tr[H\rho] - tr[\rho H] = 0$ (by cyclic)

$tr[H\rho \log \rho] - tr[\rho H \log \rho]$

$= tr[H \log \rho\, \rho]$ by cyclic

$= tr[H \rho \log \rho]$ by commuting $\log \rho$ w/ ρ

$\stackrel{!}{=} 0$

Therefore it completely vanishes

2) Consider a particle in $d=3$ with

$$H = \frac{P^2}{2m} + V(\vec{x})$$

Find $\frac{d}{dt}\langle L_z \rangle$ $\qquad \vec{L} \equiv \vec{x} \times \vec{p}$

and compare to classical mechanics

$$L_z = \cancel{x_1 p_2} \; x\, p_y - y\, p_x \equiv x_1 p_2 - x_2 p_1$$

$$\frac{d}{dt} L_z = -i [L_z, H]$$

$$[L_z, H] = \left[L_z, \frac{P^2}{2m}\right] + [L_z, V(\vec{x})]$$

First part must vanish by rotational invariance but explicitly

$$\left[L_z, \frac{P^2}{2m}\right] \propto [x p_y - y p_x, p_x^2 + p_y^2] = \left([x, p_x^2] p_y - [y, p_y^2] p_x\right)$$
$$= (2 p_x p_y - 2 p_y p_x) = 0$$

$$[L_z, V(x)] = [x p_y - y p_x, V] = x [p_y, V] - y [p_x, V(x)]$$
$$= -i (x\, \partial_y V - y\, \partial_x V)$$
$$= -i (\vec{x} \times \vec{\nabla} V)|_z$$

Torque

$$\hookrightarrow \frac{d}{dt}\langle L_z \rangle = -\langle \vec{x} \times \vec{\nabla} V \rangle|_z = \langle \vec{x} \times \vec{F} \rangle|_z = \langle \tau_z \rangle$$