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Quantum Mechanics A (Physics 212A) Fall 2018
Worksheet 7 – Solutions

Problems

1. **Euclidean Time**

   This problem explores the connection between stat mech and the path integral.

   Consider the propagator:

   $$\langle x_f|e^{-\mathbf{i}HT}|x_i\rangle = \int [\mathcal{D}x(t)]_{x(0)=x_i}^{x(T)=x_f} \; e^{\mathbf{i}S[x(t)]} \tag{1}$$

   (a) On the left hand side, define $\beta = \mathbf{i}T$ and consider $x_f = x_i \equiv x$. If you then perform an integral over all x, what is this equal to?

   $\int dx \; \langle x|e^{-\beta H}|x\rangle = \sum_n \int dx \; e^{-\beta E_n}\langle n|x\rangle\langle x|n\rangle = \sum_n e^{\beta E_n} = Z$

   This is the partition function

   (b) What do each of those steps translate to for the right hand side?

   The first step implies we should define 'Euclidean' time as $\tau = \mathbf{i}t$ and consider paths that take β long.

   The second step means we should consider only paths that begin and end at the same point; that are periodic in the τ coordinate.

   The third step says we should integrate over all such paths, regardless on end-points.

   Together the imply: $Z = \int [\mathcal{D}x(\tau)]_{x(0)=x(\beta)} \; e^{-S_E[x(\tau)]}$

   In words, the path integral is over all paths on the "thermal circle" of radius set by β

2. **A Dirty Estimate**

   Consider two identical atoms of liquid He-4. Estimate the temperature for which the free particle Euclidean action:

   $$S_E = \oint_0^{\hbar\beta} d\tau \; \frac{m}{2}(\frac{dx}{d\tau})^2 \tag{2}$$

   evaluated for interchanging two particles changes by $\Delta S_E \approx \hbar$. Compare this temperature to the critical/superfluid transition temperature $T^* \approx 2.1768K$

   Hint: Use the mean free path $\ell_f \approx (\sqrt{2}n\sigma)^{-1} = \frac{2^{3/2}}{\pi}n^{-\frac{1}{3}}$ where n is the number density.

   Our approximation will be $dx \approx \ell_f$ and $d\tau \approx \hbar\beta$ giving $\Delta S_E \approx \frac{m}{2}\frac{\ell_f^2}{\hbar\beta_c} \approx \hbar$

   Invert for temperature: $T_c \approx \frac{\pi^2\hbar^2}{4km}\frac{n^{2/3}}{m} = \frac{\pi^2\hbar^2\rho^{2/3}}{4km^{5/3}} \approx 2.1183K$

3. **The Particle on a Ring**

Consider the quantum dynamics of a particle of mass m and charge q on a circular hoop of radius R in the presence of an external magnetic field perpendicular to the plane of the hoop.

Let $\phi(t)$ be the angular coordinate of the particle such that $\phi \equiv \phi + 2\pi$

The action is

$$S[\phi] = \int \mathrm{dt}\ \frac{m}{2} R^2 \dot{\phi}^2 + (\frac{\theta}{2\pi})\dot{\phi} \tag{3}$$

The θ term is a "topological term" as we'll see.

(a) Show that if there is a azimuthally symmetric magnetic field passing through the ring, then θ is related to B by

$$\theta = q\Phi_B$$

where Φ_B is the magnetic flux through the ring.

$\vec{B} = \nabla \times \vec{A} = B\delta(\vec{x})\hat{z} \implies \vec{A} = \frac{\Phi_B}{2\pi r}\hat{\phi}$

Minimal coupling to ϕ is $-qR\vec{A} \cdot \dot{\phi} = -q\frac{\Phi_b}{2\pi}\dot{\phi}$ which determines θ.

(b) Show that θ is a periodic parameter, i.e. theories with θ and $\theta \to \theta + 2\pi$ are equivalent.

See part e

(c) Compute the canonical momentum associated with ϕ, and determine the Hamiltonian for this system.

$\Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi} + \frac{\theta}{2\pi} \equiv p_\phi + \frac{\theta}{2\pi}$

$H = \Pi_\phi \dot{\phi} - L = \frac{p_\phi^2}{2mR^2} = \frac{(\Pi_\phi - \frac{\theta}{2\pi})^2}{2mR^2} \implies \hat{H} = \frac{1}{2mR^2}(-\mathbf{i}\partial_\phi - \frac{\theta}{2\pi})^2$

(d) Show that the θ term doesn't effect the classical equation of motion for ϕ.

Euler-Lagrange: $\frac{\mathrm{d}\Pi_\phi}{\mathrm{dt}} = 0 = mR^2\ddot{\phi}$ and the θ dependence is gone

(e) Find the energies and wavefunctions by solving the Schrödinger equation. What happens to the eigenstates $|n, \theta\rangle$ and energies $E_n(\theta)$ as $\theta \to \theta + 2\pi$?

An ansatz is $p_\phi \psi(\phi) = \lambda_n \psi$ so that $E_n = \frac{\lambda_n^2}{2mR^2}$ which is satisfied by $\psi \propto e^{\mathbf{i}n\phi}$

Yielding $E_n = \frac{1}{2mR^2}(n - \frac{\theta}{2\pi})^2$ and $\psi(\phi) = \psi(\phi + 2\pi)$

Note that under $\theta \to \theta + 2\pi$ that $E_n \to E_{n-1}$ and the total spectrum is left unchanged. This corresponds to the insertion of *unit flux*

A comment on single-valuedness:

Notice that we have gauge freedom of $A \to A + \nabla\Lambda$ and $\psi \to \psi' = e^{-\mathbf{i}q\Lambda}\psi$

Nothing stops me from choosing $\nabla\Lambda = -A \implies \Lambda = -\frac{\Phi_B}{2\pi}\phi$

The Hamiltonian is then $\hat{H} = \frac{1}{2mR^2}(-\mathbf{i}\partial_\phi)^2$ but notice that the boundary condition for ϕ have changed.

$\psi'(\phi + 2\pi) = e^{\mathbf{i}\frac{\theta}{2\pi}(\phi+2\pi)}\psi(\phi + 2\pi) = e^{\mathbf{i}\theta}\psi'(\phi)$ a *twisted* boundary condition

So this entire problem is equivalent to a free particle propagating on a ring which picks up a phase of $e^{\mathbf{i}\theta}$ after a complete wind. This is the *Aharonov-Bohm phase.* If you demanded this phase be 1, completely unobservable, then $\theta = 2\pi\ell$ for $\ell \in \mathbb{Z}$ which is what Dirac did for his famous string

The rest of this problem aims to compute the partition function associated with (3) via the path integral. To do this we need to switch to "Euclidean time".

Let us define $t = -\mathbf{i}\tau$ and note that with this transformation the Euclidean paths are periodic, $\phi(0) = \phi(\tau = \beta)$, but they can have winding number ν which is an integer. Mathematically we're studying maps from $S^1_\beta \to S^1_{2\pi}$ and winding is $\pi_1(S^1) = \mathbb{Z}$

This is the simplest example of a "non-linear sigma model".

(f) Write a decomposition for $\phi_\nu(\tau) = \phi_0(\tau) + f_\nu(\tau)$ where ϕ_0 is periodic, and f_ν winds ν times around the circle, and expand $\phi_0(\tau)$ in a Fourier series.

Think of $\phi(\tau)$ as a map from $S^1_\beta \to S^1_{2\pi}$ with must satisfy the condition that $\phi(\beta) - \phi(0) = 2\pi\nu$. The particle comes back to it's original position at time β but traversed the circle (distance 2π) ν times

A function $\phi_\nu(\tau)$ which generally satisfies this is: $\phi_\nu = \phi_0(\tau) + 2\pi\nu\frac{\tau}{\beta}$

Where $\phi_0(\tau)$ is the map with no windings: $\phi_0(\beta) = \phi_0(0)$ and this possesses a regular Fourier expansion: $\phi_0(\tau) = \sum_n c_n \frac{1}{\sqrt{\beta}} e^{2\pi n \frac{\tau}{\beta}}$

(g) Transform the action (3) into Euclidean time and substitute the form of $\phi_\nu(\tau)$ into it. Evaluate any integrals you can.

The Euclidean action is $S_E[\phi] = \int_0^\beta \mathrm{d}\tau\ \frac{m}{2}R^2\dot{\phi}^2 - \mathbf{i}(\frac{\theta}{2\pi})\dot{\phi}$

We can plug and chug the above to get:

$S_E[\phi_\nu] = \int_0^\beta \mathrm{d}\tau\ \frac{m}{2}R^2\dot{\phi_0}^2 - \mathbf{i}\theta\nu + \frac{1}{\beta}2\pi^2R^2m\nu^2$

We can express the path integral as:

$$Z[\beta] = \int [\mathcal{D}\phi] e^{-S_E[\phi]} = \sum_{\nu\in\mathbb{Z}} \int [\mathcal{D}\phi_\nu] e^{-S_E[\phi_\nu]} \tag{4}$$

where the last equality is a decomposition into different winding modes.

(h) Reduce (4), using the result of part (g) above, to only one functional integral over $\phi_0(\tau)$: the coordinate with no winding

$Z[\beta] = \int [\mathcal{D}\phi] e^{-S_E[\phi]} = \sum_{\nu\in\mathbb{Z}} \int [\mathcal{D}\phi_\nu] e^{-S_E[\phi_\nu]} = \sum_\nu e^{-(-\mathbf{i}\theta\nu + \frac{1}{\beta}2\pi^2R^2m\nu^2)} \int [\mathcal{D}\phi_0] e^{-S_E[\phi_0]}$

(i) Evaluate the functional integral over $\phi_0(\tau)$. Note: BEWARE THE ZERO MODE!

Expanding $\phi_0 = \sum_n c_n f_n(\tau)$ where f_n are orthonormal eigenfunctions of the operator $\mathcal{O}$ in S_E with eigenvalue λ_n

For our case $\mathcal{O} = -mR^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}$ and $f_n = \frac{1}{\sqrt{\beta}} e^{2\pi \mathbf{i} n \frac{\tau}{\beta}}$ with $\lambda_n = \frac{4\pi^2R^2mn^2}{\beta^2}$

Then we rewrite $e^{-S_E} = e^{-\frac{1}{2}\sum_n \lambda_n c_n^2}$ and transform $\mathcal{D}\phi_0 = \prod_n \frac{\mathrm{d}c_n}{\sqrt{2\pi}}$ which upon doing the Gaussian integral over each c_n gives $\prod_n \frac{1}{\sqrt{\lambda_n}} = \frac{1}{\sqrt{\det \mathcal{O}}}$ which usually diverges in a way we can regulate.

Notice, however, that we have $\lambda_0 = 0$ in this problem. This makes the integral no longer Gaussian and we have to think a little.

Seperating this integral over c_0 out from the rest we see we're integrating $e^0 = 1$ over $\phi = \frac{c_0}{\sqrt{\beta}} \in [0, 2\pi]$ which implies $c_0 \in [0, 2\pi\sqrt{\beta}]$

The integral over the zero-mode is therefore: $\frac{2\pi\sqrt{\beta}}{\sqrt{2\pi}}$ making

$\int[\mathcal{D}\phi_0]e^{-S_E[\phi_0]} = \frac{2\pi\sqrt{\beta}}{\sqrt{2\pi}}[\prod_{n\neq 0}\frac{4\pi^2 mR^2n^2}{\beta^2}]^{-\frac{1}{2}}$

Now you might reasonably complain, that product is infinite! What is this supposed to mean? This infinity isn't 'real', it's due to the ambiguity in the normalization of the path integral. One method of computing a finite answer from such a divergent product (or sum) is known as ζ-function regularization. You might be familiar with it from the famous 'equation' $1 + 2 + 3 + \cdots = -\frac{1}{12}$

First define the function: $\zeta(s|\mathcal{O}) = \mathrm{Tr}\ \mathcal{O}^{-s} = \sum_n \frac{1}{\lambda_n^s}$

This generalizes the ζ function $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$. Such a function need not always converge but it has a unique 'analytic continuation'.

The crucial trick is: $\ln\det\mathcal{O} = -\frac{d}{ds}\zeta(s|\mathcal{O})|_{s=0}$

I'm going to do this case carefully in case you ever need to see it.

$\prod_{n\neq 0}\frac{4\pi^2 mR^2n^2}{\beta^2} \equiv \prod_{n\neq 0} c^2n^2 = (\prod_{n=1}^{\infty} c^2n^2)^2$ by symmetry.

Our trick implies: $\ln(\prod_{n=1}^{\infty} c^2n^2) = -\frac{d}{ds}[\sum_{n=1}^{\infty}(cn)^{-2s}]|_{s=0}$

Note that our function is related to the ζ function: $\sum_{n=1}^{\infty}(cn)^{-2s} = c^{-2s}\zeta(2s)$

$-\frac{d}{ds}[c^{-2s}\zeta(2s)] = 2c^{-2s}\ln(c)\zeta(2s) - 2c^{-2s}\zeta'(2s)$

To continue to $s = 0$ we need $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$

This implies: $\prod_{n=1}^{\infty} c^2n^2 = \frac{2\pi}{c}$

At the end of the day: $\int[\mathcal{D}\phi_0]e^{-S_E[\phi_0]} = \frac{2\pi\sqrt{\beta}}{\sqrt{2\pi}}\sqrt{\frac{mR^2}{\beta^2}} = \sqrt{\frac{2\pi m}{\beta}}R$

Bringing it together:

$Z[\beta] = \sqrt{\frac{2\pi mR^2}{\beta}}\sum_\nu e^{\mathbf{i}\theta\nu - \frac{1}{\beta}2\pi^2R^2m\nu^2}$

At the end of the day you should find:

$$Z[\beta] = \sqrt{\frac{2\pi mR^2}{\beta}}\sum_\nu e^{\mathbf{i}\theta\nu - \frac{1}{\beta}2\pi^2R^2m\nu^2} \tag{5}$$

which leads one to ask, how do I compare this to $Z = \sum_n e^{-\beta E_n}$?

Answer: Poisson resummation

(j) Apply the formula:

$$\sum_\nu e^{-\frac{1}{2}A\nu^2 + \mathbf{i}B\nu} = \sqrt{\frac{2\pi}{A}}\sum_\ell e^{-\frac{1}{2A}(B-2\pi\ell)^2} \tag{6}$$

and find the complete spectrum.

Clearly $B = \theta$ and $A = \frac{1}{\beta}(4\pi^2 R^2 m)$ so $Z = \sqrt{\frac{2\pi m R^2}{\beta}} \sqrt{\frac{2\pi}{A}} \sum_n e^{-\frac{1}{2A}(\frac{\theta}{R} - 2\pi n)^2}$

$Z = \sum_n e^{-\beta \frac{1}{8\pi^2 R^2 m}(\theta - 2\pi n)^2}$ so that $E_n = \frac{1}{8\pi^2 R^2 m}(\theta - 2\pi n)^2 = \frac{1}{2mR^2}(n - \frac{\theta}{2\pi})^2$