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Quantum Mechanics B (Physics 212B) Winter 2019
Worksheet 3 – Solutions

Problems

1. **Supersymmetry?**

   Consider a spin-$\frac{1}{2}$ particle on a line and in a magnetic field. The Hamiltonian is:

$$\hat{H} = [\frac{1}{2}\hat{P}^2 + V(x)]\mathbb{1} + B(x)\sigma^z \tag{1}$$

   Suppose your wrote your $V(x)$ and $B(x)$ as the following:

$$V(x) = W^2 \quad B(x) = -\frac{1}{\sqrt{2}}\partial_x W$$

   Where $W(x)$ is an arbitrary function known as the superpotential.

   Now define the following operators:

$$Q = \mathbf{i}(\frac{P}{\sqrt{2}} - \mathbf{i}W)\sigma^- \equiv A\sigma^- \quad Q^\dagger = -\mathbf{i}(\frac{P}{\sqrt{2}} + \mathbf{i}W)\sigma^+ \equiv A^\dagger\sigma^+ \tag{2}$$

   Where recall $\sigma^\pm = \frac{1}{2}(\sigma^x \pm \mathbf{i}\sigma^y)$ are raising and lowering operators.

   Q and $Q^\dagger$ are known as supercharges.

   (a) Show that $Q^2 = 0 = (Q^\dagger)^2$
   $(\boldsymbol{\sigma}^\pm)^2 = \frac{1}{4}(\boldsymbol{\sigma}^x \pm \mathbf{i}\boldsymbol{\sigma}^y)^2 = \mathbb{1} - \mathbb{1} + (\boldsymbol{\sigma}^z - \boldsymbol{\sigma}^z) = 0$

   (b) Show that $\{Q, Q^\dagger\} = H$ where recall $\{A, B\} = AB + BA$
   $QQ^\dagger + Q^\dagger Q = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}$
   $= \frac{1}{2}(A^\dagger A + AA^\dagger)\mathbb{1} + \frac{1}{2}(A^\dagger A - AA^\dagger)\sigma^z$
   $A^\dagger A = -\frac{1}{2}\partial_x^2 + (W^2 - \frac{1}{\sqrt{2}}[\partial_x, W]) = -\frac{1}{2}\partial_x^2 + (W^2 - \frac{1}{\sqrt{2}}\partial_x W)$
   $AA^\dagger = -\frac{1}{2}\partial_x^2 + (W^2 + \frac{1}{\sqrt{2}}\partial_x W)$
   Combining these and using the definition of V and B completes the proof.
   This is known as the "SUSY algebra"

   (c) Show that $[Q, \hat{H}] = 0 = [Q^\dagger, \hat{H}]$
   $[Q, \hat{H}] = ([Q, QQ^\dagger] + [Q, Q^\dagger Q]) = (Q[Q, Q^\dagger] + [Q, Q^\dagger]Q)$
   Now we expand using $Q^2 = 0 : = (-QQ^\dagger Q + QQ^\dagger Q) = 0$
   Similarly for $[Q^\dagger, \hat{H}]$

(d) We can also define $F = \boldsymbol{\sigma}^{-}\boldsymbol{\sigma}^{+}$ which is also a symmetry of (1). Show $[F, H] = 0$ What are $[F, Q]$ and $[F, Q^{\dagger}]$?
$[H, F] = 0$ follows immediately from $[\boldsymbol{\sigma}^{-}\boldsymbol{\sigma}^{+}, \boldsymbol{\sigma}^{z}] = 0$
Similarly $[F, Q] = -Q$ and $[F, Q^{\dagger}] = Q^{\dagger}$

(e) Note that the operators $Q, Q^{\dagger}$ aren't Hermitian but we can define $Q_1 = (Q + Q^{\dagger})$ and $Q_2 = \frac{1}{\mathbf{i}}(Q - Q^{\dagger})$. What algebra do they satisfy?
$Q_i^2 = H$

Now what does supersymmetry do? Consider an eigenstate $|\Psi\rangle$ with energy E.

(f) Compute $\langle\Psi|\hat{H}|\Psi\rangle$. What does this tell us about the ground state of the system?
$\langle\hat{H}\rangle = \langle\Psi|QQ^{\dagger} + Q^{\dagger}Q|\Psi\rangle = (|Q^{\dagger}|\Psi\rangle|^2 + |Q|\Psi\rangle|^2) \geq 0$
This implies that $E = 0 \implies Q|\Psi_0\rangle = 0 = Q^{\dagger}|\Psi_0\rangle$; the groundstates respect the symmetry

(g) Using this constraint construct the groundstate wavefunction $\Psi_0(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}$ in terms of $W(x)$

$Q\Psi_0 = 0 \implies \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}}\partial_x + W & 0 \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = 0 \implies \frac{1}{\sqrt{2}}\partial_x\psi_- = -W\psi_-$

Similarly: $Q^{\dagger}\Psi_0 = 0 \implies \frac{1}{\sqrt{2}}\partial_x\Psi_+ = W\Psi_+$

These are first order equations which give: $\Psi_{\pm} \propto e^{\pm\sqrt{2}\int dx' \; W(x')}$

I'm being slightly ambigious on the bounds of the integral and clearly, depending on the behavior of W, the wavefunction might not be normalizable. In this case you may only have one solution or none. This is related to the "spontaneous breaking of SUSY"

2. **Fermions and Bosons**

You may already know that SUSY mixes bosons and fermions. How does that appear in this model? Let's think about the spin Hilbert space $\mathcal{H}_2 = \text{span}\{|0\rangle, |1\rangle\}$

I encourage you to think about these vectors as being labeled by occupation number: $|0\rangle$ has no fermion (it is bosonic) and $|1\rangle$ has a single fermion.[1]

The fermionic creation and annihilation operators are then simply: $\hat{\psi}^{\dagger} \equiv \boldsymbol{\sigma}^{-} \quad \hat{\psi} \equiv \boldsymbol{\sigma}^{+}$

All states in the Hilbert space can be written as: $|\Psi\rangle = f_0(x)|0\rangle + f_1(x)\hat{\psi}^{\dagger}|0\rangle$

(a) Convince yourself the above is true and that $F = \hat{\psi}^{\dagger}\hat{\psi}$ is a number operator
We've already shown that F is a symmetry and it's clear that $F|0\rangle = 0$ and $F|1\rangle = |1\rangle$. Note also the $\psi^{\dagger}\psi^{\dagger}|0\rangle = 0$

Now let's show something non-trivial. I claim that all the excited ($E \neq 0$) states are two fold degenerate into bosonic ($F = 0$) and fermionic ($F = 1$) pairs.

Let's do this explicitly:

[1] I'm being slick with notation as $|0\rangle, |1\rangle$ are the computer science way of denoting the eigenvectors of σ^z for a spin system.

(b) Define $|b\rangle$ to be a state of the form $\Psi(x) = \begin{pmatrix} \Psi_+(x) \\ 0 \end{pmatrix}$ and $|f'\rangle \equiv Q^\dagger|b\rangle$. Show that $|f'\rangle$ is fermionic and degenerate with $|b\rangle$

$H|f'\rangle = HQ^\dagger|b\rangle = Q^\dagger H|b\rangle = E|f'\rangle$ where I've used $[H, Q] = 0$

To check it is fermionic recall $[F, Q^\dagger] = Q^\dagger$ so $F|f'\rangle = (Q^\dagger F + Q^\dagger)|b\rangle = |f'\rangle$

(c) Show also that the properly normalized states are $|f\rangle = \frac{1}{\sqrt{E}}|f'\rangle$

$\langle f'|f'\rangle = \langle b|QQ^\dagger|b\rangle = \langle b|H - Q^\dagger Q|b\rangle = E$ by using $\{Q, Q^\dagger\} = H$

3. **A Certain Magical Index**

This model gives another interesting example of an *index*

It's going to be useful to define the operator $(-1)^F$, the fermionic *parity*.

(a) Prove that $[(-1)^F, H] = 0 = \{(-1)^F, Q\}$

The first follows immediately from $[H, F] = 0$

And finally the object known as the *Witten index*

$$\mathrm{Tr}\ [(-1)^F e^{-\beta H}] \tag{3}$$

The object above is interesting as it only depends on the space of groundstates which is independent of β and is invariant[2] under deforming $W(x) \to \lambda W(x)$

(b) Show that (3) is equal to the number of bosonic groundstates minus the number of fermionic groundstates.

$\mathrm{Tr}\ [(-1)^F e^{-\beta H}] = \sum_n \langle n|(-1)^F e^{-\beta E_n}|n\rangle = n_B - n_F + n_{E_1}(e^{-\beta E_1} - e^{-\beta E_1}) + \cdots$

$= n_B - n_F$ where the pairing for states $E_n > 0$ was implied by the degeneracy

So for non-zero Witten index there must be some zero modes annihilated by the supercharges. This implies SUSY is not *spontaneously broken*.

4. **The Harmonic Oscillator Redux**

Consider $W(x) = \frac{\omega}{\sqrt{2}}x$ in the above problem.

(a) Write the Hamiltonian. What are $V(x)$ and $B(x)$?

$V(x) = \frac{\omega^2}{2}x^2$ and $B(x) = -\frac{\omega}{2}$

(b) What's the groundstate wavefunction? How does it depend on sign$[\omega]$? What's the spectrum of the Hamiltonian?

By above the spatial piece goes like $\psi(x) = e^{-\frac{|\omega|}{2}x^2}$ where sign$[\omega]$ determines if the spin is up or down.

The harmonic oscillator part of the Hamiltonian and the spin portion commute.

$H_{osc}|n\rangle = |\omega|(n + \frac{1}{2})|n\rangle$ and $H_{spin}|0/1\rangle = \pm\frac{\omega}{2}|0/1\rangle$

So $E_n = n|\omega|$

[2]In this sense it is topological.

(c) Calculate the Witten index (3) as well as the partition function $Z(\beta) \equiv \text{Tr } e^{-\beta \hat{H}}$

The Hilbert space factorizes as $\mathcal{H} = (L^2 \otimes |0\rangle) \oplus (L^2 \otimes |1\rangle) = L^2 \otimes \mathbb{C}^2$

Because the pieces of the Hamiltonian commute and the Hilbert space factorizes this way we can evaluate:

$\text{Tr } e^{-\beta \hat{H}} = \text{Tr }_{L^2} e^{-\beta H_{osc}} * \text{Tr }_{\mathbb{C}^2} e^{-\beta H_{spin}} = (\sum_n e^{-\beta\omega(n+\frac{1}{2})}) * (e^{-\beta\frac{\omega}{2}} + e^{\beta\frac{\omega}{2}})$

Therefore $Z(\beta) = \coth(\frac{\beta|\omega|}{2})$

Similarly for the Witten index notice that $(-1)^F$ only acts on $\mathbb{C}^2$ so:

$I_W = (\sum_n e^{-\beta\omega(n+\frac{1}{2})}) * (e^{-\beta\frac{\omega}{2}} - e^{\beta\frac{\omega}{2}}) = \frac{\omega}{|\omega|} = \pm 1$

There's a lot more to this story, including some very beautiful mathematics, but we'll have to pause here without the technology of the path integral.