Quantum Mechanics A (Physics 212A) Fall 2018
Worksheet 6 – Solutions

Problems

1. **Continuous Search**

   Grover's algorithm is a quantum algorithm which allows one to search and identify an element from unstructured list of N-many classical bit strings in $\mathcal{O}(\sqrt{N})$ time.

   The best classical algorithm for doing this, because the list is assumed unstructured, runs in $\mathcal{O}(N)$ time; checking all N elements. This is a quadratic quantum speed up.

   Let's consider a version of the problem in continuous time, as opposed to discrete quantum logic gates.

   Define the computational basis of a single qubit to be $\{|0\rangle, |1\rangle\}$, eigenstates of Z, and let us suppose we have n-many qubits living in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$.

   Let $|s\rangle$ be a computational basis element[1] and $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$ be an equal superposition of all such elements.

   (a) How many different classical bit strings do n-many qubits encode, assuming measurements in the computational basis?

   2^n many states of the form $|s_1, s_2, \cdots, s_n\rangle$ where $s_i \in \{0, 1\}$ so $N = 2^n$

   (b) Describe a simple way of preparing the "register" state $|\psi\rangle$, assuming creating eigenstates of Pauli matrices is easy to do for each qubit.

   Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, this is an X eigenstate. It's simple to see that a tensor product of these states gives the register state: $|\psi\rangle = |+\rangle_1 |+\rangle_2 \cdots |+\rangle_n$

   (c) What is the dimension of the subspace that $|s\rangle$ and $|\psi\rangle$ span?

   Note they aren't necessarily orthogonal.

   Write an orthonormal basis for this space. (Hint: Construct a state which is composed of $|s\rangle$ and $|\psi\rangle$ but is orthogonal to $|s\rangle$)

   The dimension is 2. We can use the basis $\{|s\rangle, |\psi'\rangle\}$ where $|\psi'\rangle$ is a superposition of $|s\rangle$ and $|\psi\rangle$ which is orthogonal to the former.

   $|\psi'\rangle = a|s\rangle + b|\psi\rangle$ where normalization demands: $|a|^2 + |b|^2 + a^* b\langle s|\psi\rangle + ab^*\langle\psi|s\rangle = 1$ and orthogonality requires $\langle s|\psi'\rangle = a + b\langle s|\psi\rangle = 0$

   By construction $\langle s|\psi\rangle = \frac{1}{\sqrt{2^n}} \to |\psi'\rangle = -\frac{b}{\sqrt{2^n}}|s\rangle + b|\psi\rangle$ where we can choose both coefficients to be real by global phase invariance.

   Our normalization condition is then $\frac{b^2}{2^n} + b^2 - \frac{2b^2}{2^n} = 1 \to b = \sqrt{\frac{2^n}{2^n-1}}$

   So finally we write $|\psi'\rangle = -\frac{1}{\sqrt{2^n-1}}|s\rangle + \sqrt{\frac{2^n}{2^n-1}}|\psi\rangle$

[1] So s would be some n-digit string of 0's and 1's

(d) Suppose our Hamiltonian is $\hat{H} = |s\rangle\langle s| + |\psi\rangle\langle\psi|$. Rewrite $\hat{H}$ using the basis above.

$|\psi\rangle\langle\psi| = \frac{1}{2^n}(|s\rangle + \sqrt{2^n-1}|\psi'\rangle)(\langle s| + \sqrt{2^n-1}\langle\psi'|)$

$= \frac{1}{2^n}[|s\rangle\langle s| + \sqrt{2^n-1}|s\rangle\langle\psi'| + \sqrt{2^n-1}|\psi'\rangle\langle s| + (2^n-1)|\psi'\rangle\langle\psi'|]$

$\to \hat{H} = \frac{1}{2^n}[|s\rangle\langle s| + \frac{1}{2^n}|s\rangle\langle s| + \sqrt{2^n-1}|s\rangle\langle\psi'| + \sqrt{2^n-1}|\psi'\rangle\langle s| + (2^n-1)|\psi'\rangle\langle\psi'|]$

(e) In this basis write $\hat{H}$ as a matrix and expand in terms of Pauli's. What is the form of $U = e^{-\mathbf{i}\hat{H}t}$ on this subspace? Recall $e^{-\mathbf{i}\theta\vec{\sigma}\cdot\hat{n}} = \cos\theta\mathbb{1} - \mathbf{i}\sin\theta(\vec{\sigma}\cdot\hat{n})$

Define $\mathbf{Z} = |s\rangle\langle s| - |\psi'\rangle\langle\psi'|$ and $\mathbf{X} = |s\rangle\langle\psi'| + |\psi'\rangle\langle s|$

$$\hat{H} = \begin{pmatrix} 1+\frac{1}{2^n} & \frac{\sqrt{2^n-1}}{2^n} \\ \frac{\sqrt{2^n-1}}{2^n} & 1-\frac{1}{2^n} \end{pmatrix} = \mathbb{1} + \frac{1}{2^n}\mathbf{Z} + \frac{\sqrt{2^n-1}}{2^n}\mathbf{X}$$

The identity portion of $\hat{H}$ produces a global phase which we can ignore. The form of $\vec{n} = \left(\frac{\sqrt{2^n-1}}{2^n}, 0, \frac{1}{2^n}\right) \to \theta = |\vec{n}| = \frac{1}{\sqrt{2^n}}$

Thus we write $U = \cos\frac{t}{\sqrt{2^n}}\mathbb{1} - \mathbf{i}\sin\frac{t}{\sqrt{2^n}}[\sqrt{\frac{2^n-1}{2^n}}\mathbf{X} + \frac{1}{\sqrt{2^n}}\mathbf{Z}]$

(f) Suppose we initialize our quantum computer to $|\psi\rangle$ and evolve by U for a time $t = T$ at which we then measure the state in the computational basis.

What is the probability for measuring $|s\rangle$ at T? At what time should be measure to maximize this probability?

(Hint: You will need to evaluate $U|\psi\rangle$ which you have in terms of Pauli's which act as they normally do on the basis vectors for the subspace we've been considering.)

$U|\psi\rangle = U\frac{1}{\sqrt{2^n}}(|s\rangle + \sqrt{2^n-1}|\psi'\rangle) = \frac{\cos\frac{t}{\sqrt{2^n}}}{\sqrt{2^n}}(|s\rangle + \sqrt{2^n-1}|\psi'\rangle) + \cdots$

$\cdots - \frac{\mathbf{i}}{\sqrt{2^n}}\sin\frac{t}{\sqrt{2^n}}[\sqrt{\frac{2^n-1}{2^n}}\mathbf{X}|s\rangle + \sqrt{\frac{2^n-1}{2^n}}\sqrt{2^n-1}\mathbf{X}|\psi'\rangle + \frac{1}{\sqrt{2^n}}\mathbf{Z}|s\rangle + \frac{1}{\sqrt{2^n}}\sqrt{2^n-1}\mathbf{Z}|\psi'\rangle]$

Note $\mathbf{X}|s\rangle = |\psi'\rangle$, $\mathbf{X}|\psi'\rangle = |s\rangle$, $\mathbf{Z}|s\rangle = |s\rangle$, and $\mathbf{Z}|\psi'\rangle = -|\psi'\rangle$ so using these and orthogonality I'm going to compute $\langle s|U|\psi\rangle =$

$= \frac{\cos\frac{t}{\sqrt{2^n}}}{\sqrt{2^n}} - \frac{\mathbf{i}}{\sqrt{2^n}}\sin\frac{t}{\sqrt{2^n}}[\sqrt{\frac{2^n-1}{2^n}}\sqrt{2^n-1} + \frac{1}{\sqrt{2^n}}] = \frac{1}{\sqrt{2^n}}(\cos\frac{t}{\sqrt{2^n}} - \mathbf{i}\sqrt{2^n}\sin\frac{t}{\sqrt{2^n}})$

So the probability at $t = T$ is $P_T = \frac{1}{2^n}\cos^2\frac{T}{\sqrt{2^n}} + \sin^2\frac{T}{\sqrt{2^n}}$

This we can differentiate with respect to T to extremum occur at $\sin\frac{2T}{\sqrt{2^n}} = 0$ which implies one should measure at $T = \sqrt{2^n}\frac{\pi}{2}(2k+1)$ for $k \in \mathbb{Z}$

(g) The run-time scaling of the algorithm is then just how T scales with the number of bit-strings when the probability is close to 1. Did you get the promised result?

We get that $T \sim \sqrt{2^n} \sim \mathcal{O}(\sqrt{N})$ as promised! This is a probabilistic algorithm and the exact result depends on how precisely you know when to measure and how good your Hamiltonian preparation is. It also relies on the usual ability to keep these qubits from error/decoherence. These are all very practical but important challenges!