130B Quantum Physics

Part 4. Phase and Gauge

Gauge Principles

■ Gauge Structure and Berry Phase

■ Phase Ambiguities

At its core, quantum mechanics is a **probability theory**. It postulate to *model* the probability density $p(\boldsymbol{x})$ by a *squared norm*

$$p(\boldsymbol{x}) = |\psi(\boldsymbol{x})|^2, \tag{1}$$

just to ensure the *positive semi-definite* property $p(\boldsymbol{x}) \geq 0$.

The wavefunction $\psi(\boldsymbol{x})$ itself serves as a **mathematical parameter** of the probability model, *not* a **physical observable**, and is therefore subject to some degree of ambiguity or redundancy.

- **Global phase ambiguity**. A *global* phase rotation of the wavefunction (where "global" means α does not depend on $\boldsymbol{x}$)

$$\boxed{\psi(\boldsymbol{x}) \to e^{i\,\alpha}\,\psi(\boldsymbol{x})} \tag{2}$$

has no consequence on the expectation value $\langle O \rangle$ of any physical observable O in any case

$$\langle O \rangle = \int \psi^*(\boldsymbol{x})\, O(\boldsymbol{x}, \boldsymbol{x}')\, \psi(\boldsymbol{x}')\, d^D\boldsymbol{x}\, d^D\boldsymbol{x}'. \tag{3}$$

Conclusion: quantum states $\in$ **projective Hilbert space**, where global phase is always *unphysical*.

- **Local phase ambiguity** (**Gauge redundancy**). We can push this idea further: if we restrict ourself to *diagonal* observables in the *position* basis, i.e., functions $f(\boldsymbol{x})$ that depends only on $\boldsymbol{x}$ (but not $\boldsymbol{p}$), then any *local* phase rotation

$$\boxed{\psi(\boldsymbol{x}) \to e^{i\,\chi(\boldsymbol{x})}\,\psi(\boldsymbol{x})} \tag{4}$$

will leave all expectation values $\langle f(\boldsymbol{x}) \rangle$ invariant,

$$\langle f(\boldsymbol{x}) \rangle := \int f(\boldsymbol{x})\, |\psi(\boldsymbol{x})|^2\, d^D\boldsymbol{x} = \int f(\boldsymbol{x})\, p(\boldsymbol{x})\, d^D\boldsymbol{x} \tag{5}$$

since the probability density $p(\boldsymbol{x})$ is unchanged.

- If $p(\boldsymbol{x}) = |\psi(\boldsymbol{x})|^2$ was the only *physical* probability distribution to be modeled, any $\psi(\boldsymbol{x})$ related by *local* phase rotation Eq. (4) should be treated as *equivalent*.
- This represents a **gauge redundancy**: multiple *mathematical descriptions* (e.g. wavefunctions) describing the same *physical reality* (e.g. position distribution).

Quantum decoherence provides a deeper reason of why only *diagonal* observables are measurable.

- **Environmental monitoring**: The environment has a natural tendency to monitor the **local density** $p(\boldsymbol{x})$ of particles in the space.
  - Effectively performing *weak continuous* **measurement** of $\boldsymbol{x}$.
  - Inducing **decoherence** in the position basis: rapid decay of *off-diagonal* coherence $\rho(\boldsymbol{x}, \boldsymbol{x}')$ of the density matrix for $\boldsymbol{x} \neq \boldsymbol{x}'$.
- Consequence 1: **Dephasing noise**.
  - The measurement *randomizes* the **phase** of $\psi(\boldsymbol{x})$ at every position *independently*.
  - Relative phase between $\psi(\boldsymbol{x})$ and $\psi(\boldsymbol{x}')$ no longer comparable, allowing *local* phase rotation as Eq. (4) to become a *redundancy*.
- Consequence 2: **Gauge projection**.
  - The measurement *collapse* the system towards the **particle number** eigenstates, suppressing the *number fluctuations*.
  - Effectively imposing a *gauge constraint* (like Gauss law), that couples the particle to an *emergent* **gauge field**, allowing particles to interact with each other through *emergent* **gauge forces**.

Gauge Transformation

If the gauge freedom is an (emergent) *redundancy*, it should have no *physical consequence*. For example, it should not affect the **quantum dynamics** governed by the **Schrödinger equation**.

However, the standard Schrödinger equation is *not* invariant under the **gauge transformation**

$$\psi(\boldsymbol{x}, t) \to e^{i\,\chi(\boldsymbol{x},t)}\,\psi(\boldsymbol{x}, t), \tag{6}$$

unless we modify it appropriately.

- Start from the free Schrödinger equation

$$i\,\hbar\,\partial_t \psi(\boldsymbol{x}, t) = -\frac{\hbar^2}{2\,m}\,\nabla^2 \psi(\boldsymbol{x}, t), \tag{7}$$

under gauge transformation $\psi \to e^{i\,\chi}\,\psi$ in Eq. (6), the derivative operators picks up extra terms involving $\partial_t \chi$ and $\nabla \chi$,

$$i\hbar\,(\partial_t + i\,\partial_t\chi(\boldsymbol{x},t))\,\psi(\boldsymbol{x},t) = -\frac{\hbar^2}{2\,m}\,(\nabla + i\,\nabla\chi(\boldsymbol{x},t))^2\,\psi(\boldsymbol{x},t), \tag{8}$$

and the equation does *not* remain invariant.

Exc 1 Show that Eq. (7) becomes to Eq. (8) under gauge transformation.

- To restore the **gauge invariance**, we introduce **gauge fields**:
  - **Scalar potential**: $\Phi(\boldsymbol{x},t)$ - a scalar field in the spacetime
  - **Vector potential**: $\boldsymbol{A}(\boldsymbol{x},t)$ - a vector field in the spacetime

  and replace derivatives by **covariant derivatives**:

$$\boxed{\begin{aligned}&\partial_t \to D_t := \partial_t + \frac{i}{\hbar}\,\Phi(\boldsymbol{x},t),\\ &\nabla \to \boldsymbol{D} := \nabla - \frac{i}{\hbar}\,\boldsymbol{A}(\boldsymbol{x},t),\end{aligned}} \tag{10}$$

Then Eq. (7) can be recast into the **gauge-invariant Schrödinger equation**

$$\boxed{i\hbar\,D_t\,\psi(\boldsymbol{x},t) = \frac{1}{2\,m}\,(-i\,\hbar\,\boldsymbol{D})^2\,\psi(\boldsymbol{x},t),} \tag{11}$$

or more explicitly,

$$i\hbar\,\partial_t\psi(\boldsymbol{x},t) = \left(\frac{1}{2\,m}\,(-i\,\hbar\,\nabla - \boldsymbol{A}(\boldsymbol{x},t))^2 + \Phi(\boldsymbol{x},t)\right)\psi(\boldsymbol{x},t). \tag{12}$$

Under **gauge transformation**, the wavefunction ψ and the gauge fields $(\Phi, \boldsymbol{A})$ must transform together as

$$\boxed{\begin{aligned}&\psi(\boldsymbol{x},t) \to e^{i\,\chi(\boldsymbol{x},t)}\,\psi(\boldsymbol{x},t),\\ &\Phi(\boldsymbol{x},t) \to \Phi(\boldsymbol{x},t) - \hbar\,\partial_t\chi(\boldsymbol{x},t),\\ &\boldsymbol{A}(\boldsymbol{x},t) \to \boldsymbol{A}(\boldsymbol{x},t) + \hbar\,\nabla\chi(x,t),\end{aligned}} \tag{13}$$

to ensure the covariance of the quantum dynamics.

Semiclassical Interpretation

In the WKB approximation, we write the wavefunction as

$$\psi = A\,e^{i\,S/\hbar}. \tag{14}$$

Plugging the WKB ansatz into the gauge-invariant Schrödinger equation Eq. (11) yields:

$$(-\partial_t S - \Phi) = \frac{1}{2\,m}\,(\nabla S - \boldsymbol{A})^2. \tag{15}$$

Given that the spacetime derivatives of the action S is associated to **energy** $E = -\partial_t S$ and

momentum $\boldsymbol{p} = \nabla S$, Eq. (15) can be written as

$$\boxed{(E - \Phi) = \frac{1}{2\,m}\,(\boldsymbol{p} - \boldsymbol{A})^2.} \tag{16}$$

This reveals the physical meaning of gauge fields:

- Scalar potential Φ: **potential energy**,
- Vector potential $\boldsymbol{A}$: **potential momentum**.

Note: Energy and Momentum each have *three* distinct forms

$$\begin{array}{c|ccccc} & \textbf{Total} & = & \textbf{Kinetic} & + & \textbf{Potential} \\ \hline \text{Energy} & E & = & \frac{1}{2}\,m\,\dot{\boldsymbol{x}}^2 & + & \Phi \\ \text{Momentum} & \boldsymbol{p} & = & m\,\dot{\boldsymbol{x}} & + & \boldsymbol{A} \end{array} \tag{17}$$

Exc 2 Show that both equations in Eq. (17) are consistent with Eq. (16).

- **Total** (or **Canonical**): appear directly in *conservation laws* and determine the *action* accumulated in spacetime.
- **Kinetic**: directly linked to the particle's *motion* (velocity $\dot{\boldsymbol{x}}$).
- **Potential**: exist independently of particle motion, contributing even when the particle is *at rest* ($\dot{\boldsymbol{x}} = 0$), representing the *interaction* with the *background field* in the spacetime.

Question: What are their dynamical consequences?

Newton's 2nd law — the *force* $\boldsymbol{F}$ causes the *kinetic momentum* ($m\,\dot{\boldsymbol{x}}$) to change in time:

$$\boldsymbol{F} = \frac{d}{d\,t}(m\,\dot{\boldsymbol{x}}) = \frac{d\,\boldsymbol{p}}{d\,t} - \frac{d\,\boldsymbol{A}}{d\,t}, \tag{18}$$

in two distinct ways:

- $d\,\boldsymbol{p}/d\,t = \partial_t\,\boldsymbol{p} + \dot{\boldsymbol{x}}\cdot\nabla\,\boldsymbol{p}$, in which
  - $\nabla\,\boldsymbol{p} = 0$, as $\boldsymbol{x}$ and $\boldsymbol{p}$ are independent variables,
  - **Maxwell relation**: $\partial_t \nabla S = \nabla \partial_t S$ implies

    $$\partial_t\,\boldsymbol{p} = -\nabla\,E = (\nabla\,\boldsymbol{A})\cdot\dot{\boldsymbol{x}} - \nabla\,\Phi. \tag{19}$$
- $d\,\boldsymbol{A}/d\,t = \partial_t\,\boldsymbol{A} + \dot{\boldsymbol{x}}\cdot\nabla\,\boldsymbol{A}$.

Put together, $\boldsymbol{F}$ takes the form as the **Lorentz force** (on a charge $q = 1$ particle)

$$\boxed{\begin{aligned} \boldsymbol{F} &= -\nabla\,\Phi - \partial_t\,\boldsymbol{A} + (\nabla\,\boldsymbol{A})\cdot\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}\cdot\nabla\,\boldsymbol{A} \\ &= \boldsymbol{E} + \dot{\boldsymbol{x}}\times\boldsymbol{B}, \end{aligned}} \tag{20}$$

as long as we define

$$\boxed{\begin{aligned} \boldsymbol{E} &= -\nabla\Phi - \partial_t \boldsymbol{A}, \\ \boldsymbol{B} &= \nabla\times\boldsymbol{A}. \end{aligned}} \tag{21}$$

Exc 3 Justify Eq. (19) and Eq. (20).

Obviously, $\boldsymbol{E}$ and $\boldsymbol{B}$ should be interpreted as **electric** and **magnetic** fields, allowing $\boldsymbol{F}$ to be consistently identified as the force exerted by the *electromagnetic field* on a *charged* particle.

Conclusion: **Gauge fields** $(\Phi, \boldsymbol{A})$ are not merely mathematical constructs to maintain *gauge invariance*; they give rise to the physical *electromagnetic interactions* among quantum particles. Remarkably, the **electromagnetic forces** familiar from our everyday *classical* experience emerge profoundly from the **local phase ambiguity** of matter at the *quantum* level.

Math Interlude: Lorentz Vectors

It is more convenient to unify time and space, as well as energy and momentum.

- **Spacetime**: Introduce $x = (x^0, x^1, x^2, x^3)$ to denote the coordinate of a spacetime point,
  - $t = x^0$: time,
  - $\boldsymbol{x} = (x^1, x^2, x^3)$: space,

  and denote these components as $x^\mu (\mu = 0, 1, 2, 3)$ jointly. x is said to be a **Lorentz vector**.
- **Spacetime derivatives**: Partial derivatives in the spacetime are defined as

$$\partial_\mu f(x) = \lim_{\delta x\to 0} \frac{f(x+\delta x) - f(x)}{\delta x^\mu}, \tag{22}$$

  - $\partial_t = \partial_0$: temporal derivative,
  - $\nabla = (\partial_1, \partial_2, \partial_3)$: spatial derivatives.
- **Energy-momentum**: Introduce $p = (p^0, p^1, p^2, p^3)$ to denote the energy and momentum,
  - $E = p^0 = -p_0$: energy,
  - $\boldsymbol{p} = (p^1, p^2, p^3) = (p_1, p_2, p_3)$: momentum.
- **Gauge field**: Introduce $A = (A^0, A^1, A^2, A^3)$ to denote the gauge field,
  - $\Phi = A^0 = -A_0$: scalar potential,
  - $\boldsymbol{A} = (A^1, A^2, A^3) = (A_1, A_2, A_3)$: vector potential.
- **Covariant derivatives**: Operators in Eq. (10) can now be unified as a single Lorentz vector operator

$$D_\mu = \partial_\mu - \frac{i}{\hbar} A_\mu, \tag{23}$$

  - $D_t = D_0$: covariant temporal derivative,

- $\boldsymbol{D} = (D_1, D_2, D_3)$: covariant spatial derivative.

Raising and **lowering** the index of a Lorentz vector is done by the **Lorentz metric** $g_{\mu\nu}$ or $g^{\mu\nu}$,

$$\boxed{a_\mu = g_{\mu\nu}\, a^\nu,\ a^\mu = g^{\mu\nu}\, a_\nu,} \tag{24}$$

where *repeated* indices are automatically summed over (**contracted**) following the **Einstein sum rule**. The Lorentz metric is given by

$$\boxed{g_{\mu\nu} = g^{\mu\nu} = \operatorname{diag}(-1, +1, +1, +1).} \tag{25}$$

Rule of thumb: In **index contraction**, the *upper* index can only contract with the *lower* index and vice versa.

$a^\mu\, b_\mu$ ✓ ok
$a^\mu\, b^\mu$ × no!
$a_\mu\, b_\mu$ × no!

More explicitly, the following expressions are all valid and equal

$$\begin{aligned} a^\mu\, b_\mu &= a^0\, b_0 + a^1\, b_1 + a^2\, b_2 + a^3\, b_3 \\ &= a^\mu\, g_{\mu\nu}\, b^\nu = -a^0\, b^0 + a^1\, b^1 + a^2\, b^2 + a^3\, b^3 \\ &= a_\mu\, g^{\mu\nu}\, b_\nu = -a_0\, b_0 + a_1\, b_1 + a_2\, b_2 + a_3\, b_3. \end{aligned} \tag{26}$$

Berry Phase

Berry phase is the *phase* accumulated by the wavefunction as a particle travels through the spacetime *adiabatically*.

- A processes is said to be **adiabatic**, if it happens *slowly* over a long time, i.e. the rates of change in physical observables tend to zero. In terms of the motion of a particle, it means the *velocity* of the particle is *almost zero* throughout the process:

  $$\dot{\boldsymbol{x}} \to 0. \tag{27}$$

  Eq. (27) is also called the *adiabatic limit*.

- In the adiabatic limit, the **action** of the particle is accumulated by the **potential** energy and momentum

  $$\begin{aligned} -\partial_t S &= E = \Phi, \\ \nabla S &= \boldsymbol{p} = \boldsymbol{A}. \end{aligned} \tag{28}$$

  as the *kinetic* energy and momentum *vanishes* when $\dot{\boldsymbol{x}} \to 0$.

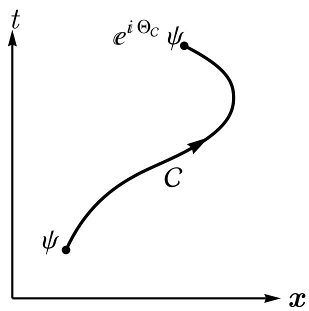
- Given that phase is related to action, the **Berry phase** that a particle accumulates along a spacetime trajectory C is given by the **path integral** that computes the accumulated action

  $$\Theta_C = \frac{S_C}{\hbar} = \frac{1}{\hbar} \int_C -\Phi\, d\, t + \boldsymbol{A} \cdot d\, \boldsymbol{x} = \frac{1}{\hbar} \int_C A_\mu\, d\, x^\mu. \tag{29}$$

Exc 4 Justify Eq. (29).

Such that *adiabatically propagating* the **wave amplitude** ψ along trajectory C will acquire the **Berry phase** shift:

$$\boxed{\psi \overset{C}{\to} e^{i\,\Theta_C}\,\psi = \exp\!\left(\frac{i}{\hbar}\int_C A_\mu\, d\,x^\mu\right)\psi.} \tag{30}$$



- For infinitesimal transportation $C: x \to x + \delta\,x$,

$$\boxed{\psi \to e^{\frac{i}{\hbar}A_\mu\,\delta x^\mu}\,\psi.} \tag{31}$$

- Under gauge transformation

$$A_\mu(x) \to A_\mu(x) + \hbar\,\partial_\mu\,\chi(x), \tag{32}$$

  - the Berry phase along an **open trajectory** C is *not* gauge invariant (hence not a physical observable):

  $$\Theta_C \to \Theta_C + \chi(x_{\text{end}}) - \chi(x_{\text{start}}), \tag{33}$$

  where x_{start}, $x_{\text{end}} = \partial C$ are the starting and ending point of C.

  - the Berry phase around a **closed loop** (Wilson loop) is gauge invariant, and is a physical observable.

▪ Gauge Field and Electromagnetism

▪ Gauge Connection

Previously, we have introduced the *covariant derivative* D_μ to make the Schrödinger equation *gauge invariant*, but is there a deeper motivation behind this? In calculus, the **derivative** of a function tells us how much the function changes between nearby points

$$\partial_x f(x) = \lim_{\delta x \to 0} \frac{f\,(x + \delta\,x) - f(x)}{\delta\,x}. \tag{34}$$

But this assumes we can *compare* the values of the function at different points directly.

- Problem: If the wavefunction $\psi(x)$ has **local phase ambiguity**,

$$\psi(x) \to e^{i\,\chi(x)}\,\psi(x), \tag{35}$$

  meaning that $\psi(x)$ at each point x is defined *up to a phase rotation*, there will be *no basis to compare* wavefunctions between *distinct points*.

  ☞ There is a similar issue in **finance**: you cannot directly compare currencies from different countries by their face values!

  - Are the following amounts of money the same?

¥100 $100 €100

  - You should first *move* (parallel transport) the money to the same place before comparing.

$108 ⟵ €100
$100
¥100 ⟶ $13.3 03/27/2025

    During the conversion, the money will be multiplied by the *exchange rate* (exponential gauge connection), e.g.

$$e^{0.076961}\,€100 = \$108,\;\; e^{-2.01741}\,¥100 = \$13.3. \tag{36}$$

- Solution: Similarly, to define a meaningful derivative for the wavefunction, we have to introduce a **gauge connection** $A_\mu(x)$ to keep track of the *phase rotation* needed to transport $\psi(x+\delta x)$ back to the point x for comparison, for every point x along any direction μ.

This allows us to (re)define the **covariant derivative**:

$$D_\mu\,\psi(x) = \lim_{\delta x \to 0} \frac{e^{-\frac{i}{\hbar}\,A_\nu(x)\,\delta x^\nu}\,\psi(x+\delta x) - \psi(x)}{\delta x^\mu}. \tag{37}$$

- Interpretation: $\psi(x+\delta x)$ has accumulated a Berry phase of $\exp\left(\frac{i}{\hbar}\,A_\nu(x)\,\delta x^\nu\right)$ compare to $\psi(x)$ as the particle travels adiabatically. So when pulling $\psi(x+\delta x)$ back to the point x, this phase should be compensated by the opposite phase factor $\exp\left(-\frac{i}{\hbar}\,A_\nu(x)\,\delta x^\nu\right)$ before comparing.

- Expressed in terms of the usual partial derivative modified by the gauge connection,

$$\boxed{D_\mu = \partial_\mu - \frac{i}{\hbar}\,A_\mu(x),} \tag{38}$$

  which exactly reproduces Eq. (23).

Exc 5 Show Eq. (38) follows from Eq. (37) by taking the limit.

- The covariant derivative commute with the **gauge transformation**

$$\boxed{\begin{aligned}&\psi(x) \to e^{i\,\chi(x)}\,\psi(x),\\ &A_\mu(x) \to A_\mu(x) + \hbar\,\partial_\mu\,\chi(x),\end{aligned}} \tag{39}$$

as adapted from Eq. (13).

- The **apparent** deformation of the wavefunction ψ from x to $x + \delta x$ can be expressed as

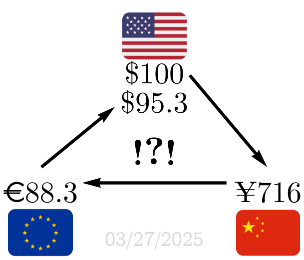
$$\psi(x + \delta x) = e^{\frac{i}{\hbar}\,A_\mu(x)\,\delta x^\mu}\, e^{\delta x^\mu\,D_\mu}\,\psi(x), \tag{40}$$

  which contains two contributions

  - the **intrinsic** deformation under **parallel transport** $\psi \to e^{\delta x^\mu\,D_\mu}\,\psi$, which is *generated* by the *covariant derivative* D_μ,
  - the **background** deformation in terms of the Berry phase $\psi \to e^{(i/\hbar)\,A_\mu(x)\,\delta x^\mu}\,\psi$ accumulated along the *gauge connection* A_μ.

■ Gauge Curvature

Exchanging currencies in cycles typically results in a loss. Why?



Because the global foreign exchange market is not *flat* — the mismatch around a *closed loop* is a measure of **curvature**.

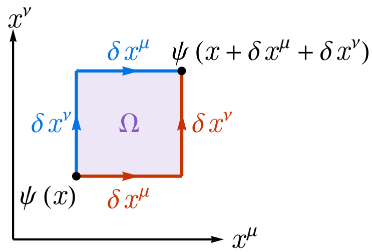
In gauge theory, the **gauge curvature** $F_{\mu\nu}$ measures the adiabatic *action* accumulated *per area* when transporting the wavefunction around the *area boundary* in spacetime.

$$S = \oint_{\partial\Omega} A_\mu\,d\,x^\mu = \int_\Omega F_{\mu\nu}\,d\,x^\mu\,d\,x^\nu. \tag{41}$$

- Operational definition: Mathematically, the gauge curvature $F_{\mu\nu}$ is defined by the *commutator* of *covariant derivatives*

$$\boxed{[D_\mu,\, D_\nu]\,\psi = -\frac{i}{\hbar}\,F_{\mu\nu}\,\psi,} \tag{42}$$

  which measures the amount of non-commutativity to transport the wavefunction along two distinct spacetime directions μ and ν.



Exc 6 Using Eq. (40), prove Eq. (42) by comparing the above two paths to transport $\psi(x)$ to $\psi(x + \delta x^\mu + \delta x^\nu)$.

- Physical meaning: In electromagnetism, $F_{\mu\nu}$ corresponds to the **electromagnetic field strength tensor**,

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.} \tag{43}$$

Exc 7 Derive Eq. (43) from Eq. (42).

- **Electric field**: $\boldsymbol{E} = (E^1, E^2, E^3)$, with $E^i := -F_{0\,i}$.

$$\boxed{\boldsymbol{E} = -\nabla\Phi - \partial_t \boldsymbol{A}.} \tag{44}$$

- **Magnetic field**: $\boldsymbol{B} = (B^1, B^2, B^3)$, with $B^i := \frac{1}{2}\epsilon^{ijk} F_{jk}$.

$$\boxed{\boldsymbol{B} = \nabla \times \boldsymbol{A}.} \tag{45}$$

Exc 8 Check that $\boldsymbol{E}$ and $\boldsymbol{B}$ are gauge invariant. Therefore, they are physical observables.

Charged Particle in Gauge Field

In quantum mechanics, the time-evolution is generated by the Hamiltonian operator $\hat{H}$.

- **Schrödinger picture**: *state* evolves in time, operator remains fixed.

$$i\hbar\,\partial_t\psi = \hat{H}\,\psi. \tag{46}$$

- **Heisenberg picture**: *operator* evolves in time, state remains fixed.

$$i\hbar\,\partial_t\,\hat{O} = \left[\hat{O}, \hat{H}\right]. \tag{47}$$

Note: Eq. (47) assumes $\hat{O}$ has no explicit time dependence, if not, its *explicit time derivative* will also contribute to the rate of change of $\hat{O}$.

Compare Eq. (46) with Eq. (12), we conclude that the Hamiltonian of the **gauge-invariant Schrödinger equation** is

$$\boxed{\hat{H} = -\frac{\hbar^2}{2\,m}\,\boldsymbol{D}^2 + \Phi,} \tag{48}$$

or more explicitly as

$$\hat{H} = H(\hat{\boldsymbol{x}},\, \hat{\boldsymbol{p}},\, t) = \frac{1}{2\,m}\,(\hat{\boldsymbol{p}} - \boldsymbol{A}(\hat{\boldsymbol{x}},\, t))^2 + \Phi(\hat{\boldsymbol{x}},\, t), \tag{49}$$

where

- $\hat{\boldsymbol{x}}$ is the **coordinate operator**.
- $\hat{\boldsymbol{p}} = -i\,\hbar\,\nabla$ is the **momentum operator**.
- They satisfy the **canonical commutation relation**

$$\boxed{\left[\hat{x}_i,\, \hat{p}_j\right] = i\,\hbar\,\delta_{ij}\,\mathbb{1}.} \tag{50}$$

Exc 9 Verify Eq. (50).

- $\hat{\Phi} = \Phi(\hat{\boldsymbol{x}},\, t)$ and $\hat{\boldsymbol{A}} = \boldsymbol{A}(\hat{\boldsymbol{x}},\, t)$ are **operator functions** of $\hat{\boldsymbol{x}}$, with *explicit* time t dependence.

Using the Heisenberg equation Eq. (47), we can compute time derivatives of the particle position operator $\hat{\boldsymbol{x}}$

- 1st order (velocity operator)

$$\partial_t\,\hat{\boldsymbol{x}} = \frac{1}{i\,\hbar}\left[\hat{\boldsymbol{x}},\, \hat{H}\right] = \frac{\hat{\boldsymbol{p}} - \hat{\boldsymbol{A}}}{m}. \tag{51}$$

- 2nd order (acceleration operator)

$$\begin{aligned} \partial_t^2\,\hat{\boldsymbol{x}} &= -\frac{1}{m}\,\partial_t\,\hat{\boldsymbol{A}} + \frac{1}{i\,\hbar}\left[\partial_t\,\hat{\boldsymbol{x}},\, \hat{H}\right] \\ &= \frac{1}{m}\left(\hat{\boldsymbol{E}} + \frac{1}{2}\left(\partial_t\,\hat{\boldsymbol{x}} \times \hat{\boldsymbol{B}} - \hat{\boldsymbol{B}} \times \partial_t\,\hat{\boldsymbol{x}}\right)\right), \end{aligned} \tag{52}$$

  where $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ operators are defined by

$$\begin{aligned} \hat{\boldsymbol{E}} &= -\nabla\hat{\Phi} - \partial_t\,\hat{\boldsymbol{A}}, \\ \hat{\boldsymbol{B}} &= \nabla \times \hat{\boldsymbol{A}}. \end{aligned} \tag{53}$$

Exc 10 Derive Eq. (51) and Eq. (52).

Eq. (52) describes the **quantum dynamics** of a *charged particle* in an *electromagnetic field* in the Heisenberg picture:

$$\boxed{m\,\partial_t^2\,\hat{\boldsymbol{x}} = \hat{\boldsymbol{E}} + \frac{1}{2}\left(\partial_t\,\hat{\boldsymbol{x}}\times\hat{\boldsymbol{B}} - \hat{\boldsymbol{B}}\times\partial_t\,\hat{\boldsymbol{x}}\right).} \tag{54}$$

In contrast, the **classical dynamics** is described by

$$m\,\ddot{\boldsymbol{x}} = \boldsymbol{F} = \boldsymbol{E} + \dot{\boldsymbol{x}}\times\boldsymbol{B}, \tag{55}$$

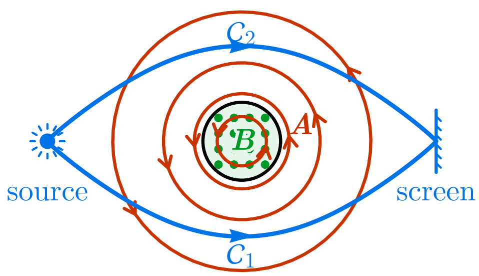
where all quantities commute. In the quantum case, however, $\partial_t\,\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{B}}$ generally do *not commute*, so their cross product must be symmetrized as in Eq. (54).

Aharonov-Bohm Effect

In quantum mechanics, the *motion* of a charged particle can be *influenced* by the **gauge fields** Φ and $\boldsymbol{A}$ through **quantum interference**, even in the *absence* of electromagnetic fields (i.e. $\boldsymbol{E} = \boldsymbol{B} = 0$) when there is no Lorentz force acting on the particle at all!

- Setup: Aharonov-Bohm Experiment
  - **Physical Arrangement**: Consider a long, thin solenoid carrying a magnetic flux ϕ_B. Outside the solenoid, the magnetic field $\boldsymbol{B}$ is zero, but the vector potential $\boldsymbol{A}$ is nonzero. For any surface $\mathcal{S}$ that fully covers the solenoid, we have

$$\phi_B = \int_{\mathcal{S}} \boldsymbol{B}\cdot d\boldsymbol{\sigma} = \oint_{\partial\mathcal{S}} \boldsymbol{A}\cdot d\boldsymbol{l}. \tag{56}$$



  - **Interferometry**: A beam of electrons is split into two paths that *encircle* the solenoid in opposite directions and then recombine to produce an *interference* pattern.
- Key idea: Even when $\boldsymbol{B} = 0$ outside the solenoid, the **vector potential $\boldsymbol{A}$** influences the *phase* of the wavefunction.
  - When an electron travels along a path C, the wavefunction acquires a **Berry phase**:

$$\psi \overset{C}{\to} \psi\,e^{i\,\Theta_C} = \psi\,\exp\!\left(\frac{i\,q}{\hbar}\int_C \boldsymbol{A}\cdot d\,\boldsymbol{x}\right), \tag{57}$$

    where $q = -e$ is recovered to represent the electron charge.
  - The **phase difference** between the two paths is

$$\Delta\Theta = \Theta_{C_1} - \Theta_{C_2} = \frac{q}{\hbar}\left(\int_{C_1} \boldsymbol{A}\cdot d\,\boldsymbol{x} - \int_{C_2} \boldsymbol{A}\cdot d\,\boldsymbol{x}\right). \tag{58}$$

  - By applying Stokes' theorem over the surface $\mathcal{S}$ enclosed by the loop $C = C_1 - C_2 = \partial\mathcal{S}$,

$$\Delta\Theta = \frac{q}{\hbar}\int_{\mathcal{S}}(\nabla\times\boldsymbol{A})\cdot d\boldsymbol{\sigma} = \frac{q\,\phi_B}{\hbar}, \tag{59}$$

where ϕ_B is the **magnetic flux** through $\mathcal{S}$ (which equals to the flux inside the solenoid as long as $\mathcal{S}$ covers the solenoid fully).

This phase shift manifests as a *shift* in the *interference fringes* when the two parts of the beam are *recombined*.

- Application: **Superconducting Quantum Interference Device** (SQUID)
  - A SQUID consists of a *superconducting loop* containing a *Josephson junction* (serving as the screen) and exploit quantum interference to detect extremely subtle changes in *magnetic flux* inside the loop.
  - By harnessing the *quantum-level sensitivity* of the Aharonov-Bohm (AB) effect, SQUIDs can measure magnetic fields as faint as 5×10^{-18}T at *microscopic* scales.
  - SQUIDs also play a pivotal role in *quantum computing*, as an approach towards *superconducting qubits*.
- Question: What Is Physical about Gauge Fields?
  - **Gauge Potentials vs. Field Strengths**: Traditionally, one might think only the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are *physical* since they are *gauge invariant* and can be measured directly by *forces*. However, the AB effect shows that the potentials Φ and $\boldsymbol{A}$ also have direct physical consequences—they affect the phase of a quantum wavefunction.
  - **Holonomy and Berry Phase**: The Berry phase around any *closed loop* is gauge invariant, and should be *physical*. All such **closed-loop Berry phases** (aka. the **holonomies**) form the *complete* set of physical observables of a gauge theory. The AB phase is an example of a holonomy: the phase accumulated around a closed loop depends on the *curvature* (here, the magnetic flux ϕ_B) enclosed by the loop.

Uniform Magnetic Field

■ Classical Dynamics

■ Cyclotron Motion

The motion of a **charged particle** in the electromagnetic field is governed by the **Lorentz force**:

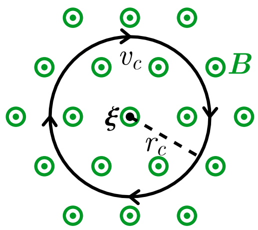
$$\boxed{m\,\ddot{\boldsymbol{x}} = \boldsymbol{F} = q\,(\boldsymbol{E}+\dot{\boldsymbol{x}}\times\boldsymbol{B}).} \tag{60}$$

- m - particle **mass**,

- q - particle **charge**.

Consider the case: **uniform magnetic field** only,

$$\boldsymbol{E} = 0, \;\; \boldsymbol{B} = B\, \boldsymbol{e}^z. \tag{61}$$



Circular motion in x-y plane:

$$\begin{aligned} \dot{\boldsymbol{x}} &= v_c(\cos(\omega_c\, t)\, \boldsymbol{e}^x - \sin(\omega_c\, t)\, \boldsymbol{e}^y), \\ \boldsymbol{x} &= r_c(\sin(\omega_c\, t)\, \boldsymbol{e}^x + \cos(\omega_c\, t)\, \boldsymbol{e}^y) + \boldsymbol{\xi}, \end{aligned} \tag{62}$$

Exc 11 Demonstrate Eq. (62) by solving the equation of motion Eq. (60).

- ω_c - **cyclotron frequency**:

$$\boxed{\omega_c = \frac{q\, B}{m}.} \tag{63}$$

- v_c - **cyclotron velocity**, set by the initial velocity of the particle,
- r_c - **cyclotron radius**,

$$\boxed{r_c = \frac{v_c}{\omega_c} = \frac{m\, v_c}{q\, B}.} \tag{64}$$

- $\boldsymbol{\xi} = \xi_x\, \boldsymbol{e}^x + \xi_y\, \boldsymbol{e}^y$ - **guiding center** (the center of the cyclotron motion). It can be reconstructed from

$$\boxed{\boldsymbol{\xi} = \boldsymbol{x} - \frac{r_c}{v_c}\, \boldsymbol{e}^z \times \dot{\boldsymbol{x}} = \boldsymbol{x} - \frac{1}{q\, B}\, \boldsymbol{e}^z \times \boldsymbol{\pi},} \tag{65}$$

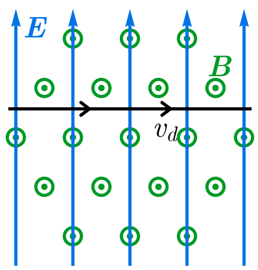
where $\boldsymbol{\pi} = m\, \dot{\boldsymbol{x}}$ denotes the **kinetic momentum**.

Exc 12 Verify Eq. (65).

■ Hall Effect

Consider the case: **uniform electric field** *perpendicular* to **uniform magnetic field**,

$$\boldsymbol{E} = E\, \boldsymbol{e}^y, \;\; \boldsymbol{B} = B\, \boldsymbol{e}^z. \tag{66}$$



The Lorentz force is balanced if

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} = v_d\, \boldsymbol{e}^x. \tag{67}$$

- **Drift velocity** of charge:

$$\boxed{v_d = \frac{E}{B}.} \tag{68}$$

- Corresponding **current density**:

$$\boldsymbol{j} = n\, q\, v_d\, \boldsymbol{e}^x, \tag{69}$$

where

  - n - carrier **density**, number of charge carriers per unit area.
  - q - carrier **charge**.

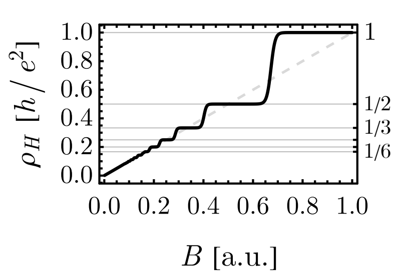
Exc 13 Justify Eq. (69).

- **Hall conductivity**:

$$\boxed{\sigma_H := \frac{j_x}{E_y} = \frac{n\, q}{B}.} \tag{70}$$

- **Hall resistivity**:

$$\rho_H := \frac{E_y}{j_x} = \frac{B}{n\, q}. \tag{71}$$



- Classical expectation: given *fixed* carrier (electron) density n, $\rho_H \propto B$ should scale *linearly* with B.
- **Quantum Hall effect**: when B is large enough, $\sigma_H = \rho_H^{-1}$ exhibits *steps* at *quantized* values:

$$\boxed{\sigma_H = \frac{\nu\, e^2}{h} \quad (\nu = 1, 2, 3, \ldots).} \tag{72}$$

  - h - Planck constant,
  - e - electron charge ($q = -e$ as charge carrier).

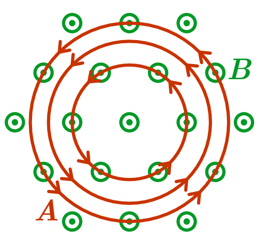
Landau Level Quantization

Gauge Field

Two-dimensional electron system in the x-y plane, with a **uniform magnetic field** perpendicular to the plane

$$\boldsymbol{B} = B\,\boldsymbol{e}^z = \nabla \times \boldsymbol{A}. \tag{73}$$

- The **vector potential** (gauge field) $\boldsymbol{A}$ circulates in the x-y plane,



$$\boxed{\boldsymbol{A} = (A_x,\ A_y) = \frac{B}{2}(-y,\ x),} \tag{74}$$

known as the **symmetric gauge**.

Exc 14 Verify Eq. (74) reproduces Eq. (73).

- However, the gauge choice is not unique. For example, the following gauge choice is also valid:

$$\boldsymbol{A} = (A_x,\ A_y) = (0,\ B\,x),$$

known as the **Landau gauge**.

We will mostly work with the circular gauge Eq. (74), following Ref. [1].

[1] David Tong. The Quantum Hall Effect (TIFR Infosys Lectures), (2016).

▪ Hamiltonian

Following Eq. (48), the Hamiltonian of the **gauge-invariant Schrödinger equation** reads

$$\hat{H} = -\frac{\hbar^2}{2\,m}\,\boldsymbol{D}^2 = -\frac{\hbar^2}{2\,m}\left(\nabla - \frac{i\,q}{\hbar}\,\hat{\boldsymbol{A}}\right)^2. \tag{75}$$

- Unit choice:
  - charge $q = 1$,
  - mass $m = 1$,
  - Planck constant $\hbar = 1$.

  Eq. (75) can be simplified to

$$\boxed{\hat{H} = \frac{1}{2}\left(\hat{\boldsymbol{p}} - \hat{\boldsymbol{A}}\right)^2 = \frac{1}{2}\,\hat{\boldsymbol{\pi}}^2.} \tag{76}$$

- $\hat{\boldsymbol{p}} = -i\,\nabla = (-i\,\partial_x, -i\,\partial_y)$ - **canonical momentum** operator.
  - **Canonical commutation** relation with **coordinate** operator $\hat{\boldsymbol{x}} = (\hat{x}, \hat{y})$:

$$\left[\hat{x}_i, \hat{p}_j\right] = i\,\delta_{ij} \quad (\text{for } i, j = x, y). \tag{77}$$

- $\hat{\boldsymbol{A}} = \boldsymbol{A}(\hat{\boldsymbol{x}}) = \left(\hat{A}_x, \hat{A}_y\right)$ - **potential momentum** operator (electromagnetic vector potential). Under symmetric gauge Eq. (74),

$$\hat{A}_x = -\frac{B}{2}\,\hat{y},\ \ \hat{A}_y = \frac{B}{2}\,\hat{x}. \tag{78}$$

- $\hat{\boldsymbol{\pi}} = (\hat{\pi}_x, \hat{\pi}_y)$ - **kinetic momentum** operator,

$$\boxed{\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{p}} - \hat{\boldsymbol{A}} = -i\,\nabla - \boldsymbol{A}(\hat{\boldsymbol{x}}) = -i\,\boldsymbol{D},} \tag{79}$$

  or, in terms of components

$$\begin{aligned} \hat{\pi}_x &= \hat{p}_x - \hat{A}_x = -i\,\partial_x - \hat{A}_x, \\ \hat{\pi}_y &= \hat{p}_y - \hat{A}_y = -i\,\partial_y - \hat{A}_y. \end{aligned} \tag{80}$$

  - They satisfy the following commutation relation

$$\boxed{[\hat{\pi}_x, \hat{\pi}_y] = i\,B,} \tag{81}$$

    which follows from the definition of **gauge curvature** in Eq. (42).

Exc 15 Verify Eq. (81).

  - They also inherit the commutation relation with the coordinate operator,

$$[\hat{x}_i, \hat{\pi}_j] = i\,\delta_{ij} \text{ (for } i, j = x, y). \tag{82}$$

▪ Guiding Center

Following Eq. (65), the **guiding center** operator $\hat{\boldsymbol{\xi}} = (\xi_x, \xi_y)$ is defined as

$$\boxed{\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{x}} - \frac{1}{B}\, \boldsymbol{e}^z \times \hat{\boldsymbol{\pi}}.} \tag{83}$$

- Under symmetric gauge Eq. (74),

$$\hat{\boldsymbol{\xi}} = \frac{1}{B}\left(\hat{\boldsymbol{p}} + \hat{\boldsymbol{A}}\right) \times \boldsymbol{e}^z, \tag{84}$$

Exc 16 Verify Eq. (84).

or, in terms of components

$$\begin{aligned} \hat{\xi}_x &= \hat{x} + \frac{1}{B}\, \hat{\pi}_y = \frac{1}{B}\left(\hat{p}_y + \hat{A}_y\right), \\ \hat{\xi}_y &= \hat{y} - \frac{1}{B}\, \hat{\pi}_x = -\frac{1}{B}\left(\hat{p}_x + \hat{A}_x\right). \end{aligned} \tag{85}$$

- They satisfy the following commutation relation

$$\boxed{\left[\hat{\xi}_x, \hat{\xi}_y\right] = \frac{1}{i\,B}.} \tag{86}$$

Exc 17 Prove Eq. (86).

- Guiding center and kinetic momentum operators commute.

$$\boxed{\left[\hat{\xi}_i, \hat{\pi}_j\right] = 0 \quad \text{(for } i, j = x, y).} \tag{87}$$

Exc 18 Prove Eq. (87).

▪ Annihilation and Creation Operators

Define two sets of annihilation and creation operators.

- **Cyclotron annihilation and creation** operators:

$$\boxed{\hat{a} = \frac{1}{\sqrt{2\,B}}\, (\hat{\pi}_x + i\,\hat{\pi}_y), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\,B}}\, (\hat{\pi}_x - i\,\hat{\pi}_y),} \tag{88}$$

such that

$$\hat{a}^\dagger \hat{a} + \frac{1}{2} = \frac{1}{2B} \hat{\boldsymbol{\pi}}^2 = \frac{1}{2B} \left(\hat{\boldsymbol{p}} - \hat{\boldsymbol{A}}\right)^2. \tag{89}$$

Exc 19 Verify Eq. (89).

- **Guiding center annihilation and creation** operators:

$$\boxed{\hat{b} = \sqrt{\frac{B}{2}} \left(\hat{\xi}_x - \boldsymbol{i}\, \hat{\xi}_y\right),\ \hat{b}^\dagger = \sqrt{\frac{B}{2}} \left(\hat{\xi}_x + \boldsymbol{i}\, \hat{\xi}_y\right),} \tag{90}$$

such that

$$\hat{b}^\dagger \hat{b} + \frac{1}{2} = \frac{B}{2} \hat{\boldsymbol{\xi}}^2 = \frac{1}{2B} \left(\hat{\boldsymbol{p}} + \hat{\boldsymbol{A}}\right)^2. \tag{91}$$

Exc 20 Verify Eq. (91).

They satisfy the following commutation relations

$$\boxed{\begin{aligned}&\left[\hat{a}, \hat{a}^\dagger\right] = 1,\ \left[\hat{b}, \hat{b}^\dagger\right] = 1,\\ &\left[\hat{a}, \hat{b}\right] = \left[\hat{a}, \hat{b}^\dagger\right] = \left[\hat{a}^\dagger, \hat{b}\right] = \left[\hat{a}^\dagger, \hat{b}^\dagger\right] = 0.\end{aligned}} \tag{92}$$

Exc 21 Prove Eq. (92).

- Implication: $\hat{a}$ and $\hat{b}$ are *annihilation operators* for two *independent* **harmonic oscillator** degrees of freedom.

- $\hat{a}^\dagger \hat{a}$ and $\hat{b}^\dagger \hat{b}$ are commuting **number operators**, and can be diagonalized *simultaneously*. Their eigenvalues correspond to two *separate* sets of **quantum numbers**, denoted as n and m respectively:

$$\boxed{\begin{aligned}&\hat{a}^\dagger \hat{a} \,|n,m\rangle = n\,|n,m\rangle,\\ &\hat{b}^\dagger \hat{b}\,|n,m\rangle = m\,|n,m\rangle,\end{aligned}} \tag{93}$$

$n,\ m = 0, 1, 2, \ldots \in \mathbb{N}$.

Landau Levels

The system has two important physical observables:

- **Hamiltonian**: using Eq. (89), Eq. (76) can be written as

$$\boxed{\hat{H} = \frac{1}{2}\hat{\boldsymbol{\pi}}^2 = B\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right).} \tag{94}$$

- **Angular momentum**:

$$\boxed{\hat{L}_z = (\hat{\boldsymbol{x}} \times \hat{\boldsymbol{p}}) \cdot \boldsymbol{e}^z = \hat{b}^\dagger\,\hat{b} - \hat{a}^\dagger\,\hat{a}.} \tag{95}$$

Exc 22

Verify Eq. (95) under symmetric gauge.

Obviously, $[\hat{H}, \hat{L}_z] = 0$, i.e. $\hat{H}$ and $\hat{L}_z$ can be simultaneously diagonalized.

- Their common eigenstates are $|n,m\rangle$:

$$\boxed{\begin{aligned} \hat{H}\,|n,m\rangle &= B\left(n + \frac{1}{2}\right)|n,m\rangle, \\ \hat{L}_z\,|n,m\rangle &= (m - n)\,|n,m\rangle, \end{aligned}} \tag{96}$$

  which are labeled by two quantum numbers:

  - n: **Landau level** index (energy level index),
  - m: **angular momentum** index (degeneracy within a Landau level).

- The energy levels are quantized

$$E_n = B\left(n + \frac{1}{2}\right), \tag{97}$$

  with $n = 0, 1, 2, \ldots \in \mathbb{N}$.

  - After restoring the energy unit,

$$\boxed{E_n = \frac{\hbar\,q\,B}{m}\left(n + \frac{1}{2}\right) = \hbar\,\omega_c\left(n + \frac{1}{2}\right).} \tag{98}$$

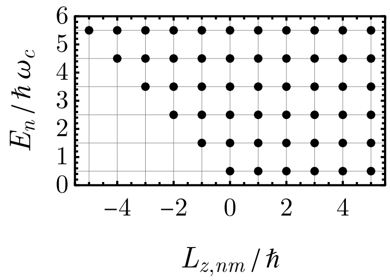
    where $\omega_c = q\,B/m$ is the **cyclotron frequency**.

  - Each level is called a **Landau level**. The $n = 0$ level is called the **lowest Landau level** (LLL).

- The angular momentum is also quantized. After restoring the unit,

$$\boxed{L_{z,nm} = \hbar\,(m - n).} \tag{99}$$

Eigenstates $|n,m\rangle$ arranged by energy v.s. angular momentum.



- **Landau level degeneracy**:
  - In an *infinite* system, each Landau level is *infinitely* degenerated.
    - Argument: The quantum number $m = 0, 1, 2, \ldots \in \mathbb{N}$ is unbounded. ⇒ Infinitely many orthogonal states $|n,m\rangle$ within the same energy level n.
  - In a realistic system of *finite* size, the Landau level degeneracy becomes *finite*, and is determined by the *total magnetic flux* measured in units of the *flux quantum*.
    - Consider electrons confined within a disk of radius R. ⇒ The *guiding center* radius ξ must satisfy $\xi \lesssim R$.
    - This puts a constraint on the operator

$$\hat{b}^\dagger\,\hat{b} + \frac{1}{2} = \frac{B}{2}\,\hat{\boldsymbol{\xi}}^2 \lesssim \frac{B\,R^2}{2}, \tag{100}$$

in terms of its eigenvalues. ⇒ $m \lesssim B\,R^2 / 2$, restoring the physical units:

$$m \lesssim \frac{e\,B\,R^2}{2\,\hbar}. \tag{101}$$

- A **flux quantum** is the amount of magnetic flux ϕ_0 that induces a $2\,\pi$ **Berry phase** for an electron braiding around it.

$$\text{phase} = \frac{\text{action}}{\hbar} = \frac{e\,\phi_0}{\hbar} = 2\,\pi, \tag{102}$$

meaning that

$$\boxed{\phi_0 = \frac{h}{e}.} \tag{103}$$

- Then Eq. (101) can be written as

$$\boxed{m \lesssim \frac{\phi_B}{\phi_0},} \tag{104}$$

where $\phi_B = \pi\,R^2\,B$ is the **total magnetic flux** through the disk.

Thus the Landau level degeneracy is set by ϕ_B / ϕ_0 — the conclusion generalizes to any shape of the system.

Complex Coordinate

Instead of using $\boldsymbol{x} = (x, y)$ to coordinate the position of the particle, it is more convenient to introduce:

- the **complex coordinate**

$$z = \sqrt{\frac{B}{2}}\,(x + i\,y),\;\; \overline{z} = \sqrt{\frac{B}{2}}\,(x - i\,y), \tag{105}$$

- and the **complex derivative**

$$\partial_z = \frac{1}{\sqrt{2\,B}}\,(\partial_x - i\,\partial_y),\;\; \partial_{\overline{z}} = \frac{1}{\sqrt{2\,B}}\,(\partial_x + i\,\partial_y). \tag{106}$$

Exc 23 Derive Eq. (106) from Eq. (105).

Using the complex notation, the creation and annihilation operators can be represented as

$$\begin{aligned} \hat{a} &= -i\left(\frac{1}{2}\,z + \partial_{\overline{z}}\right),\; \hat{a}^\dagger = i\left(\frac{1}{2}\,\overline{z} - \partial_z\right); \\ \hat{b} &= \frac{1}{2}\,\overline{z} + \partial_z,\; \hat{b}^\dagger = \frac{1}{2}\,z - \partial_{\overline{z}}. \end{aligned} \tag{107}$$

Exc 24 Verify Eq. (107).

Wave Functions

$|0{,}0\rangle$ is the **vacuum state** for both $\hat{a}$ and $\hat{b}$ bosons, defined by the condition

$$\hat{a}\,|0{,}0\rangle = \hat{b}\,|0{,}0\rangle = 0. \tag{108}$$

so the vacuum state *wave function* $\psi_{0,0}(z, \overline{z})$ should satisfy

$$\left(\frac{1}{2}\,z + \partial_{\overline{z}}\right)\psi_{0,0}(z, \overline{z}) = \left(\frac{1}{2}\,\overline{z} + \partial_z\right)\psi_{0,0}(z, \overline{z}) = 0. \tag{109}$$

- Consider the ansatz

$$\psi_{0,0}(z, \overline{z}) = f(z, \overline{z})\, e^{-\overline{z}\,z/2}, \tag{110}$$

Eq. (109) implies

$$\partial_z f(z, \overline{z}) = \partial_{\overline{z}} f(z, \overline{z}) = 0, \tag{111}$$

Exc 25

Derive Eq. (111).

meaning that $f(z, \bar{z})$ must be *independent* of both z and $\bar{z}$, i.e. it is a constant function, i.e.

$$f(z, \bar{z}) = \text{const.} \tag{112}$$

- Therefore, the vacuum state wave function reads

$$\boxed{\psi_{0,0}(z, \bar{z}) = \frac{1}{\sqrt{\pi}}\, e^{-\bar{z}\, z/2},} \tag{113}$$

which is normalized to ensure $\int d z\, d \bar{z}\, |\psi_{0,0}(z, \bar{z})|^2 = 1$.

Any other state $|n,m\rangle$ can be raised from the vacuum state $|0,0\rangle$ by applying creation operators

$$\boxed{|n,m\rangle = \frac{1}{\sqrt{n!\, m!}}\, (\hat{a}^\dagger)^n\, (\hat{b}^\dagger)^m\, |0,0\rangle,} \tag{114}$$

for n, $m = 0, 1, 2, \ldots$.

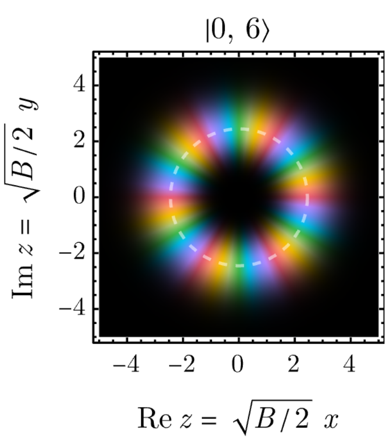
- **Lowest Landau level** (LLL): The wave functions with $n = 0$ are

$$\boxed{\psi_{0,m}(z, \bar{z}) = \frac{1}{\sqrt{\pi\, m!}}\, z^m\, e^{-\bar{z}\, z/2}.} \tag{115}$$

Exc 26

Derive Eq. (115).

The wave functions has circular symmetry, taking the **ring shape**.



- Probability density

$$|\psi_{0,m}(z,\,\bar{z})|^2 = \frac{1}{\pi\, m!}\,(\bar{z}\,z)^m\,e^{-\bar{z}\,z},\tag{116}$$

which distributes around the radius $|z| = \sqrt{m}$.

Exc 27 Estimate the radius $|z|$ that maximizes the probability.

- **Semi-classical argument**: The factor $z^m = \sqrt{B/2}\;r^m\,e^{i\,m\,\theta}$ implies that the particle accumulates $2\,\pi\,m$ *Berry phase* when travelling around a circle of *radius* r.
  - To ensure that the wave function is single-valued, m must be an integer, i.e.

$$2\,\pi\,m \in 2\,\pi\,\mathbb{Z} \Rightarrow m \in \mathbb{Z}.\tag{117}$$

  - This corresponds to a *canonical momentum* of

$$p = \frac{2\,\pi\,m}{2\,\pi\,r} = \frac{m}{r}.\tag{118}$$

  - States in the LLL should *minimize* the energy $E = (\boldsymbol{p} - \boldsymbol{A})^2 / 2$, which can be achieved if

$$p = A = \frac{\pi\,r^2\,B}{2\,\pi\,r} = \frac{r\,B}{2}.\tag{119}$$

  - Eq. (118) and Eq. (119) together sets the optimal radius

$$\frac{m}{r} = p = \frac{r\,B}{2} \Rightarrow r = \sqrt{\frac{2\,m}{B}},\tag{120}$$

    or, in terms of z variable, $|z| = \sqrt{B/2}\;r = \sqrt{m}$ (with $m \in \mathbb{Z}$).

- Any linear combination of $|0,m\rangle$ is still a state in the LLL, which takes the general form of

$$\sum_m c_m\,\psi_{0,m}(z,\,\bar{z}) = \frac{1}{\sqrt{\pi\,m!}}\left(\sum_m c_m\,z^m\right)e^{-\bar{z}\,z/2} = f(z)\,e^{-\bar{z}\,z/2}.\tag{121}$$

  Conclusion: LLL wavefunctions are **holomorphic functions** of z, multiplied by a **Gaussian envelope**.

- More generally, for higher Landau levels, the wave functions

$$\psi_{n,m}(z,\,\bar{z}) = \frac{1}{\sqrt{\pi\,n!\,m!}}\,((\bar{z} - \partial_z)^n\,z^m)\,e^{-\bar{z}\,z/2},\tag{122}$$

  can be written in terms of *associated Laguerre polynomials*.

▪ Quantum Hall Effect

▪ Filling Landau Levels

For a **single electron** confined to a *two-dimensional plane* under a *uniform perpendicular* magnetic field, the energy eigenvalues are quantized into **Landau levels**:

$$E_n = \hbar\,\omega_c\left(n + \frac{1}{2}\right) \quad (n = 0, 1, 2, \ldots). \tag{123}$$

- **Level spacing** (energy gap): $\hbar\,\omega_c$, where $\omega_c = e\,B\,/\,m$ is the *cyclotron frequency*.
- **Level degeneracy**: Given the total magnetic flux ϕ_B through the plane, degeneracy is

  $$N_\phi := \frac{\phi_B}{\phi_0} = \frac{B\,A}{\phi_0}, \tag{124}$$

  where

  - A - total area of the system,
  - $\phi_0 = h\,/\,e$ - the *magnetic flux quantum*.

Many-body system: In real materials, there is not just one single electron, but **many electrons** interacting with each other — the problem become *many-body* in nature.

- The key feature of electrons is their **fermionic statistics**.
  ⇒ **Pauli exclusion principle**: no two electrons can occupy the same quantum state.
- **Filling up energy levels**: Due to Pauli exclusion, electrons will fill up *available quantum states* starting from the lowest energy states.
- **Filling fraction** ν is the (fractional) number of Landau levels that will be filled up,

  $$\boxed{\nu = \frac{N}{N_\phi} = \frac{n\,h}{e\,B}.} \tag{125}$$

  - N - *total number* of electron in the system,
  - $n = N\,/\,A$ - **electron density**, i.e. the number of electrons per area.

  Conversely, Eq. (125) allows us to express n in terms of ν,

  $$n = \frac{\nu\,e}{h}\,B. \tag{126}$$

The phenomenology of **integer quantum Hall effect** is that the Hall conductivity σ_H takes quantized values (recall Eq. (72))

$$\sigma_H = \frac{n\,e}{B} = \frac{\nu\,e^2}{h}, \tag{127}$$

at $\nu = 1, 2, 3, \ldots$, i.e. when the **filling fraction** $\nu \in \mathbb{N}$ is an integer, corresponding to the situation where

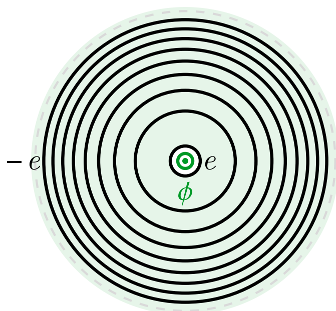
- the lowest ν **Landau levels** are completely *filled*,
- and all higher levels are completely *empty*.

This leads to an **incompressible quantum state**: electrons cannot change states without a large energy cost (the Landau level spacing $\hbar\,\omega_c$).

Message: The *quantized* Hall conductance originates from fully filling *discrete* Landau levels. — Why does each filled Landau level contribute to exactly one unit of σ_H?

Charge Pumping

Consider a *annular geometry* that allows us to probe the Hall effect via **radial charge pumping**.



- In addition to a uniform perpendicular magnetic field B, we thread an extra *magnetic flux* $\phi(t)$ through the central hole, which *slowly* increases from 0 to ϕ_0 over a *long* time T,

$$\phi(t) = \phi_0 \frac{t}{T}, \tag{128}$$

  - $\phi_0 = h / e$ - flux quantum,
  - $T \gg 1 / \omega_c$ - *adiabatic* evolution time. ($t : 0 \to T$).

- Due to the **Aharonov-Bohm effect**, each quantum state $|n,m\rangle$ will be affected — its wave function radius r_m will be deformed adiabatically,

$$r_m = \sqrt{\frac{\phi_0}{\pi B}} \sqrt{m - t / T}. \tag{129}$$

Exc 28

> Derive Eq. (129).

Under the flux insertion,

$$\boxed{\begin{array}{cccc} t: & 0 & \to & T \\ \phi(t): & 0 & \to & \phi_0 \\ r_m \propto & \sqrt{m} & \to & \sqrt{m-1} \end{array},} \tag{131}$$

the radius of every orbit shrinks from $\sqrt{m}$ to $\sqrt{m-1}$ $\Rightarrow$ pumping *one charge* e from the *outer* edge to the *inner* edge.

- During this process,
  - the radial current density:

$$j_r = \frac{1}{2\pi r}\frac{e}{T}, \tag{132}$$

- the circular electric field induced by the changing flux:

$$E_\theta = \frac{1}{2\pi r}\frac{d\phi(t)}{dt} = \frac{1}{2\pi r}\frac{\phi_0}{T}, \tag{133}$$

Hall conductivity for each fully filled Landau level:

$$\sigma_H = \frac{j_r}{E_\theta} = \frac{e}{\phi_0} = \frac{e^2}{h}. \tag{134}$$

Conclusion: *quantized* transport of charge in response to *quantized* flux insertion ⇒ *quantized* Hall conductance.

Linear Response Theory

The **Hall conductivity** σ_H characterizes how the **current density** $\boldsymbol{j}$ (observable) *responds* to the **electric field** $\boldsymbol{E}$ (perturbation).

- Current density operator for *each electron* (assuming $m = e = h = 1$) reads

$$\boxed{\hat{\boldsymbol{j}} = \frac{B}{N_\phi}\hat{\boldsymbol{\pi}}.} \tag{135}$$

Exc 29 Justify Eq. (135) based on Eq. (69).

- The *expectation value* of **current density** in the *system* is given by

$$\begin{aligned}\langle \boldsymbol{j}\rangle &= \sum_{n,m\in\text{occ}} \langle n,m|\,\hat{\boldsymbol{j}}\,|n,m\rangle\\ &= \frac{B}{N_\phi}\sum_{n,m\in\text{occ}} \langle n,m|\,\hat{\boldsymbol{\pi}}\,|n,m\rangle,\end{aligned} \tag{136}$$

where $\sum_{n,m\in\text{occ}}$ is to sum over all lowest $|n,m\rangle$ states that are *occupied* by the electron.

- Based on Eq. (88), the **kinetic momentum** operator $\hat{\boldsymbol{\pi}}$ can be expressed as

$$\boxed{\hat{\pi}_x = \sqrt{\frac{B}{2}}\left(\hat{a}^\dagger + \hat{a}\right),\ \hat{\pi}_y = \sqrt{\frac{B}{2}}\,i\left(\hat{a}^\dagger - \hat{a}\right).} \tag{137}$$

As expected, the current density $\langle \boldsymbol{j}\rangle = 0$ vanishes on the many-body ground state, in the absence of perturbation.

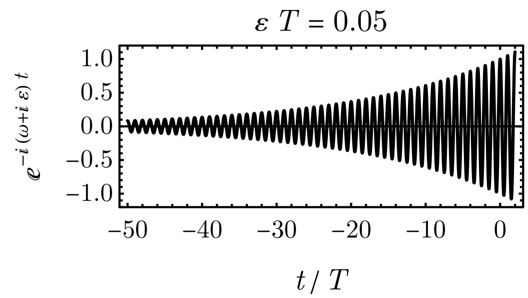
- Applying a (weak) **electric field** $\boldsymbol{E}$ to the system amounts to perturbing the **vector potential** $\boldsymbol{A}$ by a *time-dependent* perturbation $\delta\boldsymbol{A}(t)$,

$$\boldsymbol{A} \to \boldsymbol{A} + \delta\boldsymbol{A}\,(t), \tag{138}$$

with $\boldsymbol{E}(t) = -\partial_t \delta \boldsymbol{A}(t)$, according to Eq. (44).

- Assume that $\boldsymbol{E}(t) = \boldsymbol{E}\, e^{-i(\omega + i 0_+)t}$ takes an oscillatory form with frequency ω, and is adiabatically turned on from the infinite past, then

$$\delta \boldsymbol{A}(t) = \int_{-\infty}^{t} d t' \, \boldsymbol{E}(t') = -\frac{\boldsymbol{E}}{i\,\omega}\, e^{-i(\omega + i 0_+)t}. \tag{139}$$



- This means the **Hamiltonian operator** will be perturbed by

$$\hat{H} \to \hat{H} + \delta \hat{H}(t), \tag{140}$$

with the perturbation given by

$$\delta \hat{H}(t) = \frac{\partial \hat{H}}{\partial \boldsymbol{A}} \cdot \delta \boldsymbol{A}(t) \tag{141}$$

to the leading order of $\delta \boldsymbol{A}(t)$.

  - Given that $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{p}} - \boldsymbol{A}$ and

$$\hat{H} = \frac{1}{2}\,\hat{\boldsymbol{\pi}}^2 = \frac{1}{2}\,(\hat{\boldsymbol{p}} - \boldsymbol{A})^2, \tag{142}$$

  we have

$$\frac{\partial \hat{H}}{\partial \boldsymbol{A}} = -(\hat{\boldsymbol{p}} - \boldsymbol{A}) = -\hat{\boldsymbol{\pi}}. \tag{143}$$

Substitute Eq. (143) into Eq. (141), the *time-dependent perturbation* Hamiltonian reads

$$\boxed{\delta \hat{H}(t) = -\hat{\boldsymbol{\pi}} \cdot \delta \boldsymbol{A}(t) = \frac{1}{i\,\omega}\, \boldsymbol{E} \cdot \hat{\boldsymbol{\pi}}\, e^{-i(\omega + i 0_+)t}.} \tag{144}$$

The time-dependent perturbation problem can solved by the Green's function approach.

- **Dressed Green's function** (unitary time-evolution operator) can be computed from the Dyson series to the leading order

$$\boxed{\hat{G}(t, -\infty) = \hat{G}_0(t, -\infty) + (-i) \int_{-\infty}^{t} d t' \, \hat{G}_0(t, t')\, \delta \hat{H}(t')\, \hat{G}_0(t', -\infty) + \ldots,} \tag{145}$$

where the **bare Green's function** is defined by

$$\boxed{\hat{G}_0(t,\, t') = \sum_{n,m} |n,m\rangle\, e^{-i\,E_n(t-t')}\, \langle n,m|,} \tag{146}$$

with $E_n = B\,(n + 1/2)$ being the **Landau level energy** (see Eq. (97)).

- Every state will evolve in time by

$$|n,m\rangle \to \hat{G}(t,\,-\infty)\,|n,m\rangle. \tag{147}$$

As a result, by Eq. (136), the current density expectation value $\langle \boldsymbol{j}\rangle$ evolves as

$$\begin{aligned}\langle \boldsymbol{j}(t)\rangle &= \frac{B}{N_\phi} \sum_{n,m\,\in\,\mathrm{occ}} \langle n,m|\,\hat{G}(-\infty,\,t)\,\hat{\boldsymbol{\pi}}\,\hat{G}(t,\,-\infty)\,|n,m\rangle \\ &= \frac{B}{N_\phi} \sum_{n,m\,\in\,\mathrm{occ}} \langle n,m|\left(\hat{\boldsymbol{\pi}}(t) + \frac{1}{\omega}\int_{-\infty}^{t} d t'\; e^{-i\,(\omega+i0_+)\,t'}[\boldsymbol{E}\cdot\hat{\boldsymbol{\pi}}\,(t'),\,\hat{\boldsymbol{\pi}}(t)] + \ldots\right)|n,m\rangle.\end{aligned} \tag{148}$$

Exc 30 Derive Eq. (148).

where we have introduced

$$\hat{\boldsymbol{\pi}}(t) := \hat{G}_0(-\infty,\,t)\,\hat{\boldsymbol{\pi}}\,\hat{G}_0(t,\,-\infty). \tag{149}$$

- The first term in Eq. (148) is the current density in the *absence* of an electric field, which *vanishes*.

$$\sum_{n,m\,\in\,\mathrm{occ}} \langle n,m|\,\hat{\boldsymbol{\pi}}(t)\,|n,m\rangle = 0. \tag{150}$$

Exc 31 Show Eq. (150).

- The second term in Eq. (148) describes the **linear response** of the current density under the electric field, which takes the form of

$$\boxed{\langle \boldsymbol{j}(t)\rangle = \boldsymbol{E}\cdot\sigma(\omega)\,e^{-i\,(\omega+i0_+)\,t},} \tag{151}$$

with the **conductivity matrix** $\sigma_{ji}(\omega)$ given by

$$\sigma_{ji}(\omega) = \frac{B}{\omega\,N_\phi}\int_{-\infty}^{0} d t\, e^{-i\,(\omega+i0_+)\,t} \sum_{n,m\,\in\,\mathrm{occ}} \langle n,m|[\hat{\pi}_j\,(t),\,\hat{\pi}_i(0)]\,|n,m\rangle. \tag{152}$$

Exc 32 Derive Eq. (151) and Eq. (152).

Eq. (151) implies that when the applied electric field $\boldsymbol{E}\,(t) = \boldsymbol{E}\,e^{-i\,(\omega+i0_+)\,t}$ oscillates at frequency ω, the induced current $\langle \boldsymbol{j}(t)\rangle$ also oscillates at the *same* frequency. — A feature of *linear* response.

- The **Hall conductivity** corresponds to the off-diagonal component $\sigma_{xy}(\omega)$ of the conductivity matrix. In the DC limit $\omega \to 0$, it is given by the **Kobo formula**:

$$\boxed{\sigma_H := \lim_{\omega\to 0} \sigma_{xy}(\omega) = \frac{B}{i\, N_\phi} \sum_{n,m\,\in\,\text{occ}} \sum_{n'\neq n} \frac{\langle n,m|\, \hat{\pi}_x \,|n',m\rangle \langle n',m|\, \hat{\pi}_y \,|n,m\rangle - h.c.}{(E_{n'} - E_n)^2}.} \tag{153}$$

Exc 33 Derive Eq. (153).

- Using Eq. (137) to represent the kinetic momentum operator $\hat{\boldsymbol{\pi}}$ as annihilation and creation operators, and given that

$$\begin{aligned} \hat{a}\,|n,m\rangle &= \sqrt{n}\;|n-1,m\rangle, \\ \hat{a}^\dagger\,|n,m\rangle &= \sqrt{n+1}\;|n+1,m\rangle, \end{aligned} \tag{154}$$

the Hall conductivity σ_H in Eq. (153) reduces to

$$\boxed{\sigma_H = \frac{1}{N_\phi} \sum_{n,m\,\in\,\text{occ}} 1 = \sum_{n\in\text{occ}} 1.} \tag{155}$$

Exc 34 Derive Eq. (155).

- Each occupied state contributes $1/N_\phi$ to σ_H (in unit of the conductance quantum e^2/h).
- Each Landau level is N_ϕ-fold degenerated. ⇒ Fully filling each Landau level produces exactly one unit of σ_H.

Spin and Monopole

■ Classical Spin

■ Angular Momentum Decomposition

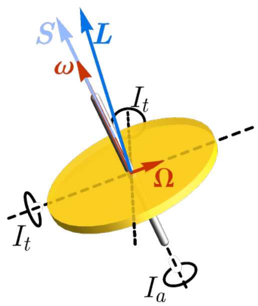
The classical motion of a **spinning top** is governed by

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt}, \tag{156}$$

where

- $\boldsymbol{\tau}$ - **torque** exerted on the top,
- $\boldsymbol{L}$ - total **angular momentum** of the top, decomposed into components *parallel* $I_a\,\boldsymbol{\omega}$ and *perpendicular* $I_t\,\boldsymbol{\Omega}$ to the spinning axis

$$\boldsymbol{L} = I_a\,\boldsymbol{\omega} + I_t\,\boldsymbol{\Omega}, \tag{157}$$



- I_a - *axial* **moment of inertia** (about the spinning axis),
- I_t - *transverse* **moment of inertia** (about either of the two equivalent transverse axes)
- $\boldsymbol{\omega}$ - *axial* **angular velocity** (along the spinning axis)
- $\boldsymbol{\Omega}$ - *transverse* **angular velocity**, describing the instantaneous rotation rate of the spinning axis itself.

Dynamics of Spinning Axis

Substitute Eq. (157) into Eq. (156),

$$\boldsymbol{\tau} = I_a\,\frac{d\boldsymbol{\omega}}{d\,t} + I_t\,\frac{d\boldsymbol{\Omega}}{d\,t}. \tag{158}$$

- **Transverse Torque Assumption**: Assume $\boldsymbol{\tau}$ has no component along the spinning axis, so the *magnitude* of $\boldsymbol{\omega}$ remains constant. Only its *direction* changes due to the rotation of the spinning axis:

$$\frac{d\boldsymbol{\omega}}{d\,t} = \boldsymbol{\Omega}\times\boldsymbol{\omega}, \tag{159}$$

Eq. (158) becomes

$$\boldsymbol{\tau} = I_a\,\boldsymbol{\Omega}\times\boldsymbol{\omega} + I_t\,\frac{d\boldsymbol{\Omega}}{d\,t}. \tag{160}$$

We are mainly interested in the *motion* of the **spinning axis**, represented by the *unit vector*

$$\boxed{\boldsymbol{n} = \frac{\boldsymbol{\omega}}{\omega}.} \tag{161}$$

Similar to Eq. (159), $\boldsymbol{n}$ also gets rotated by $\boldsymbol{\Omega}$ as

$$\frac{d\boldsymbol{n}}{d\,t} = \boldsymbol{\Omega}\times\boldsymbol{n}, \tag{162}$$

from which $\boldsymbol{\Omega}$ can be “solved” and expressed as

$$\boldsymbol{\Omega} = \boldsymbol{n} \times \frac{d\boldsymbol{n}}{dt}. \tag{163}$$

Exc 35 Derive Eq. (163) from Eq. (162).

Substitute Eq. (163) into Eq. (160), and cross product with $\boldsymbol{n}$ from right on both sides, we obtain

$$\boxed{I_t \left(\frac{d^2\boldsymbol{n}}{dt^2}\right)_\perp = \boldsymbol{\tau} \times \boldsymbol{n} - \frac{d\boldsymbol{n}}{dt} \times \boldsymbol{S},} \tag{164}$$

where

- $\boldsymbol{S} := I_a\, \boldsymbol{\omega} = S\, \boldsymbol{n}$ - **spin angular momentum** (*parallel* to the spinning axis),
- $(\ddot{\boldsymbol{n}})_\perp = \ddot{\boldsymbol{n}} - (\ddot{\boldsymbol{n}} \cdot \boldsymbol{n})\ \boldsymbol{n}$ - component of acceleration $\ddot{\boldsymbol{n}}$ in the tangent plane.

Exc 36 Derive Eq. (164).

Magnetic Monopole

Electromagnetic Analogy

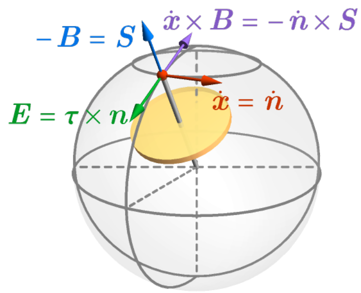
The motion of the spinning axis $\boldsymbol{n}$ (**spin dynamics**) can be interpreted as the motion of a *charged particle* on a *unit sphere* with an *magnetic monopole* inside (**charge dynamics**).

- Analogy: Compare the spin dynamics in Eq. (164) and the charge dynamics in Eq. (55)

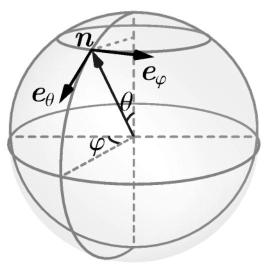
$$\begin{array}{rcccccc} \text{spin}: & I_t\,(\ddot{\boldsymbol{n}})_\perp & = & \boldsymbol{\tau}\times\boldsymbol{n} & - & \dot{\boldsymbol{n}}\times\boldsymbol{S} \\ \text{charge}: & m\,\ddot{\boldsymbol{x}} & = & \boldsymbol{E} & + & \dot{\boldsymbol{x}}\times\boldsymbol{B} \end{array} \tag{165}$$

Spin dynamics	Charge dynamics
Spin orientation : $\boldsymbol{n}$	Charge position : $\boldsymbol{x}$
Moment of inertia : I_t	Mass (interia) : m
Torque-induced force : $\boldsymbol{\tau}\times\boldsymbol{n}$	Electric field : $\boldsymbol{E}$
Spin angular momentum : $-\boldsymbol{S}$	Magnetic field : $\boldsymbol{B}$

- Similarity: Just as in electromagnetism, where the Lorentz force *deflects* a charge moving in a magnetic field, the spin-induced term $-\dot{\boldsymbol{n}}\times\boldsymbol{S}$ generates *precession* of the spinning axis.



- Difference: The constraint to the *sphere* makes the coordinate system *non-Euclidean* (curved), in which ∇ operator is defined differently. [2]



- **Spherical coordinate**: parametrize the spin axis

$$\boldsymbol{n} = (\cos\varphi \sin\theta,\ \sin\varphi \sin\theta,\ \cos\theta), \tag{166}$$

by the *polar angle* $\theta \in [0, \pi]$ and the *azimuthal angle* $\varphi \in [0, 2\pi)$.

- **Unit vectors**:

$$\begin{aligned} \boldsymbol{e}^\theta &= (\cos\varphi \cos\theta, \sin\varphi \cos\theta, -\sin\theta), \\ \boldsymbol{e}^\varphi &= (-\sin\varphi, \cos\varphi, 0). \end{aligned} \tag{167}$$

- **Surface gradient**: $\nabla_\perp \Phi$ for a scalar field Φ,

$$\nabla_\perp \Phi = \boldsymbol{e}^\theta\, \partial_\theta \Phi + \boldsymbol{e}^\varphi \frac{1}{\sin\theta}\, \partial_\varphi \Phi. \tag{168}$$

- **Surface curl**: $\nabla_\perp \times \boldsymbol{A}$ for a vector field $\boldsymbol{A} = A_\theta\, \boldsymbol{e}^\theta + A_\varphi\, \boldsymbol{e}^\varphi$,

$$\nabla_\perp \times \boldsymbol{A} = \frac{1}{\sin\theta} (\partial_\theta (A_\varphi \sin\theta) - \partial_\varphi A_\theta)\, \boldsymbol{n}. \tag{169}$$

[2] Wikipedia. Del in cylindrical and spherical coordinates.

Differential Geometry for Vector Calculus

In **vector calculus**, we often compute *gradients*, *divergences*, *curls*, and integrals over curves, surfaces, or volumes. While powerful, these operations depend heavily on the *coordinate system* and *dimension*. **Differential geometry** provide a more *geometric* and *coordinate-free* language for calculus. They unify many familiar operations and extend naturally to curved spaces (manifolds).

- **Differential Forms**: differential form is the basic object in differential geometry, it is an object that you can *integrate*:
  - A **0-form** is just a *scalar* field $f(x)$, which can be integrated (evaluated) on a **point** x.
  - A **1-form** is a *vector* field (*line* element), like $\omega(x) = \omega_i(x)\, dx^i$, something you can integrate along a **curve**.
  - A **2-form** is an *tensor* field (oriented *surface* element), e.g. $\sigma_{ij}(x)\, dx^i \wedge dx^j$, integrable over a **surface**.

  In general:
  - A k**-form** is something you integrate over a k-dimensional submanifold.
  - The **wedge product** $\wedge$ defines an oriented, antisymmetric product between forms:

  $$dx^i \wedge dx^j = -dx^j \wedge dx^i. \tag{170}$$

- **Metric**: The metric defines how distance is measured on the manifold.

  $$\boxed{ds^2 = g_{ij}(x)\, dx^i\, dx^j,} \tag{171}$$

  where
  - ds^2 is the squared infinitesimal distance element,
  - $g_{ij}(x)$ are the components of the metric tensor, forming a symmetric positive-definite matrix.
  - dx^i are the differential 1-form basis (cotangent basis)
- **Exterior Derivative**: the exterior derivative d acts on differential forms to produce forms of higher degree. Given $f = f_I\, dx^I$,

  $$\boxed{df = \partial_i f_I\, dx^i \wedge dx^I.} \tag{172}$$

  - $d^2 = 0$: the exterior derivative of an exterior derivative is always zero.
  - **Stoke's Theorem**:

  $$\boxed{\int_{\partial M} \omega = \int_M d\omega.} \tag{173}$$

- **Hodge Dual**: On an n-dimensional oriented manifold, the Hodge star operator $\star$ maps k-forms to $(n-k)$-forms.

  $$\star\left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) = \frac{|\det[g_{ij}]|^{1/2}}{(n-k)!}\, g^{i_1 j_1} \ldots g^{i_k j_k}\, \epsilon_{j_1 \ldots j_n}\, dx^{j_{k+1}} \wedge \ldots \wedge dx^{j_n}, \tag{174}$$

  where
  - $\epsilon_{j_1 \ldots j_n}$ is the **Levi-Civita symbol**,

- $g^{ij} = \langle dx^i, dx^j \rangle$ is the **inverse metric**, which tells how differential forms dx^i and $dx^j$are "angled" with respect to each other.

Exterior derivative and Hodge dual enable us to represent **vector calculus operators** in terms of differential forms in a unified manner.

Vector Calculus	Differetial Forms
grad (∇f)	$d f$
curl $(\nabla \times \boldsymbol{\omega})$	$\star\, d\omega$
div $(\nabla \cdot \boldsymbol{\omega})$	$\star\, d \star \omega$
Laplacian $(\nabla^2 f)$	$\star\, d \star d f$

(175)

See Refs. [3] for more details of the above concepts.

Application in spherical coordinates on S^2.

- **Metric**: the distance element is given by

$$d s^2 = d\theta^2 + \sin^2\theta\, d\varphi^2, \tag{176}$$

which implies

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},\ g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix}, \tag{177}$$

and $|\det[g_{ij}]|^{1/2} = \sin\theta$.

- **Unit (co)vectors**: orthonormal basis of 1-forms

$$\boxed{\begin{aligned} \boldsymbol{e}^\theta &\leftrightarrow \sqrt{g_{\theta\theta}}\, d\theta = d\theta, \\ \boldsymbol{e}^\varphi &\leftrightarrow \sqrt{g_{\varphi\varphi}}\, d\varphi = \sin\theta\, d\varphi. \end{aligned}} \tag{178}$$

This enables us to represent any *vector* field as *1-form* field, such as

$$\boldsymbol{A} = A_\theta\, \boldsymbol{e}^\theta + A_\varphi\, \boldsymbol{e}^\varphi \leftrightarrow A = A_\theta\, d\theta + A_\varphi \sin\theta\, d\varphi. \tag{179}$$

- **Gradient**: given a scalar field Φ,

$$\boxed{\operatorname{grad}\Phi = \boldsymbol{e}^\theta\, \partial_\theta \Phi + \boldsymbol{e}^\varphi\, \frac{1}{\sin\theta}\, \partial_\varphi \Phi.} \tag{180}$$

Exc 37 Derive Eq. (180) using differential geometry approach.

- **Curl**: given a vector field $\boldsymbol{A} = A_\theta\, \boldsymbol{e}^\theta + A_\varphi\, \boldsymbol{e}^\varphi$,

$$\boxed{\operatorname{curl}\boldsymbol{A} = \frac{1}{\sin\theta}\,(\partial_\theta(A_\varphi \sin\theta) - \partial_\varphi A_\theta)\, \boldsymbol{n}.} \tag{181}$$

Exc 38 Derive Eq. (181) using differential geometry approach.

- **Divergence**: given a vector field $\boldsymbol{A} = A_\theta\, \boldsymbol{e}^\theta + A_\varphi\, \boldsymbol{e}^\varphi$,

$$\boxed{\operatorname{div} \boldsymbol{A} = \frac{1}{\sin\theta}\,(\partial_\theta(A_\theta \sin\theta) + \partial_\varphi A_\varphi).} \tag{182}$$

Exc 39 Derive Eq. (182) using differential geometry approach.

- **Laplacian**: given a scalar field ψ,

$$\boxed{\nabla^2\psi = \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta\psi) + \frac{1}{\sin^2\theta}\,\partial_\varphi^2\psi.} \tag{183}$$

Exc 40 Derive Eq. (183) using differential geometry approach.

[3] Vincent Bouchard. MATH 315: Calculus IV (University of Alberta).

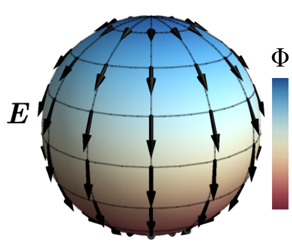
▪ Effective Gauge Field

Introduce the effective gauge field $(\Phi, \boldsymbol{A})$ on the sphere, such that

- Effective **electric field**:

$$\boxed{\boldsymbol{E}(\boldsymbol{n}) = -\nabla_\perp \Phi(\boldsymbol{n}).} \tag{184}$$

In this way, $\boldsymbol{E}$ is guaranteed to lie in the tangent plane, the same as the torque-induced force $\boldsymbol{\tau}\times\boldsymbol{n}$.



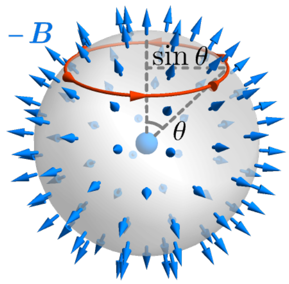
Example: a spinning top in a uniform gravity field $\Phi(\boldsymbol{n}) \propto n_z = \cos\theta$,

$$\boldsymbol{E} = -\nabla_\perp \Phi = \boldsymbol{e}_\theta \sin\theta. \tag{185}$$

- Effective **magnetic field**:

$$\boxed{\boldsymbol{B}(\boldsymbol{n}) = -S\,\boldsymbol{n} = \nabla_\perp \times \boldsymbol{A}(\boldsymbol{n}).} \tag{186}$$

where $S = I_a\,\omega$ is the spin angular momentum (magnitude). The magnetic field points towards the origin, where there is effective an **magnetic monopole**.



One common choice of vector potential $\boldsymbol{A}(\boldsymbol{n})$ to produce such magnetic field is the **Wu-Yang monopole potential** [4]

$$\boxed{\boldsymbol{A}(\boldsymbol{n}) = S\,\frac{\cos\theta - 1}{\sin\theta}\,\boldsymbol{e}_\varphi} \tag{187}$$

Exc 41 Verify that $\boldsymbol{A}(\boldsymbol{n})$ given in Eq. (187) satisfies Eq. (186).

- As a simple justification, along a *latitude loop* at the polar angle θ, the loop integral of the vector potential is

  $$\oint \boldsymbol{A}\cdot d\,\boldsymbol{l} = \int S\,\frac{\cos\theta - 1}{\sin\theta}\,\sin\theta\,d\varphi = 2\,\pi\,S\,(\cos\theta - 1), \tag{188}$$

  which indeed equals to the *magnetic flux* through the *spherical cap* from the north pole down to angle θ

  $$\int \boldsymbol{B}\cdot d\boldsymbol{\sigma} = -S\,\Omega(\theta) = -2\,\pi\,S\,(1-\cos\theta), \tag{189}$$

  where $\Omega(\theta) = 2\,\pi\,(1-\cos\theta)$ denotes the *solid angle* of the cap.

- However, as $\theta\to\pi$ near the south pole, $A_\varphi\to\infty$ diverges. What is going wrong?

[4] T. T. Wu, C. N. Yang. Dirac monopole without strings: monopole harmonics. Nuclear Physics B107 365-380 (1976).

▪ Quantization of Spin (or Monopole)

The divergence of A_φ is due to the requirement that an *infinitesimal latitude loop* near the south pole ($\theta\to\pi$) must accumulate a finite amount of **Berry phase** set by the *total magnetic flux* through the sphere

$$\Theta = \frac{1}{\hbar}\oint \boldsymbol{A}\cdot d\,\boldsymbol{l} = \frac{2\,\pi}{\hbar}\,\sin\theta\,A_\varphi = \frac{4\,\pi\,B}{\hbar} = -\frac{4\,\pi\,S}{\hbar}. \tag{190}$$

The divergence can not be avoid, unless … Θ is actually *equivalent* to 0, i.e.

$$\Theta = 2\,\pi\,n, \tag{191}$$

with $n\in\mathbb{Z}$.

- Therefore, it is only possible to avoid singular assignment of $\boldsymbol{A}(\boldsymbol{n})$ if the **spin angular momentum** S is *quantized* to

$$\boxed{S = \frac{\hbar}{2}\, n,} \tag{192}$$

  with $n = 0, 1, 2, \ldots$.

- This is also a statement about the **magnetic monopole**, that the **total magnetic flux** emitted by a magnetic monopole must be quantized to

$$\boxed{\phi_B = 4\,\pi\, B = -2\,\pi\,\hbar\, n.} \tag{193}$$

Mathematically, the singularity is avoided by using two overlapping coordinate patches (north and south hemispheres) with smooth gauge fields on each.

- On the **northern hemisphere** ($\theta \in [0, \pi/2]$, excluding the south pole):

$$\boldsymbol{A}_N\,(\boldsymbol{n}) = S\,\frac{\cos\theta - 1}{\sin\theta}\,\boldsymbol{e}_\varphi. \tag{194}$$

- On the **southern hemisphere** ($\theta \in [\pi/2, \pi]$, excluding the north pole):

$$\boldsymbol{A}_S\,(\boldsymbol{n}) = S\,\frac{\cos\theta + 1}{\sin\theta}\,\boldsymbol{e}_\varphi. \tag{195}$$

- On the **equator** ($\theta = \pi/2$) where both patches overlap, the two gauge potentials are related by a **gauge transformation**:

$$\boldsymbol{A}_N(\varphi) - \boldsymbol{A}_S(\varphi) = \hbar\,\boldsymbol{e}_\varphi\,\partial_\varphi\,\chi(\varphi), \tag{196}$$

  with $\chi(\varphi) = -2\,(S/\hbar)\,\varphi$.

Since φ and $\varphi + 2\,\pi$ correspond to the *same* point on the equator, the gauge transformation $\psi(\varphi) \to e^{i\,\chi(\varphi)}\,\psi(\varphi)$ is only consistent if

$$\begin{aligned} &e^{i\,\chi(\varphi)} = e^{i\,\chi(\varphi+2\,\pi)} \\ &\Rightarrow \exp(-i\,2\,(S/\hbar)\,\varphi) = \exp(-i\,2\,(S/\hbar)\,(\varphi + 2\,\pi)) \end{aligned} \tag{197}$$

which requires

$$\frac{2\,S}{\hbar} \in \mathbb{Z}, \tag{198}$$

reproducing the **spin quantization condition** in Eq. (192).

Quantum Spin

Hamiltonian

Consider the **spherical symmetric** case, where there is no external scalar potential

$\Phi(\boldsymbol{n}) = 0$. Similar to Eq. (48), the quantum dynamics of a spin is described by the Hamiltonian

$$\boxed{\hat{H} = -\frac{\hbar^2}{2\, I_t}\, \boldsymbol{D}_\perp^2 + \frac{S^2}{2\, I_a}.} \tag{199}$$

- $I_t = m$ - the *transverse* **moment of inertia** of the spin,
- $\boldsymbol{D}_\perp$ - **surface covariant derivative**, along tangent directions on the sphere,

$$\boxed{\boldsymbol{D}_\perp = \nabla_\perp - \frac{i}{\hbar}\, \boldsymbol{A},} \tag{200}$$

  - $\nabla_\perp = \boldsymbol{e}^\theta\, \partial_\theta + \boldsymbol{e}^\varphi\, \frac{1}{\sin\theta}\, \partial_\varphi$ - **surface gradient**,
  - $\boldsymbol{A} = A_\theta\, \boldsymbol{e}^\theta + A_\varphi\, \boldsymbol{e}^\varphi$ - **gauge connection** (vector potential) on the sphere. Take the Wu-Yang monopole potential:

$$A_\theta = 0,\ A_\varphi = S\, \frac{\cos\theta - 1}{\sin\theta}, \tag{201}$$

with the quantization condition ($n \in \mathbb{N}$),

$$S = \hbar\, s = \hbar\, \frac{n}{2}, \tag{202}$$

where $s = n/2$ is introduced as the **spin quantum number** (quantized to half integers), also characterizing the *monopole strength.*

In spherical coordinate, the Hamiltonian acts on the wave function $\psi(\theta, \varphi)$ as

$$\boldsymbol{D}_\perp^2\, \psi = \frac{1}{\sin\theta}\left(\partial_\theta(\sin\theta\, \partial_\theta) + \frac{1}{\sin\theta}\, (\partial_\varphi + i\, s\,(1 - \cos\theta))^2\right)\psi. \tag{203}$$

Exc 42 Derive Eq. (203) using differential geometry approach.

Schrödinger Equation

Solve the stationary **Schrödinger equation**:

$$\boxed{\hat{H}\, \psi(\theta, \varphi) = E\, \psi(\theta, \varphi).} \tag{204}$$

- Separation of variables: let

$$\psi(\theta, \varphi) = e^{i\,(m-s)\,\varphi}\, \psi(\theta), \tag{205}$$

Eq. (204) takes the form of the generalized associated Legendre differential equation

$$\left(\frac{1}{\sin\theta}\, \partial_\theta(\sin\theta\, \partial_\theta) - \frac{1}{\sin^2\theta}\, (m - s\cos\theta)^2 + \lambda\right)\psi(\theta) = 0 \tag{206}$$

where the eigenvalue λ is related to the energy E by

$$E = \frac{\hbar^2}{2\, I_t}\, \lambda + \frac{\hbar^2}{2\, I_a}\, s^2. \tag{207}$$

Exc 43 Justify Eq. (206) and Eq. (207).

- The equation Eq. (206) can be solved by

$$\psi(\theta) = (1-\cos\theta)^{\frac{s-m}{2}}\, (1+\cos\theta)^{\frac{s+m}{2}}\, P_{l-s}^{(s-m,s+m)}(\cos\theta), \tag{209}$$

where

- $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial,

$$P_n^{(a,b)}(x) = 2^{-n} \sum_{k=0}^{n} \frac{(n+a)!}{k!\,(n+a-k)!}\, \frac{(n+b)!}{(n-k)!\,(b+k)!}\, (x-1)^{n-k}\, (x+1)^k. \tag{210}$$

which is well-defined for $x \in [-1, 1]$ if $n,\ n+a,\ n+b \in \mathbb{N}$, implying the following quantization conditions:

  - $l = s,\ s+1,\ s+2,\ \ldots,$
  - $m = -l,\ -l+1,\ \ldots,\ l-1,\ l.$

- The corresponding eigenvalue λ is

$$\lambda = l(l+1) - s^2. \tag{211}$$

Exc 44 Verify Eq. (209) and Eq. (211).

■ Monopole Harmonics

Put together, define the **monopole harmonics** $Y_{slm}(\theta, \varphi)$ function

$$\boxed{Y_{slm}(\theta, \varphi) = \mathcal{N}\, e^{i\,(m-s)\,\varphi} (1-\cos\theta)^{\frac{s-m}{2}}\, (1+\cos\theta)^{\frac{s+m}{2}}\, P_{l-s}^{(s-m,s+m)}(\cos\theta).} \tag{215}$$

where $P_n^{(a,b)}$ is the Jacobi polynomial, and $\mathcal{N}$ is the *normalization* constant to ensure $\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, |\, Y_{slm}(\theta, \varphi)|^2 \sin\theta = 1$.

- The eigen wavefunction of $\hat{H}$ is given by

$$\boxed{\psi_{slm}(\theta, \varphi) = Y_{slm}(\theta, \varphi).} \tag{216}$$

- The corresponding eigen energy is

$$\boxed{E_{slm} = \frac{\hbar^2}{2\, I_t} \left(l(l+1) - s^2\right) + \frac{\hbar^2}{2\, I_a}\, s^2.} \tag{217}$$

In the *isotropic* limit ($I_a = I_t = I$), the eigen energy only depends on the quantum number l,

$$E_{slm} = \frac{\hbar^2}{2\,I}\; l(l+1). \tag{218}$$

- The **quantum numbers** s, l, m take values in
  - $s = 0, 1/2, 1, 3/2, \ldots \in \mathbb{N}/2$, with $S = \hbar\, s \Rightarrow$ sets the monopole strength (i.e. total **magnetic flux** through the unit sphere).
  - $l = s, s+1, s+2, \ldots \in s + \mathbb{N} \Rightarrow$ labels the **Landau levels** on the sphere.
  - $m = -l, -l+1, \ldots, l-1, l \Rightarrow$ labels the degenerated states within each Landau level $\Rightarrow$ Landau level-l has the **degeneracy** $(2\,l+1)$.

Angular Momentum

The total **angular momentum operator** is defined as

$$\boxed{\hat{\boldsymbol{L}} = \boldsymbol{n} \times (-i\,\hbar\, \boldsymbol{D}_\perp) + \boldsymbol{S}.} \tag{219}$$

where

- $\boldsymbol{S} = S\,\boldsymbol{n} = \hbar\, s\, \boldsymbol{n}$ is the **spin angular momentum** (*axial* component).
- $\boldsymbol{n} \times (-i\,\hbar\, \boldsymbol{D}_\perp)$ is the **orbital angular momentum** (*transverse* component) associated with the spinning axis precession, which is given by the *cross product* between:
  - $\boldsymbol{n}$ - axis **coordinate** (on the sphere),
  - $-i\,\hbar\, \boldsymbol{D}_\perp$ - axis **kinetic momentum** (in the tangent plane).

Explicitly,

$$\hat{\boldsymbol{L}} = \hbar\left(\boldsymbol{e}^\varphi(-i\,\partial_\theta) - \boldsymbol{e}^\theta\, \frac{1}{\sin\theta}\, ((-i\,\partial_\varphi) + s(1-\cos\theta)) + s\,\boldsymbol{n}\right). \tag{220}$$

Exc 45 | Derive Eq. (220).

According to Eq. (166) and Eq. (167),

$$\begin{aligned}
\boldsymbol{n} &= \cos\varphi \sin\theta\, \boldsymbol{e}^x + \sin\varphi \sin\theta\, \boldsymbol{e}^y + \cos\theta\, \boldsymbol{e}^z,\\
\boldsymbol{e}^\theta &= \cos\varphi \cos\theta\, \boldsymbol{e}^x + \sin\varphi \cos\theta\, \boldsymbol{e}^y - \sin\theta\, \boldsymbol{e}^z,\\
\boldsymbol{e}^\varphi &= -\sin\varphi\, \boldsymbol{e}^x + \cos\varphi\, \boldsymbol{e}^y.
\end{aligned} \tag{221}$$

$\hat{\boldsymbol{L}}$ in Eq. (220) can be decomposed in Cartesian coordinate system as

$$\boxed{\hat{\boldsymbol{L}} = \hat{L}_x\, \boldsymbol{e}^x + \hat{L}_y\, \boldsymbol{e}^y + \hat{L}_z\, \boldsymbol{e}^z,} \tag{222}$$

with

$$\hat{L}_x = \hbar\left(\sin\varphi\,(i\,\partial_\theta) + \cos\varphi\left(\cot\theta\,(i\,\partial_\varphi) + s\,\frac{1-\cos\theta}{\sin\theta}\right)\right),$$

$$\hat{L}_y = \hbar\left(-\cos\varphi\,(i\,\partial_\theta) + \sin\varphi\left(\cot\theta\,(i\,\partial_\varphi) + s\,\frac{1-\cos\theta}{\sin\theta}\right)\right),$$

$$\hat{L}_z = \hbar((-i\,\partial_\varphi) + s).$$

Exc 46 Derive Eq. (223).

They satisfy the following commutation relations:

$$\left[\hat{L}_x, \hat{L}_y\right] = i\,\hbar\,\hat{L}_z, \quad \left[\hat{L}_y, \hat{L}_z\right] = i\,\hbar\,\hat{L}_x, \quad \left[\hat{L}_z, \hat{L}_x\right] = i\,\hbar\,\hat{L}_y, \tag{224}$$

which could be summarized as $\hat{\boldsymbol{L}} \times \hat{\boldsymbol{L}} = i\,\hbar\,\hat{\boldsymbol{L}}$ in vector form. This is the defining property of any angular momentum operator.

Exc 47 Verify Eq. (224).

▪ Squared Angular Momentum

The **squared angular momentum operator** is generally defined as

$$\hat{\boldsymbol{L}}^2 := \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2. \tag{225}$$

- For our case of $\hat{\boldsymbol{L}}$ in Eq. (219), $\hat{\boldsymbol{L}}^2$ can be explicitly written out

$$\begin{aligned}\hat{\boldsymbol{L}}^2 &= -\hbar^2\,\boldsymbol{D}_\perp^2 + S^2 \\ &= \frac{-\hbar^2}{\sin\theta}\left(\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin\theta}\,(\partial_\varphi + i\,s\,(1-\cos\theta))^2\right) + \hbar^2\,s^2.\end{aligned} \tag{226}$$

- A key property of $\hat{\boldsymbol{L}}^2$ is that it commutes with any component of the angular momentum operator,

$$\left[\hat{\boldsymbol{L}}^2, \hat{L}_a\right] = 0 \quad (\text{for } a = x, y, z), \tag{227}$$

or simply written as $\left[\hat{\boldsymbol{L}}^2, \hat{\boldsymbol{L}}\right] = 0$ in vector form. It is commonly referred to as the **Casimir operator** of the $\mathfrak{so}(3)$ algebra — an element in the Lie algebra that *commutes* with all its *generators*.

Exc 48 Proof Eq. (227) based on Eq. (224).

Given $\left[\hat{\boldsymbol{L}}^2, \hat{L}_z\right] = 0$ (i.e. $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$ are commuting operators), they can be *simultaneously diagonalized* by a set of **common eigenstates**, which turns out to be the *monopole harmonics*

$Y_{slm}(\theta, \varphi)$,

$$\boxed{\begin{aligned}&\hat{\boldsymbol{L}}^2\, Y_{slm}(\theta, \varphi) = \hbar^2\, l\,(l+1)\, Y_{slm}(\theta, \varphi),\\ &\hat{L}_z\, Y_{slm}(\theta, \varphi) = \hbar\, m\, Y_{slm}(\theta, \varphi).\end{aligned}} \tag{228}$$

Exc 49

> Verify Eq. (228).

- The quantum numbers are reinterpreted as
  - $l = s,\ s+1,\ s+2,\ \ldots \in s + \mathbb{N}$ - **angular quantum number** (labeling quantized total angular momentum),
  - $m = -l,\ -l+1,\ \ldots,\ l-1,\ l$ - **magnetic quantum number** (labeling quantized z-component of angular momentum).

▪ Spin-1/2

The smallest *non-trivial* monopole strength is

$$s = 1/2, \tag{229}$$

corresponding to a quantum spinning top with *axial* **angular momentum**

$$S = \hbar\, s = \frac{\hbar}{2}. \tag{230}$$

The total angular momentum can not be smaller than S.

The *lowest* Landau level is achieved at $l = s = 1/2$, where the total angular momentum saturates its minimal value $S = \hbar/2$, realizing a spin-1/2 system. In this case:

- There are only two options for the quantum number m

$$m = \pm \frac{1}{2}, \tag{231}$$

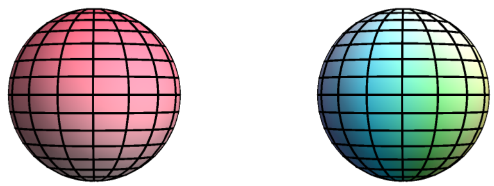
  corresponding to the *up* and *down* spin states.
- The corresponding *monopole harmonics* wave functions are

$$Y_{1/2,1/2,m}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}\, e^{i\,(m-1/2)\,\varphi} (1-\cos\theta)^{\frac{1/2-m}{2}}\, (1+\cos\theta)^{\frac{1/2+m}{2}}, \tag{232}$$

  or respectively as

$$\boxed{\begin{aligned}&Y_{1/2,1/2,+1/2}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}\, \sqrt{1+\cos\theta}\,,\\ &Y_{1/2,1/2,-1/2}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}\, e^{-i\,\varphi}\, \sqrt{1-\cos\theta}\,.\end{aligned}} \tag{233}$$

$Y_{1/2,1/2,+1/2}$ $\qquad$ $Y_{1/2,1/2,-1/2}$



- **Angular momentum** eigenvalues:

$$\begin{aligned} \hat{\boldsymbol{L}}^2\ Y_{1/2,1/2,\pm 1/2} &= \frac{3}{4}\hbar^2\ Y_{1/2,1/2,\pm 1/2}, \\ \hat{L}_z\ Y_{1/2,1/2,\pm 1/2} &= \pm\frac{1}{2}\hbar\ Y_{1/2,1/2,\pm 1/2}. \end{aligned} \tag{234}$$

- **Energy** eigenvalue (2-fold degenerated)

$$\hat{H}\ Y_{1/2,1/2,\pm 1/2} = \left(\frac{\hbar^2}{4\,I_t} + \frac{\hbar^2}{8\,I_a}\right) Y_{1/2,1/2,\pm 1/2}. \tag{235}$$

Puzzle: It seems that $Y_{1/2,1/2,-1/2}(\theta, \varphi)$ is not single-valued at the *south pole* ($\theta = \pi$), is there anything wrong?

- **Topological obstruction**: Monopole harmonics can not be *globally* single-valued in a naive coordinate sense because of the *nontrivial* **gauge curvature** induced by the monopole.
- However, the apparent multivaluedness is a **gauge artifact**, and can be removed by the *gauge transformation*.

Solution: define the monopole harmonics on different hemispheres with different gauge choices, and switch gauge choices at the equator by gauge transformation.

- For the **northern hemisphere** ($\theta \in [0, \pi/2]$, excluding the south pole):

$$\begin{aligned} \boldsymbol{A}_N(\theta, \varphi) &= \frac{\hbar}{2}\frac{\cos\theta - 1}{\sin\theta}\ \boldsymbol{e}_\varphi, \\ Y^N_{1/2,1/2,+1/2}(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}}\sqrt{1+\cos\theta}, \\ Y^N_{1/2,1/2,-1/2}(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}}\,e^{-i\varphi}\sqrt{1-\cos\theta}. \end{aligned} \tag{236}$$

- For the **southern hemisphere** ($\theta \in [\pi/2, \pi]$, excluding the north pole):

$$\boldsymbol{A}_S(\theta, \varphi) = \frac{\hbar}{2}\frac{\cos\theta + 1}{\sin\theta}\ \boldsymbol{e}_\varphi,$$

$$Y^S_{1/2,1/2,+1/2}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}\,e^{i\varphi}\,\sqrt{1+\cos\theta}\,,$$

$$Y^S_{1/2,1/2,-1/2}(\theta,\varphi)=\frac{1}{\sqrt{4\pi}}\,\sqrt{1-\cos\theta}\,.$$

- On the **equator** ($\theta=\pi/2$), the two gauge choices are related by a **gauge transformation**:

$$\begin{aligned}\boldsymbol{A}_N(\pi/2,\varphi)&=\boldsymbol{A}_S(\pi/2,\varphi)+\hbar\,\boldsymbol{e}_\varphi\,\partial_\varphi\,\chi(\varphi),\\ Y^N_{1/2,1/2,m}(\pi/2,\varphi)&=e^{i\,\chi(\varphi)}\,Y^S_{1/2,1/2,m}(\pi/2,\varphi),\end{aligned}\tag{238}$$

with $\chi(\varphi)=-\varphi$.

Exc 50 Verify Eq. (238).

Therefore, monopole harmonics can (only) be defined by piecing different gauge patches together with gauge transformations,

$$Y_{1/2,1/2,m}(\theta,\varphi)=\begin{cases}Y^N_{1/2,1/2,m}(\theta,\varphi) & \theta\in[0,\pi/2],\\ Y^S_{1/2,1/2,m}(\theta,\varphi) & \theta\in[\pi/2,\pi],\end{cases}\tag{239}$$

such that there is no singularity on any patch, and physical quantities are all well-behaved.