Machine Learning Holography

Lecture I: From tensor network to Boltzmann Machine

Referece: You, Yang, Qi, arXiv: 1709.01223
Hayden, Nezami, Qi, et.al., arXiv: 1601.01694
You, Gu, arXiv: 1803.10425

1° Entanglement Features

1.1° Quantum Entanglement – Quantum Information sharing

quantum bit: $|0\rangle, |1\rangle$ (qubit)

two qubits: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

entangled state: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \triangleq \begin{pmatrix}1\\0\\0\\1\end{pmatrix}/\sqrt{2}$

(EPR state) $\neq (\psi_0|0\rangle + \psi_1|1\rangle) \otimes (\phi_0|0\rangle + \phi_1|1\rangle)$

know everything about the system but nothing about its part

How to quantify entanglement?

1.2° Entanglement Entropy

density matrix $\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \triangleq \frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}$

reduced density matrix $|\underset{A}{\alpha}\underset{\bar{A}}{\beta}\rangle$

$$\rho_A = \mathrm{Tr}_{\bar{A}}\,\rho = \sum_\beta \langle\beta|\rho|\beta\rangle = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \triangleq \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

probability in $|0\rangle$ state $p_0 = 1/2$ (actually independent
probability in $|1\rangle$ state $p_1 = 1/2$ of basis choice)

entropy: $S = -\sum_i p_i \ln p_i = \ln 2 = 1$ bit

Von Neuman entropy

$$S_A^{(1)} = -\mathrm{Tr}\,\rho_A \ln \rho_A = -\sum_i p_i \ln p_i$$ (suppose p_i is eigenvalues of ρ_A)

Rény entropy

$$S_A^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\,\rho_A^n = \frac{1}{1-n} \ln \sum_i p_i^n$$

In fact, $S_A^{(1)} = \lim_{n\to 1} S_A^{(n)}$

$$\lim_{n\to 1} \frac{1}{1-n} \ln \sum_i p_i^n = \lim_{n\to 1} \frac{1}{\partial_n(1-n)} \partial_n \left(\ln \sum_i p_i^n\right) = \lim_{n\to 1} -\frac{\sum_i p_i^n \ln p_i}{\sum_i p_i^n}$$

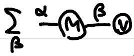
For EPR state $S_A^{(n)} = \frac{1}{1-n} \ln \left(\frac{1}{2}\right)^{n-1} = \ln 2$ For all n.

$\Rightarrow$ Rény entropy are also useful measures of entanglement

1.3° Tensor network representation

 vector  matrix  tensor

 $= v_\alpha$  $= M_{\alpha\beta}$.

$(Mv)_\alpha =$  $= \sum_\beta$  $= \sum_\beta M_{\alpha\beta} v_\beta$

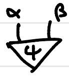
$v^T M v =$  $= \sum_{\alpha,\beta}$  $= \sum_{\alpha,\beta} v_\alpha M_{\alpha\beta} v_\beta$

$\mathrm{Tr} M =$  $= \sum_\alpha$  $= \sum_\alpha M_{\alpha\alpha}$

$\mathbb{1} =$ — as $\mathbb{1}_{\alpha\beta} = \overline{\alpha\ \beta} = \delta_{\alpha\beta}$

$\mathrm{Tr}\,\mathbb{1} =$  $= \sum_\alpha 1 = d$ ← dimension of the linear space

Quantum state as a tensor.

$|\psi\rangle = \sum_{\alpha\beta} \psi_{\alpha\beta} |\alpha\beta\rangle$ →  $= \psi_{\alpha\beta}$  $= \psi^*_{\alpha\beta}$

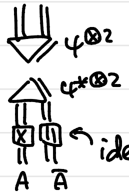
density matrix reduced density matrix

$\rho = |\psi\rangle\langle\psi| =$  $\rho_A = \mathrm{Tr}_{\bar{A}}\, \rho =$ 

let's consider 2nd Rény entropy

$S_A^{(2)} = -\ln \mathrm{Tr}\, \rho_A^2$

or $e^{-S_A^{(2)}} = \mathrm{Tr}\, \rho_A^2 = \mathrm{Tr}$  $=$  21世纪霉漫图 —程茶.

$= \mathrm{Tr}$  $= \mathrm{Tr}$  $= \mathrm{Tr}\left(|\psi\rangle\langle\psi|\right)^{\otimes 2} \mathcal{X}_{[\sigma]}$

$\mathcal{X}_{[\sigma]} = \mathcal{X}_{\sigma_1} \mathcal{X}_{\sigma_2}$ $\qquad \mathcal{X}_\sigma = \begin{cases} \| & \sigma = +1 \\ \mathrm{X} & \sigma = -1 \end{cases}$

Ising variables $\sigma_i = \pm 1$

our choice of A corresponds to $\sigma_1 = -1 \quad \sigma_2 = +1$

In general, use Ising configuration $[\sigma]$ to lable region A, s.t. $\sigma_i = \begin{cases} +1 & i \in \bar{A} \\ -1 & i \in A \end{cases}$ ←

or	$[\sigma] = (-,+)$	$A\ \bar{A}$
other	$(+,-)$	$\bar{A}\ A$
	$(+,+)$	$\bar{A}\ \bar{A}$
	$(-,-)$	$A\ A$

1.4° Entanglement Features.

$$W_\psi^{(n)}[\sigma] = e^{(1-n)S_\psi^{(n)}[\sigma]} = \mathrm{Tr}\,\rho_A^n \qquad \chi_{[\sigma]} = \prod_i \chi_{\sigma_i}$$

$$= \mathrm{Tr}\,(|\psi\rangle\langle\psi|)^{\otimes n}\,\chi_{[\sigma]} \qquad \chi_{\sigma_i} = \begin{cases} |||| & \sigma_i = +1 \\ \times\!\!\times & \sigma_i = -1 \end{cases}$$

$\left\{ \begin{array}{l} \text{all orders of Rény } n \\ \text{over all possible entanglement regions } [\sigma] \end{array} \right.$

we will focus on 2^{nd} Rény $(n=2)$ from now on.

$$W_\psi[\sigma] = e^{-S_\psi[\sigma]} = \mathrm{Tr}(|\psi\rangle\langle\psi|)^{\otimes 2}\,\chi_{[\sigma]}$$

very much like an Ising model

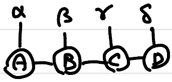
$S_\psi[\sigma]$ - energy $\quad W_\psi[\sigma]$ - Boltzman weight.

2° Random Tensor Network (RTN)

2.1° RTN: tensor network of random tensors $\qquad P[\psi] \propto e^{-\|\psi\|^2}$

example: Page State

$$|\psi\rangle = \sum_{\alpha\beta\gamma\delta} \psi_{\alpha\beta\gamma\delta}\,|\alpha\beta\gamma\delta\rangle \qquad \psi_{\alpha\beta\gamma\delta} =$$  $\sim \mathcal{N}(0,1)$

random MPS state $\qquad \psi_{\alpha\beta\gamma\delta} =$  $\left.\begin{array}{l} A_{\alpha m} \\ B_{\beta m n} \\ C_{\gamma n l} \\ D_{\delta l} \end{array}\right\} \sim \mathcal{N}(0,1)$

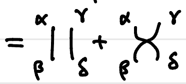
2.2° Ensemble Average

$\mathbb{E}$  $= \mathbb{E}\,\psi_\alpha\psi_\beta^* = \int D[\psi]\,\psi_\alpha\psi_\beta^*\,e^{-\|\psi\|^2} = \delta_{\alpha\beta} = |^{\alpha}_{\beta}$

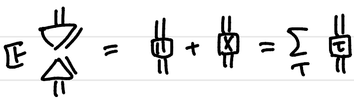
$\mathbb{E}$  $= \mathbb{E}\,\psi_\alpha\psi_\beta^*\psi_\gamma\psi_\delta^* = \int D[\psi]\,\psi_\alpha\psi_\beta^*\psi_\gamma\psi_\delta^*\,e^{-\|\psi\|^2}$

free scalar field wick theorem

$$= \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}$$

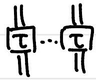


can be written as

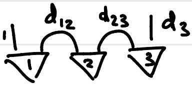


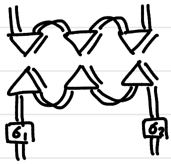
 $\equiv \chi_\tau = \begin{cases} || & \tau = +1 \\ \times & \tau = -1 \end{cases}$

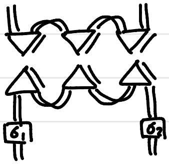
what if the tensor has more legs

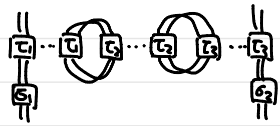
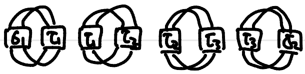
$\mathbb{E}$  $= \sum_{[\tau]}$ 

2.3° Entanglement Features of RTN states

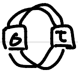
$|\psi\rangle =$ 

$W_\psi[\sigma] = \mathrm{Tr}\,(|\psi\rangle\langle\psi|)^{\otimes 2}\, \chi_{[\sigma]} = \mathrm{Tr}$ 

$\mathbb{E}\, W_\psi[\sigma] = \mathbb{E}\,\mathrm{Tr}$ 

$= \sum_{[\tau]} \mathrm{Tr}$  $= \sum_{[\tau]}$ 

ok, what is

 $= \mathrm{Tr}$  $= \mathrm{Tr}\, \chi_\tau \chi_\sigma^{T} = \mathrm{Tr}\, \chi_{\sigma\tau} = \begin{cases} \text{if } \sigma\tau = +1 & = d^2 = e^{2\ln d} \\ \text{if } \sigma\tau = -1 & = d = e^{\ln d} \end{cases}$

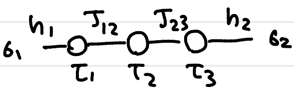
 

$= e^{(\frac{1}{2}\ln d)\,\sigma\tau + \frac{3}{2}\ln d} = e^{J\sigma\tau + E_0}$, $J = \frac{1}{2}\ln d$, $E_0 = \frac{3}{2}\ln d$

$\mathbb{E}\, W_\psi[\sigma] \propto \sum_{[\tau]} e^{-E[\sigma,\tau]}$

$E[\sigma,\tau] = -h_1\sigma_1\tau_1 - J_{12}\tau_1\tau_2 - J_{23}\tau_2\tau_3 - h_2\tau_3\sigma_2$

$= -\sum_{ij} J_{ij}\tau_i\tau_j - \sum_{i\in\partial} h_i\sigma_i\tau_i$



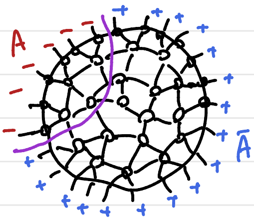
where $J_{ij} = \frac{1}{2}\ln d_{ij}$ $h_i = \frac{1}{2}\ln d_i$

$$e^{-F[\sigma]} = \sum_{[\tau]} e^{-E[\sigma,\tau]}$$

$$\mathbb{E}\, e^{-S[\sigma]} = \mathbb{E}\, W_\psi[\sigma]$$

$S[\sigma] \sim$ Free energy of the Ising model, subject to the boundary condition

2.4° Holographic Interpretation



suppose a many-body state has a holographic RTN representation.

$S(A) \propto$ Free engy of domain wall

$\propto$ geodesic in the bulk

$\Rightarrow S(A) = \frac{1}{4G_N}|\gamma_A|$ Ryu-Takanayaki

3° Entanglement Feature Learning

Given $S(A)$ over different regions, try to feagure out Ising model J_{ij}

3.1° Boltzmann Machine

generative model, unsupervised learning

Given image data set $\{[\sigma], \cdots\}$ following $P_{dat}[\sigma]$

Build a model to generate $[\sigma]$ with $P_{mdl}[\sigma]$ unknow.

Bring $P_{mdl} \to P_{dat}$ by minimizing KL divergence.

$$\mathcal{L} = KL(P_{dat} \| P_{mdl}) = \sum_{[\sigma]} P_{dat}[\sigma] \ln \frac{P_{dat}[\sigma]}{P_{mdl}[\sigma]}$$

$$= -\sum_{[\sigma]} P_{dat}[\sigma] \ln P_{mdl}[\sigma] + \text{const.}$$

Boltzmann Machine

$$P_{mdl}[\sigma] = \frac{1}{Z} e^{-F[\sigma]} = \frac{1}{Z}\sum_{[\tau]} e^{-E[\sigma,\tau]}$$

$$\hookrightarrow Z = \sum_{[\sigma]} e^{-F[\sigma]} = \sum_{[\sigma,\tau]} e^{-E[\sigma,\tau]}$$

$$\mathcal{L} = -\sum_{[\sigma]} P_{dat}[\sigma]\left(-F[\sigma] - \ln Z\right)$$

$\hookrightarrow F = -\ln Z$ unclamped. free energy

$$= \langle F[\sigma]\rangle_{dat} - F$$

clampped unclampped.

$$E[\sigma,\tau] = -\sum_{ij} J_{ij}\tau_i\tau_j - \sum_{i\in\partial} h_i \sigma_i \tau_i$$

$h_i = \frac{1}{2}\ln d$ given

J_{ij} to be trained.

$$\frac{\partial \mathcal{L}}{\partial J_{ij}} = \langle \tau_i \tau_j\rangle_{cl} - \langle \tau_i \tau_j \rangle_{uncl}$$

to minimize $\mathcal{L}$: $J_{ij} \to J_{ij} + \gamma \frac{\partial \mathcal{L}}{\partial J_{ij}}$ γ: learning rate

Pdat : Monte Carlo Sampling A weight by energy $S(A)$

3.2° After training J_{ij} → encodes the holographic bulk geometry.

Free fermion CFT with mass deformation.

$z_{cut} \sim -\log m$ depth = RG scale

as $m \to 0$ towards CFT. the AdS bulk grows deeper.

4° with dynamics → gravitational fluctuation?

$W_\psi[6] = \mathrm{Tr}(|\psi\rangle\langle\psi|)^{\otimes 2} X_{[6]}$

$W_u[6,\tau] = \mathrm{Tr}\ U^{\dagger\otimes 2} X_{[6]} U^{\otimes 2} X_{[\tau]}$

Operator formulation

$|W_\psi\rangle = \sum_{[6]} W_\psi[6] |[6]\rangle$ $\hat{W}_u = \sum_{[6,\tau]} |[6]\rangle W_u[6,\tau] \langle[\tau]|$

now $|\psi'\rangle = U|\psi\rangle$

$\Rightarrow |W_{\psi'}\rangle = \hat{W}_u \hat{W}_{\mathbb{1}}^{-1} |W_\psi\rangle$

when $U \sim \mathbb{1} + i\delta t H$ $\Rightarrow \hat{W}_u = (\mathbb{1} - t\hat{H}_{EF}) \hat{W}_{\mathbb{1}}$

$\Rightarrow \partial_t |W_\psi\rangle = -\hat{H}_{EF} |W_\psi\rangle \Rightarrow$ entanglement dynamics.

Lecture II: From Renormalization Group to Generative Model

Reference

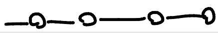
Hu, Li, Wang, You. arXiv: 1903.00804

Qi, arXiv: 1309.6282

Li, Wang. arXiv: 1802.02840

1° Exact Holographic Mapping

1.1° lattice Dirac fermion



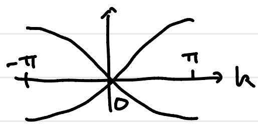
$$H = \sum_i \frac{1}{2}\left(c_{i+1}^\dagger (i\sigma^x - \sigma^y) c_i + h.c.\right) + c_i^\dagger \sigma^y c_i$$

$c_i = \begin{pmatrix} c_{i\uparrow} \\ c_{i\downarrow} \end{pmatrix}$ translation symmetry $\rightarrow c_i = \sum_{k \in BZ} e^{iki} c_k$

Momentum space

$BZ: (-\pi, \pi)$

$$H = \sum_k c_k^\dagger h(k) c_k$$



$h(k) = \sin k \sigma^x + (1-\cos k)\sigma^y$

$\epsilon(k) = \pm\sqrt{\sin^2 k + (1-\cos k)^2}$

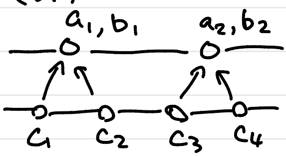
near $k \to 0$ $\quad \epsilon(k) = \pm|k|$ Dirac

1.2° EHM transformation (Real space RG)

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_{2i-1} \\ c_{2i} \end{pmatrix}$$

a: $k \sim 0$ mode, low energy

b: $k \sim \pi$ mode, high energy



enlarged unit cell $\rightarrow$ folded BZ

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \sum_i e^{-ik(2i)}\begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{BZ}{2}$$

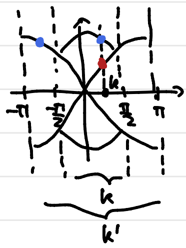
$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \sum_i \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{-2iki} \sum_{k' \in BZ}\begin{pmatrix} e^{ik'(2i-1)} \\ e^{ik'2i} \end{pmatrix} c_{k'}$$

$$= \sum_{k' \in BZ} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-ik'} \\ 1 \end{pmatrix} \sum_i e^{i2(k'-k)i} c_{k'}$$

$$\| \; 2(k'-k) \in 2\pi\mathbb{Z}$$

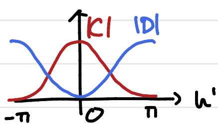
$$= \sum_{k' \in BZ} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-ik'}+1 \\ e^{-ik'}-1 \end{pmatrix} \frac{1}{\sqrt{2}} \delta_{k' = k + \pi\mathbb{Z}} c_{k'}$$

$$= \sum_{k' \sim k} \frac{1}{2}\begin{pmatrix} e^{-ik'}+1 \\ e^{-ik'}-1 \end{pmatrix} c_{k'}$$



$C(z) = \frac{1}{\sqrt{2}}(z+1) \qquad a_k = \frac{1}{\sqrt{2}}\sum_{k' \sim k} C(e^{-ik'}) c_{k'}$

$D(z) = \frac{1}{\sqrt{2}}(z-1) \qquad b_k = \frac{1}{\sqrt{2}}\sum_{k' \sim k} D(e^{-ik'}) c_{k'}$



1.3° Hamiltonian

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \sum_{k'\sim k} \frac{1}{\sqrt{2}} \begin{pmatrix} C(e^{-ik'}) \\ D(e^{-ik'}) \end{pmatrix} c_{k'} = \frac{1}{\sqrt{2}} \begin{pmatrix} C(e^{-ik}) & C(-e^{-ik}) \\ D(e^{-ik}) & D(-e^{-ik}) \end{pmatrix} \begin{pmatrix} c_k \\ c_{k+\pi} \end{pmatrix} = F(k) \begin{pmatrix} c_k \\ c_{k+\pi} \end{pmatrix}$$

↳ $k' = k$ or $k' = k+\pi$

$$\Rightarrow \begin{pmatrix} c_k \\ c_{k+\pi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} C(e^{ik}) & D(e^{ik}) \\ C(-e^{ik}) & D(-e^{ik}) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = F(k)^\dagger \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$H = \sum_{k'\in BZ} c_{k'}^\dagger (\sin k' \sigma^x + (1-\cos k')\sigma^y) c_{k'}$$

$$= \sum_{k\in \frac{BZ}{2}} c_k^\dagger (\sin k\, \sigma^x + (1-\cos k)\sigma^y) c_k + c_{k+\pi}^\dagger (-\sin k\, \sigma^x + (1+\cos k)\sigma^y) c_{k+\pi}$$

$$= \sum_{k\in \frac{BZ}{2}} (c_k^\dagger \; c_{k+\pi}^\dagger)\, H(k) \begin{pmatrix} c_k \\ c_{k+\pi} \end{pmatrix} \qquad H(k) = \begin{pmatrix} \sin k\,\sigma^x + (1-\cos k)\sigma^y & 0 \\ 0 & -\sin k\,\sigma^x + (1+\cos k)\sigma^y \end{pmatrix}$$

$$= \sum_{k\in \frac{BZ}{2}} (a_k^\dagger \; b_k^\dagger)\, \underbrace{F(k) H(k) F(k)^\dagger}_{\parallel} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$= \sum_{k\in \frac{BZ}{2}} (a_k^\dagger \; b_k^\dagger)\, \tilde{H}(k) \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$\tilde{H}(k) = \frac{1}{2} \begin{pmatrix} \sin 2k\,\sigma^x + (1-\cos 2k)\sigma^y & -i((\cos 2k - 1)\sigma^x + \sin 2k\,\sigma^y) \\ i((\cos 2k - 1)\sigma^x + \sin 2k\,\sigma^y) & -\sin 2k\,\sigma^x + (3+\cos 2k)\sigma^y \end{pmatrix}$$

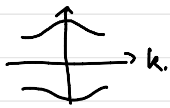
$$H_a = \sum_{k\in \frac{BZ}{2}} a_k^\dagger \frac{1}{2} (\sin 2k\,\sigma^x + (1-\cos 2k)\sigma^y) a_k$$

rescale $k \to k/2$ $\quad H_a \to H_a/2$

$$H_a = \sum_{k\in BZ} a_{k/2}^\dagger (\sin k\,\sigma^x + (1-\cos k)\sigma^y) a_{k/2}$$

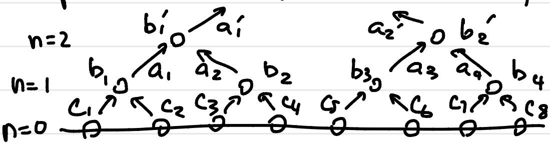
↳ Hamiltonian of the low energy freedom $a_{k/2}$ is fixed point under RG transformation

$$H_b = \sum_{k\in BZ} b_{k/2}^\dagger (-\sin k\,\sigma^x + (3+\cos k)\sigma^y) b_{k/2}$$



↳ Hamiltonian of the high energy freedom is massive.

Repeat the transformation for a_k, leaving b_k at its current layer



boundy $\quad$ bulk (1-dim higher!)

$c_i \to b_{n,i}$

layer $\quad$ site.

1.4° correlation function

$$a_k = \frac{1}{\sqrt{2}} \sum_{k'\sim k} C(e^{-ik'}) c_{k'} \qquad \longrightarrow \qquad a_{n k_n} = \frac{1}{\sqrt{2}} \sum_{k_{n-1}\sim k_n} C(e^{-i k_{n-1} 2^{n-1}}) a_{n-1\, k_{n-1}}$$

$$b_k = \frac{1}{\sqrt{2}} \sum_{k'\sim k} D(e^{-ik'}) c_{k'} \qquad \qquad b_{n k_n} = \frac{1}{\sqrt{2}} \sum_{k_{n-1}\sim k_n} D(e^{-i k_{n-1} 2^{n-1}}) a_{n-1\, k_{n-1}}$$

apply the EHM transformation recurssively

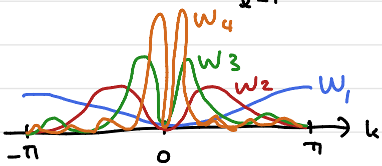
$$b_{n k_n} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \sum_{k_{n-2}\sim k_n} D(e^{-i k_{n-1} 2^{n-1}}) C(e^{-i k_{n-2} 2^{n-2}}) a_{n-2\, k_{n-2}}$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_0\sim k_n} D(e^{-i k_0 2^{n-1}}) C(e^{-i k_0 2^{n-2}}) \cdots \underbrace{a_{0,k_0}}_{=c_{k_0}}$$

$$= \sum_{k_0\sim k_n} W_n(e^{-ik_0})\, c_{k_0}$$

where

$$W(z) = D(z^{2^{n-1}}) \prod_{l=1}^{n-1} C(z^{2^{l-1}}) = \frac{1}{\sqrt{2^n}} \frac{(z^{2^{n-1}} - 1)^2}{z-1}$$

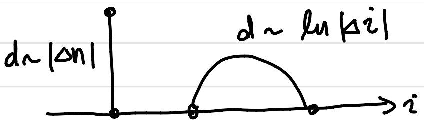


bulk correlation $\langle b_{n_1 i_1} b^\dagger_{n_2 i_2} \rangle = G(n_1 i_1, n_2 i_2)$

bounday correlation $\langle c_k c_k^\dagger \rangle = G_\partial(k)$

$$G(n_1 i_1, n_2 i_2) = \sum_{k\in BZ} G_\partial(k)\, W_{n_1}(e^{-ik}) W_{n_2}(e^{ik})\, e^{ik(2^{n_1} i_1 - 2^{n_2} i_2)}$$

$$d(n_1 i_1, n_2 i_2) = -\ln |G(n_1 i_1, n_2 i_2)|$$



consistent with hyperbolic geometry

$$ds^2 = \frac{dy^2 + dx^2}{y^2}$$

if $(x, y) = (2^n i, 2^n)$

Hilbert space preserving RG $\leftrightarrow$ Exact holographic mapping

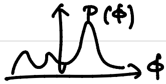
Questis: Generalization to Interacting Systems?

2° QFT and Renormalization Group

2.1° Path Integral

ϕ : real scalar field. consider "0D" QFT toy model (↙ no space time)

ϕ can fluctuate



↳ following some probability distribution

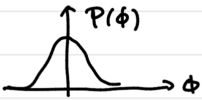
$$P(\phi) = \frac{1}{Z} e^{-S(\phi)} \quad \text{where } Z = \int d\phi \, e^{-S(\phi)}$$

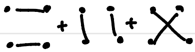
$S(\phi)$ ↓ negative log likelyhood = action (energy)

free theory (Gaussian) : $S(\phi) = \frac{1}{2\sigma^2}\phi^2$

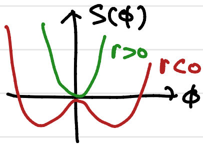
correlation function $\langle \phi\phi\cdots\phi \rangle = \frac{1}{Z}\int d\phi \, \phi\phi\cdots\phi \, e^{-S(\phi)}$

$\langle \phi \rangle = 0 \quad \langle \phi\phi \rangle = \sigma^2 \quad \langle \phi^4 \rangle = 3\sigma^4 \quad \langle \phi^{2n} \rangle = (2n-1)!!\,\sigma^{2n}$




Wick's theorem.

interacting theory : $S(\phi) = \frac{r}{2}\phi^2 + \lambda\phi^4$

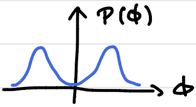


no simple formula for $\langle \phi^{2n} \rangle$

→ sample ϕ and take statistical average.

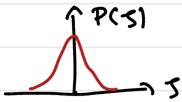
How to generate ϕ efficiently ? → random number generator

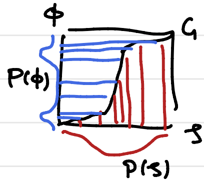
target



$\phi = G(s)$ ←

source (Gaussian)





$$P(\phi)\,d\phi = P(s)\,ds$$

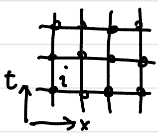
$$\Rightarrow \underset{\text{interaction theory}}{\underset{\uparrow}{P(\phi)}} = P(s)\left(\frac{d\phi}{ds}\right)^{-1} = \underset{\text{free theory}}{\underset{\uparrow}{P(s)}}\left(\frac{\partial G(s)}{\partial s}\right)^{-1}$$

G: non-linear function.

2.2° Complex Scalar Field Model

introduce spacetime (descretized Eucleandian) lattice

at each point i → define $\phi_i \in \mathbb{C}$



$$S[\phi] = -t\sum_{\langle ij\rangle}(\phi_i^*\phi_j + h.c.) + \sum_i (r|\phi_i|^2 + \lambda|\phi_i|^4)$$

Momentum space $\phi_i = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}_i}\phi_{\vec{k}}$

$$S[\phi] = \sum_{\vec{k}} \phi_{\vec{k}}^* \epsilon_{\vec{k}} \phi_{\vec{k}} + \text{interaction} \ldots$$

$\hookrightarrow \epsilon_{\vec{k}} = -2(\cos k_x + \cos \underset{\omega}{\underset{\shortparallel}{k_\tau}}) + r$

at large scale, $k_x, \omega \to 0$

$\epsilon_{\vec{k}} = \omega^2 + k_x^2 + \underbrace{(r-2)}_{m^2}$ so. $S[\phi] = \phi^*(-\partial_\tau^2 - \partial_x^2 + m^2)\phi + \lambda|\phi|^4$

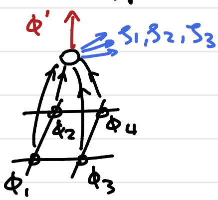
phase diagram

$$\underset{m^2<0}{\text{algebraic}} \quad \underset{KT}{|} \quad \underset{m^2>0}{\text{disordered}} \longrightarrow m^2$$

$\langle\phi_i\phi_j^*\rangle \sim r_{ij}^{-\alpha}$ $\quad\langle\phi_i\phi_j^*\rangle \sim e^{-r_{ij}/\xi}$

$\hookrightarrow$ compact boson CFT, Luttinger liquid theory.

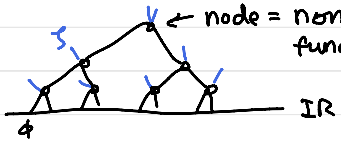
2.3° real space renormalization group



$$\underbrace{(\phi_1, \phi_2, \phi_3, \phi_4)}_{UV} \xrightarrow{RG} (\underset{\substack{IR\\(\text{relevant})}}{\phi'} ; \underbrace{\xi_1, \xi_2, \xi_3}_{\substack{\text{noise}\\(\text{irrelevant})}})$$

$$(\phi_1, \phi_2, \phi_3, \phi_4) = G(\phi'; \xi_1, \xi_2, \xi_3)$$

$\uparrow$ generator = inverse RG.



$\leftarrow$ node = non-linear function G $\leftarrow$ represented by neural net (subject to training)

Machine designed RG scheme.

(1) initiate sampling from the bulk.

$\xi_{n,i} \sim \mathcal{N}(0,1)$ i.e. $P[\xi] \propto e^{-|\xi|^2}$ Gaussian random source

(2) push through the network to the boundary,

obtain a sample $\phi = G[\xi]$

mean while can calculate $P[\phi] = P[\xi]\det(\delta_\xi G[\xi])^{-1}$

or more stable version: $\ln P[\phi] = \ln P[\xi] - \ln\det(\delta_\xi G[\xi])$

(3) objective: bring $P[\phi]$ close to $\frac{1}{Z}e^{-S[\phi]}$ by minimizing

$$KL(P[\phi] \| e^{-S[\phi]}) = \int D[\phi] P[\phi](\ln P[\phi] - \ln e^{-S[\phi]})$$

$$= \int D[\phi] P[\phi](\ln P[\xi] - \ln\det(\delta_\xi G[\xi]) + S[G[\phi]])$$

$$\mathcal{L} = KL(P[\phi] \| e^{-S[\phi]}) = \sum_{\xi \sim \mathcal{N}(0,1)} S[G[\xi]] - \ln\det(\delta_\xi G[\xi])$$

G: non-linear functions parametrized by model parameters

(4) gradient descend $G \to G - \gamma \frac{\partial \mathcal{L}}{\partial G}$

all we need is the expression of $S[\phi]$

$\to$ then the machine learns G, i.e. the optimal EHM

A few words about $\ln\det(\delta_\xi G[\xi])$

$x \xrightarrow{f} y \xrightarrow{g} z \qquad \left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right)$ in series

$$\ln\det\left(\frac{\partial z}{\partial x}\right) = \ln\det\left(\frac{\partial z}{\partial y}\right) + \ln\det\left(\frac{\partial y}{\partial x}\right)$$

$x\begin{cases} x_A \xrightarrow{f_A} y_A \\ x_B \xrightarrow{f_B} y_B \end{cases}\Big\} y \qquad \left(\frac{\partial y}{\partial x}\right) = \begin{pmatrix} \frac{\partial y_A}{\partial x_A} & 0 \\ 0 & \frac{\partial y_B}{\partial x_B} \end{pmatrix}$ in parallel.

$$\ln\det\left(\frac{\partial y}{\partial x}\right) = \ln\det\left(\frac{\partial y_A}{\partial x_A}\right) + \ln\det\left(\frac{\partial y_B}{\partial x_B}\right)$$

2.4° holographic interpretation.

$\min KL(P[\phi] \| e^{-S[\phi]})$

$Z = \int D[\phi]\, e^{-S[\phi]} \xleftrightarrow{\text{dual.}} Z = \int D[\xi] D[G]\, P[\xi] \det(\delta_\xi G[\xi])^{-1}$

CFT — AdS (gravity)

ϕ: field at criticality

ξ: massive matter field

G: gravitational background

push the boundary field back into the bulk

$\xi = G^{-1}[\phi] \qquad \xi_{ni} = G^{-1}[\phi]_{ni}$

$$\langle \xi_{n_1 i_1} \xi^*_{n_2 i_2} \rangle = \frac{1}{Z}\int D[\phi]\, e^{-S[\phi]}\, G^{-1}[\phi]_{n_1 i_1} G^{-1}[\phi]_{n_2 i_2}$$

measure the geometry by mutal information in the bulk

$$d(n_1 i_1, n_2 i_2) = -\ln|\langle \xi_{n_1 i_1} \xi^*_{n_2 i_2} \rangle|$$

$\Rightarrow ds^2 = \frac{1}{y^2}(dy^2 + dx_0^2 + dx_1^2)$ can be verified.

Future direction: gravitational fluctuation $\to$ learning dynamics

gauge fluctuation $\to$ allow equalvariant functions

Lecture III: From AdS/CFT to neural ODE

References

Hashimoto, et. al. arXiv:1802.08313, 1809.10536, 1903.04951

Hartnoll, Lucas, Sachdev arXiv: 1612.07324

McGreevy, arXiv: 0909.0518

1° AdS/CFT Correspondance.

1.1° Gubser-Klebanov-Polyakov-Witten (GKPW) Formula

Holographic Mapping ↔ Invertible RG ↔ Generative Model

Boltzmann Machine : generate Ising configuration $[\sigma]$

$$P_{dat}[\sigma] \sim e^{-F[\sigma]}$$

$$P_{mdl}[\sigma] \sim \sum_{[\tau]} e^{-E[\sigma,\tau]} \qquad E[\sigma,\tau] = -\sum_{\langle ij\rangle} J_{ij}\tau_i\tau_j - \sum_{i\in\partial} \sigma_i\tau_i$$

objective:

$$\min \ KL\left(P_{dat}[\sigma] \,\|\, P_{mdl}[\sigma]\right)$$

$$W_{QFT}[\sigma] = e^{-F[\sigma]} \xleftrightarrow{\text{dual}} W_{Grav.}[\sigma] = \sum_{[\tau]} e^{-E[\sigma,\tau]}$$

boundary field $[\sigma]$ (visible units)

bulk field $[\tau]$ (hidden units) fluctuating on background geometry → J_{ij} (conectivity)

Contineous version: Ising spins → scalar field.

$$Z_{QFT}[J] = \langle e^{-\int dx\, J(x) O(x)} \rangle_{QFT} \qquad J: \text{probe} \quad O: \text{observable}$$

$$Z_{Grav.}[J] = \int^{\varphi\to J} D[\varphi]\, e^{-S[\varphi]}$$

$$= e^{-S[\varphi^* \to J]} \quad \text{(classical limit)}$$

1.2° Anti-de Sitter Spacetime

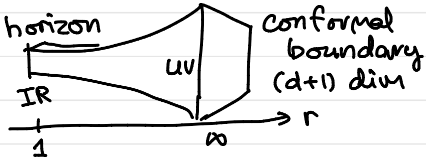
AdS : solution of Einstein equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with negative cosmological constant $\Lambda < 0$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad x^\mu = (r, t, \vec{x}) \qquad \vec{x}: d\text{-dim}$$

$$= -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\vec{x}^2$$

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & & & \\ & \frac{1}{f(r)} & & \\ & & r^2 & \\ & & & \ddots \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{f(r)} & & & \\ & f(r) & & \\ & & \frac{1}{r^2} & \\ & & & \ddots \end{pmatrix}$$

$$\sqrt{g} = |\det g_{\mu\nu}|^{1/2}$$
$$= r^d$$



Black hole solution $f(r) = r^2 - 1$

1.3° Scalar field on AdS

$$S[\varphi] = \int_M \sqrt{g}\,\left(g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + m^2\varphi^2\right)$$
$$= \underbrace{\int_M \sqrt{g}\,\varphi(-\Box + m^2)\varphi}_{\text{bulk}} + \underbrace{\int_M \partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\varphi\,\partial_\nu\varphi\right)}_{\text{boundary}}$$

$$S_{blk}[\varphi] = \int_M \sqrt{g}\,\varphi(-\Box + m^2)\varphi$$
$$\Box = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu\right)$$

$$S_{bdy}[\varphi] = \int_{\partial M}\sqrt{\gamma}\;\varphi\, n^\alpha\partial_\alpha\varphi$$ γ: induced metric, n: normal unit vector

Equation of motion

$$(-\Box + m^2)\varphi = 0$$

Assume we want to study uniform ($\partial_x = 0$) static ($\partial_t = 0$) responce.

$$\Box = \frac{1}{r^d}\partial_r\left(r^d f(r)\partial_r\right) = f(r)\partial_r^2 + \left(\frac{d\,f(r)}{r} + f'(r)\right)\partial_r$$

$$\Rightarrow\; -\frac{1}{r^d}\partial_r\left(r^d f(r)\partial_r\varphi\right) + m^2\varphi = 0$$

1.4° conformal boundary $(r\to\infty)$ $f(r)\to r^2$

$$\left(-r^2\partial_r^2 - (d+2)\,r\,\partial_r + m^2\right)\varphi = 0$$ homogeneous equation

trial $\varphi \sim r^{-\Delta}$

$$-\Delta(\Delta+1) + (d+2)\Delta + m^2 = 0 \;\Rightarrow\; \Delta_\pm = \frac{1}{2}\left(D \pm \sqrt{D^2 + 4m^2}\right)$$

$D = d+1$ spacetime dimension

so in general

$$\varphi = \frac{\varphi_-}{r^{\Delta_-}} + \frac{\varphi_+}{r^{\Delta_+}}$$

for $m^2 > 0$, $\Delta_- < 0 < \Delta_+ \Rightarrow \varphi_-$ dominates $r\to\infty$

$$J \to \varphi \;\Rightarrow\; J = \varphi_-$$

$$S_{bdy}[\varphi] = \int_{\partial M} \sqrt{\gamma}\, \varphi\, n^\alpha \partial_\alpha \varphi \qquad \sqrt{\gamma} = r^D \quad n^\alpha = (r, 0, 0 \cdots)$$

$$= \int_{\partial M} r^D \left(\frac{\varphi_-}{r^{\Delta_-}} + \frac{\varphi_+}{r^{\Delta_+}}\right) r\partial_r \left(\frac{\varphi_-}{r^{\Delta_-}} + \frac{\varphi_+}{r^{\Delta_+}}\right)$$

$$= \int_{\partial M} r^D \left(-\Delta_- \frac{\varphi_-^2}{r^{2\Delta_-}} - (\Delta_- + \Delta_+)\frac{\varphi_- \varphi_+}{r^{\Delta_- + \Delta_+}} + \cdots \right)$$

$$= \int_{\partial M} \left(-\Delta_- \frac{\varphi_-^2}{r^\nu} - D\varphi_- \varphi_+ + \cdots \right)$$

$\nu = \sqrt{D^2 + 4m^2}$

contact term to be regularized $\qquad \varphi_-(D\varphi_+)$ coupling

$\uparrow J \quad \uparrow O$

so $\quad J = \varphi_- \quad \langle O \rangle = D\varphi_+$

$$\begin{cases} \varphi = \dfrac{\varphi_-}{r^{\Delta_-}} + \dfrac{\varphi_+}{r^{\Delta_+}} \\ \partial_r \varphi = -\dfrac{\Delta_-}{r}\dfrac{\varphi_-}{r^{\Delta_-}} - \dfrac{\Delta_+}{r}\dfrac{\varphi_+}{r^{\Delta_+}} \end{cases} \Rightarrow \begin{cases} J = r^{\Delta_-} \dfrac{r\partial_r + \Delta_+}{\Delta_+ - \Delta_-}\varphi \\ \langle O \rangle = D r^{\Delta_+} \dfrac{r\partial_r + \Delta_-}{\Delta_- - \Delta_+}\varphi \end{cases}$$

1.5° near horizon $(r \to 1)$

$$f(r) = r^2 - 1 \quad \begin{cases} f(r) \to 2(r-1) \\ f'(r) \to 2 \end{cases}$$

$$\left(-2\partial_r - \frac{\omega^2}{2(r-1)} + k^2 + m^2\right)\varphi = 0$$

asymptotic solution $\quad \varphi_\omega \sim (r-1)^{\pm i\omega/2}$

in-falling $\quad \varphi = \varphi_\omega e^{-i\omega t} = \exp(-i\omega(t + \frac{1}{2}\ln(r-1)))$

$\partial_r \varphi = -\frac{i\omega}{2}(r-1)^{-i\omega/2} \to 0$ for $\omega \to 0$ static mode.

horizon condition : $\partial_r \varphi = 0$

1.6° Summary

$$-\frac{1}{r^d}\partial_r(r^d f(r)\partial_r \varphi) + m^2\varphi = 0$$

introduce $\quad \pi = \partial_r \varphi \qquad f(r)\partial_r \pi + \frac{1}{r^d}\partial_r(r^d f(r))\pi = m^2 \varphi$

$$\begin{cases} \partial_r \varphi = \pi \\ \partial_r \pi = \partial_r \ln(r^d f(r))\, \pi + \dfrac{1}{f(r)} m^2 \varphi \end{cases}$$

horizon ($r \to 1$): $\varphi \in \mathbb{R}$, $\pi = \partial_r \varphi = 0$

boundary ($r \to \infty$): $J = r^{\Delta_-} \dfrac{r\pi + \Delta_+ \varphi}{\Delta_+ - \Delta_-}$

$$\langle O \rangle = D r^{\Delta_+} \frac{r\pi + \Delta_- \varphi}{\Delta_- - \Delta_+}$$

But equation is linear → only produce linear response

introduce non-linearity by interaction.

$$\begin{cases} \partial_r \varphi = \pi \\ \partial_r \pi = \partial_r \ln(r^d f(r))\, \pi + \dfrac{1}{f(r)} \dfrac{\delta V(\varphi)}{\delta \varphi} \end{cases}$$

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 + \cdots$$

horizon ($r \to 1$): $\varphi \in \mathbb{R}$, $\pi = \partial_r \varphi = 0$

boundary ($r \to \infty$): $J = r^{\Delta_-} \dfrac{r\pi + \Delta_+ \varphi}{\Delta_+ - \Delta_-}$

$$\langle O \rangle = D r^{\Delta_+} \frac{r\pi + \Delta_- \varphi}{\Delta_- - \Delta_+}$$

loss function. $\mathcal{L} = \left(\langle O \rangle_{mdl} - \langle O \rangle_{dat}|_J\right)^2$

2° Neural ODE

2.1° Forward ODE

let $\vec{x} = \begin{pmatrix} \varphi \\ \pi \end{pmatrix}$ $x_i(t)$ governed by

$$\begin{cases} \dfrac{d}{dt} x_i(t) = v_i(\vec{x}, \theta) \\ \mathcal{L} = L(\vec{x}(1)) \end{cases}$$

integrated from $t=0 \to t=1$.

↖ model parameters

to train θ → need $\dfrac{\partial \mathcal{L}}{\partial \theta}$

2.2° Backward ODE

first need to know $a_i(t) = \dfrac{\partial \mathcal{L}}{\partial x_i(t)}$

$$a_i(t) = \frac{\partial \mathcal{L}}{\partial x_j(t+dt)} \frac{\partial x_j(t+dt)}{\partial x_i(t)}$$

$$= a_j(t+dt) \frac{\partial}{\partial x_i(t)} \left(x_j(t) + v_j(\vec{x}, \theta)\, dt\right)$$

$$= a_j(t+dt)\left(\delta_{ij} + \partial_{x_i} v_j(\vec{x},\theta)\, dt\right)$$

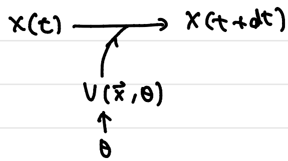
$$= a_j(t+dt) + a_j(t+dt)\,\partial_{x_i} v_j \; dt.$$

$$\Rightarrow \quad a_j(t+dt) = a_j(t) - a_j\, \partial_{x_i} v_j \; dt$$

$$\Rightarrow \quad \frac{d}{dt} a_j = - a_j\, \partial_{x_i} v_j \qquad \text{starting from} \quad a_i(1) = \frac{\partial L}{\partial x_i(1)}$$

parameter gradient

$$\frac{\partial L}{\partial \theta} = \int_0^1 \frac{\partial L}{\partial x_i(t)} \; \frac{\partial\left(x_i(t) - x_i(t-dt)\right)}{\partial \theta}$$



$$= \int_0^1 a_i(t)\, \partial_\theta v_i$$

2.3° Summary

forward $\quad \frac{d}{dt} x_i = v_i \qquad$ starting from $x_i(0) \quad \rightarrow x_i(1)$

backward

$$\frac{d}{dt}\begin{pmatrix} x_i \\ a_i \\ \partial_\theta L \end{pmatrix} = \begin{pmatrix} v_i \\ - a_j\, \partial_{x_i} v_j \\ - a_j\, \partial_\theta v_j \end{pmatrix} \qquad \text{starting from} \quad \begin{pmatrix} x_i \\ a_i \\ \partial_\theta L \end{pmatrix}_{t=1} = \begin{pmatrix} x_i(1) \\ \partial_{x_i(1)} L \\ 0 \end{pmatrix}$$

update $\quad \theta \rightarrow \theta - \gamma\, \partial_\theta L$