

# Quantum Mechanics

## Second Quantization

---

### Second Quantization

#### ■ Quantum Many-Body States

#### ■ Identical Particles

**Second quantization** is a formalism used to describe quantum many-body systems of **identical particles**.

- **Classical mechanics:** each particle is labeled by a distinct position  $\mathbf{r}_i \Rightarrow$  any different configuration of  $\{\mathbf{r}_i\}$  correspond to a different classical many-body state.
- **Quantum mechanics:** particles are *identical*, such that *exchanging* two particles ( $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$ ) does *not* lead to a different quantum many-body state.

**Permutation symmetry** of identical particles  $\Rightarrow$  **joint probability distribution** must be *invariant* under permutation:

$$p(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots) = p(\dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots), \quad (1)$$

where the probability distribution  $p$  is related to the **many-body wave function**  $\Psi$  by

$$p(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots) = |\Psi(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots)|^2. \quad (2)$$

The wave function can only change up to an *overall phase factor*.

$$\Psi(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots) = e^{i\varphi} \Psi(\dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots). \quad (3)$$

It forms as a **one-dimensional representation** of the **permutation group**. Mathematical fact: there are only *two* 1-dim representations for any permutation group,

- **trivial** representation  $\Rightarrow$  **bosons**

$$\Psi_B(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots) = +\Psi_B(\dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots) \quad (4)$$

- **sign** representation  $\Rightarrow$  **fermions**

$$\Psi_F(\dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots) = -\Psi_F(\dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots) \quad (5)$$

#### ■ Dirac Notations

Let us rephrase this using *Dirac ket-state notation* (more concise). Consider a complete set of **single-particle states**  $|\alpha\rangle$  (labeled by  $\alpha$ )

$$|\alpha\rangle = \int d^d \mathbf{r} \psi_\alpha(\mathbf{r}) |\mathbf{r}\rangle, \quad (6)$$

where  $\psi_\alpha(\mathbf{r})$  is the wave function representing the state.

- A two-particle state with the 1st particle in  $|\alpha_1\rangle$  and the 2nd particle in  $|\alpha_2\rangle$  will be described by

$$\begin{aligned} |\alpha_1\rangle \otimes |\alpha_2\rangle &= \int d^d \mathbf{r}_1 \int d^d \mathbf{r}_2 \psi_{\alpha_1}(\mathbf{r}_1) \psi_{\alpha_2}(\mathbf{r}_2) |\mathbf{r}_1\rangle \otimes |\mathbf{r}_2\rangle \\ &= \int d^d \mathbf{r}_1 \int d^d \mathbf{r}_2 \Psi(\mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1\rangle \otimes |\mathbf{r}_2\rangle. \end{aligned} \quad (7)$$

$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_{\alpha_1}(\mathbf{r}_1) \psi_{\alpha_2}(\mathbf{r}_2)$  is identified as the two-body wave function.

- Exchanging  $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$  in the wave function  $\Psi(\mathbf{r}_1, \mathbf{r}_2)$  leads to a new wave function  $\Psi'(\mathbf{r}_1, \mathbf{r}_2)$

$$\Psi'(\mathbf{r}_1, \mathbf{r}_2) = \Psi(\mathbf{r}_2, \mathbf{r}_1) = \psi_{\alpha_1}(\mathbf{r}_2) \psi_{\alpha_2}(\mathbf{r}_1) = \psi_{\alpha_2}(\mathbf{r}_1) \psi_{\alpha_1}(\mathbf{r}_2), \quad (8)$$

which corresponds to a new state

$$\begin{aligned} &\int d^d \mathbf{r}_1 \int d^d \mathbf{r}_2 \Psi'(\mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1\rangle \otimes |\mathbf{r}_2\rangle \\ &= \int d^d \mathbf{r}_1 \int d^d \mathbf{r}_2 \psi_{\alpha_2}(\mathbf{r}_1) \psi_{\alpha_1}(\mathbf{r}_2) |\mathbf{r}_1\rangle \otimes |\mathbf{r}_2\rangle \\ &= |\alpha_2\rangle \otimes |\alpha_1\rangle, \end{aligned} \quad (9)$$

describing a two-particle state with the 1st particle in  $|\alpha_2\rangle$  and the 2nd particle in  $|\alpha_1\rangle$ .

**Conclusion:** exchanging the positions of two particles ( $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ )  $\Leftrightarrow$  exchanging the labels of the single-particle state ( $\alpha_1 \leftrightarrow \alpha_2$ ).

## ■ First-Quantized States

First-quantization approach:

- Suppose the **single-particle Hilbert space** is  $D$  dimensional, spanned by the **single-particle basis states**  $|\alpha\rangle$  ( $\alpha = 1, 2, \dots, D$ ).
- The **many-body Hilbert space** of  $N$  particles will be  $D^N$  dimensional, spanned by the **many-body basis states**

$$|[\alpha]\rangle \equiv |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \dots \otimes |\alpha_N\rangle, \quad (10)$$

where  $\alpha_i = 1, 2, \dots, D$  labels the state of the  $i$ th particle.

- A generic **first-quantized state** is a linear superposition of these basis states

$$|\Psi\rangle = \sum_{[\alpha]} \Psi[\alpha] |[\alpha]\rangle, \quad (11)$$

where the coefficient  $\Psi[\alpha] \in \mathbb{C}$  is also called the **many-body wave function** (as a more general function of labels  $\alpha_i$  not positions  $\mathbf{r}_i$ ).

Most of the first-quantized states are *not* qualified to describe systems of *identical particles*.

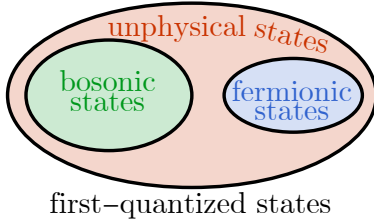
- For identical **bosons**,  $\Psi[\alpha]$  must be **symmetric**

$$\Psi_B(\dots, \alpha_i, \dots, \alpha_j, \dots) = +\Psi_B(\dots, \alpha_j, \dots, \alpha_i, \dots) \quad (12)$$

- For identical **fermions**,  $\Psi[\alpha]$  must be **antisymmetric**

$$\Psi_F(\dots, \alpha_i, \dots, \alpha_j, \dots) = -\Psi_F(\dots, \alpha_j, \dots, \alpha_i, \dots) \quad (13)$$

These states only span a *subspace* of the *first-quantized* Hilbert space.



We would like to pick out (or construct) the **basis states** for the **bosonic** and **fermionic** subspaces. Starting from a generic basis state  $|\alpha\rangle$ , we can construct

- **bosonic** states by **symmetrization**

$$\begin{aligned} \mathcal{S}|\alpha\rangle &= \mathcal{S}|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \dots \otimes |\alpha_N\rangle \\ &\equiv \sum_{\pi \in S_N} |\alpha_{\pi(1)}\rangle \otimes |\alpha_{\pi(2)}\rangle \otimes \dots \otimes |\alpha_{\pi(N)}\rangle, \end{aligned} \quad (14)$$

- **fermionic** states by **antisymmetrization**

$$\begin{aligned} \mathcal{A}|\alpha\rangle &= \mathcal{A}|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \dots \otimes |\alpha_N\rangle \\ &\equiv \sum_{\pi \in S_N} (-)^\pi |\alpha_{\pi(1)}\rangle \otimes |\alpha_{\pi(2)}\rangle \otimes \dots \otimes |\alpha_{\pi(N)}\rangle, \end{aligned} \quad (15)$$

$\pi$  denotes an  $S_N$  group element and  $(-)^\pi$  is the **permutation sign** of  $\pi$ .

$$(-)^\pi = \begin{cases} +1 & \text{if } \pi \text{ has even number of inversions} \\ -1 & \text{if } \pi \text{ has odd number of inversions} \end{cases} \quad (16)$$

An inversion is a pair  $(x, y)$  such that  $x < y$  and  $\pi(x) > \pi(y)$ . Take the  $S_3$  group for example:

$$\begin{array}{cccccc} \pi(123) & 123 & 231 & 312 & 321 & 213 & 132 \\ (-)^\pi & +1 & +1 & +1 & -1 & -1 & -1 \end{array} \quad (17)$$

- Examples of **bosonic** states (unnormalized):

$$\begin{aligned} \mathcal{S}|\alpha\rangle \otimes |\beta\rangle &= |\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle, \quad (\text{assuming } \alpha \neq \beta) \\ \mathcal{S}|\alpha\rangle \otimes |\alpha\rangle &= |\alpha\rangle \otimes |\alpha\rangle. \end{aligned} \quad (18)$$

- Examples of **fermionic** states (unnormalized):

$$\begin{aligned} \mathcal{A}|\alpha\rangle \otimes |\beta\rangle &= |\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle, \quad (\text{assuming } \alpha \neq \beta) \\ \mathcal{A}|\alpha\rangle \otimes |\alpha\rangle &= 0 \Rightarrow \text{no such fermionic state.} \end{aligned} \quad (19)$$

**Pauli exclusion principle:** two (or more) identical fermions can not occupy the same state

simultaneously.

Originally  $|\alpha\rangle \otimes |\beta\rangle$  and  $|\beta\rangle \otimes |\alpha\rangle$  (for  $\alpha \neq \beta$ ) are two orthogonal first-quantized states, under either *symmetrization* or *antisymmetrization*, they correspond to the same state (up to  $\pm 1$  overall phase)

$$\begin{aligned}\mathcal{S} |\alpha\rangle \otimes |\beta\rangle &= \mathcal{S} |\beta\rangle \otimes |\alpha\rangle, \\ \mathcal{A} |\alpha\rangle \otimes |\beta\rangle &= -\mathcal{A} |\beta\rangle \otimes |\alpha\rangle.\end{aligned}\tag{20}$$

- The first-quantized Hilbert space is *redundant*  $\Rightarrow$  there are *fewer* basis states in the bosonic and fermionic subspaces.
- Consider  $N$  particles, each can take one of  $D$  different single-particle states,
  - the **dimension of bosonic subspace**:

$$\mathcal{D}_B = \frac{(N + D - 1)!}{N! (D - 1)!}.\tag{21}$$

- the **dimension of fermionic subspace**:

$$\mathcal{D}_F = \frac{D!}{N! (D - N)!}.\tag{22}$$

It turns out that  $\mathcal{D}_B + \mathcal{D}_F \leq D^N$  as long as  $N > 1 \Rightarrow$  the remaining basis states in the first-quantized Hilbert space are *unphysical* (for identical particles).

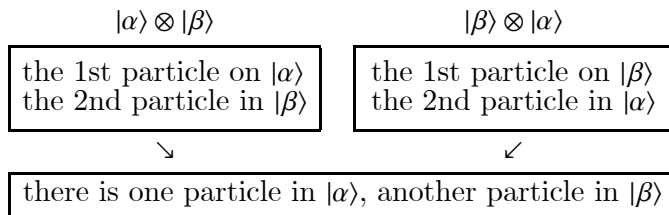
These unphysical states are annoying: we can not combine the states in the Hilbert space freely. We must always remember to symmetrized/antisymmetrized the state.  $\Rightarrow$  Is there a better way to organize the many-body Hilbert space, such that all states in the space are physical?

## ■ Second-Quantized States (Fock States)

Sometimes *difficulties* in physics arise from the *inappropriate language* we used. There are two different ways to describe many-body states:

- In **first-quantization**, we ask: *Which particle is in which state?*
- In **second-quantization**, we ask: *How many particles are there in every state?*

The question we ask in first-quantization is inappropriate: if the particles are *identical*, it will be impossible to tell which particle is which in the first place. We need to switch to a new language



The new description does not require the labeling of particles.  $\Rightarrow$  It contains no redundant information.  $\Rightarrow$  It leads to a more precise and succinct description.

In the second-quantization approach,

- Each **basis state** in the many-body Hilbert space is labeled by a set of **occupation numbers**  $n_\alpha$  (for  $\alpha = 1, 2, \dots, D$ )

$$|[n]\rangle \equiv |n_1, n_2, \dots, n_\alpha, \dots, n_D\rangle, \quad (23)$$

meaning that there are  $n_\alpha$  particles in the state  $|\alpha\rangle$ .

$$n_\alpha = \begin{cases} 0, 1, 2, 3, \dots & \text{bosons,} \\ 0, 1 & \text{fermions.} \end{cases} \quad (24)$$

- For **bosons**,  $n_\alpha$  can be any non-negative integer.
- For **fermions**,  $n_\alpha$  can only take 0 or 1, due to the *Pauli exclusion principle*.
- The occupation numbers  $n_\alpha$  sum up to the total number of particles, i.e.  $\sum_\alpha n_\alpha = N$ .
- The states  $|[n]\rangle$  are also known as **Fock states**.
- All Fock states form a complete set of basis for the many-body Hilbert space, or the **Fock space**.
- Any generic **second-quantized** many-body state is a linear combination of *Fock states*,

$$|\Psi\rangle = \sum_{[n]} \Psi[n] |[n]\rangle. \quad (25)$$

## ■ Representation of Fock States

The **first-** and the **second-quantization** formalisms can both provide *legitimate* description of identical particles. (The first-quantization is just awkward to use, but it is still valid.)

Every *Fock state* has a **first-quantized representation**.

- The Fock state with all occupation numbers to be zero is called the **vacuum state**, denoted as

$$|0\rangle \equiv |\dots, 0, \dots\rangle \quad (26)$$

It corresponds to the *tensor product unit* in the first-quantization, which can be written as

$$|0\rangle_B = |0\rangle_F = 1. \quad (27)$$

We use a subscript  $B/F$  to indicate whether the Fock state is **bosonic** ( $B$ ) or **fermionic** ( $F$ ).

For vacuum state, there is no difference between them.

- The Fock state with only one *non-zero* occupation number is a **single-mode Fock state**, denoted as

$$|n_\alpha\rangle = |\dots, 0, n_\alpha, 0, \dots\rangle \quad (28)$$

In terms of the first-quantized states

$$\begin{aligned} |1_\alpha\rangle_B &= |1_\alpha\rangle_F = |\alpha\rangle, \\ |2_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle, \\ |3_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle, \end{aligned} \quad (29)$$

$$|n_\alpha\rangle_B = \underbrace{|\alpha\rangle \otimes |\alpha\rangle \otimes \dots \otimes |\alpha\rangle}_{n_\alpha \text{ factors}} \equiv |\alpha\rangle^{\otimes n_\alpha}.$$

- For **multi-mode Fock states** (meaning more than one single-particle state  $|\alpha\rangle$  is involved), the first-quantized state will involve appropriate symmetrization depending on the particle statistics. For example,

$$\begin{aligned} |1_\alpha, 1_\beta\rangle_B &= \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle), \\ |1_\alpha, 1_\beta\rangle_F &= \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle). \end{aligned} \tag{30}$$

Note the difference between bosonic and fermionic Fock states (even if their occupation numbers are the same). Here are more examples

$$\begin{aligned} |2_\alpha, 1_\beta\rangle_B &= \frac{1}{\sqrt{3}} (|\alpha\rangle \otimes |\alpha\rangle \otimes |\beta\rangle + |\alpha\rangle \otimes |\beta\rangle \otimes |\alpha\rangle + |\beta\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle), \\ |1_\alpha, 1_\beta, 1_\gamma\rangle_F &= \frac{1}{\sqrt{6}} (|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle + |\beta\rangle \otimes |\gamma\rangle \otimes |\alpha\rangle + \\ &\quad |\gamma\rangle \otimes |\alpha\rangle \otimes |\beta\rangle - |\gamma\rangle \otimes |\beta\rangle \otimes |\alpha\rangle - |\beta\rangle \otimes |\alpha\rangle \otimes |\gamma\rangle - |\alpha\rangle \otimes |\gamma\rangle \otimes |\beta\rangle). \end{aligned} \tag{31}$$

Ok, you get the idea. In general, the Fock state can be represented as

- for **bosons**,

$$|[n]\rangle_B = \left( \frac{\prod_\alpha n_\alpha!}{N!} \right)^{1/2} \mathcal{S} \otimes_\alpha |\alpha\rangle^{\otimes n_\alpha}. \tag{32}$$

- for **fermions**,

$$|[n]\rangle_F = \frac{1}{\sqrt{N!}} \mathcal{A} \otimes_\alpha |\alpha\rangle^{\otimes n_\alpha}. \tag{33}$$

$\mathcal{S}$  and  $\mathcal{A}$  are symmetrization and antisymmetrization operators defined in Eq. (14) and Eq. (15).

## ■ Creation and Annihilation Operators

### ■ State Insertion and Deletion

The **creation** and **annihilation operators** are introduced to *create* and *annihilate* particles in the quantum many-body system, as indicated by their names. The first step towards defining them is to understand how to *insert* and *delete* a single-particle state from the first-quantized state in a *symmetric* (or *antisymmetric*) manner.

Let us first declare some notations:

- Let  $|\alpha\rangle, |\beta\rangle$  be single-particle states.
- Let  $1$  be the *tensor identity* (meaning that  $|\alpha\rangle \otimes 1 = 1 \otimes |\alpha\rangle = |\alpha\rangle$ ).
- Let  $|\Psi\rangle, |\Phi\rangle$  be generic *first-quantized states* as in Eq. (11).

Now we define the **insertion operator**  $\triangleright_{\pm}$  and **deletion operator**  $\triangleleft_{\pm}$  by the following rules:

- Linearity (for  $a, b \in \mathbb{C}$ )

$$\begin{aligned} |\alpha\rangle \triangleright_{\pm} (a |\Psi\rangle + b |\Phi\rangle) &= a |\alpha\rangle \triangleright_{\pm} |\Psi\rangle + b |\alpha\rangle \triangleright_{\pm} |\Phi\rangle, \\ |\alpha\rangle \triangleleft_{\pm} (a |\Psi\rangle + b |\Phi\rangle) &= a |\alpha\rangle \triangleleft_{\pm} |\Psi\rangle + b |\alpha\rangle \triangleleft_{\pm} |\Phi\rangle. \end{aligned} \quad (34)$$

- Vacuum action

$$\begin{aligned} |\alpha\rangle \triangleright_{\pm} 1 &= |\alpha\rangle, \\ |\alpha\rangle \triangleleft_{\pm} 1 &= 0. \end{aligned} \quad (35)$$

- Recursive relation

$$\begin{aligned} |\alpha\rangle \triangleright_{\pm} |\beta\rangle \otimes |\Psi\rangle &= |\alpha\rangle \otimes |\beta\rangle \otimes |\Psi\rangle \pm |\beta\rangle \otimes (|\alpha\rangle \triangleright_{\pm} |\Psi\rangle), \\ |\alpha\rangle \triangleleft_{\pm} |\beta\rangle \otimes |\Psi\rangle &= \langle \alpha | \beta \rangle |\Psi\rangle \pm |\beta\rangle \otimes (|\alpha\rangle \triangleleft_{\pm} |\Psi\rangle). \end{aligned} \quad (36)$$

$\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$  if  $|\alpha\rangle$  and  $|\beta\rangle$  are orthonormal basis states. The subscript  $\pm$  of the insertion or deletion operators indicates whether symmetrization (+) or antisymmetrization (−) is implemented.

## ■ Boson Creation and Annihilation

- The **boson creation operator**  $b_{\alpha}^{\dagger}$  adds a boson to the single-particle state  $|\alpha\rangle$ , *increasing* the occupation number by one  $n_{\alpha} \rightarrow n_{\alpha} + 1$ . It acts on a  $N$ -particle first-quantized state  $|\Psi\rangle$  as

$$b_{\alpha}^{\dagger} |\Psi\rangle = \frac{1}{\sqrt{N+1}} |\alpha\rangle \triangleright_{+} |\Psi\rangle, \quad (37)$$

where  $|\alpha\rangle \triangleright_{+}$  *inserts* the single-particle state  $|\alpha\rangle$  to  $N+1$  possible insertion positions *symmetrically*.

- The **boson annihilation operator**  $b_{\alpha}$  removes a boson from the single-particle state  $|\alpha\rangle$ , *reducing* the occupation number by one  $n_{\alpha} \rightarrow n_{\alpha} - 1$  (while  $n_{\alpha} > 0$ ). It acts on a  $N$ -particle first-quantized state  $|\Psi\rangle$  as

$$b_{\alpha} |\Psi\rangle = \frac{1}{\sqrt{N}} |\alpha\rangle \triangleleft_{+} |\Psi\rangle, \quad (38)$$

where  $|\alpha\rangle \triangleleft_{+}$  *removes* the single-particle state  $|\alpha\rangle$  from  $N$  possible deletion positions *symmetrically*.

## □ Single-Mode Fock States

Based on these definitions, we can show that the creation and annihilation operators acting on single-mode Fock states as

$$\begin{aligned}
b_\alpha^\dagger |n_\alpha\rangle &= \frac{1}{\sqrt{n_\alpha + 1}} |\alpha\rangle \triangleright_+ |\alpha\rangle^{\otimes n_\alpha} \\
&= \frac{n_\alpha + 1}{\sqrt{n_\alpha + 1}} |\alpha\rangle^{\otimes (n_\alpha + 1)} \\
&= \sqrt{n_\alpha + 1} |n_\alpha + 1\rangle. \\
b_\alpha |n_\alpha\rangle &= \frac{1}{\sqrt{n_\alpha}} |\alpha\rangle \triangleleft_+ |\alpha\rangle^{\otimes n_\alpha} \\
&= \frac{n_\alpha}{\sqrt{n_\alpha}} |\alpha\rangle^{\otimes (n_\alpha - 1)} \\
&= \sqrt{n_\alpha} |n_\alpha - 1\rangle.
\end{aligned} \tag{40}$$

Thus we conclude

$$\begin{aligned}
b_\alpha^\dagger |n_\alpha\rangle &= \sqrt{n_\alpha + 1} |n_\alpha + 1\rangle, \\
b_\alpha |n_\alpha\rangle &= \sqrt{n_\alpha} |n_\alpha - 1\rangle.
\end{aligned} \tag{41}$$

- Especially, when acting on the vacuum state

$$\begin{aligned}
b_\alpha^\dagger |0_\alpha\rangle &= |1_\alpha\rangle, \\
b_\alpha |0_\alpha\rangle &= 0.
\end{aligned} \tag{42}$$

- Using Eq. (41), we can show that

$$b_\alpha^\dagger b_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle, \tag{43}$$

meaning that  $b_\alpha^\dagger b_\alpha$  is the **boson number operator** of the  $|\alpha\rangle$  state.

All the single-mode Fock state can be constructed by the boson creation operator from the vacuum state

$$|n_\alpha\rangle = \frac{1}{\sqrt{n_\alpha!}} (b_\alpha^\dagger)^{n_\alpha} |0_\alpha\rangle. \tag{44}$$

## □ Generic Fock States

The above result can be generalized to any Fock state of bosons

$$\begin{aligned}
b_\alpha^\dagger |\dots, n_\beta, n_\alpha, n_\gamma, \dots\rangle_B &= \sqrt{n_\alpha + 1} |\dots, n_\beta, n_\alpha + 1, n_\gamma, \dots\rangle_B, \\
b_\alpha |\dots, n_\beta, n_\alpha, n_\gamma, \dots\rangle_B &= \sqrt{n_\alpha} |\dots, n_\beta, n_\alpha - 1, n_\gamma, \dots\rangle_B.
\end{aligned} \tag{45}$$

These two equations can be considered as the **defining properties** of boson creation and annihila-



tion operators.

### □ Operator Identities

Eq. (45) implies the following operator identities

$$\boxed{[b_\alpha^\dagger, b_\beta^\dagger] = [b_\alpha, b_\beta] = 0, [b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}.} \quad (46)$$

These relations can be considered as the **algebraic definition** of boson creation and annihilation operators.

### ■ Fermion Creation and Annihilation

- The **fermion creation operator**  $c_\alpha^\dagger$  adds a fermion to the single-particle state  $|\alpha\rangle$ , *increasing* the occupation number by one  $n_\alpha \rightarrow n_\alpha + 1$  (while  $n_\alpha = 0$ ). It acts on a  $N$ -particle first-quantized state  $|\Psi\rangle$  as

$$\boxed{c_\alpha^\dagger |\Psi\rangle = \frac{1}{\sqrt{N+1}} |\alpha\rangle \triangleright_- |\Psi\rangle,} \quad (47)$$

where  $|\alpha\rangle \triangleright_-$  *inserts* the single-particle state  $|\alpha\rangle$  to  $N+1$  possible insertion positions *antisymmetrically*.

- The **fermion annihilation operator**  $c_\alpha$  removes a fermion from the single-particle state  $|\alpha\rangle$ , *reducing* the occupation number by one  $n_\alpha \rightarrow n_\alpha - 1$  (while  $n_\alpha = 1$ ). It acts on a  $N$ -particle first-quantized state  $|\Psi\rangle$  as

$$\boxed{c_\alpha |\Psi\rangle = \frac{1}{\sqrt{N}} |\alpha\rangle \triangleleft_- |\Psi\rangle,} \quad (48)$$

where  $|\alpha\rangle \triangleleft_-$  *removes* the single-particle state  $|\alpha\rangle$  from  $N$  possible deletion positions *antisymmetrically*.

### □ Single-Mode Fock States

Based on these definitions, we can show that the creation and annihilation operators acting on single-mode Fock states as

$$\begin{aligned} c_\alpha^\dagger |0_\alpha\rangle &= |\alpha\rangle \triangleright_- 1 = |\alpha\rangle = |1_\alpha\rangle \\ c_\alpha^\dagger |1_\alpha\rangle &= \frac{1}{\sqrt{2}} |\alpha\rangle \triangleright_- |\alpha\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\alpha\rangle - |\alpha\rangle \otimes |\alpha\rangle) = 0 \end{aligned} \quad (49)$$

$$\begin{aligned} c_\alpha |0_\alpha\rangle &= 0 \\ c_\alpha |1_\alpha\rangle &= |\alpha\rangle \triangleleft_- |\alpha\rangle = 1 = |0_\alpha\rangle. \end{aligned} \quad (50)$$

Thus we conclude (note that  $n_\alpha = 0, 1$  only take two values)

$$\begin{aligned} c_\alpha^\dagger |n_\alpha\rangle &= \sqrt{1 - n_\alpha} |1 - n_\alpha\rangle, \\ c_\alpha |n_\alpha\rangle &= \sqrt{n_\alpha} |1 - n_\alpha\rangle. \end{aligned} \quad (51)$$

- Using Eq. (51), we can show that

$$c_\alpha^\dagger c_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle, \quad (52)$$

meaning that  $c_\alpha^\dagger c_\alpha$  is the **fermion number operator** of the  $|\alpha\rangle$  state.

All the single-mode Fock state can be constructed by the boson creation operator from the vacuum state

$$|n_\alpha\rangle = (c_\alpha^\dagger)^{n_\alpha} |0_\alpha\rangle. \quad (53)$$

### □ Generic Fock States

The above result can be generalized to any Fock state of bosons

$$\begin{aligned} c_\alpha^\dagger |\dots, n_\beta, n_\alpha, n_\gamma, \dots\rangle_F &= (-)^{\sum_{\beta < \alpha} n_\beta} \sqrt{1 - n_\alpha} |\dots, n_\beta, 1 - n_\alpha, n_\gamma, \dots\rangle_F, \\ c_\alpha |\dots, n_\beta, n_\alpha, n_\gamma, \dots\rangle_F &= (-)^{\sum_{\beta < \alpha} n_\beta} \sqrt{n_\alpha} |\dots, n_\beta, 1 - n_\alpha, n_\gamma, \dots\rangle_F. \end{aligned} \quad (54)$$

These two equations can be considered as the **defining properties** of fermion creation and annihilation operators.

### □ Operator Identities

Eq. (54) implies the following operator identities

$$\{c_\alpha^\dagger, c_\beta^\dagger\} = \{c_\alpha, c_\beta\} = 0, \quad \{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}. \quad (55)$$

These relations can be considered as the **algebraic definition** of fermion creation and annihilation operators.