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Quantum Mechanics A (Physics 212A) Fall 2018
Worksheet 5 – Solutions

Problems

1. **Schmidt Decomposition**

   In the problems below, consider a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with the factors not necessarily of the same dimension. Consider a state of this system

$$|a\rangle = \sum_{i=1}^{N}\sum_{r=1}^{M} a_{ir}|i\rangle_A \otimes |r\rangle_B \ .$$

   (a) We defined the *Schmidt number* of the state $|a\rangle$ to be the rank of the matrix a_{ir}. Show that this is the same as the number of nonzero eigenvalues of

$$\boldsymbol{\rho}_A = \text{tr }_B|a\rangle\langle a| \ .$$

   Show that it's *also* the same as the number of nonzero eigenvalues of

$$\boldsymbol{\rho}_B = \text{tr }_A|a\rangle\langle a| \ .$$

   [Hint: To solve this problem, it's useful to take advantage of our ability to change basis. In particular, the problem does not depend on what basis we choose for $\mathcal{H}_A$ and $\mathcal{H}_B$. If w were a Hermitian matrix, we could find a basis where it was diagonal, by using its eigenvectors as the basis elements. w is not Hermitian (and not even square in general), but there is still something we can do: such a matrix has right eigenvectors (elements of $\mathcal{H}_B$) and left eigenvectors (elements of $\mathcal{H}_A$ and they can be used to choose basis cleverly. In particular, any matrix has a *singular value decomposition* (SVD) of the form

$$w = U^T \Lambda V \qquad w_{ir} = U^T_{ij}\Lambda_{js}V_{sr} \ .$$

   where U and V are unitary (basis transformations) and Λ is a diagonal matrix (the entries are the 'singular values', which are in fact real and positive). Notice that Λ is not square. It looks like

$$\Lambda_{js} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{js} \quad \text{if } N > M \quad \text{or} \quad \Lambda_{js} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \end{pmatrix}_{js} \text{ if } M > N.$$

(In the examples I chose $(M, N) = (3, 5)$ and $(5, 3)$ respectively.) And U is $N \times N$ and V is $M \times M$. So we can choose a new basis for $\mathcal{H}_a$ which is the image of our old one under U: $|i'\rangle \equiv U|i\rangle$. Similarly for $\mathcal{H}_b$: $|r'\rangle \equiv V|r\rangle$. In this basis, we hvae

$$|w\rangle = \sum_{i'r'} \Lambda_{i'r'}|i'\rangle \otimes |r'\rangle$$

and Λ is diagonal.]

I'm going to adopt a slightly different notation. Let $|\psi\rangle$ be a pure state in $\mathcal{H}$.

I claim that there exist $|i_A\rangle$, $|i_B\rangle$ orthonormal basis states of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively such that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$ where $\sum_i \lambda_i^2 = 1$.

Note this immediately implies $\rho_A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$ and $\rho_B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$

This is known as the *Schmidt decomposition*. Many of the results below will follow from the existence of this decomposition. The proof is taken from Nielson and Chuang.

Let $|j\rangle$ and $|k\rangle$ be arbitrary orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Most generally $|\psi\rangle = \sum_{jk} a_{jk}|j\rangle|k\rangle$ for some complex coefficients a_{jk}.

By the singular value decomposition below we can diagonalize the matrix of coefficients A by $A = U^T \Lambda V$ for some $N \times N$ unitary U and an $M \times M$ unitary V defined on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

Note that the range of the indicies j and k need not be the same corresponding to the case of $N \neq M$. Thus A needn't be square. For the following discussion to make sense I'm going to define the index $i = min(j, k)$; it picks out the diagonal submatrix entries.

We then rewrite $|\psi\rangle = \sum_{ijk} u_{ji}\lambda_{ii}v_{ik}|j\rangle|k\rangle$ and do a redefinition:

$|i_A\rangle \equiv \sum_j u_{ji}|j\rangle$ and $|i_B\rangle \equiv \sum_k v_{ik}|k\rangle$ bringing $|\psi\rangle$ to the form previously claimed!

That $|i_A\rangle$ and $|i_B\rangle$ are othonormal bases follows from the unitarity of U and V and our assumption that $|j\rangle$ and $|k\rangle$ were.

Now we can get a few things for free. The first is that the Schmidt number, the rank of A, has to be the same as the rank of Λ. This is the number of *non-zero* values λ_i. Clearly this is also the same between ρ_A and ρ_B by the form above.

2. **Entanglement cannot be created locally**

   Define a *local unitary* to be an operator of the form $\mathbf{U}_A \otimes \mathbf{U}_B$ where $\mathbf{U}_{A,B}$ acts only on $\mathcal{H}_{A,B}$. These are the operations that can be done by actors with access only to A *or* B. Show that by acting on a state of $\mathcal{H}$ with a local unitary we cannot change the Schmidt number or the entanglement entropy of either factor. Consider both the case of a pure state of $\mathcal{H}$ and a mixed state of $\mathcal{H}$; note that the action of a unitary $\mathbf{U}$ on a density matrix $\boldsymbol{\rho}$ is

   $$\boldsymbol{\rho} \to \mathbf{U}\boldsymbol{\rho}\mathbf{U}^\dagger.$$

   [We conclude from this that to create an entangled state from an unentangled state, we must bring the two subsystems together and let them interact, resulting in a more general unitary evolution than a local unitary.]

A local unitary has the form $U = U_A \otimes U_B$ such that

$U|\psi\rangle = \sum_{j,k} a_{jk} U_A|j\rangle_A \otimes U_B|k\rangle_B \equiv \sum_{j,k} a_{jk}|j'\rangle|k'\rangle$ which can then be brought into a Schmidt decomposition exactly as before; we absorb the unitary into the basis transformation.

The set of λ_i don't change. So the number of them which are 0 (Schmidt number) and the trace of ρ_ψ for computing the entanglement entropy remain the same.

What if I started directly with a mixed state ρ rather than a pure state $|\psi\rangle$? To address this I'm going to introduce a perhaps needlessly fancy construction: the *purification* of ρ.

This says that every density matrix ρ can be written *non-uniquely* as a reduced density matrix for some pure state $|\phi\rangle$.

That is, $|\phi\rangle$ is a state in an even larger Hilbert space $\mathcal{H}^{BIG} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{fake}$ so that $\rho = \text{tr}\ _{fake}[|\phi\rangle\langle\phi|]$ associating it with $\mathcal{H}_A \otimes \mathcal{H}_B$

Given this fact I could just imagine proving the problem for a general mixed state by constructing its purification, acting locally on it with $U_A \otimes U_B$ which doesn't involve the fake space, and then relying on the proof above for pure states.

Such a method of proof is quite common/useful in quantum information theory and is jokingly referred to as 'going to the Church of the Larger Hilbert space'.

3. **Quantum interference versus measurement of which-way information**

Consider a double-slit interference experiment, described by a quantum system with two orthonormal states (call them $|\uparrow\rangle$ and $|\downarrow\rangle$), representing the possible paths taken by the particles. A particle emerging in state $|\uparrow\rangle$ produces a wavefunction at the screen of the form $\psi_\uparrow(x)$, (where x is a coordinate along the screen) while a particle emerging in state $|\downarrow\rangle$ produces the wavefunction $\psi_\downarrow(x)$. The evolution from the wall with the slits to the screen is linear in the input state.

As the source repeatedly spits out particles, the screen counts how many particles hit at each location x.

Suppose, for simplicity, that $\psi_\uparrow(x) = e^{\mathbf{i}k_\uparrow x}$, $\psi_\downarrow(x) = e^{\mathbf{i}k_\downarrow x}$ where $k_{\uparrow,\downarrow}$ are some constants.

(a) If the particles are all spat out in the state $|\uparrow\rangle$, what is the x-dependence of the resulting pattern $P_\uparrow(x)$?
$P_\uparrow(x) = |\langle x|\uparrow\rangle|^2 = e^{-\mathbf{i}k_\uparrow x} e^{\mathbf{i}k_\uparrow x} = 1$

(b) If the particles are all spat out in the (normalized) state $|\psi\rangle = \mu|\uparrow\rangle + \lambda|\downarrow\rangle$ what is the x-dependence of the resulting pattern, $P_\psi(x)$? Assume μ, λ are real.
Similar $P_\psi(x) = |\langle x|\psi\rangle|^2 = 1 + 2\lambda\mu\cos[(k_\uparrow - k_\downarrow)x]$

Now we wish to take into account interactions with the environment, which we will model by another two-state system, with Hilbert space $\mathcal{H}_E$. Suppose these interactions are described by the hamiltonian

$$\hat{H} = \boldsymbol{\sigma}^z \otimes \hat{M} \tag{1}$$

acting on $\mathcal{H}_2 \otimes \mathcal{H}_E$ where $\boldsymbol{\sigma}^z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ acting on the Hilbert space $\mathcal{H}_2$ of particle paths, and $\hat{M}$ is an operator acting on the Hilbert space of the environment.

Suppose that the initial state of the whole system is:

$$|\Psi_0\rangle = (\mu|\uparrow\rangle + \lambda|\downarrow\rangle) \otimes |\uparrow\rangle_E \tag{2}$$

and that $\hat{M} = m\boldsymbol{\sigma}^x = m(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)_E$

(c) Find $|\Psi(t)\rangle$, the state of the whole system at time t

$U_t = e^{-\mathbf{i}\hat{H}t}$ and $|\Psi(t)\rangle = U_t|\Psi_0\rangle = \mu|\uparrow\rangle \otimes e^{-\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E + \lambda|\downarrow\rangle \otimes e^{+\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E$

Using $e^{-\mathbf{i}\theta(\hat{n}\cdot\vec{\boldsymbol{\sigma}})} = \cos\theta\mathbb{1} - \sin\theta(\hat{n}\cdot\vec{\boldsymbol{\sigma}})$ we can write the above as:

$|\Psi(t)\rangle = \mu\cos(mt)|\uparrow\uparrow\rangle - \mu\sin(mt)|\uparrow\downarrow\rangle + \lambda\cos(mt)|\downarrow\uparrow\rangle + \lambda\sin(mt)|\downarrow\downarrow\rangle$

(d) How does the interference pattern depend on x and t? For simplicity, $\mu = \frac{1}{\sqrt{2}} = \lambda$

First $\langle x|\Psi(t)\rangle = \frac{\cos(mt)}{\sqrt{2}}(e^{-\mathbf{i}k_\uparrow x} + e^{-\mathbf{i}k_\downarrow x})|\uparrow\rangle_E + \frac{\sin(mt)}{\sqrt{2}}(e^{-\mathbf{i}k_\downarrow x} - e^{-\mathbf{i}k_\uparrow x})|\downarrow\rangle_E$

$P_\Psi(x,t) = |\langle x|\Psi(t)\rangle|^2$ which can be computed easily by writing the above as a 2 component vector in the $|\uparrow\rangle_E, |\downarrow\rangle_E$ basis and taking the inner product.

$P_\Psi(x,t) = 1 + \cos(2mt)\cos[(k_\uparrow - k_\downarrow)x]$

Notice at $t = 0$ or $m = 0$ we get the answer from part (b) and that as time evolves the inteference decays with $\tau \sim \frac{1}{m}$

(e) Interpret the previous result in terms of the time-dependence of the entanglement between the two qbits.

This can be answered by looking at $|\Psi(t)\rangle$. At $t = 0$ the particle and the environment are in a product state and thus have no entanglement. This is when there is interference between the slits.

However at $t = \frac{\pi}{4m}$, the point where the interference is completely gone, the spins are maximally entangled with $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\leftarrow\rangle_E + |\downarrow\rangle|\rightarrow\rangle_E)$

Now for this model of an environment the entanglement will oscillate. However a more realistic model will not; imagine a photon coming into causal contact very briefly before rushing off to Alpha Centauri.

(f) What would happen if instead the initial state of the environment were an eigenvector of $\hat{M}$?

Take $|\Psi_0\rangle = |\rightarrow\rangle|\rightarrow\rangle_E$ then $|\Psi_t\rangle = \frac{e^{-\mathbf{i}mt}}{\sqrt{2}}|\uparrow\rangle|\rightarrow\rangle + \frac{e^{\mathbf{i}mt}}{\sqrt{2}}|\downarrow\rangle|\rightarrow\rangle$ and there's no decoherence.