University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics A (Physics 212A) Fall 2018 Worksheet 8 – Solutions

Problems

1. **Conformal Quantum Mechanics**

   Let's think about transformations of the form $\hat{x}(t) \to x'(t') = e^{\frac{\lambda}{2}} x(t)$ where λ is real. [1]

   This is known as a *scale* transformation.

   Quantum mechanically we should be able to realize this transformation with a unitary operator: $U = e^{\mathbf{i}\lambda\hat{D}}$ where $\hat{D}$ is hermitian.

   This is known as the *dilatation* generator. Let's look at the time independent version of the above first.

   (a) The above implies that $U^\dagger \hat{x} U = e^{\frac{\lambda}{2}}\hat{x}$. Use the BCH expansion to derive $[\mathbf{i}D, x]$

   The LHS of the equation gives $U^\dagger x U = x - \lambda[\mathbf{i}D, x] + \frac{\lambda^2}{2}[\mathbf{i}D, [\mathbf{i}D, x]] + \cdots$

   The RHS gives $e^{\frac{\lambda}{2}} x = (1 + \frac{\lambda}{2} + \frac{1}{2}\frac{\lambda^2}{4} + \cdots)x$

   We can equate the expressions term by term to give that $[\mathbf{i}D, x] = -\frac{1}{2}x$

   (b) Use the Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ to derive $[\mathbf{i}\hat{D}, \hat{p}]$

   $[[\mathbf{i}\hat{D}, \hat{p}], \hat{x}] + [[\hat{p}, \hat{x}], \mathbf{i}\hat{D}] + [[\hat{x}, \mathbf{i}\hat{D}], \hat{p}] = 0$

   $= [[\mathbf{i}D, p], x] + 0 + \frac{1}{2}[x, p]$ so $[x, [\mathbf{i}D, p]] = \frac{1}{2}\mathbf{i}$

   This is the canonical conjugate relation! Therefore $[\mathbf{i}D, p] = \frac{1}{2}p$

   (c) Check that $\hat{D} = -\frac{1}{4}(\hat{x}\hat{p} + \hat{p}\hat{x})$ satisfies these relations

   This is just direct calculation.

   Now consider a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ for which $\mathbf{i}\hat{D}V = V$

   Such a potential would be *scale invariant*

   (d) Calculate $[\hat{H}, \hat{D}]$ explicitly using the above form

   $[H, D] = -[D, \hat{H}] = -[D, \frac{p^2}{2m}] - [D, V]$. Let's go term by term.

   $-[D, V] = \frac{1}{4}([xp, V] + [px, V]) = \frac{1}{4}(x[p, V] + [p, V]x) = \frac{1}{4\mathbf{i}}(x\partial_x V + \partial_x V x) = -\hat{D}V$

   But by assumption $DV = -\mathbf{i}V$ so $-[D, V] = \mathbf{i}V$

   $-[D, \frac{p^2}{2m}] = \frac{1}{8m}([xp, p^2] + [px, p^2]) = \frac{1}{8m}([x, p^2]p + p[x, p^2]) = \mathbf{i}(\frac{p^2}{2m})$

   Where in the last line I used $[x, p^2] = 2\mathbf{i}p$

   Together $[H, D] = \mathbf{i}H$

[1] The factor of $\frac{1}{2}$ here is because $\hat{x}$ is a primary operator with dimension $\Delta = -\frac{1}{2}$. This is energy dimension.

(e) Show that $V(x) = \frac{g}{x^2}$ is an example of such a potential
$\mathbf{i}DV = -\frac{g}{4}(x\partial_x V + \partial_x V x)$ but $\partial_x(\frac{1}{x^2}) = -\frac{2}{x^3}$ so $DV = \frac{g}{4}(\frac{4}{x^2}) = V$

2. **Building Bloch's Theorem**

Consider a 1D Hamiltonian with a periodic potential $V(x) = V(x + na)$ for $n \in \mathbb{Z}$ and a the lattice spacing.

(a) Define the operator T^n by $T^n|x\rangle = |x + na\rangle$. Show this is a symmetry.
We would need that $[H, T] = 0$ or equivalently $H' = T^\dagger H T = H$. Computing $H'\psi(x) = \langle x|H'|\psi\rangle$. Writing H' in the position basis: $H' = -\frac{1}{2}\partial^2_{x'} + V(x')$.
$V(x') = V(x + a) = V(x)$ by assumption. One computes the Jacobian: $\frac{\partial x'}{\partial x} = \frac{\partial x+a}{\partial x} = \frac{\partial x}{\partial x} = 1$ to see the kinetic part is also unchanged. Thus H' is invariant and the result is shown!

(b) Assuming H has no shared degeneracy with T, show that any eigenfunctions of this system can be chosen to obey

$$\psi_k(x - a) = e^{-\mathbf{i}ka}\psi_k(x) \tag{1}$$

Recall that $T|k\rangle = e^{-\mathbf{i}ka}|k\rangle$ and $\langle x|k\rangle \equiv \psi_k(x)$.
Since T is a symmetry, and there is no degeneracy, I can diagonalize H in the basis of T: $|k\rangle$. The above implies that $T|k\rangle = e^{-\mathbf{i}ka}|k\rangle$ and that $H|k\rangle = E_k|k\rangle$.
If there were degeneracy then not every eigenvector of H need be an eigenvector of T. This is called *spontaneous symmetry breaking*
The proof then is very easy: $\langle x - a|k\rangle = \langle x|T|k\rangle = e^{-\mathbf{i}ka}\psi_k(x)$

(c) Infer from (1) that one can then write $\psi_k(x) = e^{\mathbf{i}kx}u_k(x)$ where $u_k(x) = u_k(x + a)$
This amounts to showing the claimed form for ψ_k satisfies (1)
$\psi_k(x - a) = e^{\mathbf{i}k(x-a)}u_k(x - a) = e^{-\mathbf{i}ka}e^{\mathbf{i}kx}u_k(x)$ done.
Note that because not every a is a valid symmetry transformation the function u_k is not generically a constant.

Note that k is different from our usual momentum. It's a *crystal momentum*!

(d) Show explicitly that for $P = -\mathbf{i}\partial_x$ that $P\psi_k(x) \neq k\psi_k(x)$
$-\mathbf{i}\partial_x(\psi_k(x)) = ke^{\mathbf{i}kx}u_k(x) - \mathbf{i}e^{\mathbf{i}kx}u'_k(x)$

(e) Show that $\frac{-\pi}{a} \leq k \leq \frac{\pi}{a}$. What is $k + \frac{2\pi}{a}$?
Consider $\psi_{k+\frac{2\pi}{a}}(x) = e^{\mathbf{i}kx}e^{\mathbf{i}\frac{2\pi}{a}x}u_{k+\frac{2\pi}{a}}(x)$ and notice that the product $e^{\mathbf{i}\frac{2\pi}{a}x}u_{k+\frac{2\pi}{a}}(x)$ is still a periodic function of x with period a.
Therefore this is just a relabelling of the function u_k. Call $v_k(x) \equiv e^{\mathbf{i}\frac{2\pi}{a}x}u_{k+\frac{2\pi}{a}}(x)$
Then $\psi_{k+\frac{2\pi}{a}} = e^{\mathbf{i}kx}v_k(x)$; this transformation on k did nothing! Therefore $k + \frac{2\pi}{a} \equiv k$ and that's why we only need the finite region.