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Quantum Mechanics B (Physics 212B) Winter 2019 Worksheet 8 – Solutions

Problems

First let's derive a useful result in time-dependent perturbation theory known as 'Fermi's Golden Rule'

Consider a time-*independent* perturbation $V(t) = V$ which is turned on at $t = 0$. We're interested in the question: what is the rate at which transitions from an initial state $|i\rangle$ to some set of final states $|f\rangle$ occur?

The first order expression for the amplitude can be written as:

$$A_{i\to f} = -\mathbf{i}\int_0^t dt'\ e^{-\mathbf{i}(E_f-E_i)t'}\langle f|V|i\rangle = \frac{e^{-\mathbf{i}(E_f-E_i)t}-1}{E_f-E_i}V_{if} \tag{1}$$

which implies the probability $P_{i\to f} = |A_{i\to f}|^2$ is:

$$P_{i\to f} = 4\frac{\sin^2\frac{\omega_{if}t}{2}}{\omega_{if}^2}|V_{if}|^2 \tag{2}$$

where $\omega_{if} = E_f - E_i$. This is clearly peaked around $\omega_{if} = 0$.

We're interested in the rate $\Gamma_{i\to f} = \lim_{t\to\infty}\frac{P_{i\to f}}{t}$. In this limit we can use the fact this function is becoming peaked to express it in terms of a delta function.

$$\Gamma_{i\to f} = 2\pi\delta(E_f - E_i)|V_{if}|^2 \tag{3}$$

This is one instance of the Fermi Golden rule. The delta function effectively conserves conservation of energy, as those are the transitions that can happen for $t\to\infty$.

Suppose there are multiple states $|f\rangle$ we can transition into. Then the full transition rate would involve a sum $\Gamma = \sum_f \Gamma_{i\to f}$.

If there's many of such states, such as a continuum of free particle states, we can approximate this sum as $\sum_f \approx \int dE_f\ \rho(E_f)$ where $\rho(E_f)$ is the "density of states".

Let's do this and write the final expression of Fermi's Golden Rule:

$$\Gamma_{i\to f} = 2\pi\rho(E_f)|V_{if}|^2 \tag{4}$$

1. **Spin Flips**

Consider a spin-$\frac{1}{2}$ particle moving in the following potential:

$$V = \frac{1}{2}x^2 \otimes |\downarrow\rangle\langle\downarrow| \tag{5}$$

At $t < 0$ the state of the state is $|i\rangle = |0\rangle|\downarrow\rangle$ where $|0\rangle$ is the ground state of the SHO.

Now suppose at $t = 0$ another potential is turned on:

$$W = \Omega\sigma^x \tag{6}$$

where $\Omega \ll 1$ such that this can be treated as a perturbation.

Compute the rate of transitions out from $|i\rangle$.

The original potential describes bound states for spin down states and a continuum of free particle states for spin up.

The perturbation allows for spin flips, because it is σ^x, but otherwise doesn't effect the position space features. So the idea is the perturbation can flip the spin and then effectively shut off the original potential, forcing it to transition to some continuum plane wave $|f\rangle = |k\rangle|\uparrow\rangle$

The density of states for a free particle in $d = 1$ using $E = \frac{k^2}{2}$ is $\rho(E) = \frac{1}{\sqrt{2E}}$

We know the initial energy is $E_i = \frac{1}{2}$ from the SHO. Now we need the matrix element

The spin piece is easy $\langle\uparrow|\Omega\sigma^x|\downarrow\rangle = \Omega$

The spatial piece involves an integral as $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}}e^{\mathbf{i}kx}$ and $\langle x|0\rangle = \frac{1}{\pi^{1/4}}e^{-\frac{1}{2}x^2}$

So then $\langle i|f\rangle \propto \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2}e^{-\mathbf{i}kx} = \frac{e^{-\frac{k^2}{2}}}{\sqrt[4]{\pi}} = \frac{e^{-\frac{1}{2}}}{\sqrt[4]{\pi}}$ where in the last step I used $E_i = \frac{1}{2} = \frac{k^2}{2}$

So then $\Gamma = 2\pi\frac{1}{\sqrt{\pi}}\frac{1}{e}\Omega^2$