

# 140B Statistical Physics

## Part 2. Quantum Gases

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### Thermal Quantum Gases

#### ■ Bosons and Fermions

#### ■ Quantum Many-Body System

Imagine our university as a **quantum many-body system**, where

- Each person represents a **particle**,
- Each role they take (be it a student, professor, staff, or service personnel) is akin to a **quantum state**.
- Each role describes a person's social functionality, just as each quantum state describes a particle's **physical properties**, such as *momentum, spin, charge, color, flavor*, etc.

Each particle *occupies* a quantum state. (like each person takes a role)

- The **occupation number**  $n_\alpha = 0, 1, 2, \dots$  represents the *number* of particles occupying the  $\alpha$ th quantum state (denoted as  $|\alpha\rangle$ ). (like the number of people in each role)
- Each quantum state  $|\alpha\rangle$  also has an associated **energy**  $\epsilon_\alpha$  -- the cost to keep a particle in the state. (like each role in a university comes with a salary -- the cost to keep a person in their position)

#### ■ Single-Particle v.s. Many-Body State

It is important to distinguish quantum states at two different levels:

- **Single-particle state:** the state or mode of a particle (like the role of a person), denoted as  $|\alpha\rangle$  (with  $\alpha = 1, 2, 3, \dots$  being the mode index).
- **Many-body state:** the state of a system of particles (like the role of an institute), denoted as  $|n\rangle$  with

$$n := \{n_\alpha \in \mathbb{N} \mid \alpha = 1, 2, 3 \dots\}, \quad (1)$$

which is an integer vector that encodes occupation numbers on all available *single-particle* states  $|\alpha\rangle$ .

The many-body state  $|n\rangle$  serves as the **microstate** of the quantum system in the statistical mechanics context, for which we can define:

- The **total number of particles** in the system (like the total number of people in our university) is

$$N(\mathbf{n}) = \sum_{\alpha} n_{\alpha}. \quad (2)$$

- The **total energy** of the system (like the total salary expenditure of our university) is

$$E(\mathbf{n}) = \sum_{\alpha} \epsilon_{\alpha} n_{\alpha}. \quad (3)$$

[Note: to differentiate the two energies  $E$  and  $\epsilon_{\alpha}$ , we often call  $E$  the **many-body energy**, and  $\epsilon_{\alpha}$  the **single-particle energy**.]

## ■ Identical Particles

In the quantum world, particles can be made **identical**, such that they can not be distinguished from one another. (This is akin to a scenario where Personal Identifiable Information (PII), like names or IDs, is protected and kept confidential. In this case, we can only talk about how many students are there in a class without revealing who they are.)

Using the math language:

- For **distinguishable particles**, a two-particle system can be described by the *many-body* quantum state  $|\alpha\rangle \otimes |\beta\rangle$ , where the 1st particle is in *single-particle* state  $|\alpha\rangle$  and the 2nd in  $|\beta\rangle$ .
- For **identical particles**, *many-body* states like  $|\alpha\rangle \otimes |\beta\rangle$  and  $|\beta\rangle \otimes |\alpha\rangle$  are indistinguishable:

|   |   |
|---|---|
| $ \alpha\rangle \otimes  \beta\rangle$  | $ \beta\rangle \otimes  \alpha\rangle$                                      |
| the 1st particle in $ \alpha\rangle$<br>the 2nd particle in $ \beta\rangle$         | the 1st particle in $ \beta\rangle$<br>the 2nd particle in $ \alpha\rangle$ |
| ↓   | ↖   |
| there is one particle in $ \alpha\rangle$ , and another particle in $ \beta\rangle$ |   |
| $ 1_{\alpha} 1_{\beta}\rangle$  |   |

The only legitimate description is  $|1_{\alpha} 1_{\beta}\rangle$ , as an equal-weight linear combination of  $|\alpha\rangle \otimes |\beta\rangle$  and  $|\beta\rangle \otimes |\alpha\rangle$ . But there are still two possible ways to combine them:

- Symmetric** combination  $\Rightarrow$  **bosons**

$$|1_{\alpha} 1_{\beta}\rangle_B = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle), \quad (4)$$

- Anti-symmetric** combination  $\Rightarrow$  **fermions**

$$|1_{\alpha} 1_{\beta}\rangle_F = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle). \quad (5)$$

What if we try to bring both particles to the  $|\alpha\rangle$  state?

- For **bosons**,  $|\alpha\rangle\otimes|\alpha\rangle$  is still a meaningful quantum state, with the state norm enhanced by  $\sqrt{2}$  due to a *constructive interference*:

$$\frac{1}{\sqrt{2}} (|\alpha\rangle\otimes|\alpha\rangle + |\alpha\rangle\otimes|\alpha\rangle) = \sqrt{2} |\alpha\rangle\otimes|\alpha\rangle = \sqrt{2} |2_\alpha\rangle_B. \quad (6)$$

**Boson enhancement factor:** the probability to transfer a boson to a state that has already been occupied by  $n$  bosons will get enhanced by a factor of  $n+1$ .

⇒ Bosons *like* to stay in the *same* state (exhibiting an *extroverted* personality).

- For **fermions**, the quantum state vanishes due to *destructive interference*.

$$\frac{1}{\sqrt{2}} (|\alpha\rangle\otimes|\alpha\rangle - |\alpha\rangle\otimes|\alpha\rangle) = 0. \quad (7)$$

**Pauli exclusion principle:** two (or more) fermions can not occupy the same state simultaneously.

⇒ Fermions *hate* to stay in the *same* state (exhibiting an *introverted* personality).

This leads to different sets of possible values that the **occupation number**  $n_\alpha$  can take:

$$n_\alpha = \begin{cases} 0, 1, 2, \dots & \text{for bosons,} \\ 0, 1. & \text{for fermions,} \end{cases} \quad (8)$$

which results in different thermodynamic behaviors between bosons and fermions.

## ■ Examples of Bosons and Fermions

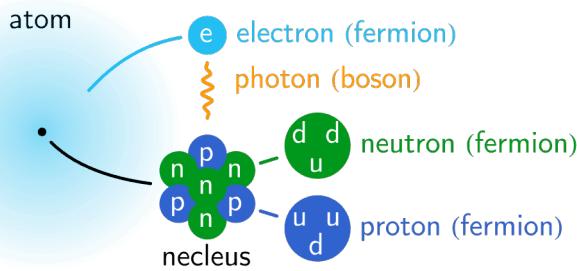
- Elementary particles (the Standard Model).

| Fermions<br>(matter constituent)                               |  |  | Bosons<br>(force mediator) |                           |                     |
|--|--|--|----------------------------|---------------------------|---------------------|
| I<br>mass<br>charge<br>spin<br>2.2MeV<br>2/3<br>1/2<br>u<br>up | II<br>1.28GeV<br>2/3<br>1/2<br>c<br>charm                      | III<br>173GeV<br>2/3<br>1/2<br>t<br>top                        | 0<br>0<br>1<br>g<br>gluon  | 0<br>0<br>0<br>H<br>higgs |                     |
| QUARKS   |  |  |                            |                           |                     |
| II<br>4.7MeV<br>-1/3<br>1/2<br>d<br>down                       | III<br>96MeV<br>-1/3<br>1/2<br>s<br>strange                    | IV<br>4.18GeV<br>-1/3<br>1/2<br>b<br>bottom                    | 0<br>0<br>1<br>γ<br>photon |                           |                     |
| LEPTONS  |  |  |                            |                           |                     |
| I<br>0.511MeV<br>-1<br>1/2<br>e<br>electron                    | II<br>106MeV<br>-1<br>1/2<br>μ<br>muon                         | III<br>1.78GeV<br>-1<br>1/2<br>τ<br>tauon                      | 0<br>±1<br>W<br>W boson    |                           |                     |
|  |  |  |                            |                           | SCALAR BOSONS       |
| II<br><1eV<br>0<br>1/2<br>ν <sub>e</sub><br>electron neutrino  | III<br><0.17MeV<br>0<br>1/2<br>ν <sub>μ</sub><br>muon neutrino | IV<br><18.2MeV<br>0<br>1/2<br>ν <sub>τ</sub><br>tauon neutrino | 0<br>1<br>Z<br>Z boson     |                           | GAUGE VECTOR BOSONS |

- Fermions are of *half-integer* spins.

- Bosons are of *integer* spins.

- Inside an atom ( ${}^7\text{Li}$  isotope):



- Rules (due to the multiplication rule of *fermion sign*):

$$\begin{aligned} \text{boson} + \text{boson} &= \text{boson}, \\ \text{fermion} + \text{boson} &= \text{fermion}, \\ \text{fermion} + \text{fermion} &= \text{boson}. \end{aligned} \tag{9}$$

- Total number of fermions in an atom

$$N_F = N_p + N_n + N_e \stackrel{\text{neutral}}{=} 2 N_p + N_n. \tag{10}$$

- A neutral atom is a fermion (boson) iff the number of *neutrons*  $N_n$  is odd (even).

## ■ Statistical Distributions

### ■ What is Ideal Gas?

**Ideal gas** is a system of **non-interacting** (free) particles.

- Its energy is a simple sum of energies of particles in each single-particle mode

$$E(\mathbf{n}) = \sum_{\alpha} \epsilon_{\alpha} n_{\alpha}. \tag{11}$$

In real gases or more complex systems (liquids, solids), there are **interactions** between particles.

- The energy function will contain higher-order interaction terms

$$E(\mathbf{n}) = \sum_{\alpha} \epsilon_{\alpha} n_{\alpha} + \underbrace{\sum_{\alpha\beta} \epsilon_{\alpha\beta} n_{\alpha} n_{\beta}}_{\text{two-body interaction}} + \underbrace{\sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} n_{\alpha} n_{\beta} n_{\gamma}}_{\text{three-body interaction}} + \dots \tag{12}$$

- The interaction, say  $\epsilon_{\alpha\beta}$ , is

- **Attractive**, if  $\epsilon_{\alpha\beta} < 0$ : energy is *reduced* when particles attract each other. (like people in a cooperative environment will reduce operational cost)
- **Repulsive**, if  $\epsilon_{\alpha\beta} > 0$ : energy is *increased* when particles repel each other. (like people in a confrontational environment will increase operational cost)

- If the interaction is weak  $\epsilon_{\alpha\beta}, \epsilon_{\alpha\beta\gamma}, \dots \ll \epsilon_\alpha$ , the system can remain in the gas phase. If the interaction gets strong enough, the system may transition into liquid or solid phases.

These ideas applies to quantum particles. An **ideal quantum gas** refers to a system of **non-interacting** (free) *bosons* or *fermions*.

## ■ Grand Partition Function

The **grand partition function** is particularly convenient to study the statistical mechanics of quantum gases.

- Each **microstate** of a quantum many-body system is labeled by

$$\mathbf{n} := \{n_\alpha \mid \alpha = 1, 2, \dots\}. \quad (13)$$

- Following Eq. (3) and Eq. (2), the **energy** and **particle number** functions are

$$E(\mathbf{n}) = \sum_{\alpha} \epsilon_{\alpha} n_{\alpha}, \quad N(\mathbf{n}) = \sum_{\alpha} n_{\alpha}. \quad (14)$$

- The **probability**  $p(\mathbf{n})$  to observe the system in the microstate is

$$p(\mathbf{n}) = \frac{1}{Z} e^{-\beta(E(\mathbf{n}) - \mu N(\mathbf{n}))}, \quad (15)$$

where **grand partition function**  $Z$  is given by

$$Z = \sum_{\mathbf{n}} e^{-\beta(E(\mathbf{n}) - \mu N(\mathbf{n}))}. \quad (16)$$

Given that there is *no interaction* among single-particle states for **ideal gas**, the probability distribution  $p(\mathbf{n})$  can be *factorized* (meaning that the distribution of each  $n_\alpha$  is *independent*)

$$p(\mathbf{n}) = \prod_{\alpha} p(n_{\alpha}), \quad (17)$$

where

- The probability to observe  $n_\alpha$  particles in the single-particle state is

$$p(n_{\alpha}) = \frac{1}{Z_{\alpha}} e^{-\beta(\epsilon_{\alpha} - \mu) n_{\alpha}}. \quad (18)$$

- With the single-mode grand partition function

$$Z_{\alpha} = \sum_{n_{\alpha}} e^{-\beta(\epsilon_{\alpha} - \mu) n_{\alpha}}. \quad (19)$$

The full grand partition function is also factorized

$$Z = \prod_{\alpha} Z_{\alpha}, \quad (20)$$

**Exc 1** Derive Eq. (17 - 20) from Eq. (15, 16).

Or written in the logarithm form

$$\log \mathcal{Z} = \sum_{\alpha} \log \mathcal{Z}_{\alpha}. \quad (21)$$

#### □ Bosons

For **bosons**,  $n_{\alpha} = 0, 1, 2, \dots$  can take all natural numbers.

$$\mathcal{Z}_{\alpha} = \sum_{n_{\alpha}=0}^{\infty} e^{-\beta(\epsilon_{\alpha}-\mu) n_{\alpha}} = \frac{1}{1 - e^{-\beta(\epsilon_{\alpha}-\mu)}}. \quad (22)$$

**Exc 2** Evaluate the summation in Eq. (22).

Or written in the logarithm form

$$\log \mathcal{Z}_{\alpha} = -\log(1 - e^{-\beta(\epsilon_{\alpha}-\mu)}). \quad (23)$$

Note: the partition function is only well-defined for  $\beta(\epsilon_{\alpha}-\mu) > 0$  (i.e.  $\mu < \epsilon_{\alpha}$ ).

#### □ Fermions

For **fermions**,  $n_{\alpha} = 0, 1$  has only two possibilities due to the *Pauli exclusion principle*.

$$\mathcal{Z}_{\alpha} = \sum_{n_{\alpha}=0,1} e^{-\beta(\epsilon_{\alpha}-\mu) n_{\alpha}} = 1 + e^{-\beta(\epsilon_{\alpha}-\mu)}. \quad (24)$$

Or written in the logarithm form

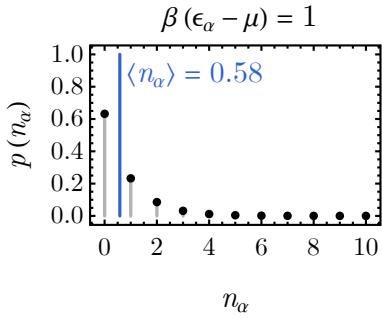
$$\log \mathcal{Z}_{\alpha} = \log(1 + e^{-\beta(\epsilon_{\alpha}-\mu)}). \quad (25)$$

### ■ Bose-Einstein Distribution

Substitute the boson partition function  $\mathcal{Z}_{\alpha}$  in Eq. (22) into the probability distribution  $p(n_{\alpha})$  in Eq. (18),

$$p(n_{\alpha}) = (1 - e^{-\beta(\epsilon_{\alpha}-\mu)}) e^{-\beta(\epsilon_{\alpha}-\mu) n_{\alpha}}. \quad (26)$$

- The distribution is tuned by a joint parameter  $\beta(\epsilon_{\alpha}-\mu)$ .

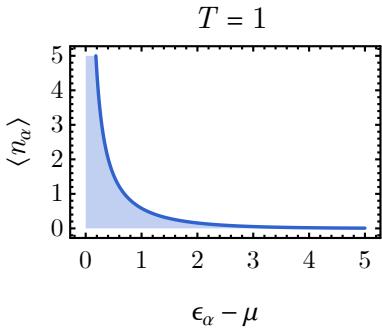


Based on the probability distribution Eq. (26), the **average boson number**  $\langle n_\alpha \rangle$  occupying the single-particle state is

$$\langle n_\alpha \rangle = \sum_{n_\alpha=0}^{\infty} n_\alpha p(n_\alpha) = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}. \quad (27)$$

**Exc 3** Evaluate the summation in Eq. (27).

This is also known as the **Bose-Einstein distribution**.



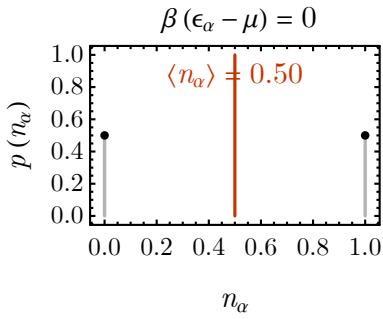
- The Bose-Einstein distribution is well-defined only for  $\epsilon_\alpha - \mu > 0$ .

## ■ Fermi-Dirac Distribution

Substitute the fermion partition function  $\mathcal{Z}_\alpha$  in Eq. (24) into the probability distribution  $p(n_\alpha)$  in Eq. (18),

$$p(n_\alpha) = \frac{e^{-\beta(\epsilon_\alpha - \mu) n_\alpha}}{1 + e^{-\beta(\epsilon_\alpha - \mu)}} = \begin{cases} \frac{1}{e^{-\beta(\epsilon_\alpha - \mu)} + 1} & n_\alpha = 0, \\ \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1} & n_\alpha = 1. \end{cases} \quad (28)$$

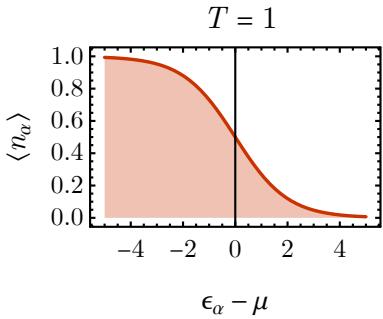
- The distribution is tuned by a joint parameter  $\beta(\epsilon_\alpha - \mu)$ .



Based on the probability distribution Eq. (28), the **average fermion number**  $\langle n_\alpha \rangle$  occupying the single-particle state is

$$\langle n_\alpha \rangle = \sum_{n_\alpha=0,1} n_\alpha p(n_\alpha) = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1}. \quad (29)$$

This is also known as the **Fermi-Dirac distribution**.

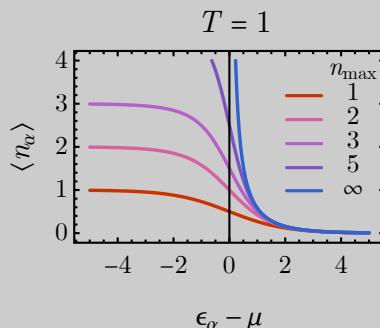


- The Fermi-Dirac distribution is applicable for any  $\epsilon_\alpha - \mu \in \mathbb{R}$ .

Imagine a hypothetical type of identical particles (neither boson nor fermion), with the defining property that its occupation number  $n_\alpha = 0, 1, \dots, n_{\max}$  can only take values up to some integer  $n_{\max}$ .

- (i) Calculate the grand partition function  $\mathcal{Z}_\alpha = \sum_{n_\alpha=0}^{n_{\max}} e^{-\beta(\epsilon_\alpha - \mu)n_\alpha}$  for such hypothetical particles on a single-particle state of energy  $\epsilon_\alpha$ .
- (ii) Construct the probability distribution  $p(n_\alpha)$  of the occupation number  $n_\alpha$ , and prove that the expectation value  $\langle n_\alpha \rangle$  can be computed from the following derivative:  $\langle n_\alpha \rangle := \sum_{n_\alpha=0}^{n_{\max}} n_\alpha p(n_\alpha) = \beta^{-1} \partial_\mu \log \mathcal{Z}_\alpha$ .
- (iii) Use the derivative trick to compute  $\langle n_\alpha \rangle$  based on the result of  $\mathcal{Z}_\alpha$  obtained in problem (i).
- (iv) Compare the result with Eq. (27) and Eq. (29), and show that the hypothetical particles behave like bosons as  $n_{\max} \rightarrow \infty$ , and fermions as  $n_{\max} = 1$ .

**HW  
1**



## ■ Maxwell-Boltzmann Distribution

Interestingly, the **average occupation number**  $\langle n_\alpha \rangle$  assumes a unified form

$$\langle n_\alpha \rangle = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - \eta}. \quad (30)$$

where  $\eta$  denotes the **fermion sign**:

$$\eta = \begin{cases} +1 & \text{for bosons,} \\ -1 & \text{for fermions.} \end{cases} \quad (31)$$

The inverse temperature  $\beta = 1/T$  is always a positive real number, and the chemical potential  $\mu$  should obey:

- For **bosons**, the chemical potential  $\mu$  can not exceed the lowest energy level (i.e.  $\forall \alpha : \mu < \epsilon_\alpha$ ).
- For **fermions**, there is no restriction on the chemical potential.

Is there a “classical limit” where bosons and fermions behave the same? -- Yes, this is the limit when the gas is dilute (i.e.  $N/V \rightarrow 0$ ).

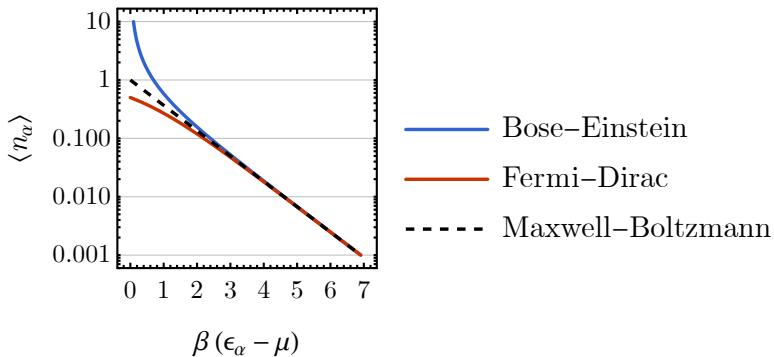
- In the **dilute limit**, the average occupancy of each single-particle state is close to zero (compare to 1), i.e.  $\langle n_\alpha \rangle \rightarrow 0$  (such that  $N = \sum_\alpha \langle n_\alpha \rangle$  is also small).

- Based on Eq. (30), this limit is achieved when  $e^{\beta(\epsilon_\alpha - \mu)} \rightarrow \infty$ , such that the fermion sign  $\eta$  in the denominator is negligible, and the average occupation number

$$\langle n_\alpha \rangle \simeq e^{-\beta(\epsilon_\alpha - \mu)} \quad (32)$$

will be close to zero.

This, Eq. (32), is known as the **Maxwell-Boltzmann distribution**.



There will be no distinction between Bose and Fermi gases in their *dilute* limit (also called the *classical* limit, as the quantum effect that distinguishes bosons and fermions is no longer important).

## ■ Energy and Particle Number

The average occupation number  $\langle n_\alpha \rangle$  enables us to compute

- The average total **energy** (defined as  $E := \sum_{\mathbf{n}} E(\mathbf{n}) p(\mathbf{n})$ ):

$$E = \sum_{\alpha} \epsilon_{\alpha} \langle n_{\alpha} \rangle = \sum_{\alpha} \frac{\epsilon_{\alpha}}{e^{\beta(\epsilon_{\alpha} - \mu)} - \eta}. \quad (33)$$

- The average total **particle number** (defined as  $N := \sum_{\mathbf{n}} N(\mathbf{n}) p(\mathbf{n})$ ):

$$N = \sum_{\alpha} \langle n_{\alpha} \rangle = \sum_{\alpha} \frac{1}{e^{\beta(\epsilon_{\alpha} - \mu)} - \eta}. \quad (34)$$

The sign  $\eta$  depends on whether the particles are bosons ( $\eta = +1$ ) or fermions ( $\eta = -1$ ).

## ■ Continuum Limit

### ■ Particles in Free Space

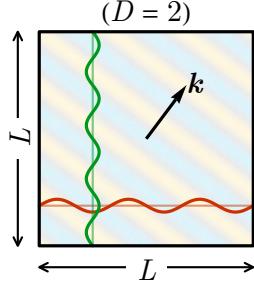
So far, we have not explained what labels the single-particle state and what determines its energy  $\epsilon_{\alpha}$ .

Consider non-interacting particles in the **free space** (assuming translation symmetry),

energy eigenstates are labeled by **momentum**  $\mathbf{k}$  (as a good quantum number) and **spin**  $\sigma$

$$\alpha = (\mathbf{k}, \sigma), \quad (35)$$

- **Momentum**  $\mathbf{k} = (k_1, k_2, \dots, k_D)$  is a  $D$ -dimensional *vector* for particles in  $D$ -dimensional space.



In quantum mechanics, particles are *waves*, and the momentum  $k_i$  is set by the *wave length*  $\lambda_i$  via the **De Broglie relation** in the corresponding direction,

$$k_i = \frac{2\pi}{\lambda_i} = \frac{2\pi m_i}{L} \quad (m_i \in \mathbb{Z}), \quad (36)$$

which must be **quantized** in a box of finite size  $L$ .

- The volume of the system (the box) will be

$$V = L^D. \quad (37)$$

- **Spin**  $\sigma$  labels the *internal* degrees of freedom of the particle. The number of spin states is called the **spin degeneracy**, denoted as  $g_s$ .

- Electron (spin-1/2 fermion):

$$g_s = 2. \quad (38)$$

Two spin states:  $\sigma = \pm 1/2$ .

- Photon (spin-1 *gauge* boson):

$$g_s = D - 1. \quad (39)$$

$(D - 1)$  spin (polarization) states:  $\sigma = \pm 1$  for  $D = 3$ . The  $\sigma = 0$  photon (longitudinal) photon is forbidden by the gauge invariance.

- Phonon (spin-1 *scalar* boson):

$$g_s = D. \quad (40)$$

$D$  spin (polarization) states:  $\sigma = 0, \pm 1$  for  $D = 3$ .

Usually (without internal symmetry breaking), the energy  $\epsilon_\alpha = \epsilon_{\mathbf{k}, \sigma}$  does not depend on the spin degrees of freedom  $\sigma$ , so different spin states are *degenerated* in energy.

## ■ Dispersion Relations

**Dispersion relation** tells us how the energy  $\epsilon_{\mathbf{k}}$  of a particle (or a wave) is related to its

momentum  $\mathbf{k}$ .

- **Non-relativistic** particles (electrons in metal):

$$\epsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}, \quad (41)$$

with  $m$  - the **inertial mass** of the particle.

- **Relativistic** particles (photons, acoustic phonons, electrons in graphene):

$$\epsilon_{\mathbf{k}} = c|\mathbf{k}|. \quad (42)$$

with  $c$  - the **limiting speed** of the particle, e.g. the speed light/sound or fermi-velocity.  
(In these expressions, we have set  $\hbar = 1$ ).

## ■ Density of States

**Density of state** (DOS)  $g(\epsilon)$  counts the number of single-particle states per volume and per energy interval  $d\epsilon$  near the energy  $\epsilon$  in the **continuum limit**, defined as the *integration measure* (weighting function) required to convert the *state summation* to an *energy integration* for any function  $f(\epsilon)$ :

$$\forall f: \sum_{\alpha} f(\epsilon_{\alpha}) = V \int d\epsilon g(\epsilon) f(\epsilon). \quad (43)$$

For example, we can use it to compute the energy  $E$  and particle number  $N$  in integral forms (following Eq. (33) and Eq. (34))

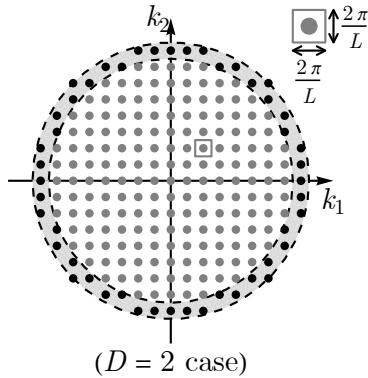
$$E = V \int d\epsilon \frac{\epsilon g(\epsilon)}{e^{\beta(\epsilon-\mu)} - \eta}. \quad (44)$$

$$N = V \int d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} - \eta}. \quad (45)$$

Similar approach applies to many other thermodynamic properties whenever a *state sum* is to be evaluated.

## □ General Principle

Counting number of states in the momentum space:



- In this picture, each point corresponds to a state in the momentum space with  $g_s$ -fold spin degeneracy. The momentum is discretized according to Eq. (36).
- The continuum limit refers to the limit  $L \rightarrow \infty$ , such that the states are densely distributed, forming a continuum.
- Consider a ball in the momentum space of radius  $k$ , its volume is

$$\begin{aligned}\Omega &= \int_{|\mathbf{k}| < k} d^D \mathbf{k} \\ &= \int_0^k A_D k^{D-1} dk \\ &= \frac{A_D}{D} k^D,\end{aligned}\tag{46}$$

where  $A_D$  is the area of a  $(D - 1)$ -dimensional hypersphere of unit radius (in the  $D$ -dimensional space), given by

$$A_D = \frac{2\pi^{D/2}}{\Gamma(D/2)},\tag{47}$$

where  $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$  is the Euler Gamma function.

$$\begin{array}{c|ccccccccccccc} D & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots \\ \hline A_D & 2 & 2\pi & 4\pi & 2\pi^2 & \frac{8\pi^2}{3} & \pi^3 & \frac{16\pi^3}{15} & \frac{\pi^4}{3} & \frac{32\pi^4}{105} & \frac{\pi^5}{12} & \dots \end{array}$$

- The number of states  $N$  in the ball is

$$N_k = \sum_a 1 = g_s \sum_{|\mathbf{k}| < k} 1 = g_s \frac{\Omega}{\left(\frac{2\pi}{L}\right)^D} = \frac{g_s A_D V}{(2\pi)^D D} k^D.\tag{48}$$

- On the other hand, we expect the DOS function  $g(\epsilon)$  to be such that

$$N_k = \sum_a 1 = V \int_0^{\epsilon_k} d\epsilon g(\epsilon).\tag{49}$$

Therefore, comparing Eq. (48) and Eq. (49),  $g(\epsilon)$  should be given by

$$\begin{aligned}
g(\epsilon) &= \frac{1}{V} \left( \frac{\partial N_k}{\partial \epsilon_k} \right)_{\epsilon_k=\epsilon} = \frac{1}{V} \left( \frac{\partial N_k}{\partial k} \frac{\partial k}{\partial \epsilon_k} \right)_{\epsilon_k=\epsilon} \\
&= \frac{g_s A_D}{(2\pi)^D} \left( k^{D-1} \frac{\partial k}{\partial \epsilon_k} \right)_{\epsilon_k=\epsilon},
\end{aligned} \tag{50}$$

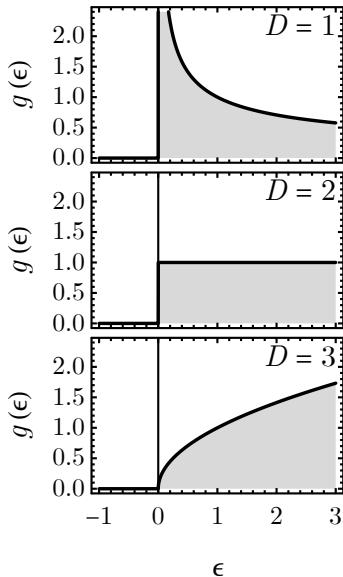
which is fully determined by the dispersion relation  $\epsilon_k$ .

#### □ Specific Examples

- For **non-relativistic** particles (with the inertial mass  $m$ )

$$g(\epsilon) = \frac{g_s A_D}{2(2\pi)^D} (2m)^{D/2} \epsilon^{D/2-1}. \tag{51}$$

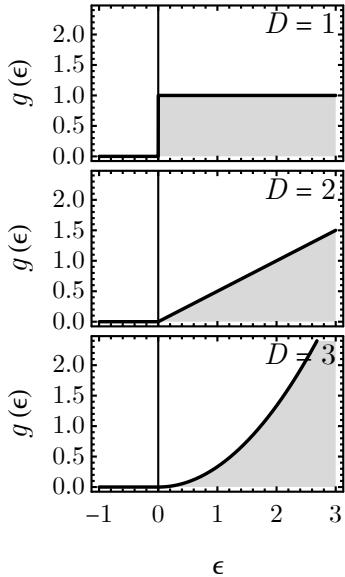
**Exc 4** Derive Eq. (51) given the dispersion relation Eq. (41).



- For **relativistic** particles (with the limiting speed  $c$ )

$$g(\epsilon) = \frac{g_s A_D}{(2\pi)^D} c^{-D} \epsilon^{D-1}. \tag{52}$$

**Exc 5** Derive Eq. (51) given the dispersion relation Eq. (42).



In these examples, for particles in free space, the DOS always scales with the particle energy  $\epsilon$  in a **power-law** manner. We might as well introduce a **characteristic energy**  $\epsilon_0$  to express the DOS uniformly as

$$g(\epsilon) = \frac{\Delta \epsilon^{\Delta-1}}{\epsilon_0^\Delta}. \quad (53)$$

where the exponent

$$\Delta = D/\zeta > 0 \quad (54)$$

depends on:

- the **dimension of space**  $D$ ,
- the **dynamical exponent**  $\zeta$  (such that  $\epsilon_k \sim k^\zeta$ ):
  - $\zeta = 1$  for *relativistic* particles with *linear* dispersion,
  - $\zeta = 2$  for *non-relativistic* particles with *quadratic* dispersion.

| System                                       | $D$ | $\zeta$ | $\Delta$ |
|--|-----|---------|----------|
| Electrons in metal                           | 3   | 2       | 3/2      |
| Electrons in graphene                        | 2   | 1       | 2        |
| Electrons in field-effect transistors        | 2   | 2       | 1        |
| Electrons in nanotube                        | 1   | 2       | 1/2      |
| Electrons on quantum Hall insulator boundary | 1   | 1       | 1        |
| Phonons in solid                             | 3   | 1       | 3        |
| Atomic gas                                   | 3   | 2       | 3/2      |
| Cold atoms in 2D optical trap                | 2   | 2       | 1        |

Eq. (53) is designed in such a way that the integration measure takes a simple form

$$g(\epsilon) d\epsilon = d\left(\frac{\epsilon}{\epsilon_0}\right)^\Delta, \quad (55)$$

assuming the single-particle energy  $\epsilon \geq 0$  is non-negative. The energy scale  $\epsilon_0$  depends on the kinetic model of the free particle,

$$\frac{1}{\epsilon_0^\Delta} = \begin{cases} \frac{g_s}{\Gamma(\Delta/2+1)} \left(\frac{1}{2\pi^{1/2} c}\right)^\Delta & \zeta = 1, \\ \frac{g_s}{\Gamma(\Delta+1)} \left(\frac{m}{2\pi}\right)^\Delta & \zeta = 2. \end{cases} \quad (56)$$

**Exc 6** Derive Eq. (56).

## ■ Thermodynamic Properties

### ■ Free Energy

The **free energy** plays a central role in statistical mechanics, bridging the microscopic and macroscopic realms. It is defined by the logarithmic partition function

$$\mathcal{F} = -T \log \mathcal{Z}. \quad (57)$$

- According to Eq. (21), for ideal gases, the free energy arises from the independent contributions of each single-particle mode.

$$\begin{aligned} \mathcal{F} &= -T \sum_{\alpha} \log \mathcal{Z}_{\alpha} \\ &= \eta T V \int_0^{\infty} d\epsilon g(\epsilon) \log(1 - \eta e^{-\beta(\epsilon-\mu)}). \end{aligned} \quad (58)$$

It will be convenient to introduce the **fugacity**

$$z = e^{\beta\mu} = e^{\mu/T}. \quad (59)$$

- For bosons:  $z \in (0, 1)$  (as  $\mu < 0$ ).
- For fermions:  $z \in (0, +\infty)$ .

Using the general form of the DOS  $g(\epsilon)$  in Eq. (53), the free energy can be written as

$$\mathcal{F} = -T V \left( \frac{T}{\epsilon_0} \right)^\Delta I_\eta^\Delta(z), \quad (60)$$

where  $I_\eta^\Delta(z)$  is a dimensionless function of the fugacity  $z = e^{\mu/T}$ , as defined by the following integral

$$I_\eta^\Delta(z) := \int_0^\infty dx \frac{x^\Delta}{z^{-1} e^x - \eta}. \quad (61)$$

**Exc 7** Derive and verify Eq. (60, 61).

#### □ Mathematical Properties of $I_\eta^\Delta(z)$

- Expression:  $I_\eta^\Delta(z)$  can be expressed in terms of the Euler Gamma function  $\Gamma(n)$  and the polylogarithm function  $\text{Li}_n(z)$

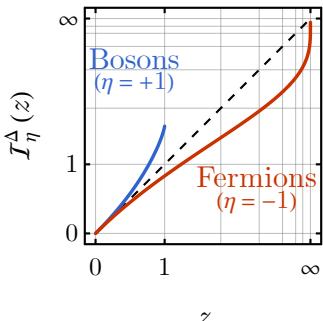
$$I_\eta^\Delta(z) = \frac{1}{\eta} \Gamma(\Delta + 1) \text{Li}_{\Delta+1}(\eta z), \quad (62)$$

where  $\Gamma(n)$  and  $\text{Li}_n(z)$  are defined by

$$\begin{aligned} \Gamma(n) &= \int_0^\infty x^{n-1} e^{-x} dx, \\ \text{Li}_n(z) &= \sum_{k=1}^\infty \frac{z^k}{k^n}. \end{aligned} \quad (63)$$

**Exc 8** Verify Eq. (62) given the definition Eq. (63).

- Graph: Behavior of  $I_\eta^\Delta(z)$  for  $\eta = \pm 1$ .



- For bosons ( $\eta = +1$ ):  $z \in (0, 1)$  and  $I_+^\Delta(z)$  stops at  $z = 1$ .
- For fermions ( $\eta = -1$ ):  $z \in (0, +\infty)$  and  $I_-^\Delta(z)$  monotonically increases towards  $+\infty$  as  $z \rightarrow +\infty$ .
- When  $z \rightarrow 0$ ,  $I_\eta^\Delta(z)$  for both bosons and fermions converges to the same linear behavior  $I_\eta^\Delta(z) \propto z$ .
- Derivative:  $I_\eta^\Delta(z)$  behaves like  $(\log z)^\Delta$  under differentiation

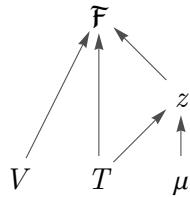
$$z \frac{\partial I_\eta^\Delta(z)}{\partial z} \equiv \frac{\partial I_\eta^\Delta(z)}{\partial \log z} = \Delta I_\eta^{\Delta-1}(z). \quad (64)$$

**Exc 9**

Verify Eq. (64).

## ■ Free Energy Derivatives

As shown in Eq. (60), the (grand) free energy  $\mathcal{F}$  is a function of  $V, T, \mu$  (or  $z = e^{\mu/T}$ ), as illustrated by the following graph



It obeys the following thermodynamic identity

$$d\mathcal{F} = -P dV - S dT - N d\mu, \quad (65)$$

which enables us to compute the following thermodynamic quantities.

- Pressure

$$P = -\left(\frac{\partial \mathcal{F}}{\partial V}\right)_{T,\mu} = T \left(\frac{T}{\epsilon_0}\right)^\Delta I_\eta^\Delta(z). \quad (66)$$

- Entropy

$$S = -\left(\frac{\partial \mathcal{F}}{\partial T}\right)_{V,\mu} = V \left(\frac{T}{\epsilon_0}\right)^\Delta \left( (\Delta + 1) I_\eta^\Delta(z) - \frac{\Delta \mu}{T} I_\eta^{\Delta-1}(z) \right). \quad (67)$$

**Exc 10**

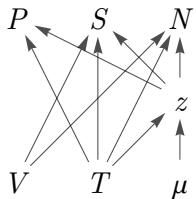
Derive Eq. (67).

- Particle number

$$N = - \left( \frac{\partial \mathcal{F}}{\partial \mu} \right)_{V,T} = \Delta V \left( \frac{T}{\epsilon_0} \right)^\Delta I_\eta^{\Delta-1}(z). \quad (68)$$

**Exc 11** Derive Eq. (68).

The variable dependence among these thermodynamic quantities is summarized by the following graph:



Based on these, the **energy**  $E$  can be reconstructed

$$\begin{aligned} E &= \mathcal{F} + T S + \mu N \\ &= \Delta T V \left( \frac{T}{\epsilon_0} \right)^\Delta I_\eta^\Delta(z). \end{aligned} \quad (69)$$

**Exc 12** Derive Eq. (69).

Comparing Eq. (66) and Eq. (69), we conclude that

$$E = \Delta P V, \quad (70)$$

which holds under all conditions regardless of bosons or fermions. For example, in  $D = 3$  dimensional space,

- For gas of non-relativistic particles,  $E = \frac{3}{2} P V$ .
- For gas of relativistic particles,  $E = 3 P V$ .

Let us try a different way to compute the energy  $E$  and the particle number  $N$  for the ideal quantum gas.

(i) Starting from Eq. (33) and Eq. (34), convert the state summation to energy integration, and prove that  $E$  and  $N$  can be expressed as

$$E = V \int_0^\infty d\epsilon g(\epsilon) \frac{\epsilon}{e^{\beta(\epsilon-\mu)} - \eta},$$

$$N = V \int_0^\infty d\epsilon g(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)} - \eta}.$$

(ii) Plugging in the density of state  $g(\epsilon)$  in Eq. (53), manipulate the integrations to the standard form in Eq. (61), and show that the results are matches Eq. (69) and Eq. (68) .

**HW 2**

## ■ Equation of State

The **equation of state** is an equation that relates the thermodynamic properties together, like pressure  $P$ , volume  $V$ , particle number  $N$ , and temperature  $T$ .

- A well-known example is the (classical) **ideal gas law**:

$$P V = N T, \quad (71)$$

where the Boltzmann constant has been set to  $k_B = 1$ . However, it only describes the ideal gas in the *classical limit* (low density and high temperature).

- Comparing Eq. (66) and Eq. (68), we found the **equation of state** for **ideal quantum gas** (free bosons and fermions)

$$P V = k_{\text{eff}} N T, \quad (72)$$

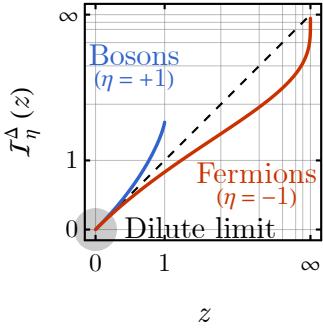
where the  $P V / N T$  ratio is a function of *fugacity*  $z = e^{\mu/T}$  only, denoted as

$$k_{\text{eff}} := \frac{P V}{N T} = \frac{I_{\eta}^{\Delta}(z)}{\Delta I_{\eta}^{\Delta-1}(z)}. \quad (73)$$

It can be viewed as an effective Boltzmann constant (we will see that  $k_{\text{eff}} = 1$  for ideal classical gas as  $z \rightarrow 0$ ).

### □ Dilute (Classical) Limit

In the **dilute limit** (the *classical limit*), we expand  $I_{\eta}^{\Delta}(z)$  around  $z \rightarrow 0$  (or  $\mu / T \rightarrow -\infty$ ).



$$I_{\eta}^{\Delta}(z) \stackrel{z \rightarrow 0}{=} \Gamma(\Delta + 1) \left( z + \frac{\eta}{2^{\Delta+1}} z^2 + \dots \right). \quad (74)$$

**Exc  
13**

Verify Eq. (74).

Using Eq. (74), around  $z \rightarrow 0$ ,  $k_{\text{eff}}$  and  $N$  behaves as

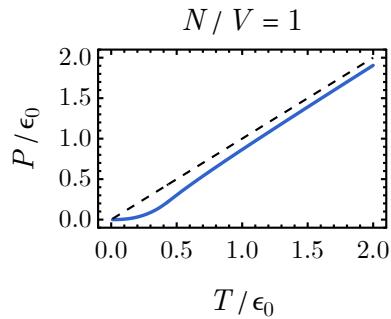
$$\begin{aligned} k_{\text{eff}} &= 1 - \frac{\eta z}{2^{\Delta+1}} + \dots, \\ N &= V \left( \frac{T}{\epsilon_0} \right)^\Delta \Gamma(\Delta+1) z + \dots \end{aligned} \tag{75}$$

**Exc 14** Verify the expansions in Eq. (75).

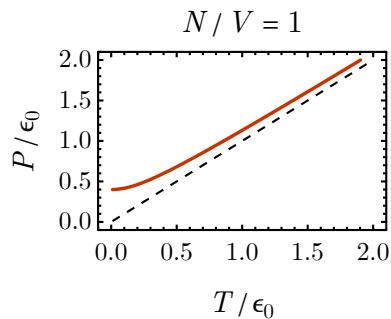
We can solve  $z$  in terms of  $N / V$  and substitute the solution to  $k_{\text{eff}}$ , the expansion becomes

$$\frac{P V}{N T} = \left( 1 - \frac{\eta}{2^{\Delta+1} \Gamma(\Delta+1)} \frac{N}{V} \left( \frac{\epsilon_0}{T} \right)^\Delta + \dots \right) \tag{76}$$

- The deviation from the ideal *classical* gas behavior ( $P V = N T$ ) occurs when  $(N / V)(\epsilon_0 / T)^\Delta \gg 1$ , i.e. when
  - the temperature is low ( $T \rightarrow 0$ ),
  - or the density is high ( $N / V \rightarrow \infty$ ).
- Bose statistics ( $\eta = +1$ ) causes an *decrease* in pressure compare to the classical limit, as if there is an effective *attraction* between *bosons*.



- Fermi statistics ( $\eta = -1$ ) causes an *increase* in pressure compare to the classical limit, as if there is an effective *repulsion* between *fermions*.



## ■ Isothermal Process

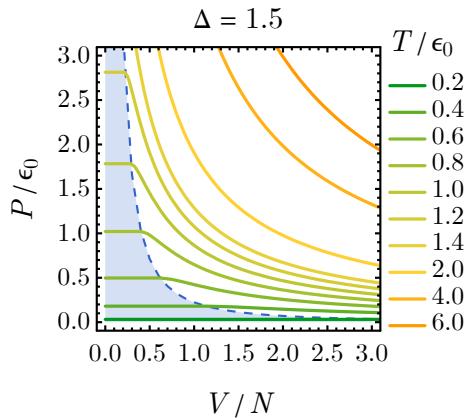
**Isothermal** (constant-temperature) process is the thermodynamic process where  $T$  and  $N$  are fixed, i.e. the system in quasi-equilibrium with a thermal bath ( $dT = 0$ ) and there is no particle exchange ( $dN = 0$ ).

Given a fixed particle number  $N$ , Eq. (66) and Eq. (68) tell us how to compute pressure  $P$  and volume  $V$  as a function of temperature  $T$  and fugacity  $z$

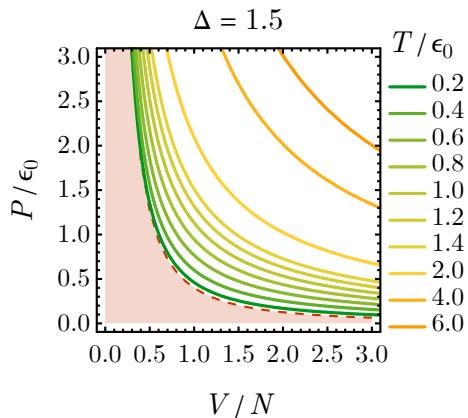
$$\begin{aligned} P(T, z) &= T \left( \frac{T}{\epsilon_0} \right)^\Delta I_\eta^\Delta(z), \\ V(T, z) &= N \left( \Delta \left( \frac{T}{\epsilon_0} \right)^\Delta I_\eta^{\Delta-1}(z) \right)^{-1}. \end{aligned} \quad (77)$$

Fix  $T$ , and vary  $z$ , the point  $(V(T, z), P(T, z))$  traces out a curve in the  $P$ - $V$  plane, called the **isothermal curve**. The system will move along the isothermal curve in the isothermal process.

- For bosons ( $\eta = +1$ ), the isothermal curves are like



- For fermions ( $\eta = -1$ ), the isothermal curves are like



In the classical limit, isothermal curves follow  $P V = N T = \text{const}$ . However, as  $N / V \rightarrow \infty$  or

$T \rightarrow 0$ , isothermal curves deviates from the classical behavior in distinct ways,

- **Degenerated Bose gas:** the curves flatten out at low volumes, suggesting transitions to *Bose-Einstein condensates* (BEC), where pressure cease to increase as more and more bosons condense to a *zero-momentum* state that does not contribute to pressure.
- **Degenerated Fermi gas:** even at zero temperature, fermions can sustain a finite pressure, called the *degeneracy pressure*, due to the *Pauli exclusion principle* preventing fermions from occupying the same quantum state.

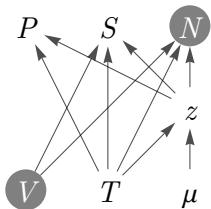
## ■ Specific Heat

**Specific heat** is the amount of *heat* (per particle) required to change the *temperature* of a system. There are two types of specific heat for systems of fixed particle number:

- **Isochoric** (constant-volume) specific heat

$$c_V = \frac{T}{N} \left( \frac{\partial S}{\partial T} \right)_{N,V}. \quad (78)$$

It can be computed by evaluating  $S$  as a function of  $T$  given  $N, V$  fixed.

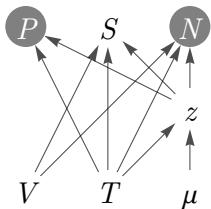


- Given  $N, V$ , input  $T$ .
- Solve for  $z$  from  $N(V, T, z) = N$ .
- Output  $S = S(V, T, z)$ .

- **Isobaric** (constant-pressure) specific heat

$$c_V = \frac{T}{N} \left( \frac{\partial S}{\partial T} \right)_{N,P}. \quad (79)$$

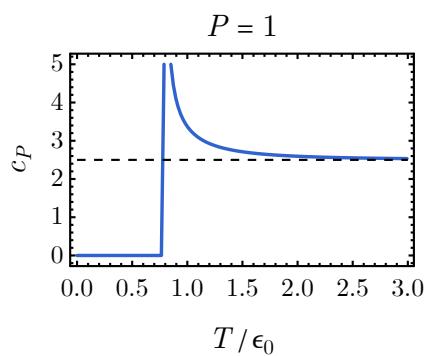
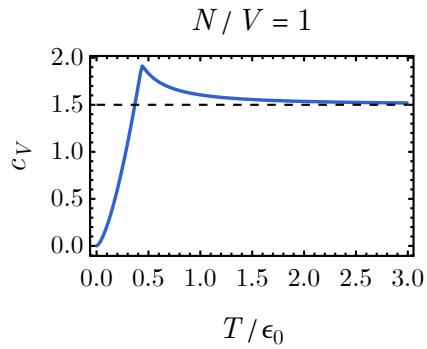
It can be computed by evaluating  $S$  as a function of  $T$  given  $N, P$  fixed.



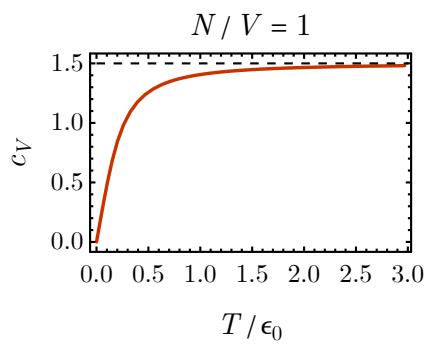
- Given  $N, P$ , input  $T$ .

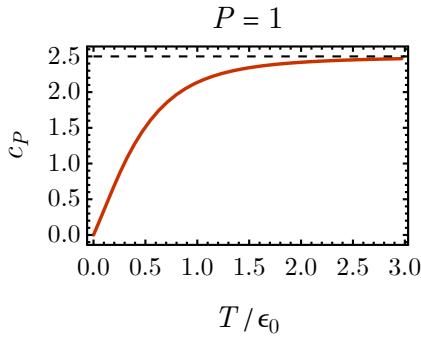
- Solve for  $z$  from  $P(T, z) = P$ .
- Solve for  $V$  from  $N(V, T, z) = N$ .
- Output  $S = S(V, T, z)$ .

For Bose gas, their behaviors are like:



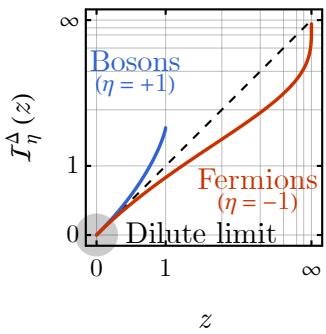
For Fermi gas, their behaviors are like:





#### □ Dilute (Classical) Limit

In the **dilute limit** (the classical limit), we expand  $I_\eta^\Delta(z)$  around  $z \rightarrow 0$  (or  $\mu/T \rightarrow -\infty$ ),



Using the expansion formula Eq. (74), we find

$$P = T \left( \frac{T}{\epsilon_0} \right)^\Delta \Gamma(\Delta + 1) z + \dots, \quad (80)$$

$$S = V \left( \frac{T}{\epsilon_0} \right)^\Delta \Gamma(\Delta + 1) (1 + \Delta - \log z) z + \dots, \quad (81)$$

$$N = V \left( \frac{T}{\epsilon_0} \right)^\Delta \Gamma(\Delta + 1) z + \dots. \quad (82)$$

**Exc 15** Verify the expansions in Eq. (80, 81, 82).

- To compute the *isochoric* (constant *volume*) specific heat,  $N$  and  $V$  are held fixed. First solve  $z$  from Eq. (82),

$$z = \frac{1}{\Gamma(\Delta + 1)} \frac{N}{V} \left( \frac{\epsilon_0}{T} \right)^\Delta, \quad (83)$$

then substitute Eq. (83) to Eq. (81), to the leading order,

$$S = N (\Delta \log T + \log V - \log N) + \text{const.} \quad (84)$$

**Exc 16** | Derive Eq. (84).

therefore,  $c_V = (\partial S / \partial \log T)_{N,V} / N$  is

$$c_V = \Delta. \quad (85)$$

- To compute the *isobaric* (constant *pressure*) specific heat,  $N$  and  $P$  must be fixed. First solve  $z$  from Eq. (80),

$$z = \frac{1}{\Gamma(\Delta+1)} \frac{P}{T} \left( \frac{\epsilon_0}{T} \right)^\Delta, \quad (86)$$

then substitute Eq. (86) to Eq. (81), to the leading order,

$$S = N ((\Delta+1) \log T - \log P) + \text{const.} \quad (87)$$

**Exc 17** | Derive Eq. (87).

therefore,  $c_P = (\partial S / \partial \log T)_{N,P} / N$  is

$$c_P = \Delta + 1. \quad (88)$$

Following the same approach, one may also compute  $c_V$  and  $c_P$  to the next order in  $z$ ,

$$\begin{aligned} c_V &= \Delta + \frac{\eta}{2^{\Delta+1} \Gamma(\Delta-1)} \frac{N}{V} \left( \frac{\epsilon_0}{T} \right)^\Delta + \dots, \\ c_P &= \Delta + 1 + \frac{\eta(\Delta+1)}{2^{\Delta+1} \Gamma(\Delta)} \frac{P}{T} \left( \frac{\epsilon_0}{T} \right)^\Delta + \dots \end{aligned} \quad (89)$$

**Exc 18** | Derive Eq. (89).

The relations like  $c_P - c_V = 1$  and  $c_P / c_V = 1 + 1/\Delta$  are only valid in the dilute (classical) limit, which do not apply to quantum gases in general.

## ■ Adiabatic Process

**Adiabatic** process is the thermodynamic process where  $S$  and  $N$  are fixed, i.e. no heat transfer ( $dS = 0$ ) and no particle exchange ( $dN = 0$ ).

The adiabatic process of ideal gas is described by

$$P V^\gamma = \text{const.}, \quad (90)$$

where  $\gamma$  is the **adiabatic exponent**:

$$\gamma = 1 + 1/\Delta. \quad (91)$$

**Exc 19** Prove the statement Eq. (90, 91).

It holds beyond the classical limit and applies to generic ideal quantum gases. For example, when  $\Delta = 3/2$ , we have  $\gamma = 5/3$  -- a familiar result for ideal gases.

## ■ Summary

|               | Bose gas  | Fermi gas                           | Classical limit              |
|---------------|---|-------------------------------------|------------------------------|
| Energy        | $E = \Delta P V$  | ✓                                   | ✓                            |
| Eq. of state  | $P V = k_{\text{eff}} N T$                                | $k_{\text{eff}} < 1$                | $k_{\text{eff}} > 1$         |
| Specific heat | $c_V$   | $(\sim T^\Delta)_{T \rightarrow 0}$ | $(\sim T)_{T \rightarrow 0}$ |
|               | $c_P$   | $0_{T \rightarrow 0}$               | $(\sim T)_{T \rightarrow 0}$ |
|               |   |                                     | $\Delta + 1$                 |
| Isochoric     | $V = \text{const.}$                                       | ✓                                   | ✓                            |
| Isobaric      | $P = \text{const.}$                                       | ✓                                   | ✓                            |
| Isothermal    | $P V = \text{const.}$                                     | ✗                                   | ✗                            |
| Adiabatic     | $P V^\gamma = \text{const.}$<br>$(\gamma = 1 + 1/\Delta)$ | ✓                                   | ✓                            |

- In reality, gases are consist of quantum particles (either bosons or fermions) at the microscopic level.  $\Rightarrow$  The classical gas is not real.
- However, in the dilute limit, both Bose and Fermi gases converge to a universal behavior described by the classical gas. Given  $z \propto (N/V)(\epsilon_0/T)^\Delta$  [see Eq. (83)], the **classical limit** ( $z \rightarrow 0$ ) is characterized by

$$\frac{N}{V} \left( \frac{\epsilon_0}{T} \right)^\Delta \ll 1. \quad (92)$$

In the following, we will move away from the classical limit, and explore the intriguing realm of degenerated quantum gases.

# Degenerated Quantum Gases

## ■ Degenerated Bose Gas

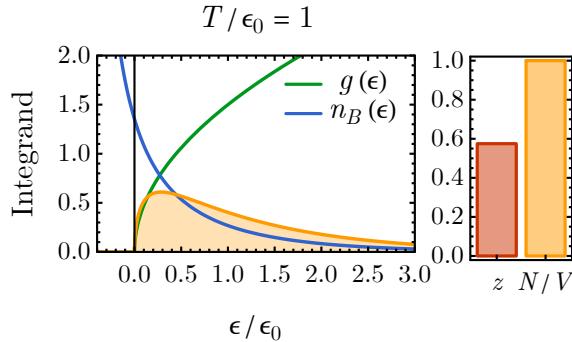
## ■ Unsustainable Thermal Bosons

Let us now understand the elephant in the room: what happens to the Bose gas at low temperature?

Consider  $N$  bosons in a volume  $V$ , recalling Eq. (45), the number density  $N/V$  should be given by the following integral

$$\frac{N}{V} = \int_0^\infty d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} - 1}. \quad (93)$$

- $g(\epsilon) = \Delta \epsilon^{\Delta-1} / \epsilon_0^\Delta$  is the (per volume) density of states.
- As the temperature  $T = 1/\beta$  lowers, the chemical potential  $\mu$  (or the fugacity  $z = e^{\mu/T}$ ) will adjust to maintain the number density  $N/V$  (say  $N/V = 1$ ).



However, the constraint  $\mu < 0$  (or  $z < 1$ ) sets a limit for the adjustment.

- It is possible that the integral can not provide the required  $N/V$ , even if the chemical potential is tuned to  $\mu = 0$ . To see this, we investigate the divergent behavior of the integrand near  $\epsilon \rightarrow 0$ ,

$$\frac{N}{V} \propto \int_0^\infty d\epsilon \frac{\epsilon^{\Delta-1}}{e^{\epsilon/T} - 1} \underset{\epsilon \rightarrow 0}{\sim} T \int_0^\infty \frac{d\epsilon}{\epsilon^{2-\Delta}}. \quad (94)$$

- If  $\Delta > 1$ , the integral will converge (even if  $\mu = 0$ ), which can not provide the required  $N/V$  as  $T$  lowers to a certain point.
- If  $\Delta \leq 1$ , the integral can diverge (if  $\mu = 0$ ), which enables the adjustment of  $\mu$  to reach the required  $N/V$  no matter how low  $T$  is.

It seems that bosons will disappear at low temperature. Where do the “missing” bosons go?

- The integral Eq. (93) only accounts for the *thermal* bosons (bosons occupying the *excited states*).
- Due to the vanishing DOS as  $\epsilon \rightarrow 0$ , the bosons occupying the lowest-energy *ground state* ( $\epsilon = 0$ ) was not taken into account.

To be more careful, the number of bosons should contain two parts

$$N = N_{\text{th}} + N_0 \quad (95)$$

- $N_{\text{th}}$ : the **thermal boson number**, defined by the integral Eq. (93), whose result is given by Eq. (68),

$$N_{\text{th}} = \Delta V \left( \frac{T}{\epsilon_0} \right)^\Delta I_+^{\Delta-1}(z). \quad (96)$$

- $N_0$ : the **condensed boson number** (in the ground state), given by the Bose-Einstein distribution  $n_B(\epsilon) = (z^{-1} e^{\beta\epsilon} - 1)^{-1}$  at  $\epsilon = 0$ :

$$N_0 = \frac{1}{z^{-1} - 1}, \quad (97)$$

meaning that the fugacity will be set by  $z = N_0 / (N_0 + 1)$ .

Now there is no problem keeping  $N$  fixed: when  $N_{\text{th}}$  becomes insufficient to host all  $N$  bosons by thermal excitations, the  $N_0 = N - N_{\text{th}}$  remaining bosons will *condense* to the ground state.

## ■ Bose-Einstein Condensation

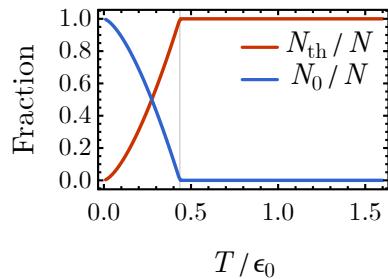
**Bose-Einstein condensation** (BEC) is the phenomenon that a *macroscopic* number of bosons occupies the lowest-energy *single-particle* state at low temperature. The corresponding *state of matter* is called **Bose-Einstein condensate**.

- By saying that the ground state occupation number  $N_0$  is *macroscopic*, we mean

$$\lim_{V \rightarrow \infty} \frac{N_0}{V} \neq 0. \quad (98)$$

This is unusual, as for other excited states, the occupation number does not scale with the system size  $V$  (i.e.  $\langle n_a \rangle / V \rightarrow 0$  as  $V \rightarrow \infty$ ).

- The *fraction* of thermal v.s. condensed (ground-state) bosons:



The condensate fraction in the BEC phase

$$\frac{N_0}{N} = 1 - \left( \frac{T}{T_c} \right)^\Delta. \quad (99)$$

**Exc  
20**

Derive Eq. (99).

- The **BEC transition temperature**  $T_c$  is set by the temperature at which  $z \rightarrow 1$ ,

$$T_c = \epsilon_0 \left( \frac{1}{\Gamma(\Delta + 1)} \frac{N}{\zeta(\Delta)} \frac{V}{V} \right)^{1/\Delta}, \quad (100)$$

where  $\zeta(n) := \text{Li}_n(1)$  is the Riemann zeta function, defined by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

**Exc  
21**

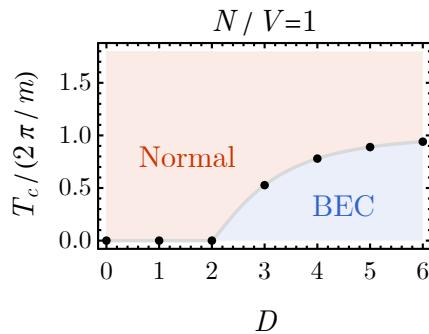
Derive Eq. (100).

- For non-relativistic particles,

$$T_c = \frac{2\pi}{m} \left( \frac{1}{\zeta(D/2)} \frac{N}{V} \right)^{2/D}. \quad (101)$$

To *increase* the BEC transition temperature, one can

- either *reduce* the boson *mass*  $m$ ,
- or *increase* the boson *density*  $N / V$ .
- Dimension dependence of  $T_c$  (following Eq. (101))



BEC transition only happens in 3D and higher dimensions (for non-relativistic bosons).

- In general, BEC transition only happens in bosonic systems with  $\Delta = D/\zeta > 1$ . According to Eq. (100),  $T_c \rightarrow 0$  as  $\Delta \rightarrow 1_+$  due to the divergence of the Riemann zeta function  $\zeta(\Delta \rightarrow 1) \rightarrow \infty$ .

**HW  
3**

In the BEC phase (when  $T < T_c$ ), the pressure  $P$  of the Bose gas will gradually decrease to zero as the temperature  $T$  is lowered. Show that they follow a power law  $P = A T^\lambda$  with (i) a proportionality constant  $A$  independent of  $N, V$  (which is very *different* from the  $P V = N T$  behavior of the ideal classical gas), and (ii) determine the power  $\lambda$  in terms of the exponent  $\Delta = D/\zeta$  that appears in the density of state.

## ■ Ultracold Atoms

The Bose-Einstein condensate (BEC) of weakly interacting atoms was observed in **ultracold atom** systems, where atoms are trapped by lasers and cooled to extremely low temperatures.

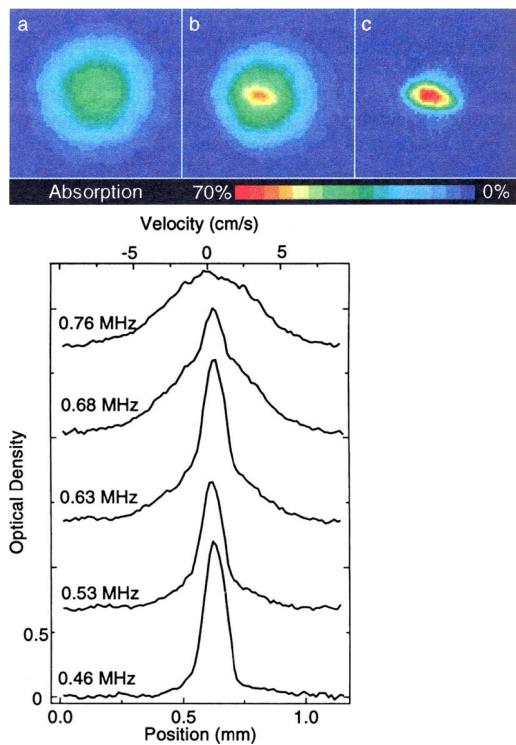
- Examples are found in *bosonic* atoms, such as Rb, Na, Li, of a number  $N \sim 10^4 \rightarrow 10^7$  below the temperature  $T_c \sim 10^{-7}$  K. [2001 Nobel Prize]

How do we know whether the temperature was really that low?

Traditional thermometers become ineffective in measuring such extremely low temperature, leading researchers to *infer* temperature by observing the Bose-Einstein distribution directly!

A **time-of-flight** (TOF) experiment was performed, involving the following steps:

- **Release of Atoms:** Initially, the atoms are confined in an optical trap. The trap (trapping laser) is tuned off, allowing atoms to expand freely.
- **Free Expansion:** The atoms expand for a set duration. During this time, their positions evolve according to their initial momenta: atoms with *larger momenta* will travel *further*.
- **Imaging:** Following the expansion period, the spatial distribution of the atoms is captured using imaging techniques. This image reflects the **momentum distribution** of the atoms.



- [1] K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Bose-Einstein Condensation in a Gas of Sodium Atoms, Phys. Rev. Lett. **75**, 3969 (1995)

The expected number of atoms carrying momentum  $\mathbf{k}$  is given by the Bose-Einstein distribution:

$$N_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}, \quad (102)$$

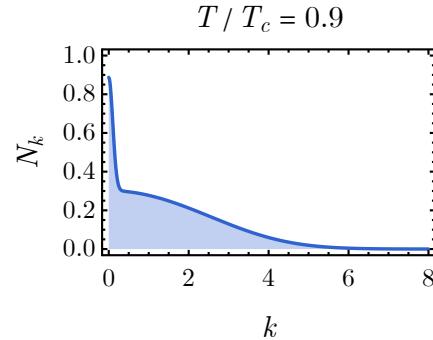
where  $\epsilon_k = k^2 / (2 m)$  is kinetic energy of the atom.

- Aggregating within each momentum shell  $4 \pi k^2 dk$  gives the distribution for the **momentum magnitude**  $k \equiv |k|$ :

$$N_k = \frac{4 \pi k^2}{\exp\left(\frac{1}{T} \left(\frac{k^2}{2m} - \mu\right)\right) - 1}. \quad (103)$$

- However, this only accounts for the thermal boson. One should also include the potential contribution from  $N_0$  bosons condensed in the ground state.

$$N_k = \frac{4 \pi k^2}{\exp\left(\frac{1}{T} \left(\frac{k^2}{2m} - \mu\right)\right) - 1} + N_0 \delta(k). \quad (104)$$

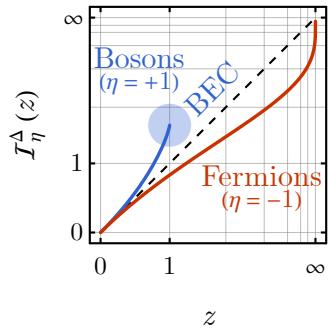


The peak at  $k = 0$  (artificially broadened for visualization) represents the condensed bosons in the lowest-energy state.

## ■ Condensation Transition

The **BEC transition** refers to the *phase transition* between the *BEC phase* (where  $N_0 / V \neq 0$ ) and the *normal phase* (where  $N_0 / V = 0$ ) in the thermodynamic limit  $V \rightarrow \infty$ .

- Near the BEC transition, the fugacity is close to one. So we expand  $I_+^\Delta(z)$  around  $z \rightarrow 1$ .



$$I_{+}^{\Delta}(z) \stackrel{z \rightarrow 1}{=} \begin{cases} -\log(1-z) & \text{if } \Delta = 0, \\ \frac{\pi^2}{6} + (1-z)(\log(1-z) - 1) + \dots & \text{if } \Delta = 1, \\ \Gamma(\Delta+1)(\zeta(\Delta+1) - \zeta(\Delta)(1-z)) - \frac{\pi(1-z)^{\Delta}}{\sin \pi \Delta} + \dots & \text{otherwise.} \end{cases} \quad (105)$$

**Exc 22** Verify Eq. (105).

Let us focus on the case of non-relativistic bosons in  $D = 3$  dimension, corresponding to  $\Delta = D/2 = 3/2$ .

- The expansion Eq. (105) enables us to compute the thermal boson number  $N_{\text{th}}$  [see Eq. (68)] and the energy  $E$  [see Eq. (69)]:

$$\begin{aligned} N_{\text{th}} &= \frac{3\sqrt{\pi}\zeta(3/2)}{4} V \left( \frac{T}{\epsilon_0} \right)^{3/2} \left( 1 - \frac{2\sqrt{\pi}}{\zeta(3/2)} \sqrt{1-z} + \dots \right), \\ E &= \frac{9\sqrt{\pi}\zeta(5/2)}{8} T V \left( \frac{T}{\epsilon_0} \right)^{3/2} \left( 1 - \frac{\zeta(3/2)}{\zeta(5/2)} (1-z) + \dots \right). \end{aligned} \quad (106)$$

**Exc 23** Derive Eq. (106).

- These expressions in Eq. (106) are not very friendly. Let us simplify them by introducing:
- The BEC transition temperature  $T_c$  (adapt from Eq. (100) or Eq. (101)),

$$T_c = \epsilon_0 \left( \frac{4}{3\sqrt{\pi}\zeta(3/2)} \frac{N}{V} \right)^{2/3}. \quad (107)$$

- The energy  $E_c$  of the system at  $T = T_c$ ,

$$E_c = \frac{9\sqrt{\pi}\zeta(5/2)}{8} T_c V \left( \frac{T_c}{\epsilon_0} \right)^{3/2} = \frac{3}{2B} N T_c. \quad (108)$$

- Two numerical constants  $A$  and  $B$ ,

$$A = \frac{2\sqrt{\pi}}{\zeta(3/2)} \approx 1.35697,$$

$$B = \frac{\zeta(3/2)}{\zeta(5/2)} \approx 1.94737.$$

Then Eq. (106) can be organized into a relatively neat form:

$$\begin{aligned} \frac{N_{\text{th}}}{N} &= \left(\frac{T}{T_c}\right)^{3/2} (1 - A \sqrt{1-z}), \\ E &= E_c \left(\frac{T}{T_c}\right)^{5/2} (1 - B(1-z)). \end{aligned} \quad (110)$$

- The behavior of the fugacity  $z$  near the transition can be analyzed as follows:

- When  $T > T_c$ ,  $N_{\text{th}} = N$  (thermal excitations can host all bosons), then  $z$  is given by the equation

$$1 = \left(\frac{T}{T_c}\right)^{3/2} (1 - A \sqrt{1-z}), \quad (111)$$

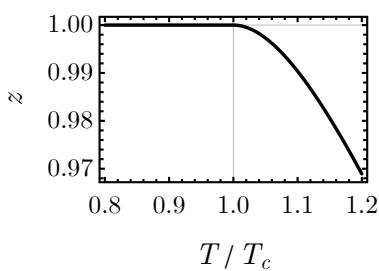
whose solution is

$$z = 1 - \frac{1}{A^2} \left(1 - \left(\frac{T_c}{T}\right)^{3/2}\right)^2. \quad (112)$$

- When  $T < T_c$ , it is no longer possible for  $N_{\text{th}}$  to match  $N$ . The remaining  $N_0 = N - N_{\text{th}}$  bosons will condense. In this case,  $z = N_0/(N_0 + 1) \rightarrow 1$  will become very close to 1 (given that  $N_0$  is macroscopically large).

This behavior can be summarized as

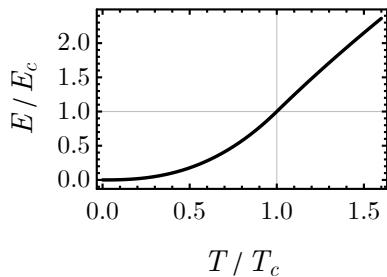
$$z = 1 - \frac{1}{A^2} \max\left(0, 1 - \left(\frac{T_c}{T}\right)^{3/2}\right)^2. \quad (113)$$



Note that the *piecewise* nature of  $z$  (as a function of  $T$ ) is the fundamental reason that leads to *singularities* in various thermodynamic functions at  $T = T_c$ , as will be seen later.

- Plugging the solution Eq. (113) to Eq. (110), we obtain the energy  $E$  as a function of temperature  $T$  (under fixed particle number  $N$  and volume  $V$ )

$$E = E_c \left( \frac{T}{T_c} \right)^{5/2} \left( 1 - \frac{B}{A^2} \max\left(0, 1 - \left( \frac{T_c}{T} \right)^{3/2}\right)^2 \right). \quad (114)$$

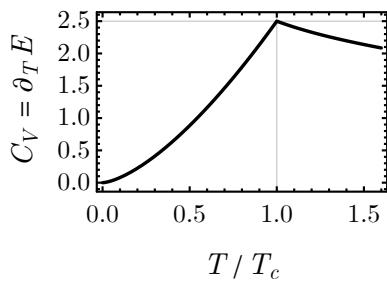


The key indicator of a **phase transition** is hidden in the *higher order derivatives* of the **energy** with respect to the phase transition **driving parameter** (which, in this case, is the temperature  $T$ ).

- 1st order derivative: (isochoric) **heat capacity** -- the amount of *heat* (per particle) required to change the *temperature*.

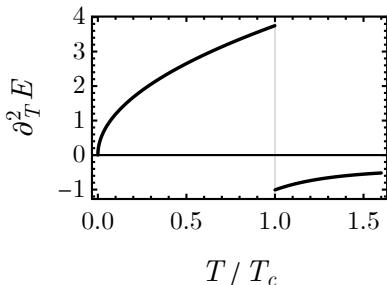
$$C_V = \left( \frac{\partial E}{\partial T} \right)_{N,V}. \quad (115)$$

**Exc 24** Show that the heat capacity  $C_V$  in Eq. (115) is related to the specific heat  $c_V$  in Eq. (78) by  $C_V = c_V N$ .



The heat capacity exhibits a cusp at  $T = T_c$ , which is rooted in the similar cusp behavior in  $(\partial z / \partial T) \sim \max(0, 1 - (T_c / T)^{3/2})$ .

- 2nd order derivative: starts to exhibit a *discontinuity* at  $T = T_c$ .



### Classification of Phase Transitions:

An ***n*th order phase transition** is a phase transition at which the *energy* (or free energy) starts to exhibit **singularity** (discontinuity or divergence) at its *n*th order *derivative* with respect to the *driving parameter*.

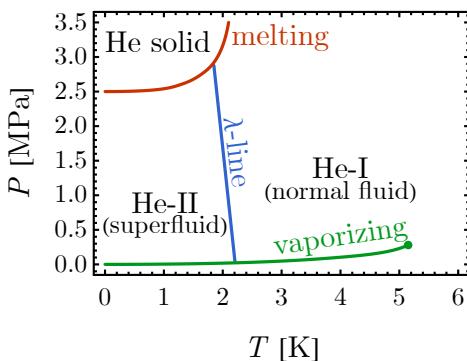
- 1st order phase transitions:
  - Boiling of water
  - Melting of ice
  - Structural transitions in some materials
- 2nd order (or continuous) phase transitions:
  - BEC transitions
  - Superconducting transitions
  - Magnetic transitions in some materials
  - Liquid crystal transitions
- 3rd order phase transitions:
 

(They are less common or more theoretical.)

  - Certain topological quantum phase transitions (e.g. mass generation transition of 2D Dirac fermions)
- Infinite order phase transitions:
  - Berezinskii-Kosterlitz-Thouless (BKT) transitions (e.g. superfluid transition in 2D) [2016 Nobel Prize]

## ■ Superfluid Helium

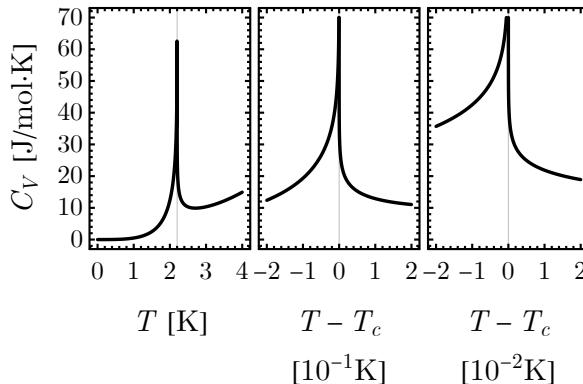
${}^4\text{He}$  (the most abundant isotope of helium) is a *bosonic* atom. It has two liquid phases:



- A **classical** liquid (normal fluid) phase, called He-I: finite viscosity, limited heat conductivity, staying in its container normally.

- A **quantum** liquid (**superfluid**) phase, called He-II: zero viscosity, nearly infinite thermal conductivity, can spontaneously climb walls and escape its container.

The phase transition between He-I and He-II phases is called the  $\lambda$ -transition, due to the  $\lambda$  shape (logarithmic divergence) of the specific heat at the critical point.



There is a close connection between BEC and superfluidity:

- **BEC** refers to the condensate of **non-interacting** bosons (gas),
- **Superfluid** is the condensate of (repulsively) **interacting** bosons (liquid).

They share some common features, such as *macroscopic boson occupation* in the lowest-energy state. However, the *interaction* is essential to maintain the **quantum coherence** among condensed bosons, such that they can propagate collectively in the form of a **macroscopic matter wave** -- the key feature of superfluidity.

## ■ Black-Body Radiation

### ■ Radiations Carried by Bosons

**Radiation** is the emission or transmission of *energy* in the form of *waves* (= *particles*) through the space.

- **Electromagnetic** radiation (light) is carried by **photons** (spin-1).
- **Acoustic** radiation (sound) is carried by **phonons** (spin-0).
- **Spin wave** radiation is carried by **magnons** (spin-1).
- **Gravitational** radiation is carried by **gravitons** (spin-2).
- \***Dark matter** radiation could *hypothetically* be carried by **axions** (spin-0).

All these particles are **bosons** (given their *integer* spins).

**Black-body radiation** is an **ideal Bose gas** of radiation-carrying bosons in thermal equilibrium with a *heat bath* (i.e. the “*black body*” that can absorb and emit radiation).

- The physical laws of black-body radiation applies to all different types of radiations, although we will mainly focus on **photons gas** in the following discussions.

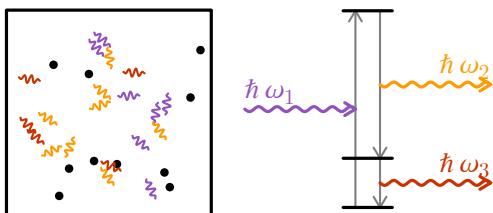
Examples:

| System                            | $T$                 | Dominant radiation  |
|-----------------------------------|---------------------|---------------------|
| Cosmic microwave background (CMB) | 2.7K                | Microwave           |
| Cold interstellar dusts           | $\sim 10\text{K}$   | Terahertz           |
| Night hemisphere of the Earth     | 300K                | Infrared            |
| Electric stove/oven               | 500K                | Infrared            |
| Incandescent light                | 2500K               | Infrared to Visible |
| The Sun                           | 6000K               | Visible             |
| Black hole accretion disk         | $\sim 10^6\text{K}$ | X-ray               |

## ■ Zero Chemical Potential for Radiations

A key feature of **radiations** is that there is *no particle number conservation* for the radiation-carrying bosons.

- This is because a heat bath could absorb one high-energy boson and emit two lower-energy bosons (causing boson number to +1), or vice versa (causing boson number to -1).
- For example, atoms can serve as heat bath to absorb and emit photons.



- In such processes,
  - while **energy** remains *conserved*  $\Rightarrow \langle E \rangle = E$ ,
  - the **particle number** does *not*  $\Rightarrow \langle N \rangle = ??$ .

- If the particle number is not a conserved quantity, we should not require a definitive expectation value for the number of particles in the system. Therefore, the *grand canonical ensemble* is *not* a good starting point for our discussion.

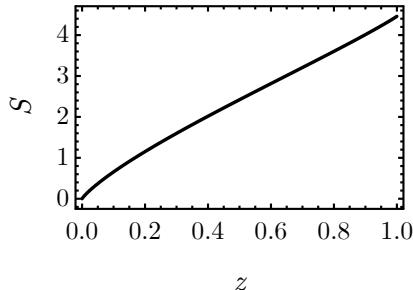
However, having already embarked on a journey with the *grand canonical ensemble*, do we need to start all over again?

-- No panic. In statistical mechanics, whenever faced with questions we can't answer, we take the answer that **maximizes entropy**.

- Our problem: how to infer the chemical potential  $\mu$  with *no knowledge* about the particle number  $N$ .
- According to Eq. (67), the entropy  $S$  is related to chemical potential  $\mu$  (or the fugacity  $z = e^{\mu/T}$ ) by

$$S = V \left( \frac{T}{\epsilon_0} \right)^\Delta \left( (\Delta + 1) I_+^\Delta(z) - \Delta I_+^{\Delta-1}(z) \log z \right). \quad (116)$$

Given  $V$  and  $T$  fixed,  $S$  is a *monotonically increasing* function of  $z \in (0, 1)$  for any choice of  $\Delta > 0$ .



**Exc  
25**

Prove that  $S$  is a monotonically increasing function of  $z \in (0, 1)$ .

The entropy  $S$  is always *maximized* at  $z = 1$ , corresponding to

$$\mu = 0. \quad (117)$$

- The **chemical potential** is always **zero** for **radiation-carrying bosons**.
- In fact, there is no need to introduce the chemical potential  $\mu$  as a Lagrangian multiplier in the first place, if the statistical ensemble is not constrained by the particle number  $N$ .

## ■ Spectral Energy Density

The **spectral energy density**  $\rho(\omega)$  describes how the radiation **energy density**  $E / V$  is distributed over different **frequencies**  $\omega$ , as defined by

$$\frac{E}{V} = \int \rho(\omega) d\omega. \quad (118)$$

- In quantum mechanics, due to the *wave-particle duality*, the radiation **frequency**  $\omega$  is related to the boson **single-particle energy**  $\epsilon$  by

$$\epsilon = \hbar \omega. \quad (119)$$

(We may set  $\hbar = 1$ ).

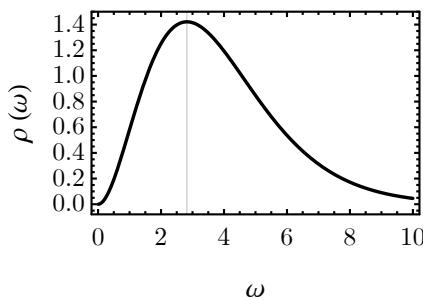
- Knowing that radiations are carried by *bosons* ( $\eta = +1$ ) of *zero chemical potential* ( $\mu = 0$ ), Eq. (44) implies that the spectral energy density for the black-body radiation should take the following form

$$\begin{aligned}\rho(\omega) &= \frac{\omega g(\omega)}{e^{\beta\omega} - 1} \\ &= \frac{\Delta}{\epsilon_0^\Delta} \frac{\omega^\Delta}{e^{\beta\omega} - 1}.\end{aligned}\tag{120}$$

Further assuming relativistic gauge bosons in  $D = 3$  dimensional space (such as photons), we obtain the **Planck radiation law**:

$$\rho(\omega) = \frac{1}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\omega} - 1}.\tag{121}$$

**Exc 26** | Derive Eq. (120) and Eq. (121).



The spectral distribution  $\rho(\omega)$  has a maximum at the frequency

$$\omega_m \approx 2.821 T.\tag{122}$$

**Exc 27** | Derive Eq. (122).

Thus the dominant frequency of thermal radiation will shift towards higher frequency linearly with the raising temperature, known as the **Wien's displacement law**.

## ■ Thermodynamic Properties

Given the zero chemical potential  $\mu = 0$  (or unit fugacity  $z = 1$ ), thermodynamic quantities of the black-body radiation takes simple forms [see Eq. (66-69) and Eq. (60)].

- **Pressure**

$$P = \kappa T^{\Delta+1}.\tag{123}$$

- **Entropy**

$$S = (\Delta + 1) \kappa V T^\Delta.\tag{124}$$

- **Particle number**

$$N = \frac{\zeta(\Delta)}{\zeta(\Delta + 1)} \kappa V T^\Delta. \quad (125)$$

- Energy

$$E = \Delta \kappa V T^{\Delta+1}. \quad (126)$$

- Free energy ( $F = E - TS = \mathcal{F} + \mu N$ )

$$F = \mathcal{F} = -\kappa V T^{\Delta+1}, \quad (127)$$

The constant  $\kappa$  is defined as

$$\kappa = \frac{1}{\epsilon_0^\Delta} \Gamma(\Delta + 1) \zeta(\Delta + 1). \quad (128)$$

**Exc  
28**

Derive Eq. (123-128).

Based on these results, we can conclude a few important thermodynamic relations for black-body radiations.

- Equation of state:

$$P V = \frac{\zeta(\Delta + 1)}{\zeta(\Delta)} N T, \quad (133)$$

which deviated from ideal classical gas behavior ( $P V = N T$ ) by a factor  $\zeta(\Delta + 1) / \zeta(\Delta)$ . For relativistic bosons in three dimensional space, we have  $\Delta = 3$ , and  $\zeta(4) / \zeta(3) \approx 0.9004$ , a bit smaller than 1.

- In terms of energy  $E$ ,

$$E = \Delta \kappa V T^{\Delta+1}, \quad (134)$$

we can express

$$\begin{aligned} P V &= \frac{1}{\Delta} E, \\ TS &= \left(1 + \frac{1}{\Delta}\right) E, \\ F &= -\frac{1}{\Delta} E. \end{aligned} \quad (135)$$

- Thermodynamics processes:

| Process    | Relation                     | Notes   |
|------------|------------------------------|---|
| Isobaric   | $P = \text{const.}$          |   |
| Isothermal | $T = \text{const.}$          | as $P$ and $T$ are locked together by $P = \kappa T^{\Delta+1}$ |
| Isochoric  | $V = \text{const.}$          |   |
| Adiabatic  | $P V^\gamma = \text{const.}$ | with $\gamma = 1 + 1/\Delta$                                    |

## ■ Specific Heat

Since particle number is not conserved in radiation gas, it is only meaningful to define the specific heat as the heat capacity per *volume*.

- **Isochoric** (constant-volume) specific heat

$$c_V = \frac{T}{V} \left( \frac{\partial S}{\partial T} \right)_V = \left( 1 + \frac{1}{\Delta} \right) \kappa T^\Delta. \quad (136)$$

It exhibits a  $c_V \sim T^\Delta$  behavior, a important feature of radiation gas.

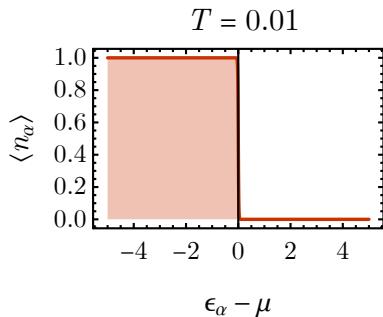
- Isobaric (constant-pressure) specific heat can not be defined as making  $P$  constant prevents us to change  $T$ .

## ■ Degenerated Fermi Gas

### ■ Fermi Energy

In the  $T \rightarrow 0$  low-temperature limit, the Fermi-Dirac distribution is simply a step function:

$$\langle n_\alpha \rangle = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1} \xrightarrow{T \rightarrow 0} \begin{cases} 1 & \text{for } \epsilon_\alpha < \mu, \\ 0 & \text{for } \epsilon_\alpha > \mu. \end{cases} \quad (137)$$



- Each fermion added to the system will just fill up the available single-particle states from low energy to high energy.
- The energy of the last filled state is called the **Fermi energy**, defined by the chemical potential at  $T = 0$ ,

$$\epsilon_F := \mu(T = 0). \quad (138)$$

The Fermi energy is set by the **fermion density**  $N / V$  in the system. To see this, we compute the number of fermions  $N$  following Eq. (45),

$$\begin{aligned} N &= V \int d\epsilon \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \\ &\stackrel{T \rightarrow 0}{=} V \int_0^{\epsilon_F} d\epsilon g(\epsilon). \end{aligned} \quad (139)$$

For free particles, recall Eq. (55):  $g(\epsilon) d\epsilon = d(\epsilon/\epsilon_0)^\Delta$ , the energy integral simply results in

$$N = V \left( \frac{\epsilon_F}{\epsilon_0} \right)^\Delta, \quad (140)$$

which determines the Fermi energy,

$$\epsilon_F = \epsilon_0 \left( \frac{N}{V} \right)^{1/\Delta}. \quad (141)$$

- While  $\epsilon_F$  is the highest energy that a fermion can take, the average energy  $E / N$  carried by a fermion is a fraction of  $\epsilon_F$ , given by

$$E = \frac{\Delta}{\Delta + 1} N \epsilon_F. \quad (142)$$

**Exc  
29**

Derive Eq. (142).

It goes as  $E = \frac{3}{5} N \epsilon_F$  for  $\Delta = \frac{3}{2}$  (for non-relativistic fermions in three dimensional space).

- Given  $E = \Delta P V$  [recall Eq. (70)], the pressure  $P$  of the degenerated Fermi gas (at  $T = 0$ ) is

$$P = \frac{1}{\Delta + 1} \frac{N}{V} \epsilon_F. \quad (143)$$

Amazingly,  $P$  does not vanish at zero temperature for Fermi gas (unlike the ideal classical gas  $P V = N T$ ).

- The finite pressure of Fermi gas at low temperature is called the **degeneracy pressure**.
- It is a consequence of Pauli exclusion principle and is important in understanding the stability of white dwarf stars and neutron stars in astrophysics.

Although these results were derived in the  $T \rightarrow 0$  limit, the picture hold as the temperature  $T \ll T_F$  is low compared to the **Fermi temperature**  $T_F$ , defined as the temperature scale corresponding to the Fermi energy:

$$T_F = \frac{\epsilon_F}{k_B}. \quad (144)$$

- Under the setting of  $k_B = 1$ , temperature and energy are essentially the same unit, with

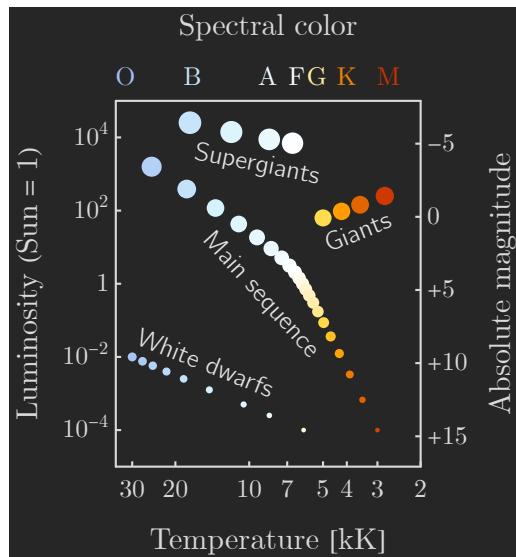
$$1.60218 \times 10^{-19} \text{ J} = 1 \text{ eV} = 11604.5 \text{ K} \approx 10^4 \text{ K}. \quad (145)$$

- $T_F$  can vary by orders of magnitudes in different systems:

| System                           | $\epsilon_F$           | $T_F$                    | $T$                        |
|----------------------------------|------------------------|--------------------------|----------------------------|
| Ultracold ${}^6\text{Li}$ atoms  | $\sim 0.1 \text{ neV}$ | $\sim 10^{-6} \text{ K}$ | $\sim 10^{-7} \text{ K}$   |
| Superfluid ${}^3\text{He}$ atoms | $\sim 0.1 \text{ meV}$ | $\sim 1 \text{ K}$       | $\sim 10^{-3} \text{ K}$   |
| Electrons in metal               | $\sim 1 \text{ eV}$    | $\sim 10^4 \text{ K}$    | $300 \text{ K}$            |
| Electrons in white dwarf stars   | $\sim 0.1 \text{ MeV}$ | $\sim 10^9 \text{ K}$    | $\sim 10^{6-8} \text{ K}$  |
| Nucleons in atomic nucleus       | $\sim 10 \text{ MeV}$  | $\sim 10^{11} \text{ K}$ | --                         |
| Neutrons in neutron stars        | $\sim 0.1 \text{ GeV}$ | $\sim 10^{12} \text{ K}$ | $\sim 10^{6-11} \text{ K}$ |

## ■ Compact Stars

**Compact stars**, such as *white dwarfs*, *neutron stars*, and *black holes*, are endpoints of stellar evolution. In the **Hertzsprung-Russell (HR) Diagram** of astrophysics, stars are plotted according to their *luminosity* and *temperature*:



- **Main sequence stars:** the majority of stars, including the Sun, in stable phase of stellar evolution, fusing H into He in their cores.
- **White dwarf stars:** remnants of low to medium mass stars, like our Sun, no nuclear fusion anymore and are supported against gravitational collapse by **electron degeneracy pressure**.
- **Neutron stars:** remnants of supernova from more massive stars, gravity is strong enough to push electrons into nucleus,



leaving neutrons alone to fight against gravitational collapse by **neutron degeneracy pressure**. (Too faint to be visible on the HR diagram.)

- **Black holes:** extremely dense remnants formed from the gravitational collapse of very massive stars post-supernova. Nothing can sustain against gravity, even light can not escape.

For a white dwarf star of mass  $M$  and radius  $R$ , its energy consists of

$$E = E_{\text{gas}} + E_{\text{grav}} \quad (147)$$

- The energy of the degenerated electron gas

$$E_{\text{gas}} = \frac{\Delta}{\Delta + 1} N \epsilon_F \propto \frac{M^{1+1/\Delta}}{R^{3/\Delta}}. \quad (148)$$

**Exc 30** Verify Eq. (148).

- The gravitational energy

$$E_{\text{grav}} \propto -\frac{M^2}{R}. \quad (149)$$

Put together, the balance is achieved when the energy  $E$  in Eq. (147) is minimized, which gives a relationship between the radius  $R$  and the mass  $M$  of the star:

$$R \sim M^{\frac{1-\Delta}{3-\Delta}}. \quad (150)$$

**Exc 31** Derive Eq. (150).

- For **non-relativistic** electrons:  $\Delta = 3/2$ , Eq. (150) predicts

$$R \sim M^{-1/3}, \quad (151)$$

meaning that the star will *shrink* as it gets more *massive*.

- Because as  $M$  increases, gravity compresses the star more, forcing electrons to get closer, generating a higher degeneracy pressure (Fermi energy) to balance the stronger gravity, thus resulting in a smaller star radius  $R$ .
- However, this trend does not sustain for ever. As  $\epsilon_F$  grows to the point that it becomes comparable to the electron *rest mass* energy  $m_e c^2$ , *relativity* effects sets in.
- For **relativistic** electrons:  $\Delta = 3$ , Eq. (150) predicts

$$R \sim M^\infty, \quad (152)$$

a strange expression ... but it just means  $M \sim R^0 = \text{const.}$ , i.e.  $M$  will top out when  $R$  reduces to the point that electrons become relativistic.

- In the relativistic regime, the degeneracy pressure ( $P \sim R^{-1}$ ) does not increase as rapidly as it does in the non-relativistic regime ( $P \sim R^{-2}$ ). Consequently, there comes a point where the pressure can no longer counterbalance the gravity for a more massive star.

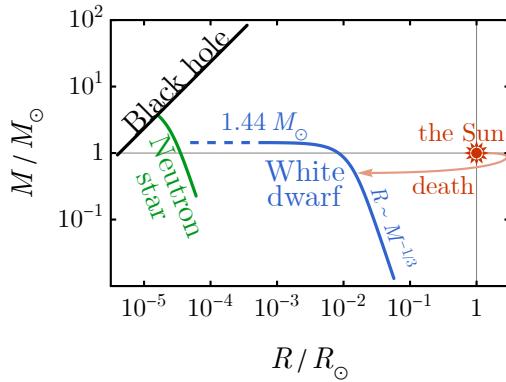
This leads to a mass limit, called the **Chandrasekhar limit**, about 1.44 of the solar mass  $M_\odot$ ,

$$M_{\text{limit}} \approx 1.44 M_\odot. \quad (153)$$

Beyond this limit, the white dwarf can no longer sustain itself against gravitational collapse, potentially leading to:

- a type Ia *supernova* (the “standard candle” in astronomy),
- or a more compact object, like a *neutron star* or a *black hole*.

Similar analysis can be made for **neutron stars**. While for **black holes**, one needs a different analysis to see  $R = 2 G M / c^2$  as predicted by *General Relativity*. Put together the mass-radius relations for different compact stars:



This helps us understand the fate of the Sun. After exhausting its nuclear fuel, it will:

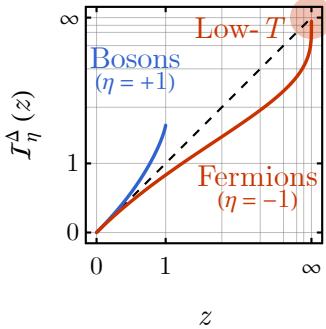
- first expand into a red giant (swallowing the Earth),
- then blow away its outer layers,
- leaving behind a white dwarf (cooling over time).

## ■ Low-Temperature Expansion

To better understand the low-temperature ( $T \ll T_F$ ) behavior of Fermi gas, we should study the function  $I_-^\Delta(z)$  in the  $z \rightarrow \infty$  limit. Because when  $T/\epsilon_F \rightarrow 0$ , we have

$$z = e^{\mu/T} \simeq e^{\epsilon_F/T} \rightarrow +\infty. \quad (154)$$

Expanding  $I_-^\Delta(z)$  in the  $z \rightarrow \infty$  limit is called the **low-temperature expansion** (or the Sommerfeld expansion).



$$I_-^\Delta(z) \stackrel{z \rightarrow \infty}{=} (\log z)^{\Delta+1} \left( \frac{1}{\Delta+1} + \frac{\pi^2 \Delta}{6 (\log z)^2} + \frac{7 \pi^4 \Delta (\Delta-1) (\Delta-2)}{360 (\log z)^4} + \dots \right). \quad (155)$$

**Exc 32**

Verify Eq. (155).

- The expansion Eq. (155) enables us to compute the fermion number  $N$  [see Eq. (68)] and the energy  $E$  [see Eq. (69)] at low temperature,

$$\begin{aligned} N &= V \left( \frac{\mu}{\epsilon_0} \right)^\Delta \left( 1 + \frac{\pi^2 \Delta (\Delta-1)}{6} \left( \frac{T}{\mu} \right)^2 + \dots \right), \\ E &= \frac{\Delta}{\Delta+1} \mu V \left( \frac{\mu}{\epsilon_0} \right)^\Delta \left( 1 + \frac{\pi^2 \Delta (\Delta+1)}{6} \left( \frac{T}{\mu} \right)^2 + \dots \right). \end{aligned} \quad (156)$$

**Exc 33**

Derive Eq. (156).

- For fixed fermion number  $N$ , we can solve for the chemical potential  $\mu$ ,

$$\mu = \epsilon_F \left( 1 - \frac{\pi^2 (\Delta-1)}{6} \left( \frac{T}{\epsilon_F} \right)^2 + \dots \right), \quad (157)$$

where  $\epsilon_F = \epsilon_0 (N / V)^{1/\Delta}$  is the Fermi energy.**Exc 34**

Derive Eq. (157).

- We can see  $\mu = \epsilon_F$  at  $T = 0$ , as expected.
- As  $T$  increases,  $\mu$  will deviates from  $\epsilon_F$  slowly, in quadratic order of  $T / \epsilon_F$ .
- Substitute the solution of  $\mu$  in Eq. (157) to the expression of  $E$  in Eq. (156), we obtain  $E$  as a function of  $T$  at fixed  $N$  and  $V$ .

$$E = \frac{\Delta}{\Delta+1} N \epsilon_F \left( 1 + \frac{\pi^2 (\Delta+1)}{6} \left( \frac{T}{\epsilon_F} \right)^2 + \dots \right). \quad (158)$$

**Exc 35** | Derive Eq. (158).

## ■ Specific Heat

The energy function  $E$  in Eq. (158) enables us to compute the (isochoric) **specific heat**. Following the definition

$$c_V = \frac{1}{N} \left( \frac{\partial E}{\partial T} \right)_{N,V}, \quad (159)$$

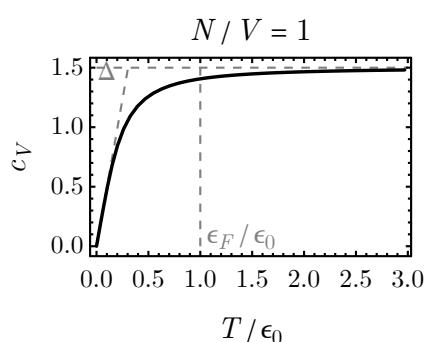
the result reads

$$c_V = \frac{\pi^2 \Delta}{3} \left( \frac{T}{\epsilon_F} \right) + \dots \quad (160)$$

**Exc 36** | Derive Eq. (160).

- The specific heat of a Fermi gas is *linear* in  $T$  at low temperature.
  - Recall Eq. (85) that  $c_V = \Delta$  is the result for a classical gas. Compare to that, the specific heat of a degenerated Fermi gas is smaller by a factor of  $T/\epsilon_F$  (or  $T/T_F$ ).
- The specific heat essentially counts the number of modes (per particle) that can be *thermally excited* to participate in energy exchange.
- In **classical gas**, every particle can participate in energy exchange with the heat bath.
  - In **degenerated Fermi gas**, only a  $T/\epsilon_F$  fraction of the fermions near the Fermi surface are excited and can contribute to the energy exchange. The remaining fermions deep in the Fermi sea are effectively frozen.

The specific heat will deviate from the low-temperature linear- $T$  behavior if  $T$  approaches to  $\epsilon_F$  (or  $T_F$ ), and eventually saturate to the classical limit.



## ■ Susceptibilities

**Number susceptibility** is the (relative) rate that the particle number  $N$  changes with the chemical potential  $\mu$ , defined by

$$\chi_N = \frac{1}{N} \left( \frac{\partial N}{\partial \mu} \right)_{V,T}. \quad (161)$$

To the leading order in the  $T \rightarrow 0$  limit,

$$\chi_N = \frac{\Delta}{\epsilon_F} + \dots \quad (162)$$

**Exc 37** | Derive Eq. (162).

The number susceptibility forms a basis to compute other susceptibilities of the Fermi gas in response to different external fields.

Electron is a fermion that carries both **charge** ( $-e$ ) and **spin**  $\hbar/2$  (which corresponds to a **magnetic moment**  $\mu_B = e\hbar/(2m_e)$ , known as a Bohr magneton). These properties enables electron gas to response to both *electric* and *magnetic* fields.

- **Charge susceptibility:** the rate that the charge density  $Q/N$  changes with the electric potential  $\mathcal{E}$ .

$$\chi_Q = \frac{1}{N} \left( \frac{\partial Q}{\partial \mathcal{E}} \right)_{V,T}. \quad (163)$$

- The total charge of the electron gas is

$$Q = -eN. \quad (164)$$

- Varying in electric potential  $d\mathcal{E}$  will cause a change in the electron chemical potential by

$$d\mu = -e d\mathcal{E}. \quad (165)$$

- Therefore,  $\chi_Q$  is related to  $\chi_N$  by

$$\chi_Q = e^2 \chi_N = \frac{\Delta e^2}{\epsilon_F} + \dots \quad (166)$$

- **Spin susceptibility:** the rate that the magnetization density  $M/N$  changes with the magnetic field  $B$ .

$$\chi_M = \frac{1}{N} \left( \frac{\partial M}{\partial B} \right)_{V,T}. \quad (167)$$

- The total magnetization of the electron gas is

$$M = \mu_B(N_\uparrow - N_\downarrow), \quad (168)$$

where  $N_\uparrow$  and  $N_\downarrow$  are respectively the number of up-spin and down-spin electrons in a mag-

netic field.

- Varying the magnetic field  $dB$  will cause a change in the chemical potential for up-spin and down-spin electrons,

$$\begin{aligned} d\mu_{\uparrow} &= \mu_B dB, \\ d\mu_{\downarrow} &= -\mu_B dB. \end{aligned} \quad (169)$$

- Therefore,  $\chi_M$  is related to  $\chi_N$  by

$$\chi_M = \mu_B^2 \chi_N = \frac{\Delta \mu_B^2}{\epsilon_F} + \dots \quad (170)$$

This is also known as the **Pauli susceptibility**, which (partially) explains the **paramagnetism** of many metals.

The *charge* and *spin* susceptibilities can be measured in experiments by putting the electron gas in an *electromagnetic field*, thereby providing a feasible approach to infer the number susceptibility (which is otherwise not directly measurable).

Note: the specific heat  $c_V$  and different susceptibilities  $\chi_Q, \chi_M$  are all proportional to  $\Delta/\epsilon_F$  in the  $T \rightarrow 0$  limit:

$$\begin{aligned} c_V &= \frac{\pi^2 \Delta}{3} \left( \frac{T}{\epsilon_F} \right) + \dots, \\ \chi_Q &= \frac{\Delta e^2}{\epsilon_F} + \dots, \\ \chi_M &= \frac{\Delta \mu_B^2}{\epsilon_F} + \dots, \end{aligned} \quad (171)$$

By taking ratios among them, the non-universal feature like  $\epsilon_F$  can be eliminated, resulting in an *universal ratio*, call the **Wilson ratio**.

$$R_W = \frac{\pi^2}{3} \frac{\chi_Q T}{e^2 c_V} = \frac{\pi^2}{3} \frac{\chi_M T}{\mu_B^2 c_V} = 1. \quad (172)$$

- For non-interacting electrons, the Wilson ratio is expected to be unity. This prediction holds for most **metals**, where electron-electron interactions are well *screened* and thus negligible.
- However, *deviations* from unity in the Wilson ratio are indicative of significant *electron-electron interactions* within the materials. -- A defining feature of **strongly correlated electron systems**, including many unconventional **high-temperature superconductors**.

**HW  
4**

Thermoelectric (Seebeck) effect is an interesting phenomenon that *temperature* difference across a metal can induce a *voltage* difference. The Seebeck coefficient  $\mathcal{S}$  characterizes how electric potential  $\mathcal{E}$  raises with temperature  $T$ , defined as

$$\mathcal{S} = \left(\frac{\partial \mathcal{E}}{\partial T}\right)_{Q,V}.$$

- (i) Express the Seebeck coefficient in terms of the temperature derivative of chemical potential.
- (ii) Use Eq. (157) to calculate the Seebeck coefficient  $\mathcal{S}$  to the leading order of  $T/\epsilon_F$  at low-temperature.
- (iii) Show that the ratio  $R_S = e\mathcal{S}/c_V$  between the Seebeck coefficient  $\mathcal{S}$  and the specific heat  $c_V$  is a universal ratio (that does not depend on the Fermi energy  $\epsilon_F$  to the leading order in  $T/\epsilon_F$ ).