费曼物理配套题解

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Problem Set for Feynman's Lecture I

Chap. 1 Atoms in Motion

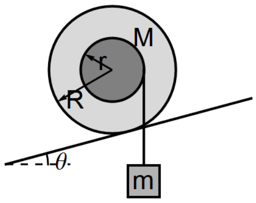
1. Avogadro Constant

List two methods to measure Avogadro constant. You are encouraged to use the library and internet resources, but do remember to list your references.

Chap. 4 Conservation of Energy

1. Principle of Virtual Work (I)

A bobbin of mass M, consists of a central cylinder of radius r and two end plates of R. It is placed on a slotted incline on which it will roll but not slip, and a mass m is suspended from a cord wound around the bobbin. It is observed that the system is in static equilibrium. What is the angle of tilt θ of the incline?



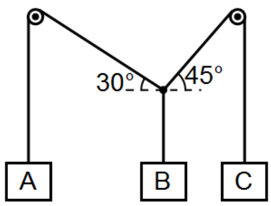
Hint: use the *principle of virtual work* to solve this problem. Recall Feynman's words on p.4-4, "if we say it is just balanced, it is reversible and so can move up and down." So consider what the change is in gravitational potential energy when you imagine the block m hanging down and the cylinder rolling up a little bit more.

Solution:

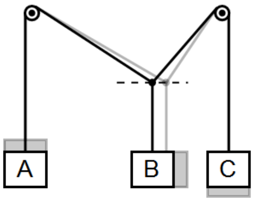
$$\theta = \arcsin \frac{mr}{(M+m)R}.$$

2. Principle of Virtual Work (II)

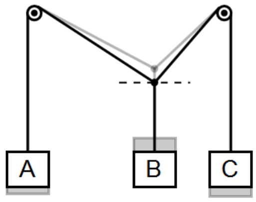
The system shown is in static equilibrium. Use the principle of virtual work to find the weights B and C, given the weight of A is W_A. Neglect the weight of the strings and the friction in the pulleys.



Hint: you may consider the following two variations, and calculate the consequence changes in gravitational potential energy respectively.



Variation 1: the joint virtually displaced by a small amount horizontally.



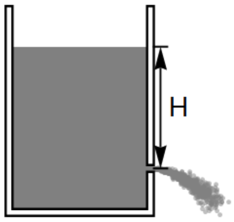
Variation 2: the joint virtually displaced by a small amount vertically.

Solution:

$$W_B = \frac{1+\sqrt{3}}{2} W_A,$$
$$W_C = \frac{\sqrt{6}}{2} W_A.$$

3. Conservation of Energy in Hydrodynamics

A tank of cross-sectional area A contains a liquid having density ρ. The liquid squirts freely from a small hole of area a at a distance H below the free surface of the liquid. If the liquid has no internal friction (viscosity), with what speed does it emerge?



Solution:
Suppose the liquid emerge with the speed v, then the liquid surface will drop with a speed Av/a, so according to conservation of energy

$$p_0 + \rho g H + \frac{1}{2}\rho\left(\frac{Av}{a}\right)^2 = p_0 + \frac{1}{2}\rho v^2.$$

Therefore the solution is

$$v = \left(\frac{2gH}{1 - a^2/A^2}\right)^{1/2}.$$

Some may assume the limit $a \ll A$, and obtain $v = \sqrt{2gH}$, also reasonable.

4. Different Forms of Energy

In quantum mechanics, the energy of a photon is related to the photon frequency ν through the Planck constant $E = h\nu$. In statistical mechanics, the energy scale of a thermodynamic process is related to the temperature T by the Boltzmann constant $E = k_B T$. Give a rough estimate of the temperature of our universe from the observed 160 GHz cosmic microwave background radiation.

Solution: $T = h\nu/k_B \sim 7.7\text{K}$.

Need to tell the student that because photons can move in three directions, its thermal energy is $\langle h\nu \rangle = 3k_B T$, so in fact $T \sim 2.7K$.

Chap. 5 Time and Distance

1. Carbon-14 Dating

After the organism dies, carbon-14 continues to decay without being replaced with fresh ^{14}C. The amount of ^{14}C is compared to the amount of ^{12}C, the stable form of carbon, to determine how much radiocarbon has decayed, thereby dating the artifact. A scrap of paper taken from the Dead Sea Scrolls was found to have a $^{14}C/^{12}C$ ratio of 0.795 times that found in plants living today. Estimate the age of the scroll. Given that the half-life of ^{14}C is 5730 years.
Note: Assuming that the radiocarbon levels have remained relatively constant in most of the biosphere due to the metabolic processes in living organisms.

Solution:
In this problem the $^{14}C/^{12}C$ ratio is 0.795 times that found in plants living today, meaning that 0.795 times the original amount of ^{14}C remains in the scroll up to now.
With the exponential decay formula

$$A = A_0(1/2)^{t/\tau},$$

where A is the present amount of the radioactive isotope, A_0 is the original amount of the radioactive isotope that is measured in the same units as A. t is the time it takes to reduce the original amount of the isotope to the present amount, and τ is the half-life of the isotope. In this problem $\tau = 5730\,\text{years}$, and $A/A_0 = 0.795$. Therefore one can determine $t = 1900$ years.

2. Dimension Analysis (I): Planck Time and Planck Length

Use the three universal constants: the gravitational constant G, the reduced Planck constant $\hbar$ and the speed of light in vacuum c, to construct a quantity t_P of time dimension and a quantity l_P of length dimension.
Note: t_P and l_P are known as the Planck time and Plank length, named after Max Planck, who was the first to propose it. They are the scale of space-time fluctuations in speculative theories of quantum gravity, which also set the fundamental units of space-time, if any.

Solution:

$$t_P = \left(\frac{\hbar G}{c^5}\right)^{1/2} \approx 5.4 \times 10^{-44}\text{s}.$$

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} \approx 1.6 \times 10^{-35}\text{m}.$$

3. Dimension Analysis (II): Classical Radius of Electron

According to Einstein's famous energy-mass relation, $m_e c^2$ is the energy of an electron in rest (where m_e is the mass of electron). The classical electron radius r_e is roughly the size

the electron would need to have for its mass to be completely due to its electrostatic potential energy - not taking quantum mechanics into account. Find r_e by constructing a quantity of the length dimension.
Note: We now know that *quantum mechanics*, indeed quantum field theory, is needed to understand the behavior of electrons at such short distance scales, thus the classical electron radius is no longer regarded as the actual size of an electron. The classical electron radius is roughly the length scale at which *renormalization* becomes important in quantum electrodynamics.

Solution:

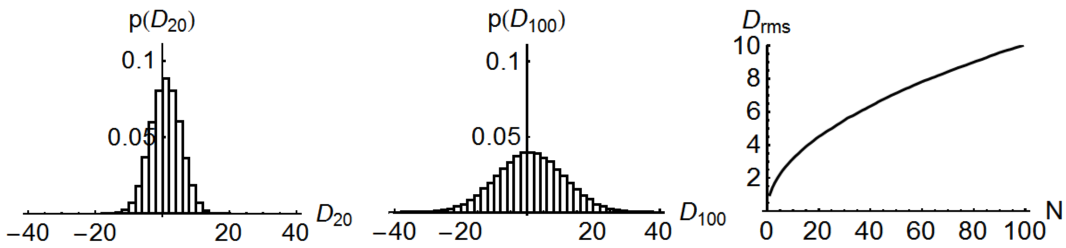
$$r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e c^2} \approx 2.8 \times 10^{-15}\text{m}.$$

Some may take it seriously, and calculate by shell or ball model to obtain the pre-factor, but that is not necessary.

Chap. 6 Probability

1. Computer Simulation of Random Walk

Design a program to simulate random walk on a 1D lattice, i.e. to release a particle from $D_0 = 0$ and in each step $D_N = D_{N-1} \pm 1$, where the choice is taken randomly with same probability. By running the simulation for more than 10000 times, do the following statistics:
(a) Show the histogram of probability distribution for D_{20} and for D_{100}.
(b) Plot the root-mean-square distance D_{rms} as a function of the number of steps N.
You are supposed to obtained the following figures.



Note: You can see the distributions are like Gaussian functions, with the standard deviations σ increasing with the number of steps N in the manner $\sigma \propto N^{1/2}$.
Hint: If you are programming with *Mathematica*, type “RandomChoice” in the Document Center, then you will see an example of 1D random walk in the “Applications” section.

2. Drifting Random Walk

Now let us consider a different kind of random walk on 1D lattice. We still start from $D_0 = 0$, and move to the nearest neighboring site in each step $D_N = D_{N-1} \pm 1$. But the probability of moving forward $(D_N = D_{N-1} + 1)$ and backward $(D_N = D_{N-1} - 1)$ are different. Let the forward probability be p, and hence the backward probability will be $(1-p)$. In this case, the random walk will not center around the origin, but will drift away as if driven by an external force.

(a) Determine how the root-mean-square distance D_{rms} depends on the number of steps N for this drifting random walk.

In condensed matter physics, the drifting random walk is a simple model to study the transport of electron in a disorder media under external electric field. The random walk is a Markov process, which means the choices of the subsequent steps are independent of the choices made in the previous steps. That is to say, electron does not remember its history. However in reality, it takes certain amount of time for the electron to forget its history, known as the relaxation time τ, the typical time to thermalize an electron in the material. If we observe the electron by snap shots with time interval τ, then the motion of electron can be considered as Markov process, and hence nicely modeled by the drifting random walk. Each step takes the time τ, in which the electron moves forward or backward by the distance of mean free path ξ. In the presence of external electric field of the strength E, it can be given by statistical mechanics that the probability for the electron to move one step forward is $p = (1 + \tanh \beta\xi eE)/2$, where $\beta = 1/(k_B T)$ is the inverse of the product of the Boltzmann constant and temperature, and e is the electron charge. Suppose the number of electron in the material per unit volume is n. Calculate the electric current density j given the electric field E. In the week field limit $E \to 0$, we are expected to produce Ohm's Law, which states that the current density is proportional to the applied electric field, where the factor is called the conductivity σ defined by $j = \sigma E$.

(b) Work out this conductivity in the limit $E \to 0$. With increasing temperature, does the conductivity increase or drop?

Solution:

(a) Starting from $D_0 = 0$. In each step, the random choice is made with different probability:

$$D_N = \begin{cases} D_{N-1} + 1, & p \\ D_{N-1} - 1, & 1 - p \end{cases}.$$

So for the squares,

$$D_N^2 = \begin{cases} D_{N-1}^2 + 2D_{N-1} + 1, & p \\ D_{N-1}^2 - 2D_{N-1} + 1, & 1 - p \end{cases}.$$

The expectation of D_N is given by the following recurrence equation

$$\langle D_N \rangle = \langle D_{N-1} \rangle + 2p - 1.$$

Given $\langle D_0 \rangle = 0$, we obtain $\langle D_N \rangle = N(2p - 1)$. The expectation of the net distance is not zero for the drifting random walk, but increasing with the number of steps linearly. Now we go on with the expectation for D_N^2,

$$\langle D_N^2 \rangle = \langle D_{N-1}^2 \rangle + 2\langle D_{N-1} \rangle (2p - 1) + 1 = \langle D_{N-1}^2 \rangle + 2(N - 1)(2p - 1)^2 + 1.$$

The solution of this recurrence equation is

$$\langle D_N^2 \rangle = N(1 + (N - 1)(2p - 1)^2).$$

Therefore the root-mean-square distance D_{rms} goes as

$$D_{\text{rms}} = N^{1/2}(1 + (N - 1)(2p - 1)^2)^{1/2}.$$

When N is small, $D_{\text{rms}} \sim N^{1/2}$, the random walk behaves as a diffusion. As N increases, $D_{\text{rms}} \sim N$, it becomes a ballistic transport. The cross over from diffusive to ballistic transport happens when $N \sim N_c = (2p - 1)^{-2}$.

(b) This demonstrates how electrons transports (classically) in a resistance with applied voltage. The drifting velocity of electron is

$$v = \frac{\xi\langle D_N\rangle}{N\tau} = \frac{\xi}{\tau}(2p-1) = \frac{\xi}{\tau}\tanh\beta\xi eE.$$

The current density is

$$j = nev = \frac{ne\xi}{\tau}\tanh\beta\xi eE.$$

In the week field limit,

$$j = \frac{ne^2\xi^2}{\tau k_B T}E,$$

which gives a conductivity decreasing with temperature,

$$\sigma = \frac{ne^2\xi^2}{\tau k_B T}.$$

Chap. 7 The Theory of Gravitation

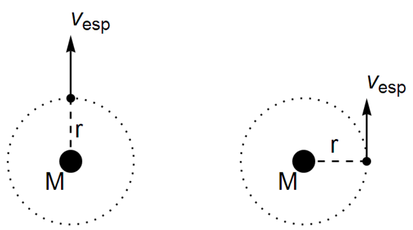
1. Falling to the Sun

Suppose one day, the Earth suddenly stopped orbiting around the Sun (i.e. its orbital speed dropped to zero). How much time it would take for the Earth to leave its present orbit and crash into the Sun?
Hints: use Kepler's Third Law, no need to solve the equation of motion.

Solution: 0.176 year, or 64.5 days.
Kepler's Third Law states that the period of a complete orbit is proportional to the semimajor axis of that orbit $T \propto a^{3/2}$. The Earth is orbiting around the Sun roughly on a circle, so the semimajor axis is the radius, say 1 a.u., and the period is 1 year. If the Earth were to stop orbiting and to fall directly towards the sun, it would be effectively going on an extremely squeezed ellipse orbit with semimajor axis $a = 1/2$ a.u. and semiminor axis $b = 0$, so the corresponding period of this orbit should be $(1/2)^{3/2}$ year. However, the Earth is not able to finish this orbit. It will finish half of the orbit, and then crash into the Sun. So the time taken is $(1/2)^{3/2} \times (1/2) = 0.176$ year, roughly two months.

2. Escape Schemes for Satellite

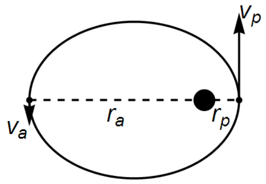
In this problem, we will discuss whether a satellite that was moving perpendicular to its radius to the planet of the mass M, at a distance r and with the escape velocity, would, in fact, escape – because it is not self-evident. It would be, if it were headed straight out, radially (scheme A); but whether it would make it or not if it were headed perpendicular to the radius (scheme B), is another question.



Scheme A Scheme B

(a) Determine the escape velocity v_{esp} in scheme A by conservation of energy. (By "escape" we mean that the satellite can in principle travel to the place infinitely distant away from the planet.)

If we remember Kepler's First Law, we can figure out that if the satellite failed to escape by scheme B, it would make an ellipse, and we can figure out how far away it would get.



(b) If the perihelion of the ellipse is r_p, use the conservation of energy and Kepler's Second Law to determine the aphelion r_a.

(c) Let $r_p = r$, what should v_p be, such that $r_a \to \infty$ (i.e. the satellite would escape)? Is it the same escape velocity as that in scheme A?

Solution:

(a) Escape velocity for scheme A

$$v_{esp} = \left(\frac{2GM}{r}\right)^{1/2}.$$

(b) The aphelion

$$r_a = \frac{r_p}{\frac{2GM}{r_p v_p^2} - 1}.$$

(c) The escape velocity is the same for scheme B.

Chap. 8 Motion

1. Bouncing Table Tennis Ball

A table tennis ball was initially dropped from a height of 30cm above the table. On each bounce the downward speed of the ball arriving at the table was reduced by a factor e (the coefficient of restitution) in the rebound, i.e.

$$v_{\text{upward}} = ev_{\text{downward}}.$$

If 4 seconds after the ball was released, the silencing of the sound indicated that all bouncing had ceased. What was the value of e?

Solution:

Released from the height h, the ball will reach the table with the speed

$$v_0 = \sqrt{2gh}.$$

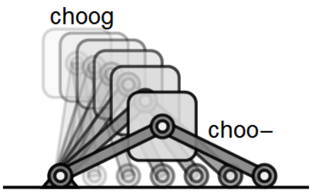
Then it rebounds with velocity $v_1 = ev_0$, and will fall back in time $t_1 = 2v_1/g$. To generalize, the nth bounce starts with velocity $v_n = e^n v_0$ and will complete in time $t_n = 2v_n/g = 2e^n v_0/g$. Sum up the time of each bounce and the initial falling, we obtained the time for all bouncing to cease,

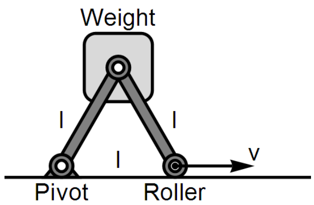
$$t = \frac{v_0}{g} + \sum_{n=1}^{\infty} \frac{2e^n v_0}{g} = \frac{2\sqrt{2h}}{\sqrt{g}}\left(\frac{1}{1-e} - \frac{1}{2}\right) = \left(\frac{2h}{g}\right)^{1/2} \frac{1+e}{1-e}.$$

In this problem, $h = 30\text{cm}$ and $t = 4\text{s}$, so $e = 0.88$.

2. The Choo-choog Machine (I): Acceleration

Suppose we have a machine of some kind, as illustrated in the following figure.





It has got two rods connected by a pivot (like an elbow joint) with a big weight on it. The end of one rod is connected to the floor by a stationary pivot, and the end of the other rod has a rolling pivot that rolls along the floor. See it is going choo-choog: the roller is going back and forth, the weight is going up and down, and so on. Now let us say the length of both rods are l, and the roller is being driven forth horizontally at a constant velocity v starting from the pivot position. Find the *acceleration* (both the horizontal and vertical components) of the weight when the distance between the roller and the pivot is also l.

Solution:

The displacement of the weight

$$x = vt/2,$$

$$y = \sqrt{l^2 - x^2} = \sqrt{l^2 - \left(\frac{vt}{2}\right)^2}.$$

By time derivative,

$$v_x = v/2,$$

$$v_y = -\frac{v^2}{4}\frac{t}{\sqrt{l^2 - (vt/2)^2}} = -\frac{v}{2}\frac{x}{\sqrt{l^2 - x^2}},$$

$$a_x = 0,$$

$$a_y = -\frac{v^2 l^2}{4(l^2 - (vt/2)^2)^{3/2}} = -\frac{v^2 l^2}{4(l^2 - x^2)^{3/2}}.$$

When the distance between the roller and the pivot is l $(x = l/2)$, the time $t = l/v$, thus

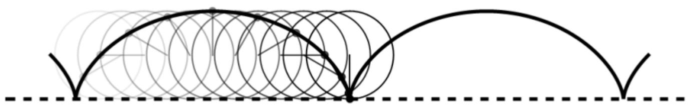
$$a_y = -\frac{2}{3\sqrt{3}}\frac{v^2}{l} = -\frac{2\sqrt{3}}{9}\frac{v^2}{l}.$$

Or some may obtain the answer in polar system,

$$\boldsymbol{a} = -\frac{v^2}{3l}\boldsymbol{e}_r - \frac{v^2}{3\sqrt{3}l}\boldsymbol{e}_\theta.$$

3. Cycloid Motion

A cycloid is the curve defined by the path of a point on the edge of circular wheel as the wheel rolls along a straight line.



Suppose the wheel is rolling with uniform speed. The point starting from the origin trace out a cycloid described by

$$\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t) \end{cases},$$

where r is the radius of the wheel and t is the time. The displacement of the point $\boldsymbol{r} = (x, y)$ is thus given as a function of time, which contains the full information about the kinematics of the point.

(a) Find the velocity of the point $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ as a function of time.

(b) Find the acceleration of the point $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ as a function of time.

(c) What is the acceleration a_n in the normal direction (the direction perpendicular to the velocity)?

The *radius of curvature* at a point P on a given curve is simply the radius of the *osculating circle* at that point, which, the osculating circle, is a unique circle which most closely approximates the curve near P.

(d) Find the radius of curvature ρ for the cycloid.

(e) Show that the formula of centripetal acceleration $a_n = v^2/\rho$ holds for the cycloid motion.

Solution:

(a) $v_x = r(1 - \cos t) = 2r\sin^2(t/2)$, $v_y = r\sin t = 2r\sin(t/2)\cos(t/2)$.

(b) $a_x = r\sin t$, $a_y = r\cos t$.

(c) The direction of velocity (tangent direction) is $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}| = (\sin(t/2), \cos(t/2))$. The normal direction is perpendicular to that, which reads $\boldsymbol{n} = (\cos(t/2), -\sin(t/2))$. Thus the normal acceleration is

$$a_n = |\boldsymbol{a} \cdot \boldsymbol{n}| = r\left|\sin t \cos\frac{t}{2} - \cos t \sin\frac{t}{2}\right| = r\left|\sin\frac{t}{2}\right| = r\left(\frac{1 - \cos t}{2}\right)^{1/2}.$$

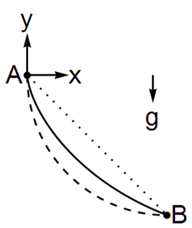
(d) Radius of curvature

$$\rho = 4r\left|\sin\frac{t}{2}\right| = 2\sqrt{2}r\sqrt{1 - \cos t}.$$

(e) $v = 2r\sin(t/2)$, so $v^2/\rho = r|\sin(t/2)| = a_n$.

4. The Curve of Fastest Descent (I): The Descent Time

The curve of fastest descent is the curve between two points A and B that is covered in the least time by a body that starts at the point A with zero speed and is constrained to move along the curve to the point B, under the action of constant gravity and assuming no friction.



The straight line segment connecting A and B (the dotted line) is the curve of shortest distance but surely not of the fastest descent. Galileo in 1638 had studied the problem in his famous work *Discourse on Two New Sciences*. His conjecture was the arc of a quarter circle (the dashed curve). However the correct solution was obtained in 1697, that the curve of fastest descent is a segment of the upside down cycloid (the solid curve).

To demonstrate that the descent time of the cycloid is indeed less than that of the arc and the straight line, let us consider a concrete example. Let the coordinate of A and B be $A(0,0)$ and $B(1,-1)$. The straight line is given by the linear equation $y=-x \quad (0 \le x \le 1)$. The arc is given by the equation of a circle $(x-1)^2+y^2=1 \quad (0 \le x \le 1, -1 \le y \le 0)$. The fasted descent curve is given by the equation of inverted cycloid,

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(\cos\theta - 1) \end{cases},$$

with $r = 0.572917$ and θ ranging from 0 to 2.41201.

(a) Find the descent time of the straight line.

(b) Find the descent time of the arc of a quarter circle.

(c) Find the descent time of the inverted cycloid.

Hints: The speed of the body can be obtained from conservation of energy. The following integral will be useful

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x^2)}} = 2.62206.$$

Solution:

In general, the speed of the body is $v=\sqrt{2g|y|}$. The displacement element is

$$\mathrm{d}s = -\left(\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1\right)^{1/2} \mathrm{d}y.$$

Then the descent time can be obtained by integral

$$T = \int \mathrm{d}t = \int \frac{\mathrm{d}s}{v}.$$

(a) $\mathrm{d}s = -\sqrt{2}\mathrm{d}y$, so

$$T = \frac{1}{\sqrt{g}} \int_0^{-1} \frac{-\mathrm{d}y}{\sqrt{|y|}} = 2g^{-1/2}.$$

(b) For $(x-1)^2 + y^2 = 1$,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{y}{\sqrt{1-y^2}},$$

so

$$\mathrm{d}s = -\frac{\mathrm{d}y}{\sqrt{1-y^2}}.$$

$$T = \frac{1}{\sqrt{2g}} \int_0^{-1} \frac{-\mathrm{d}y}{\sqrt{|y|(1-y^2)}} = \frac{\sqrt{2\pi}\Gamma(5/4)}{\Gamma(3/4)} g^{-1/2} = 1.85407 g^{-1/2}.$$

(c) From the cycloid equations

$$\mathrm{d}x = r(1-\cos\theta)\mathrm{d}\theta,$$
$$\mathrm{d}y = -r\sin\theta\, \mathrm{d}\theta,$$

so

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = r\sqrt{2(1 - \cos\theta)}\mathrm{d}\theta.$$

While

$$v = \sqrt{2g|y|} = \sqrt{gr}\sqrt{2(1 - \cos\theta)},$$

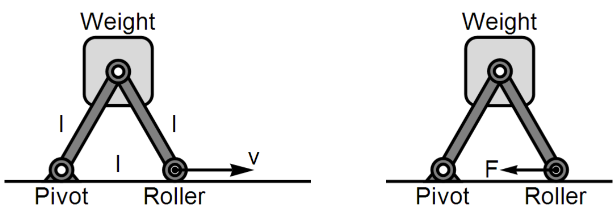
therefore the descent time is

$$T = \int \frac{\mathrm{d}s}{v} = \int_0^{\theta_\mathrm{B}} \frac{r\mathrm{d}\theta}{\sqrt{gr}} = \sqrt{r}\theta_B g^{-1/2} = 1.82568 g^{-1/2}.$$

Chap. 9 Newton's Law of Dynamics

1. The Choo-choog Machine (II): Force

Let us look at the choo-choog machine again. There are two pivoted rods which carry a weight of the mass M. At the right the roller is being driven back or forth by some machinery at a constant speed v. In order to keep the roller moving with constant speed, the machinery must exert a horizontal force F on the roller.



According to Newton's law of dynamics, what is the force F required when the distance between the roller and the pivot is l.

Solution:

The vertical acceleration of the weight

$$\ddot{y} = -\frac{2v^2}{3\sqrt{3}l}.$$

The up-hold force on the weight should be

$$T = Mg + M\ddot{y},$$

and the horizontal force on the roller is

$$F = \frac{1}{\sqrt{3}}\frac{T}{2} = \frac{Mg}{2\sqrt{3}} - \frac{Mv^2}{9l}.$$

2. Computer Simulation of Planetary Motion

Design a computer program to simulate the motion of plane. The equations follow from Newton's Second Law:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{GMmx}{r^3},$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{GMmy}{r^3},$$

$$r = \sqrt{x^2 + y^2}.$$

Rescale the units so as to make $GM = 1$. The initial position is fixed at $x(0) = 1,\ y(0) = 0$. The initial velocity is always along y direction (i.e. $v_x(0) = 0$). Run the simulation for the following initial velocities:

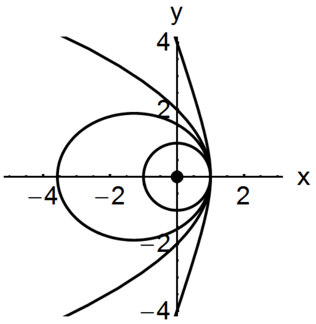
(a) $v_y(0) = 1$,

(b) $v_y(0) = 5/4$,

(c) $v_y(0) = \sqrt{2}$,

(d) $v_y(0) = 2$.

Hints: You are supposed to obtained the following orbits. In order to obtained the whole branch of the orbit, your simulation should not only go forward in time but also backward in time.



(e) For the case (a) and (b), use the simulation to determine the period of the orbital motion. Does that satisfy Kepler's Third Law?

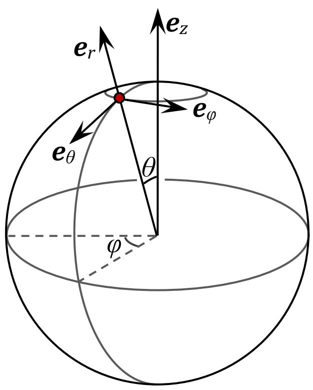
3. Deviation of Falling Body

The "gravity" experienced by a particle of the mass m on the Earth is

$$\boldsymbol{G} = m\boldsymbol{g} - 2m\boldsymbol{\omega} \times \boldsymbol{v},$$

where $\boldsymbol{g}$ is the effective gravity acceleration with centrifugal acceleration been absorbed, $\boldsymbol{\omega}$ is the angular velocity of the Earth rotation (which is measured in a inertial frame out of the Earth), and $\boldsymbol{v}$ is the velocity of the particle with respect the Earth surface. The last term $-2m\boldsymbol{\omega} \times \boldsymbol{v}$ is the Coriolis "force" on the moving object, which is solely a kinematic effect due to the transformation of coordinate system.

The Coriolis force influences the motion of falling bodies on the earth. For example, the falling body always deviates eastward. To see this, let us first decompose the Coriolis force in the spherical frame.



(a) Let $\boldsymbol{\omega} = \omega \boldsymbol{e}_z$ and $\boldsymbol{v} = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta + v_\varphi \boldsymbol{e}_\varphi$, write down the Coriolis force $-2m\boldsymbol{\omega} \times \boldsymbol{v}$

in terms of $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\varphi$.

We can see that the Coriolis force on the Earth surface shows a dependence on the latitude θ. You can think of the difference for the Coriolis force in north and south hemisphere.

(b) According to Newton's law $\boldsymbol{G} = m\boldsymbol{a}$, what are the equations of motion for v_r, v_θ and v_φ respectively.

The angular velocity of the Earth rotation is $\omega = 7.27 \times 10^{-5} \mathrm{rad} \cdot \mathrm{s}^{-1}$. For a falling body of velocity $10 \mathrm{m} \cdot \mathrm{s}^{-1}$, the Coriolis acceleration is of the order $10^{-4} \mathrm{m} \cdot \mathrm{s}^{-2}$ which is far smaller than the gravity acceleration $g = 9.8 \mathrm{m} \cdot \mathrm{s}^{-2}$. Therefore the Coriolis force can be treated as perturbation, which means that we can solve the equations of motion as if $\omega = 0$ and then make corrections to our solution by turning on the Coriolis term.

(c) In the absence of the Coriolis force (i.e. $\omega = 0$), what is the solution of $\boldsymbol{v}$ (by each component) for a falling body released at rest from the height h? Also obtain the displacement by integration $\boldsymbol{x} = \int \mathrm{d}t\, \boldsymbol{v}$, from which determine the falling time T.

(d) Calculate the corrections to $\boldsymbol{v}$ and $\boldsymbol{x}$ to the first order of ω. Show that the falling body will deviate eastwards by $(1/3)\omega g (2h/g)^{3/2} \sin\theta$.

Note: the deviation is positive definite for all θ, meaning that the falling body will always deviate eastwards no matter in which hemisphere. This provides a global definition of *east*.

(e) Will an upward-thrown body (with initial velocity v_0) fall back to the starting point? If not, determine the displacement (both the magnitude and the direction, up to the first order of ω) of the landing point with respect to the starting point. Does it make any difference in the different hemisphere?

Hint: Use the perturbation approach as you have done in problems (c) and (d).

Solution:

(a) $-2m\boldsymbol{\omega} \times \boldsymbol{v} = 2m\omega v_\varphi \sin\theta\, \boldsymbol{e}_r + 2m\omega\cos\theta \left(v_\varphi \boldsymbol{e}_\theta - v_\theta \boldsymbol{e}_\varphi\right) - 2m\omega v_r \sin\theta\, \boldsymbol{e}_\varphi$.

(b) Equations of motion

$$\dot{v}_r = -g + 2\omega v_\varphi \sin\theta,$$
$$\dot{v}_\theta = 2\omega v_\varphi \cos\theta,$$
$$\dot{v}_\varphi = -2\omega v_\theta \cos\theta - 2\omega v_r \sin\theta.$$

(c) Velocity $v_r = -gt$, $v_\theta = v_\varphi = 0$. Displacement $x_r = h - gt^2/2$, $x_\theta = x_\varphi = 0$. Falling time $T = \sqrt{2h/g}$.

(d) From $\dot{v}_\varphi = -2\omega v_\theta \cos\theta - 2\omega v_r \sin\theta = 2\omega g t \sin\theta$, we have

$$v_\varphi = \omega g t^2 \sin\theta,$$

and therefore

$$x_\varphi = \frac{1}{3}\omega g t^3 \sin\theta.$$

While $v_\theta = 0$ and $x_\theta = 0$ still holds up to the first order correction. The landing position is

$$\boldsymbol{x} = \frac{1}{3}\omega g (2h/g)^{3/2} \sin\theta\, \boldsymbol{e}_\varphi,$$

which deviates eastward both in north and south hemisphere.

(e) First solve the equation of motion in the case of $\omega = 0$. The solution is $v_r = v_0 - gt$, $v_\theta = v_\varphi = 0$. The falling time is $T = 2v_0/g$.

With the Coriolis force term,

$$v_r = v_0 - gt + O(\omega^2)$$

$$v_\varphi = -\omega(2v_0 t - gt^2)\sin\theta + O(\omega^3),$$
$$v_\theta = O(\omega^2),$$

by integration

$$x_r = v_0 t - gt^2/2 + O(\omega^2),$$
$$x_\varphi = -\omega(v_0 t^2 - gt^3/3)\sin\theta + O(\omega^3),$$
$$x_\theta = O(\omega^2).$$

The correction in falling time is of the order ω^2,

$$T = 2v_0/g + O(\omega^2).$$

Thus up to the first order correction,

$$\boldsymbol{x} = x_\varphi(T)\boldsymbol{e}_\varphi = -\frac{4\omega v_0^3}{3g^2}\sin\theta\,\boldsymbol{e}_\varphi.$$

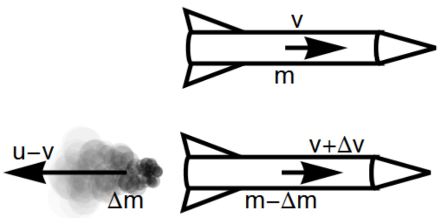
So the upward-thrown body will deviate westwards in both north and south hemisphere.

Some may not be clear about the idea of perturbation. Perturbation is done order by order. Higher order terms are not necessarily to be included. Some may not understand the physical reason of the result, and insists that the body should fall to the original place, due to cancelation of inversed process. However, because the rotation of the Earth breaks time reversal symmetry, such argument does not hold. As the body rise to the top, it acquires a velocity to the west, which allows it to continue moving westward in the falling.

Chap. 10 Conservation of Momentum

1. Rocket Propulsion

This is a more detailed model of rocket propulsion than the one in the Lecture. We are going to take a rocket floating around in the empty space first – forgetting all about gravity, and so on. The rocket is built to hold a lot of fuel, and it has got some kind of engine by which it squirts fuel out the back. From the point of view of the rocket, it is always squirting it out at constant speed, say u, relative to the rocket itself. We will suppose the initial mass of the rocket is m_0, and the stuff is squirted out at a rate μ (that is the mass per second, and should not be confused with the speed u). Now the question is, how much velocity will the rocket accumulate in the time t?



Hints: If we watch the rocket for a small interval of time Δt, what do we see? Well, there is a certain mass $\Delta m = \mu\,\Delta t$ that goes out with a certain speed. Meanwhile, the rocket losses the same amount of mass and gains some velocity. Use the conservation of momentum to determine the unknowns.

Solution:

According to the conservation of momentum

$$mv = (m - \Delta m)(v + \Delta v) - \Delta m(u - v),$$

thus

$$\frac{\Delta v}{u} = \frac{\Delta m}{m} = \frac{\mu \Delta t}{m_0 - \mu t}.$$

The solution is

$$v = u \ln \frac{m_0}{m_0 - \mu t}.$$

2. Satellite Retardation: Origin of Air Friction

An earth satellite of mass 10kg and average cross-sectional area $0.50\mathrm{m}^2$ is moving in a circular orbit at 200km altitude where the molecular mean free paths are many meters and the air density is about $1.6 \times 10^{-10} \mathrm{kg\,m^{-3}}$. Under crude assumption that the molecular impacts with the satellite are effectively *inelastic* (however the molecules do not literally stick to the satellite but drop away from it with very low relative velocity).
(a) Let the velocity of the satellite be v, calculate the retarding force that the satellite would experience due to the air friction. How should such a force vary with velocity?
(b) Would the satellite's speed decrease as a result of the net force on it? Hint: Check the speed of a circular satellite orbit versus height.

Solution:
(a) $f = S\rho v^2 = 8 \times 10^{-11} \mathrm{kg \cdot m^{-1}} v^2$
Some mistake the dimension here.
(b) The energy of the satellite on circular orbital is $E = T + V = -mv^2/2$. The air friction leads to decrease of energy

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}mv^2\right) = -mv\frac{\mathrm{d}v}{\mathrm{d}t} = -fv,$$

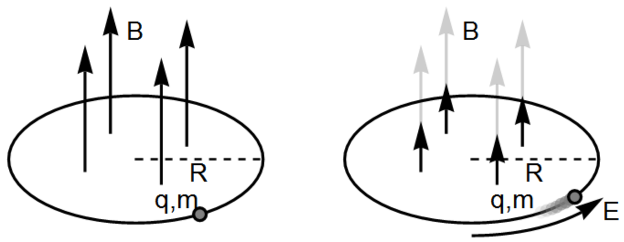
so

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{f}{m} > 0,$$

meaning the satellite will speed up.
Some do not take into account the effect of gravity, and consider the friction is too small to slow down the satellite. But this is not the right argument expected.

3. Hidden Momentum in the Magnetic Field

Consider a ring of the radius R, threading on which is a small ball with mass m and electric charge q. Inside the ring, there is an uniform magnetic field B perpendicularly penetrating the plane of the ring. Now everything is at rest. If we suddenly turn off the magnetic field, what happens to the little charged ball? You may have learned in the electromagnetism course that according to Lentz's Law, the vanishing magnetic field will induce a eddy electric field which accelerates the charge ball. So here comes the problem.
(a) When the magnetic field is turned off, what is the velocity v of the ball circling around the ring?
Hint: suppose the magnetic field is turned off in a short time Δt. Determine the eddy electric field E according to Faraday's Law, then determine the electric force on the charge, calculate the acceleration and then you will be able to find out the velocity.



(b) What is the (kinetic) momentum of the small ball before and after the magnetic field being shut down?

We have already known that charge in the electric field processes the potential energy, which is a hidden form of energy. The conservation of energy holds only if we include the contribution of this potential energy. So you may ask is there also a counterpart for the momentum, that some kind of *potential momentum* is hidden in the electromagnetic field, and the total momentum will conserve only if we take it into account? The answer is yes. Unlike the potential energy hidden in the electric field, the potential momentum is hidden in the *magnetic field*.

Upon the removal of magnetic field, this potential momentum is released and transforms into the kinetic momentum mv of the ball. That is why the ball suddenly acquires a velocity when the magnetic field is shut off.

The potential momentum per unit charge in the magnetic field is called the vector potential (or gauge potential) $A = mv/q$. In this problem, when the magnetic field is on, the vector potential is circling around the ring in the same direction as the eddy electric field.

(c) Calculate the strength of vector potential A on the ring. If the radius of the ring changes, the vector potential on the ring also changes. So the vector potential is in fact a field in the space. Can you write down the vector potential $\boldsymbol{A}(\boldsymbol{r})$ (with both components) as a function of the space coordinate $\boldsymbol{r}$?

(d) Show that the vector potential is related to the magnetic flux density through $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ for this particular problem.

Note: In general, given a magnetic field (not necessarily uniform), the vector potential can be determined by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ up to some *gauge freedom* (which we should not discuss here). The point is, once the vector potential is known, it can be used to determine the potential momentum (or the electromagnetic momentum) of a charge through $\boldsymbol{p}_{EM} = q\boldsymbol{A}$, and the total momentum (or the canonical momentum) is the sum of kinetic momentum and potential momentum $\boldsymbol{p} = m\boldsymbol{v} + q\boldsymbol{A}$ which is the only momentum conserved in the electromagnetic field. This is just as in the case of the electric field, where the conserved energy is the total of kinetic and potential energy $(1/2)mv^2 + qV$, but not any of the individual one.

Solution:

(a) $v = BqR/2m$.

(b) $p = BqR/2$.

(c) $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{B} \times \boldsymbol{r}/2$.

Chap. 11 Vectors

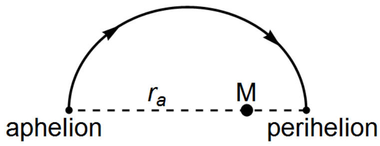
1. Vector Calculus: Line Integral of Vector

To integrate a vector $\boldsymbol{F} = (F_x, F_y, F_z)$ along a curve by differential displacement $\mathrm{d}\boldsymbol{s} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$,

$$W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s},$$

just do the dot product honestly $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z$, then finish the integral one by one. Example: calculate the work done by gravity on a mass m when it orbits around M from the aphelion to the perihelion. The gravitational force is given by

$$\boldsymbol{F} = \left(-\frac{GMmx}{r^3}, -\frac{GMmy}{r^3}, -\frac{GMmz}{r^3}\right).$$



The orbit is described by

$$\boldsymbol{s} = \left(\frac{r_a \cos\theta}{2 + \cos\theta}, \frac{r_a \sin\theta}{2 + \cos\theta}, 0\right),$$

with r_a fixed and θ going from π to 0. Find out the work $W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ in this case. Is your result consistent with that obtained from the potential energy difference?

Solution:

From the orbital equation

$$\boldsymbol{s} = \left(\frac{r_a \cos\theta}{2 + \cos\theta}, \frac{r_a \sin\theta}{2 + \cos\theta}, 0\right),$$

we obtained the differential displacement by taking derivative

$$\mathrm{d}\boldsymbol{s} = \frac{r^2 \mathrm{d}\theta}{r_a}(-2\sin\theta, 1 + 2\cos\theta, 0),$$

where we have defined $r = r_a/(2 + \cos\theta)$. The gravity in the orbital plane is

$$\boldsymbol{F} = -\frac{GMm}{r^2}(\cos\theta, \sin\theta, 0),$$

therefore the dot product reads

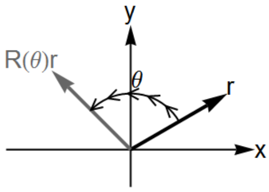
$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\frac{GMm}{r_a}\sin\theta\, \mathrm{d}\theta.$$

Finish the integral, and we obtain the work

$$W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = -\frac{GMm}{r_a}\int_\pi^0 \sin\theta\, \mathrm{d}\theta = \frac{2GMm}{r_a}.$$

2. Rotation Matrix (I): Closure of SO(2) Group

Multiplication of a matrix with a vector is like applying an operation (known as the linear transform) to the vector that transforms the original vector into a new one. Therefore the matrix can be used to represent the operation. We have learnt that the rotation in 2-dimensional space can be represent by a 2×2 rotation matrix $R(\theta)$, which, when applying on a 2-dimensional vector, will rotate it (counterclockwise) by the angle θ.



(a) Write down the rotation matrix $R(\theta)$ for arbitrary angle θ.
Starting from a (column) vector $\boldsymbol{r}_0 = [x_0 \quad y_0]^T$, rotate it first by θ_1 to $\boldsymbol{r}_1 = R(\theta_1)\boldsymbol{r}_0$ and then by θ_2 to $\boldsymbol{r}_2 = R(\theta_2)\boldsymbol{r}_1$. The two successive rotations transform the original vector $\boldsymbol{r}_0$ to $\boldsymbol{r}_2 = R(\theta_2)R(\theta_1)\boldsymbol{r}_0$.
(b) Show that $R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2)$.
This means that the combined operation of two successive rotations is still a rotation. This property is known as the closure of SO(2) group.

Solution:
(a)

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(b)

$$R(\theta_2)R(\theta_1) = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 \\ \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{bmatrix} = R(\theta_1 + \theta_2).$$

3. Rotation Matrix (II): Diagonalization of Quadratic Form

In general, a conic can be described by the quadratic equation

$$[x \quad y] M \begin{bmatrix} x \\ y \end{bmatrix} = 1. \tag{1}$$

where the matrix M reads

$$M = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}.$$

For example, if $A = a^{-2}$, $C = b^{-2}$ and $B = 0$, Eq. (1) will reduce to the standard form of an ellipse $a^{-2}x^2 + b^{-2}y^2 = 1$ with semi-major axis a and semi-minor axis b. It is just because of $B = 0$ (i.e. the matrix M in Eq. (1) is *diagonal*) that we are lucky to have the major and minor axes falling on the x and y axes respectively. However it is generally not the case.
When $B \neq 0$, the major/minor-axis of the ellipse will be rotated away of the axes of the coordinate system. To see how the ellipse is rotated, let us consider a rotation matrix $R(\theta)$ that rotates the vector $[x \quad y]^T$ (counterclockwise) by the angle θ to $[x' \quad y']^T$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix},$$

under which the conic equation becomes

$$[x' \quad y'] M' \begin{bmatrix} x' \\ y' \end{bmatrix} = 1,$$

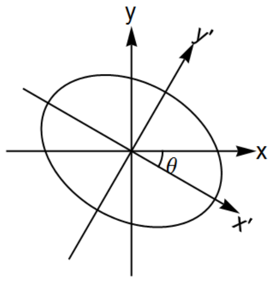
where

$$M' = \begin{bmatrix} A' & B'/2 \\ B'/2 & C' \end{bmatrix}.$$

(a) Work out A', B' and C' in terms of A, B, C and θ. Note: you should work out the most general case here. Neither M nor M' is diagonalized.

(b) Show that $\operatorname{Tr} M = \operatorname{Tr} M'$ and $\det M = \det M'$, i.e. the trace and determinant of the matrix M is invariant under rotation.

The transform of coordinates is effectively the same as the transformation of the reference frame in the *reverse* manner. If in the x'- y' frame, the major/minor axis falls on the axes of the frame, we say that the quadratic form is diagonalized in the x'- y' frame.



(c) What θ should be to diagonalize the quadratic form in the x'- y' frame?

When the quadratic form is diagonalized, the semi-major/minor axis can be determined from the diagonal elements.

(d) Find the semi-major axis a and semi-minor axis b of a conic described by Eq. (1). Give your result in terms of $\operatorname{Tr} M$ and $\det M$.

(e) Show that the conic will be an ellipse (or circle) if $\det M > 0$.

Note: a 2×2 real symmetric matrix M can always be diagonalized by a SO(2) transform $R(\theta)$ (i.e. a rotation in the 2-dimensional space). The geometric meaning of diagonalization is to rotate the conic so that its major/minor axis coincides with the frame axes. After diagonalization, the diagonal elements are known as the eigen values of the original matrix, which are invariant under rotation, providing information of the shape of the conic.

Solution:

(a) From $M' = R(\theta) M R(-\theta)$,

$$A' = A\cos^2\theta - B\sin\theta\cos\theta + C\sin^2\theta,$$
$$B' = A\sin 2\theta + B\cos 2\theta - C\sin 2\theta,$$
$$C' = A\sin^2\theta + B\sin\theta\cos\theta + C\cos^2\theta.$$

(b) $\operatorname{Tr} M' = A' + C' = A + C = \operatorname{Tr} M$.

$\det M' = A'C' - B'^2/4 = AC - B^2/4 = \det M$.

(c) $B' = 0$, so

$$\theta = \frac{1}{2}\arctan\frac{B}{C - A}.$$

(d) The axes

$$a, b = \left(\frac{1}{2}\left(\operatorname{Tr} M \pm \sqrt{(\operatorname{Tr} M)^2 - 4\det M}\right)\right)^{-1/2}.$$

Chap. 12 Characteristics of Force

1. Plot the Force v. s. Distance Curve

Use a computer software to plot the following force F as a function of distance r.
(a) Elastic force of a spring: $F = -kr$,
(b) Molecular force: $F = -k_1 r^{-7} + k_2 r^{-13}$,
(c) Electrostatic force: $F = q_1 q_2/(4\pi\epsilon_0 r^2)$,
(d) Nuclear force: $F = -\left(1/r^2 + 1/(rr_0)\right)\exp(-r/r_0)$.

2. Vector Potential and Lorentz Force

Chap. 13 Work and Potential Energy (A)

1. Gravitational Pull from Interstellar Dust

Consider a planet of mass m is orbiting around the Sun (whose mass is M) on a circle of radius r. Assume that there is uniform interstellar dust of the density ρ (mass per volume) throughout the whole solar system. The planet will experience an extra gravitational force F' provided by the dust.
(a) Show that the gravity force F' by the dust is a linear restoring force in the form of $F' = -kr$. Find the factor k. (Neglect the resistance due to collisions with the dust.)
(b) If the angular momentum of the planet is L, what is the radius r of the circular orbit, up to the first order correction of ρ?

Solution:
(a) $k = (4\pi/3)m\rho G$.
(b) From

$$\begin{cases} L = mvr, \\ \dfrac{mv^2}{r} = F = \dfrac{GMm}{r^2} + \dfrac{4\pi}{3}Gm\rho r, \end{cases}$$

we have the equation for r,

$$r = \frac{L^2}{GMm^2}\left(1 + \frac{4\pi}{3}\frac{\rho r^3}{M}\right)^{-1}.$$

In the small ρ limit, this equation can be solved by perturbation approach,

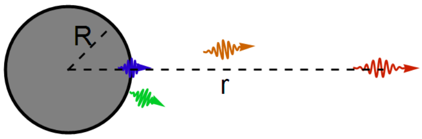
$$r = \frac{L^2}{GMm^2}\left(1 - \frac{4\pi}{3}\frac{\rho L^6}{G^3 M^4 m^6}\right).$$

Some have difficulties with solving the equation. They should be told that appropriate approximations are encouraged.

2. Gravitational Red Shift (Newtonian Limit)

Gravitational red shift refers to the phenomena that light emitted from stars undergoes frequency shift to red-end due to gravitation. According to quantum mechanics, the energy of a photon is $E = h\nu$, where h is the Planck constant and ν the photon frequency.

(a) What is the inertial (and gravitational) mass of photon?



(b) Given the photon frequency ν and the mass of the star M, what is the potential energy of photon in the gravitational field of the star (according to Newton's Law of Gravity), if the photon is r distant away from the center of the star?

The increase of the potential energy as the photon climbs out of the potential well of the star should be compensated by the decrease of its kinetic energy such that the total energy is conserved. Since the kinetic energy of the photon is proportional to its frequency, this will leads to the drop of photon frequency as it flies away from the star, and hence the gravitational red shift.

(c) According to the conservation of energy, deduce the photon frequency ν as a function of r, given that the frequency when the photon is emitted from the surface of the star ($r = R$) is $\nu(R)$.

(d) Find the red shift $\Delta\nu/\nu$ of light of a far star given its mass M and radius R, where $\Delta\nu = \nu(\infty) - \nu(R)$.

(e) It is observed[1] that the light from the white dwarf Sirius B suffers the gravitational red shift $\Delta\nu/\nu = 2.7 \times 10^{-4}$. The mass of Sirius B is roughly the same as the solar mass, estimate the radius and hence the mass density of Sirius B from its gravitational red shift.

Note: The calculation of gravitational red shift in this problem is based on Newton's Law of Gravity, which does not take into account the effect of general relativity. The general theory of relativity is known as a more correct theory of gravity, especially when the gravitational field is strong or fast varying. To estimate the influence of general relativity, we introduce the Schwarzschild radius $R_S = 2GM/c^2$. A star will becomes a black hole if its radius is less than its Schwarzschild radius, which means that light cannot escape from the surface of the star. From the viewpoint of gravitational red shift, this means light emitted from a black hole will red shift infinitely, i.e. its frequency drops to zero before travelling far. In this case, general relativity is important to calculate the red shift correctly. The precise result given by general relativity is $\nu(\infty) = \nu(R)(1 - R_S/R)^{1/2}$. Our Newtonian approximation is valid only in the limit when $R \gg R_S$, say the radius of the star is far greater than the radius of a black hole of the same mass.

Reference

[1] M.A. Barstow, H. E. Bond, J.B. Holberg, M.R. Burleigh, I. Hubeny and D. Koester, Mon. Not. R. Astron. Soc. **000**, 1-11 (25-May-2005).

Solution:

(a) Photon mass: $h\nu/c^2$.

(b) Potential energy

$$V = -\frac{GMh\nu}{c^2 r}.$$

(c) Change in the potential energy

$$\mathrm{d}V = \frac{GMh\nu}{c^2r^2}\mathrm{d}r - \frac{GMh}{c^2r}\mathrm{d}\nu.$$

Conservation of energy:

$$\mathrm{d}E = h\mathrm{d}\nu + \mathrm{d}V = h\mathrm{d}\nu + \frac{GMh\nu}{c^2r^2}\mathrm{d}r - \frac{GMh}{c^2r}\mathrm{d}\nu = 0.$$

Define the Schwarzschild radius $R_S = 2GM/c^2$, then

$$\frac{\mathrm{d}\nu}{\nu} = -\frac{R_S}{2r^2 - R_S r}\mathrm{d}r.$$

Integrate on both sides

$$\ln\nu = -\ln\left(1 - \frac{R_S}{2r}\right),$$

thus

$$\nu(r) = \nu(R)\frac{2Rr - R_S r}{2Rr - R_S R} = \nu(R)\frac{1 - GM/c^2R}{1 - GM/c^2r}.$$

(d) From distant star $r \to \infty$,

$$\nu(\infty) = \nu(R)\left(1 - \frac{R_S}{2R}\right) = \nu(R)\left(1 - \frac{GM}{c^2R}\right).$$

$$\frac{\Delta\nu}{\nu} \simeq -\frac{R_S}{2R} = -\frac{GM}{c^2R}.$$

(e) Radius of Sirius B $0.0079R_\odot = 5.5 \times 10^3\mathrm{km}$. The density $2.9 \times 10^9\mathrm{kg \cdot m^{-3}}$.

Chap. 14 Work and Potential Energy (B)

1. Potentials of Conservative Forces

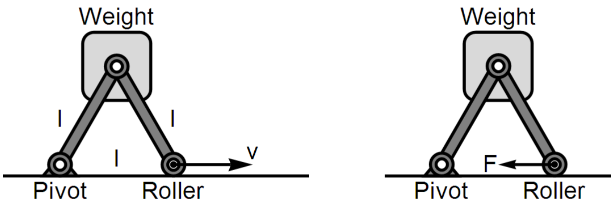
Calculate the potential of the following forces. I should also tell you what is the conventional choice of the potential zero point.

(a) Gravity, near the Earth's surface: $F = -mg$, potential zero point: $V(z = 0) = 0$.
(b) Gravity, between masses: $F = -Gm_1m_2/r^2$, potential zero point: $V(r = \infty) = 0$.
(c) Electrostatic force, between charges: $F = q_1q_2/(4\pi\epsilon_0 r^2)$, potential zero point: $V(r = \infty) = 0$.
(d) Ideal spring: $F = -kx$, potential zero point: $V(x = 0) = 0$.
(e) Nuclear force: $F = -\left(1/r^2 + 1/(rr_0)\right)\exp(-r/r_0)$, potential zero point: $V(r = \infty) = 0$.

Solution:
(a) mgz.
(b) $-Gm_1m_2/r$.
(c) $(4\pi\epsilon_0)^{-1}q_1q_2/r$.
(d) $kx^2/2$.
(e) $-r^{-1}e^{-r/r_0}$.

2. The Choo-choog Machine (III): Work and Energy

Let us go back to the choo-choog machine once more. A weight of mass M is held up by the elbow like pivoted rods of the lengths l. The roller is being driven by some machinery at a constant speed v.



When the distance between the roller and the pivot is l,
(a) What is the energy E (both kinetic and potential) of the weight?
(b) What is the changing rate of the energy $\mathrm{d}E/\mathrm{d}t$?
(c) What is the power input by the machinery calculated from $P = Fv$? Show that it is the input power that causes the energy to change, $P = \mathrm{d}E/\mathrm{d}t$.

Solution:
(a) $E = (1/6)Mv^2 + (\sqrt{3}/2)Mgl$.
(b)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{Mv^3}{9l} - \frac{\sqrt{3}}{6}Mgv.$$

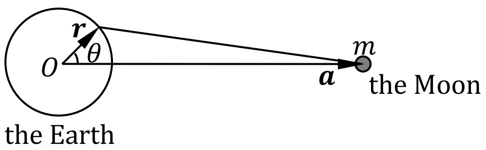
(c) The force

$$F = \frac{Mv^2}{9l} - \frac{\sqrt{3}}{6}Mg.$$

So the power $P = Fv$ is the same as $\mathrm{d}E/\mathrm{d}t$.

3. Tidal Force and Tidal Phenomena

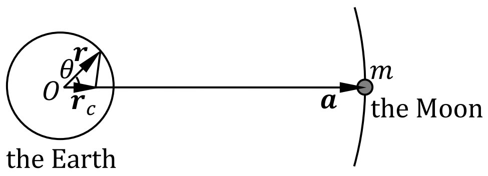
Tidal force is the cause of tides. It mainly comes from the Moon and the Sun. Let us first consider the influence of the Moon. As shown in the following figure, the Moon m is away from the Earth center by the displacement $\boldsymbol{a}$.



(a) What is the gravitational potential energy $V_m(\boldsymbol{r})$ at the surface of the Earth $\boldsymbol{r}$, corresponding to the gravitational field of the Moon.
The Earth-Moon distance a is about sixty times of the radius r of the Earth. To approximate the Moon's gravitational potential on the Earth's surface, we can Taylor expand $V_m(\boldsymbol{r})$ with respect to (r/a) up to the second order

$$V_m(\boldsymbol{r}) = V_{m0} + V_{m1}\frac{r}{a} + V_{m2}\left(\frac{r}{a}\right)^2 + \cdots$$

(b) Find V_{m0}, V_{m1} and V_{m2} as functions of the angle θ.
We should not forget that the Earth-Moon system is rotating about its center of mass $\boldsymbol{r}_c$. Suppose the Moon's orbital is circular, and its angular velocity is ω.



(c) Where is the center of mass of the Earth-Moon system? (Find the length of r_c in terms of G, m, ω and a)

Both the Earth and the Moon are rotating with respect to their center of mass (COM). Therefore the Earth itself is an non-inertial frame (even if we forget about the spinning of the Earth), in which all objects will experience the inertial force. If the object is not moving with high speed, the inertial force it experienced is basically the centrifugal force.

(d) What is the potential $V_\omega(\boldsymbol{r})$ corresponding to the centrifugal force due to rotation of the Earth with respect to the COM.

Now let us put $V_m(\boldsymbol{r})$ and $V_\omega(\boldsymbol{r})$ together with the potential of the Earth itself $V_M(\boldsymbol{r}) = -GMr^{-1}$ (where M is the mass of the Earth), thus the total potential on the surface of the Earth is

$$V(\boldsymbol{r}) = V_M(\boldsymbol{r}) + V_m(\boldsymbol{r}) + V_\omega(\boldsymbol{r}).$$

Make the Taylor expansion again,

$$V(\boldsymbol{r}) = -\frac{GM}{r} + V_0 + V_1\frac{r}{a} + V_2\left(\frac{r}{a}\right)^2 + \cdots.$$

The term V_0 is a constant which contributes nothing to the strength of the gravity field.

(e) Show that V_1 vanishes due to the cancelation between $V_m(\boldsymbol{r})$ and $V_\omega(\boldsymbol{r})$.

(f) Find out V_2, which should be a function of θ only.

Now the total potential can be simply written as

$$V(\boldsymbol{r}) = V_M(\boldsymbol{r}) + V_{\text{tidal}}(\boldsymbol{r}),$$

where $V_M(\boldsymbol{r})$ is the gravitational potential of the Earth, and $V_{\text{tidal}}(\boldsymbol{r}) = V_2(r/a)^2$ is the potential that give rise to the tidal force. It is clear that $V_{\text{tidal}}(\boldsymbol{r})$ origins from the gravitaty of the Moon.

The gravitational field strength on the surface of the Earth can be obtained from the potential by $\boldsymbol{g} = -\nabla V(\boldsymbol{r})$,

$$\boldsymbol{g} = g_r\boldsymbol{e}_r + g_\theta\boldsymbol{e}_\theta.$$

(g) Find out g_r and g_θ.

g_r contains both the free-falling acceleration and the radial tidal acceleration, while g_θ is the transverse tidal acceleration. They are responsible for the tidal phenomena. Unlike the gravity force, the tidal force is inversely proportional to the cube (not square) of the Earth-Moon distance.

(h) Given that $M = 5.98 \times 10^{24}$kg, $m = 7.36 \times 10^{22}$kg and $a/r = 60.3$, how many orders of magnitude is the tidal force smaller than the gravity force on the Earth's surface.

Suppose the Earth is a sphere of the radius R. Let h be the sea level (i.e. $r = R + h$ is the distance of the ocean surface to the center of the Earth). The surface of the ocean is an equal-potential surface given by $mg_rh = \text{const.}$

(i) What is the sea level h as a function of θ? How many times of flood tides are there in a day?

Solution:

(a) The potential of the Moon

$$V_m(\boldsymbol{r}) = -\frac{Gm}{|\boldsymbol{r}-\boldsymbol{a}|} = -\frac{Gm}{\sqrt{r^2+a^2-2ra\cos\theta}}.$$

(b) The expansion

$$V_{m0} = -\frac{Gm}{a},$$

$$V_{m1} = -\frac{Gm}{a}\cos\theta,$$

$$V_{m2} = -\frac{Gm}{a}\frac{1}{2}(3\cos^2\theta - 1).$$

(c) Suppose the mass of the Earth is M, then for the Moon on circular orbit

$$\frac{Mm}{M+m}\omega^2 a = \frac{GMm}{a^2},$$

thus

$$M+m = \frac{\omega^2 a^3}{G}.$$

The center of mass by definition satisfies

$$ma = (M+m)r_c = \frac{\omega^2 a^3}{G}r_c,$$

so

$$r_c = \frac{Gm}{\omega^2 a^2}.$$

(d) The centrifugal potential

$$V_\omega(\boldsymbol{r}) = -\frac{1}{2}\omega^2(\boldsymbol{r}-\boldsymbol{r}_c)^2 = -\frac{1}{2}\omega^2(r^2+r_c^2-2rr_c\cos\theta).$$

Substitute the expression for r_c,

$$V_\omega(\boldsymbol{r}) = -\frac{1}{2}\omega^2\left(r^2+\frac{G^2m^2}{\omega^4a^4}\right)+\frac{Gm}{a}\frac{r}{a}\cos\theta.$$

The expansion

$$V_{\omega0} = -\frac{G^2m^2}{2\omega^2a^4},$$

$$V_{\omega1} = \frac{Gm}{a}\cos\theta,$$

$$V_{\omega2} = -\frac{1}{2}\omega^2a^2.$$

(e) The first order term $V_1 = V_{m1} + V_{\omega1} = 0$.

(f) The second order term,

$$V_2 = -\frac{Gm}{a}\frac{1}{2}(3\cos^2\theta - 1) - \frac{1}{2}\omega^2 a^2.$$

(g) The gravity field,

$$g_r = -\frac{\partial}{\partial r}V(\boldsymbol{r}) = g_{\text{eff}} + \frac{Gm}{a^2}\frac{r}{a}(3\cos^2\theta - 1),$$

$$g_r = -\frac{1}{r}\frac{\partial}{\partial\theta}V(\boldsymbol{r}) = -\frac{Gm}{a^2}\frac{r}{a}(3\cos\theta\sin\theta).$$

where g_{eff} is the free-falling acceleration

$$g_{\mathrm{eff}} = -\frac{GM}{r^2} + \omega^2 r.$$

Some may not understand to separate g_{eff} out, and take the whole g_r as the radial tidal acceleration. Some forget to take the sign in front of $-\partial_r V$.

(h) $f = (m/M)(r/a)^3 = 5.6\times 10^{-8}$.

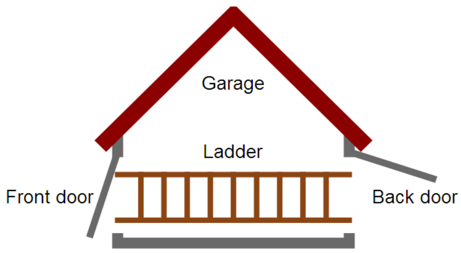
(i)

$$h = \frac{\text{const.}}{1+f(3\cos^2\theta - 1)} \sim h_0\big(1+f(3\cos^2\theta-1)\big).$$

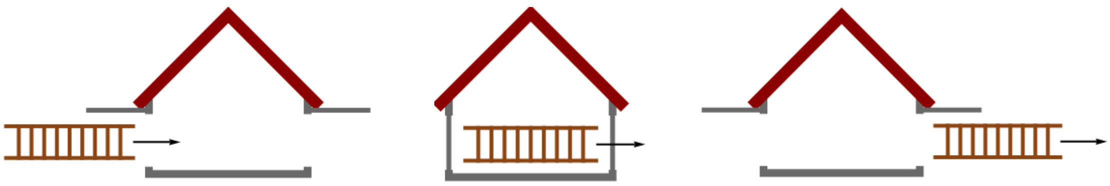
Chap. 15 The Special Theory of Relativity

1. Ladder Paradox

A garage has two doors: a front door and a back door. A ladder is placed horizontally inside. The ladder and the garage are of the same length L, when both of them are at rest.



The ladder is now set in motion at a speed of $0.6c$ relative to the garage. From the garage's reference frame, due to relativistic length contraction, the ladder appears shorter than the garage, suggesting that there is a moment when the ladder can fit completely inside the garage, and during that period, both front and back doors of the garage can be simultaneously closed and then opened, without hitting the ladder.

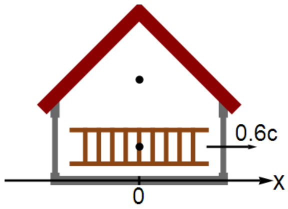


Ladder enters from the front door. | Both doors shut simultaneously. | Both doors soon open again and let the ladder out.

However, from the ladder's frame of reference, now it is the garage that appears shorter than the ladder. Therefore, there is no moment in time when the ladder is completely inside the garage; and at no moment one can close both doors without it hitting the ladder. This is an apparent paradox.

(a) To solve this paradox, first work out what is the length of the ladder from the frame of the garage, and what is the length of the garage from the frame of the ladder.

(b) From the garage frame, let $t = 0$ be the moment when two doors close and open. At this moment, the ladder is right at the middle of the garage which is set to be the origin of coordinate x.



Mark the following figures on the space-time (t v.s. x) diagram from the garage frame:

(1) the world line of the front door,

(2) the world line of the back door,

(3) the world line of the left end of the ladder,

(4) the world line of the right end of the ladder,

(5) the event A: front door closes and opens,

(6) the event B: back door closes and opens,

(7) the event C: the left end of the ladder passes by the front door,

(8) the event D: the right end of the ladder passes by the back door.

(c) For the above events A, B, C and D, write down their space-time coordinates (in the form of (x, t)) from the garage frame of reference.

(d) Use Lorentz transformation to transform the space-time coordinates events A, B, C and D into the ladder frame of reference.

The solution to the ladder paradox lies in the fact that what one observer (e.g. the garage) considers as simultaneous does not correspond to what the other observer (e.g. the ladder) considers as simultaneous (relative simultaneity). You will find that events A and B are simultaneous from the garage frame, but not simultaneous from the ladder frame. Between A and B, which event happens earlier from the ladder frame?

(e) Now mark the world lines and evens listed in problem (b) on the space-time diagram from the ladder frame.

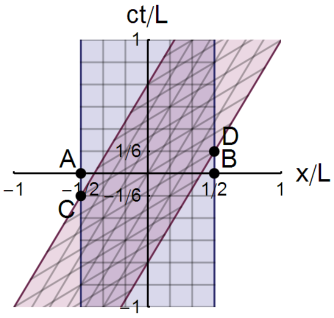
(f) What is the scenario from the ladder's point of view?

(g)* What if the back door is closed permanently and does not open again? Will the ladder been trapped in the garage? What is the scenarios from both the garage viewpoint and the ladder viewpoint then? Note: you do not need to answer the questions in the problem (g) in your homework. They are just for you to think of in your spare time.

Solution:

(a) From the garage frame the ladder undergoes Lorentz contraction, and in the ladder frame the garage also undergoes Lorentz contraction. According to the formula of Lorentz contraction, $L' = L\sqrt{1-\beta^2} = 4L/5$, given $\beta \equiv u/c = 3/5$.

(b) Space-time diagram from the frame of the garage.



(c) From the garage frame: A(-1/2, 0), B(1/2, 0), C(-1/2, -1/6), D(1/2, 1/6).

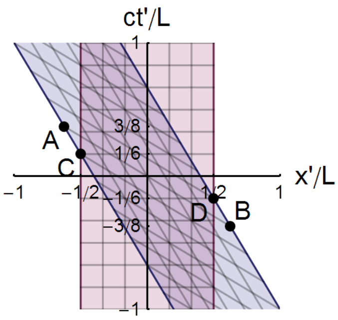
(d) Lorentz transform

$$\begin{cases} x' = \gamma(x - \beta t) \\ t' = \gamma(t - \beta x)' \end{cases}$$

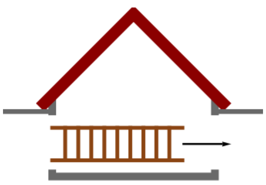
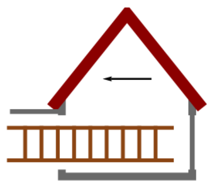
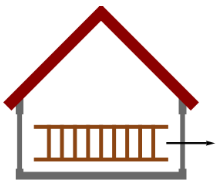
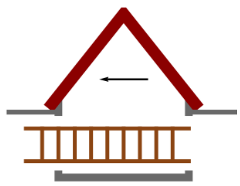
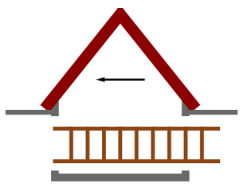
where $\beta = 3/5$ and $\gamma = 5/4$.

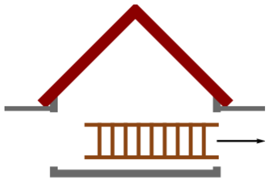
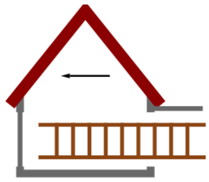
From the ladder frame: A(-5/8, 3/8), B(5/8, -3/8), C(-1/2, 1/6), D(1/2, -1/6).

(e) Space time diagram from the frame of the ladder.



(f) Scenario from the ladder frame compared with that from the garage frame.

Garage Frame		Ladder Frame	
The event C: the left end of the ladder passes by the front door.		The event B: back door closes and opens.	
The event A: front door closes and opens. The event B: back door closes and opens.		The event D: the right end of the ladder passes by the back door.	
		The event C: the left end of the ladder passes by the front door.	

The event D: the right end of the ladder passes by the back door. 	The event A: front door closes and opens. 

2. Charge Accelerated in Uniform Electric Field

A charge q will experience a force $F = Eq$ in a uniform electric field of the strength E. This law still holds true in relativistic cases (otherwise you can use it to determine the absolute velocity of a charge). Now you have learnt from the lecture, that the Newtonian law $F = \mathrm{d}(mv)/\mathrm{d}t$ is modified in relativity that the mass is no longer a constant,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

Let m_0 be the rest mass of the charge.

(a) Write down the equation of motion for this charge in the relativistic case.

(b) The charge is released at rest when $t = 0$. Determine the velocity of the charge as a function of time by solving the equation of motion. You are encouraged to plot the curve of $v(t)$ out.

(c) If you forget about relativity for the moment, and treat this problem with classical mechanics, then what is the velocity $v_{\mathrm{cl}}(t)$ as a function of time.

Note: In classical mechanics, a particle driven by a constant force will accelerate with constant acceleration. Thus after a while, the speed of the particle will exceed the speed of light. However in relativity this is not going to happen, since the particle will become more and more massive as its speed approaching to the speed of light, therefore it becomes harder and harder to keep the acceleration. In this problem, you will find that as $t \to \infty$, the velocity tends to (but will never reach) the speed of light. This is explains that why you cannot push a massive object by constant force to the speed of light. On the other hand, you will see that at the beginning when the velocity is small, the classical solution $v_{\mathrm{cl}}(t)$ is a good approximate of the relativistic one $v(t)$. This is why classical mechanics works well in our everyday life.

Solution:

(a) The equation of motion is

$$F = Eq = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right).$$

(b) Let $\beta = v/c$ and $a = Eq/m_0$, we can rearrange the equation into

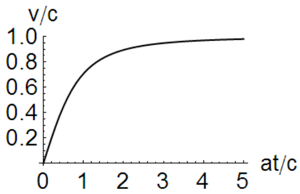
$$\frac{a}{c}\mathrm{d}t = \mathrm{d}\left(\frac{\beta}{\sqrt{1 - \beta^2}}\right).$$

Integrate on both sides, and take into account the initial condition that $\beta(t = 0) = 0$, we find

$$\frac{at}{c} = \frac{\beta}{\sqrt{1 - \beta^2}},$$

from which v can be solved,

$$v = \frac{c}{\sqrt{1 + c^2/(a^2 t^2)}}.$$



You will never accelerate the charge to exceed the speed of light.

(c) The classical solution is $v_{\mathrm{cl}} = at$, which is the first order approximation of the relativistic solution in the small t limit.

Chap. 16 Relativistic Energy and Momentum

1. Twin Paradox

You have learnt the twin paradox in the Lecture. Now let us look at an specific example and see how the paradox is solved.

A pair of twins Peter and Paul are born on the day when the spaceship Axiom is leaving for Gliese 876, which is a star 15 light years away from the Earth. Paul boards the Axiom and starts his journey, while Peter stays on the Earth. The spaceship Axiom travels at constant speed $0.6c$, (it is assumed to immediately attain its full speed upon departure), and return with the same speed as soon as it arrives at Gliese 876.

(a) How many years it takes for the Axiom to finish a round trip from the frame of the Earth?

(b) How far is Gliese 876 away from the Earth from the frame of the Axiom? Then how many years it takes for a round trip from the frame of the Axiom?

(c) When the Axiom returns, how old is Peter (the one staying on the Earth), and how old is Paul (the one travelling with the Axiom)?

As Peter grows up, he shows extraordinary talent in physics. He enters California Institute of Technology at the age of 16 (16 years after his birth) for his undergraduate study. He becomes a full professor at Princeton University when he is 34. Peter has made great achievements in physics, and is finally awarded the Nobel Prize when he is 50.

(d) Write down the space-time coordinates of the following events in the frame of the Earth, and mark them on the space-time diagram. (The even A is chosen as the origin).

(1) the event A: Peter and Paul are born,

(2) the event B: Peter enters California Institute of Technology,

(3) the event C: Peter becomes a full professor at Princeton,

(4) the event D: Peter is awarded the Nobel Prize,

(5) the event E: Paul arrive at the star Gliese 876.

Also draw the world line of Peter and Paul respectively on the same space-time diagram.

(e) For the events A, B and E, use Lorentz transformation to transform their space time coordinates from the frame of the Earth to that of the outward Axiom.

For the events C and D, we should no longer use the ordinary Lorentz transform to switch to Paul's frame. Because Paul is using two inertial frames: one on the way towards Gliese 876 and the other on the way back to the Earth. When the Axiom turns around at Gliese 876, Paul is switching from one frame to another, and thus his clock should be calibrated again by evens D and E.

(f) Write down the space-time coordinates of events D and E in the frame of the returning Axiom.

The Lorentz transform from the Earth frame to the returning Axiom frame has a shift in time due to the re-calibration of clock. It takes the following form,

$$\begin{cases} x' = \gamma\big(x + \beta(t - t_1)\big) \\ t' - t_2 = \gamma\big((t - t_1) + \beta x\big) \end{cases}.$$

(g) Use the space-time coordinates of events D and E you have obtained from both frames to determine the shifts t_1 and t_2. Use this shifted Lorentz transformation (known as Poincaré tansform) to transform the even C into the frame of returning Axiom.

(h) Mark the events A, B, C, D and E on the space-time diagram in Paul's frame. Then draw the world line of Peter and Paul respectively on the same space-time diagram. Try to explain the scenario from Paul's point of view.

Hint: Einstein, Born and Møller invoked gravitational time dilation to solve the twin paradox. The asymmetry between Peter and Paul lies in the effect of acceleration when the Axiom turns around. In general relativity, such an accelerating frame is equivalent to a inertial frame in a gravitational field. The clock farther away from the source of gravity tends to go faster. This effect becomes so remarkable at large distance that it is even possible for the time on the Earth to elapses 18 years within several minutes on the Axiom. This explains why evens B and C are simultaneous in Paul's frame.

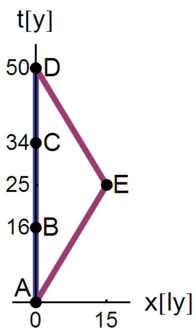
Solution:

(a) The distance between the Earth and Gliese 876 is $d = 15$ light years, and the velocity of the Axiom is $v = 3c/5$. So a round trip will take $t = 2d/v = 50$ years in the Earth's time.

(b) From the frame of the spaceship Axiom, both the Earth and the star Gliese 876 are moving relative to the spaceship at speed v during the trip. The distance between the Earth and Gliese 876 is $d\sqrt{1 - v^2/c^2} = 12$ light years (length contraction), for both the outward and return journeys. Each half of the journey takes $d/v = 20$ years, and the round trip takes $2 \times 20 = 40$ years.

(c) Peter will age 50 years, and Paul will age 40 years.

(d) The space-time diagram from the frame of the Earth.



From the frame of the Earth: A(0, 0), B(0, 16), C(0, 34), D(0, 50), E(15, 25).

(e) Lorentz transform with $\beta = 3/5$ and $\gamma = 5/4$,

$$\begin{cases} x' = \gamma(x - \beta t) \\ t' = \gamma(t - \beta x) \end{cases}.$$

From the frame of outward Axiom: A(0, 0), B(-12, 20), E(0, 20).

(f) From the frame of returning Axiom: D(0,40), E(0,20).

(g) Under the shifted Lorentz transform D: (0, 50)→(0, 40) and E: (15, 25)→(0, 20).

$$\begin{cases} 0 = \gamma\big(0 + \beta(50 - t_1)\big) \\ 40 - t_2 = \gamma\big((50 - t_1) + 0\big) \end{cases},$$

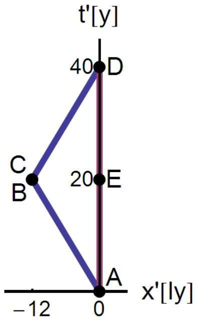
$$\begin{cases} 0 = \gamma\big(15 + \beta(25 - t_1)\big) \\ 20 - t_2 = \gamma\big((25 - t_1) + 15\beta\big) \end{cases}.$$

We can determine $t_1 = 50$, $t_2 = 40$. Therefore the transform becomes

$$\begin{cases} x' = \gamma\big(x + \beta(t - 50)\big) \\ t' - 40 = \gamma\big((t - 50) + \beta x\big) \end{cases},$$

from which we obtain the space-time coordinate of the event C in the frame of the returning Axiom: C(-12, 20).

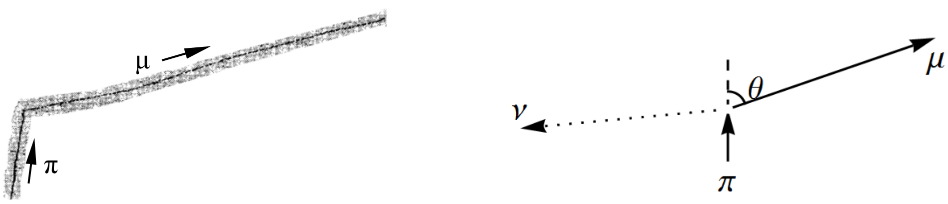
(h) The space-time diagram from the frame of Paul.



The event B and C coincide in Paul's frame. At the age of 20, Paul arrive at Gliese 876. As the Axiom turn around, Peter will suddenly grow from 16 to 34, (however Paul will not actually "see" this for lights needs time to travel). This is the effect of acceleration.

2. Determine the Rest Mass of π Messon

The pion (π meson) was first discovered on photographic plates in which there were tracks of muons (μ): some unknown particle had come in and decayed, and where its track ended, there was a new track coming off whose properties were found to be those of a muon. It was presumed that a neutrino went off in the opposite direction (leaving no track, because it is nertral). A typical decay process: $\pi^+ \to \mu^+ + \nu_\mu$. The tracks are shown in the following figure.



The rest energy of the muon was known to be 105.7MeV, and its kinetic energy was found from the properties of the track to be 4.5MeV. The neutrino has almost zero rest mass (<0.19MeV).

(a) Suppose that the pion decays at rest, how can we find the rest mass of the pion? Hints: first find the momentum of the muon and the neutrino.

(b) If the poin decays when it is still moving (with some momentum we do not know), we will not be able to determine its rest mass now. But from the photograph we can measure the angle between the muon track and the pion track, which gives $\theta = 71°$. With this experimental fact, can you give a lower bound of the pion rest mass (i.e. what is the rest mass at least)?

Solution:
(a) The energy of the muon is $E_\mu = m_\mu + T_\mu = 105.7\text{MeV} + 4.5\text{MeV} = 110.2\text{MeV}$, then the momentum of the muon is

$$p_\mu = \sqrt{E_\mu^2 - m_\mu^2} = 31.2\text{MeV}.$$

The neutrino will have the same amount of momentum but in opposite direction. $|p_\nu| = p_\mu = 31.2\text{MeV}$. Because the neutrino has zero rest mass, its energy is equal to its momentum $E_\nu = |p_\nu| = 31.2\text{MeV}$. Now the energy of the muon is 110.2MeV and the energy of the neutrino is 31.2MeV, so the total energy liberated in the reaction was 141.4MeV, which all comes from the rest mass of the pion:

$$m_\pi = E_\mu + E_\nu = 141.4\text{MeV}.$$

(b) The momentum of the muon is still $p_\mu = 31.2\text{MeV}$, but now we cannot determine the momentum of the neutrino because the pion does not decay at rest. However, we can at least pin down the horizontal component of the neutrino $p_{\nu x} = -p_{\mu x} = -p_\mu \sin\theta$, while we set the vertical component to be $p_{\nu y} = k$. Formally the energy of the neutrino is

$$E_\nu = \sqrt{p_{\nu x}^2 + p_{\nu y}^2} = \sqrt{p_\mu^2 \sin^2\theta + k^2}$$

Thus with respect to energy conservation, the energy of pion should be $E_\pi = E_\mu + E_\nu$. The momentum of pion can be determined from the conservation of vertical momentum $p_\pi = p_{\mu y} + p_{\nu y} = p_\mu \cos\theta + k$. Therefore the rest mass of pion can be given by

$$m_\pi^2 = E_\pi^2 - p_\pi^2 = \left(E_\mu + \left(p_\mu^2 \sin^2\theta + k^2\right)^{1/2}\right)^2 - \left(p_\mu \cos\theta + k\right)^2.$$

m_π^2 has a minimum when k is chosen appropriately, which, the choice, is given by the equation $\partial m_\pi^2/\partial k = 0$ with the solution

$$k = -\frac{p_\mu^2 \cos\theta \sin\theta}{\left(m_\mu^2 + p_\mu^2 \sin^2\theta\right)^{1/2}}.$$

Substitute the numbers, we arrive at $k = 2.7\text{MeV}$, and the rest mass of pion is at least 139.2MeV.

Chap. 17 Space-Time

1. Four-vector Form of Relativistic Mechanics

Many confusions in our study of relativity arise from the inappropriate human language. In the Lecture, we have leant that both the length contraction and the time dilation are "apparent" effects, if we perceive Lorentz transformation as a kind of rotation in space-time, then each length and each time interval we measured individually are just apparent length and apparent time, which are not fundamental properties of the object. Therefore in

relativity, we must always keep in mind that space and time is a unity. To carry through this idea, we insist the 4-vector notation in any possible cases, such as the space-time position - $x_\mu = (x, y, z, t)$ and momentum-energy $p_\mu = (p_x, p_y, p_z, E)$.
But when we come to the concept of velocity, a 3-vector defined as the time derivative of the space, we feel uncomfortable with the unequal treatment of space and time (why time should be the one to differentiate). It is necessary to extend velocity to a 4-vector form, known as the proper velocity, defined as the space-time x_μ differentiated by the proper time τ,

$$v_\mu = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}, \frac{\mathrm{d}z}{\mathrm{d}\tau}, \frac{\mathrm{d}t}{\mathrm{d}\tau}\right).$$

Here the original time derivative is replaced by that of proper time, which is the invariant interval between space time points,

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

(a) With the above definitions, write down the components of proper velocity for an object moving along x direction at the (traditional) speed v.
Note: it will be interesting to think of the physical meaning of the time component of the proper velocity. Is it the speed that an object "moving" in the time direction? If you look at the world line of a rest object, you will find that the rest object is indeed moving along the time direction from past to future. This is the famous saying that "the existence itself is already a kind of motion".
Along this line of thought, we can invent the proper acceleration $a_\mu = \mathrm{d}v_\mu/\mathrm{d}t$ and the proper force $F_\mu = \mathrm{d}p_\mu/\mathrm{d}\tau$, both are the 4-vector generalizations of their 3-vector forms. Then with m being the rest mass, our familiar old friend: Newton's Second Law $F_\mu = ma_\mu$ never breaks down in relativity. Hence we see that there is no need to introduce the "relativistic mass" if we define physical quantities "properly".
To give you a concrete example, let us consider a charge q of rest mass m moving in an uniform magnetic field of the strength B directed along z axis. How much proper force the magnetic field is to exert on the charge? In the classical electromagnetism, we have learnt the formula of Lorentz force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, or in its component form

$$\begin{cases} F_x = qv_yB \\ F_y = -qv_xB \end{cases}.$$

These two equations still holds in relativity when both F_μ and v_μ are the components of the proper 4-vectors, (with two additional ones $F_z = F_t = 0$).
(b) Write down the equation of motion for the charge in the magnetic field.
(c) With the initial condition given in the problem (a), find the proper velocity $v_\mu(\tau)$ as a function of proper time τ by solving the differential equations.
(d) Integrate $v_\mu(\tau)$ with respect to τ, you will obtain the equation for the world line of the charge. The charge will undergo circular motion. Determine the radius R of the circular orbit and the period T (in time not in proper time) for one rotation.
(e) Compare with the classical cycling radius and period you have learnt in the high school, what are the difference? Does this mean that the charge becomes more massive with higher speed?
Note: The concept of relativistic mass has caused a lot of controversies in the early years of relativity. Until today, it is still the most intensively attacked concept by the folk physicists.

This is why the contemporary physicists like Taylor and Wheeler[1] avoid using the this concept altogether: "The concept of "relativistic mass" is subject to misunderstanding. That's why we don't use it. First, it applies the name mass – belonging to the magnitude of a 4-vector – to a very different concept, the time component of a 4-vector. Second, it makes increase of energy of an object with velocity or momentum appear to be connected with some change in internal structure of the object. In reality, the increase of energy with velocity originates not in the object but in the geometric properties of space-time itself."

[1] Taylor, E. F., Wheeler, J. A. (1992), *Spacetime Physics, second edition*, New York: W.H. Freeman and Company.

Solution:

(a) From the definition of proper time, we learn that

$$\mathrm{d}\tau = \sqrt{1 - v^2/c^2}\,\mathrm{d}t = \mathrm{d}t/\gamma.$$

Thus using $\mathrm{d}t/\mathrm{d}\tau = \gamma$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma v,$$

similar for the y and z components, $\mathrm{d}y/\mathrm{d}\tau = \gamma \times 0 = 0$, $\mathrm{d}z/\mathrm{d}\tau = \gamma \times 0 = 0$. So the proper velocity is $v_\mu = (\gamma v, 0,0, \gamma)$.

(b) The equation of motion

$$m\frac{\mathrm{d}v_x}{\mathrm{d}\tau} = F_x = qv_y B,$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}\tau} = F_y = -qv_x B,$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}\tau} = F_z = 0,$$

$$m\frac{\mathrm{d}v_t}{\mathrm{d}\tau} = F_t = 0.$$

(c) Solution with initial condition that $v_\mu(\tau = 0) = (\gamma v, 0,0, \gamma)$ is

$$v_x = \gamma v \cos \omega\tau,$$
$$v_y = -\gamma v \sin \omega\tau,$$
$$v_z = 0,$$
$$v_t = \gamma.$$

Here $\omega = qB/m$.

(d) Integrate with respect to τ,

$$x = (\gamma v/\omega) \sin \omega\tau,$$
$$y = (\gamma v/\omega) \cos \omega\tau,$$
$$z = 0,$$
$$t = \gamma\tau.$$

Rewrite x and y as a function of time t,

$$x = \frac{\gamma v}{\omega} \sin \frac{\omega\tau}{\gamma},$$

$$y = \frac{\gamma v}{\omega} \cos \frac{\omega\tau}{\gamma}.$$

This shows that the charge undergoes a circular motion in the magnetic field with the cycling radius

$$R = \frac{\gamma v}{\omega} = \frac{mv}{qB\sqrt{1 - v^2/c^2}},$$

and the period

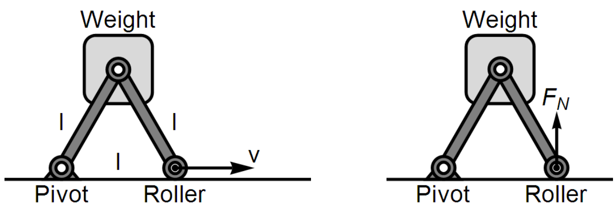
$$T = \frac{2\pi\gamma}{\omega} = \frac{2\pi m}{qB\sqrt{1 - v^2/c^2}}.$$

(e) Compare with the classical radius $R = mv/(qB)$ and $T = 2\pi m/(qB)$, it is just as if the mass has been enhance by a factor of γ. But this is just a dynamic effect, which does not imply any increase in the quantity of the constituent material of the object.

Chap. 18 Rotation in Two Dimensions

1. The Choo-choog Machine (IV): Torque and Angular Momentum

This will be the last time that we look at the choo-choog machine. A weight of mass M is held up by the elbow like pivoted rods of the lengths l. We have already known that to keep the roller moving with a constant speed v, there must be a horizontal force on the roller. However there must also be a vertical force F_N to support the roller on the floor. So we are going to find it.

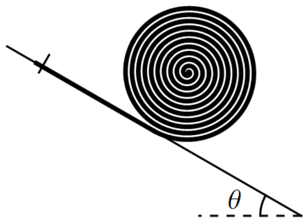


When the distance between the roller and the pivot is l,
(a) What is the angular momentum L of the weight with respect to the pivot?
(b) What is the changing rate of the angular momentum $\mathrm{d}L/\mathrm{d}t$?
(c) Consider the torque τ with respect to the pivot due to the forces exerted on the roller, and determine the force F_N.

Solution:
(a) $L = Mvl/\sqrt{3}$.
(b) $\mathrm{d}L/\mathrm{d}t = Mv^2/3\sqrt{3}$.
(c) $F_N = Mg/2 - Mv^2/3\sqrt{3}l$.

2. Flexible Tape Rolling Down

A flexible tape of the length l is tightly wound. It is then allowed to unwind as it rolls down a steep incline that makes an angle θ with the horizon. The upper end of the tape is tacked down. Find the time it takes for the tape to unwinds completely. Note: you may consider the thickness of the tape.



Solution: Take the contact point of the winded tape with the incline to be the reference point. Let M be the mass of the winded tape, and R be its radius. With respect to the reference point, the tape experiences the torque due to gravity

$$\tau = MgR\sin\theta,$$

which leads to the angular acceleration $\ddot{\phi}$ if the thickness d of the tape is neglectable

$$\tau = I\ddot{\phi},$$

where $I = (1/2)MR^2 + MR^2 = (3/2)MR^2$ is the moment of inertia. Putting together the above equations yields

$$\frac{2}{3}g\sin\theta = \ddot{\phi}R$$

Since the acceleration of the center of the tape is $\ddot{x} = \ddot{\phi}R = (2/3)g\sin\theta$, the time for unwinding is therefore

$$T_0 = \sqrt{2l/\ddot{x}} = \left(\frac{3l}{g\sin\theta}\right)^{1/2}.$$

If the thickness d of the tape is comparable to the size R, we may introduce the mass density σ such that $M = \sigma\pi R^2$. We are facing the problem of the dynamics with varying moment of inertia. According to the Law of Angular Momentum,

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t},$$

where the torque is still provided by gravity,

$$\tau = MgR\sin\theta = \pi\sigma gR^3\sin\theta,$$

while the angular momentum L is given by

$$L = I\dot{\phi} = \frac{3}{2}MR^2\dot{\phi} = \frac{3\pi\sigma}{2}R^4\dot{\phi}.$$

Therefore the equation of motion reads

$$gR^3\sin\theta = \frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(R^4\dot{\phi}\right).$$

There is a geometric constrain that the volume of the tape must conserve, i.e.

$$xd + \pi R^2 = \text{const.}$$

Differentiate with respect to time once,

$$\dot{x}d + 2\pi R\dot{R} = 0,$$

while the velocity $\dot{x}$ is related to the angular velocity $\dot{\phi}$ by $\dot{x} = R\dot{\phi}$, so we find

$$\dot{\phi} = -\frac{2\pi}{d}\dot{R}.$$

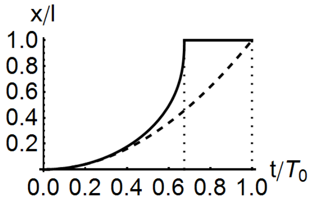
The physical meaning is that R will reduce by the thickness d when ϕ makes a round. Substituting into the equation of motion yields

$$gR^3 \sin\theta = -\frac{3\pi}{d}\frac{\mathrm{d}}{\mathrm{d}t}\left(R^4\dot{R}\right),$$

or written as (with $T_0^2 = 3l/g\sin\theta$ and $\pi R_0^2 = ld$)

$$R\ddot{R} + 4\dot{R}^2 + \left(\frac{R_0}{T_0}\right)^2 = 0.$$

R_0 is the initial radius of the winded tape and T_0 is the time to unwind if thickness of the tape is not taken into account. If the thickness were not considered, the equation of motion for R would be $R\ddot{R} + \dot{R}^2 + (R_0/T_0)^2 = 0$, which missed the factor 4 in the front of the damping term, due to the absence of the varying moment of inertial effect. This effect may be neglected when the tape starts to unwind, for the small $\dot{R}/R$. But it becomes prominent at the end of unwinding when $R \to 0$. The reducing moment of inertial speed up the unwinding and leads to a shorter unwinding time. The following figure shows the difference of the two dynamics: no thickness (dashed line), thickness taken into account (solid line). They behaves the same at the beginning of unwinding, but deviates latter when R gets small.



Given the initial condition $R(t=0) = R_0$, the solution shows that $R(t)$ drops to zero when $t = 0.675T_0$. So if the thickness is taken into account, the unwinding time is reduced to

$$T = 0.675T_0,$$

and this result is independent of the thickness d.

3. Hidden Angular Momentum in the Magnetic Field

Momentum $\boldsymbol{p}$ and angular momentum $\boldsymbol{L}$ are closely related by $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$. In the homework problem 3 of Chap. 10, we have discovered the potential momentum (i.e. the electromagnetic momentum) hidden in the magnetic field. Now it is time to generalize the potential momentum to the *potential angular momentum* (i.e. the electromagnetic angular momentum),

$$\boldsymbol{L}_{EM} = q\boldsymbol{r}\times\boldsymbol{A}.$$

Remember the vector potential $\boldsymbol{A}$ can be written as (by symmetric gauge)

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r},$$

for uniform magnetic field $\boldsymbol{B}$.

(a) Consider a charge q cycling in a uniform magnetic field B on a circular orbit of radius r (magnetic field perpendicular to the orbital plane). What is the strength of vector potential A on the orbit? How much potential angular momentum will the charge processes on the orbital?

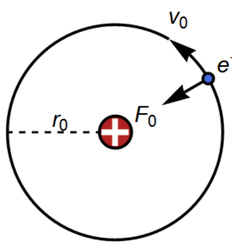
To give an example of the physical consequence of the potential angular momentum, let us look at the *Landau diamagnetic effect*. Diamagnetism is a very weak form of magnetism that is only exhibited in the presence of an external magnetic field. It is the result of changes in the orbital motion of electrons due to the external magnetic field. The induced magnetic moment is very small and in a direction opposite to that of the applied field. Diamagnetism is found in all materials; however, because it is so weak it can only be observed in materials that do not exhibit other forms of magnetism.

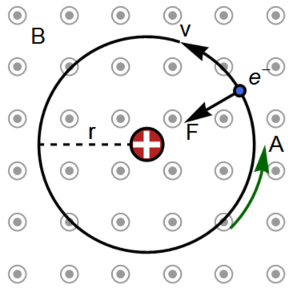
As a simple model, we consider an representative atom in the material. There is one electron orbiting around the nuclear on circular orbit of radius r_0. The electron charge is $-e$ and the nuclear charge is e (like a hydrogen atom). The electron mass is m.

(b) What is the orbital velocity v_0 of the electron? What about the cycling period?

The magnetic moment of a current circuit is defined as $M = IS$ with its direction given by right-hand rule, where I is the current running in the orbit and S is the area enclosed by the orbit.

(c) Then what is the magnetic moment due to the electron's orbital motion around the nuclear?





We adiabatically apply the magnetic field B perpendicular to the electron orbital plane. Both the radius of the orbit and the velocity of the electron may change in the presence of the magnetic field. But the shape of the orbit still remains circular.

(d) What is the relation between the new orbital radius r and velocity v so as to keep the electron on a circular orbit? Hints: you need to take into account both the Coulomb force and Lorentz force.

(e) What is the (kinetic) angular momentum of the electron (with respect to the nuclear) before and after applying the magnetic field (in terms of r_0, v_0, r and v)? How much potential angular momentum does the electron have in the magnetic field?

(f) The conservation of angular momentum states that the total angular momentum keeps the same before and after applying the magnetic field. Use this law to relate r and v to r_0 and v_0.

In the week field limit $B \to 0$, the orbital radius r and velocity v does not change much from r_0 and v_0, thus they can be expanded around r_0 and v_0 by $r = r_0 + \delta r$ and $v = v_0 + \delta v$. Treat δr and δv as very small quantities of the same order as B.

(g) Determine δr and δv by first order perturbation. Calculate the corresponding change in the magnetic momentum. Hint: Taylor expand both sides of your equations to the first order of δr, δv and B, and solve the linear equation.

The *magnetic susceptibility* for the single atom is defined as

$$\chi = \mu_0 \frac{\partial M}{\partial B},$$

where $\mu_0 = 4\pi \times 10^{-7} \mathrm{N} \cdot \mathrm{A}^{-2}$ is the magnetic constant (the vacuum permeability).

(h) According to this definition, find the magnetic susceptibility of the atom.

The negative susceptibility means the applied magnetic field will induce a magnetic moment in the opposite direction, which in turn weakens the magnetic field in the material, hence the diamagnetism.

(i) Make a rough estimate of the diamagnetic susceptibility of N_2 gas in the air (the overall susceptibility per unit volume), given the molecule size of N_2 $r_0 = 155\text{pm}$ and its mass density in the atmosphere $\rho = 0.91\text{g} \cdot \text{m}^{-3}$.

Solution:
(a) $A = Br/2$, so angular momentum is $qAr = qBr^2/2$.
(b) Original equation

$$\frac{mv_0^2}{r_0} = \frac{e^2}{4\pi\epsilon_0 r_0^2}.$$

Thus the velocity

$$v_0 = \left(\frac{e^2}{4\pi\epsilon_0 m r_0}\right)^{1/2},$$

and the period

$$T = \frac{2\pi r_0}{v_0} = 2\pi\left(\frac{4\pi\epsilon_0 m}{e^2}\right)^{1/2} r_0^{3/2}.$$

(c) Magnetic moment

$$M_0 = -\frac{e}{2} r_0 v_0 = -\frac{e^2}{2}\left(\frac{r_0}{4\pi\epsilon_0 m}\right)^{1/2}.$$

(d) On the circular orbital,

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} + evB.$$

(f) Angular momentum conservation

$$mv_0 r_0 = m\, v\, r - \frac{1}{2} e r^2 B.$$

(g) For small B,

$$\frac{2mr_0 v_0 \delta v - mv_0^2 \delta r}{r_0^2} = -\frac{2e^2 \delta r}{4\pi\epsilon_0 r_0^3} + ev_0 B,$$

$$mr_0 \delta v + mv_0 \delta r - \frac{1}{2} e r_0^2 B = 0.$$

The solution is $\delta r = 0$ (the variation is in the second order) and

$$\delta v = \frac{e r_0 B}{2m}.$$

Therefore the magnetic moment

$$M = -\frac{e}{2} r v = M_0 - \frac{e^2 r_0^2 B}{4m}.$$

(h) Thus we see the diamagnetic effect, with magnetic susceptibility

$$\chi = -\frac{\mu_0 e^2 r_0^2}{4m}$$

(i) For N_2 gas

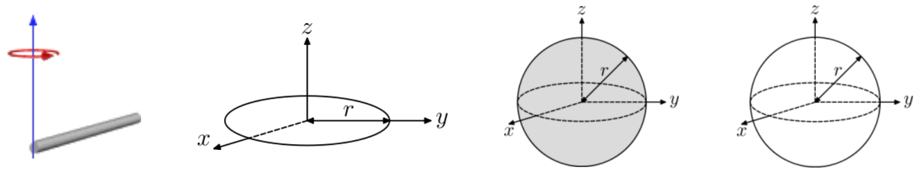
$$\chi = -\frac{\mu_0 e^2 r_0^2}{4m}\frac{N_A \rho}{\mu_{N_2}} \simeq -4 \times 10^{-9}.$$

$e = 1.6 \times 10^{-19}\text{C}$, $m = 9.1 \times 10^{-31}\text{kg}$, $N_A = 6.0 \times 10^{23}\text{mol}^{-1}$, $\mu_{N_2} = 28\text{g} \cdot \text{mol}^{-1}$.
The actual magnetic susceptibility for N_2 gas is -5.06×10^{-9}.

Chap. 19 Center of Mass; Moment of Inertia

1. Finding the Moment of Inertia

(a) Rod of length L and mass m. Find the moment of inertia about the axis of rotation at the end of the rod.
(b) Thin circular disk of radius r and mass m. Find the moments of inertia I_x, I_y and I_z about the three axes respectively.
(c) Solid sphere of radius r and mass m. Find the moment of inertia about the axis through the center.
(d) Hollow sphere of radius r and mass m. Find the moment of inertia about the axis through the center.

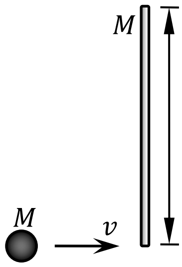


(a) (b) (c) (d)

Solution:
(a) $I = mL^2/3$.
(b) $I_x = I_y = mr^2/4,\ I_z = mr^2/2$.
(c) $I = 2mr^2/5$.
(d) $I = 2mr^2/3$.

2. Flying Putty Sticks to a Rod (I)

A thin rod of mass M and length L rests on a horizontal frictionless surface. A small piece of putty, also of mass M, and with velocity v directed perpendicularly to the rod, strikes one end and sticks, making an inelastic collision of very short duration.



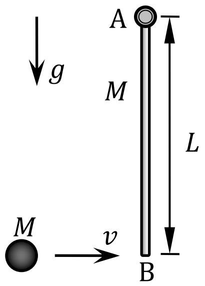
(a) What is the velocity of the center of mass of the putty-rod system before and after the collision.
(b) What is the angular momentum of the putty-rod system about its center of mass just before the collision.
(c) What is the angular velocity (about the center of mass) just after the collision?
(d) How much kinetic energy is lost in the collision?

Solution:

(a) $v_c = v/2$.
(b) $L = MvL/4$.
(c) $\omega = 6v/5L$.
(d) $\Delta E = mv^2/10$.

3. Flying Putty Sticks to a Rod (II)

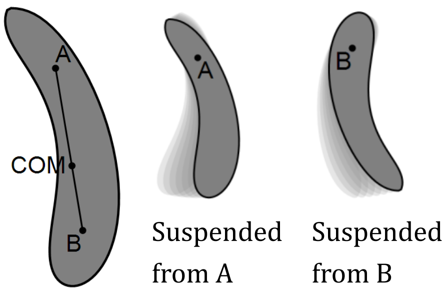
A thin uniform rod AB of mass M and length L is free to rotate in a vertical plane about a horizontal axle at the end A. A piece of putty also of mass M is thrown with velocity v horizontally at the lower end B while the bar is at rest. The putty sticks to the bar. What is the minimum velocity of the putty before impact that will make the bar rotate *all the way around* A?



Solution: $\sqrt{8gL}$

4. Physical Pendulum

The physical pendulum in the following figure has two possible pivot points A and B. Point A has a fixed position, and point B is adjustable along the line through point A and the center of mass (COM) of the pendulum. The period of the pendulum when suspended from A is found to be T. The pendulum is then reversed and suspended from B, which is moved until the pendulum again has the period T. If the distance between A and B is found to be l for equal periods, what is the freefall acceleration g at the pendulum's location?



Note: this method can be used to measure g without knowledge of the rotational inertia of the pendulum or any of its dimensions except l.

Solution: Let l_A be the distance between A and COM, and l_B be the distance between B and COM. The equation of motion reads,

$$(I_0 + ml_A^2)\ddot{\theta} = mgl_A\theta,$$
$$(I_0 + ml_B^2)\ddot{\theta} = mgl_B\theta.$$

Now the equal period condition requires,

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{mgl_A}{I_0 + ml_A^2} = \frac{mgl_B}{I_0 + ml_B^2}.$$

Thus

$$\frac{4\pi^2}{T^2} = \frac{mg(l_A - l_B)}{(I_0 + ml_A^2) - (I_0 + ml_B^2)} = \frac{g}{l_A + l_B},$$

so (with $l = l_A + l_B$)

$$g = \frac{4\pi^2 l}{T^2}$$

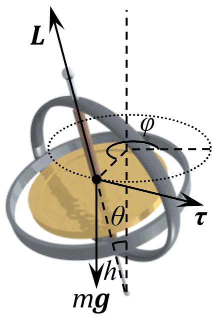
Chap. 20 Rotation in Space

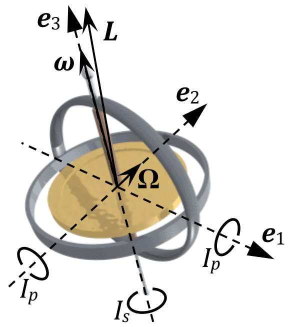
1. Precession and Nutation of Gyroscope

A gyroscope exhibits a number of behaviors including precession and nutation. The fundamental equation describing the behavior of the gyroscope is:

$$\boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t},$$

where the vectors $\boldsymbol{\tau}$ and $\boldsymbol{L}$ are the torque on the gyroscope and its angular momentum respectively. The torque is supplied by a couple of forces: gravity acting downwards on the device's centre of mass, and an equal force acting upwards to support one end of the device.





(a) Consider the gyroscope of mass m with its spinning axis inclined from the vertical direction by the angle θ. If its center of mass is h distant away from the supporting end, what is the torque exerted on the gyroscope (with respect to the supporting end)?

(b) If the magnitude of the angular momentum of the spinning gyroscope is L, find the angular velocity of the precession Ω_P.

Due to the precession motion, the angular momentum does not exactly falls along the direction of the spinning axis. To show this, let us break the angular velocity vector into components $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ perpendicular and parallel to the spinning wheel of the gyroscope. The physical meaning of $\boldsymbol{\omega}$ is the angular velocity of the wheel, while $\boldsymbol{\Omega}$ stands for the angular velocity of the axis (as the axis rotating about the supporting end). With this decomposition, the precession and the nutation motion is separated from the general motion of the gyroscope.

$\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the unit vectors along the three principle axes (one of which is the spinning axis and the other two lies in the plane of the wheel). The moments of inertia about the principle axes are different: I_s is the moment of inertia about the spinning axis, and I_p is moment of inertia about either of the other two perpendicular principal axes. The two I_p's should be almost the same, due to the approximate symmetry of the gyroscope.

(c) So what is the angular momentum $\boldsymbol{L}$ in terms of the angular velocities $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$? Is it parallel to the spinning axis?

For a realistic gyroscope, the precession or nutation motion of the axis is much slower compared to the rapid spinning of the wheel about the axis. This is to say that ω is usually orders of magnitude greater than Ω, and is almost a constant independent of the motion of the axis.

(d) By carry out the time derivative $\mathrm{d}\boldsymbol{L}/\mathrm{d}t$, prove that the equation of motion becomes,

$$\boldsymbol{\tau} = I_s \boldsymbol{\Omega} \times \boldsymbol{\omega} + I_p \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}.$$

Hints: the vector $\boldsymbol{\omega}$ only changes in its direction, and its magnitude is kept constant.

If we denote the component of angular momentum along the spinning axis by $\boldsymbol{L}_0 = I_s \boldsymbol{\omega}$, then the above equation can be casted into

$$\boldsymbol{\tau} = \boldsymbol{\Omega} \times \boldsymbol{L}_0 + I_p \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}.$$

We have learned from the Lecture that the first term on the right-hand-side describes the precession. The extra second term on the right-hand-side will give rise to the nutation.

However we are not very interested in the dynamics of $\boldsymbol{\Omega}$ itself. What we concern more is the dynamics of the axis of the gyroscope. Let us define the unit vector of the axis by $\boldsymbol{n} \equiv \boldsymbol{\omega}/\omega$. The axis rotates about the supporting end with the angular velocity $\boldsymbol{\Omega}$, so that

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{n}.$$

(e) Show that $\boldsymbol{\Omega}$ can be "solved" from the above relation,

$$\boldsymbol{\Omega} = \boldsymbol{n} \times \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}.$$

(f) Substituting into the equation of motion for $\boldsymbol{\Omega}$, show that $\boldsymbol{n}$ satisfies the following equation of motion,

$$I_p \frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}t^2} = \boldsymbol{\tau} \times \boldsymbol{n} - \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} \times \boldsymbol{L}_0.$$

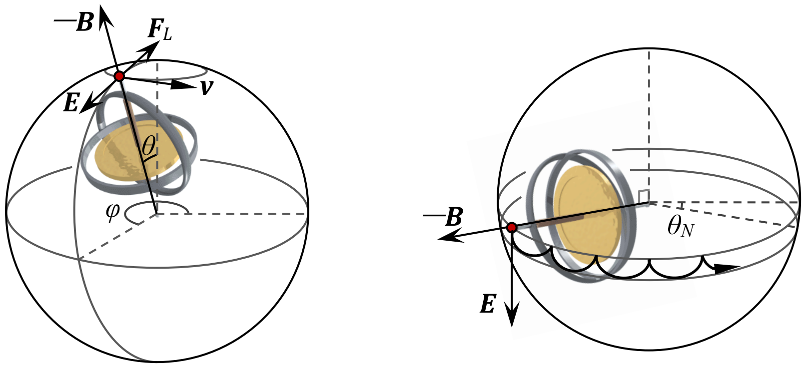
However, we are not going to solve this equation directly, because it has no analytic solution in general, and we do not wish to see our homework becoming an absolute miracle. Let us make some physical interpretations. Recall the equation of motion for a charge in the electromagnetic field, which reads $m\boldsymbol{a} = \boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}$, or written as

$$\frac{m}{q}\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \boldsymbol{E} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \times \boldsymbol{B}.$$

Compare with the equation we have just obtained for $\boldsymbol{n}$, we can make the following analog,

Axis Dynamics	Charge Dynamics
Position of the free end: $\boldsymbol{n}$	Position of the charge: $\boldsymbol{x}$
Moment of inertia: I_p	Inertia mass per charge: m/q
Force projected to the sphere: $\boldsymbol{\tau} \times \boldsymbol{n}$	Electric field: $\boldsymbol{E}$
Angular momentum of spinning: $-\boldsymbol{L}_0$	Magnetic field: $\boldsymbol{B}$

The dynamics of the free end of the gyroscope is just like a charge moving in a electric field $\boldsymbol{E} = \boldsymbol{\tau} \times \boldsymbol{n}$ and a magnetic field $\boldsymbol{B} = -\boldsymbol{L}_0$. The only difference is that the motion of is constrain on a unit sphere (known as the *Bloch sphere*), which is 2-dimensional *curved space*.



(g) Calculate the magnitude and describe the direction of $\boldsymbol{E}$ and $\boldsymbol{B}$ for all range of θ and ϕ. If the electric force is balanced by the magnetic force, the gyroscope will not fall but precess with steady velocity $\mathrm{d}\boldsymbol{n}/\mathrm{d}t$. For a given latitude θ, determine the precession speed $|\mathrm{d}\boldsymbol{n}/\mathrm{d}t|$. Is it the same as the precession speed Ω_P you have obtained in the problem (b)?

Now let us understand the physical meaning of this electromagnetic field. The gravity always wants to lay down the gyroscope. Projecting the gravitational force onto the Bloch sphere leads to the electric field. So the electric field reflects the tendency of falling of the gyroscope. Then what about the magnetic field? As the wheel of the gyroscope starts to spin, the magnetic field will be set up over the Bloch sphere as if there were a magnetic monopole lying at the center of the sphere. The strength of the magnetic field is proportional to the angular velocity ω of the wheel. Therefore the magnetic field is an effect brought by the spinning of the wheel. Without the spinning, the gyroscope will surely fall as any of other objects in the gravitational field. But with the spinning, the falling of the gyroscope leads to the acceleration in the precession direction. This exactly mimic the effect of Lorentz force that the velocity in one direction generates the force in the perpendicular direction. It is just because of this magnetic field effect that the gyroscope could start precession after release.

Now it is easier to understand nutation. If the spinning gyroscope is hold still horizontally ($\theta = \pi/2$) and then release, it is just like releasing a rest charge in the electromagnetic field. Since the electric and magnetic field are perpendicular to each other, the motion of the charge will trace out a cycloid (the curve that is generated by a point on the circumference of a circle as it rolls along a straight line). To constrain the motion on the Bloch sphere, the cycloid should be bent around the equator, thus giving the track of nutation motion.

(h) According to the above analysis, calculate the amplitude (the nutation angle θ_N) and the period of nutation, in the large ω limit.

Solution:

(a) $\tau = mgh\sin\theta$.

(b) $\Omega_p = mgh/L$.

(c) $\boldsymbol{L} = I_s\boldsymbol{\omega} + I_p\boldsymbol{\Omega}$.

(d) The equation of motion gives

$$\boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = I_s\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + I_p\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}.$$

The change of the direction of $\boldsymbol{\omega}$ is due to the rotation of the spinning axis about $\boldsymbol{\Omega}$, so

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{\omega},$$

hence

$$\boldsymbol{\tau} = I_s \boldsymbol{\Omega} \times \boldsymbol{\omega} + I_p \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}.$$

(e) Crossed by $\boldsymbol{n}$ from left

$$\boldsymbol{n} \times \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \boldsymbol{n} \times (\boldsymbol{\Omega} \times \boldsymbol{n}) = \boldsymbol{\Omega}(\boldsymbol{n} \cdot \boldsymbol{n}) - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{\Omega}) = \boldsymbol{\Omega},$$

for $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ and $\boldsymbol{n} \cdot \boldsymbol{\Omega} = 0$ (by definition of $\boldsymbol{\Omega}$: the component of angular velocity perpendicular to the axis).

$(\boldsymbol{n} \cdot \boldsymbol{\Omega}) \neq 0$!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

(f) By substitution,

$$\boldsymbol{\tau} = \left(\boldsymbol{n} \times \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}\right) \times \boldsymbol{L}_0 + I_p \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{n} \times \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}\right) = -\boldsymbol{n}\left(\boldsymbol{L}_0 \cdot \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}\right) + \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}(\boldsymbol{n} \cdot \boldsymbol{L}_0) + I_p \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} \times \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} + I_p \boldsymbol{n} \times \frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}t^2}$$
$$= L_0 \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} + I_p \boldsymbol{n} \times \frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}t^2}.$$

Crossed by $\boldsymbol{n}$ from right

$$\boldsymbol{\tau} \times \boldsymbol{n} = L_0 \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} \times \boldsymbol{n} + I_p \left(\boldsymbol{n} \times \frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}t^2}\right) \times \boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} \times \boldsymbol{L}_0 + I_p \frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}t^2}.$$

Hence the equation of motion

$$I_p \frac{\mathrm{d}^2\boldsymbol{n}}{\mathrm{d}t^2} = \boldsymbol{\tau} \times \boldsymbol{n} - \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} \times \boldsymbol{L}_0.$$

(g) The effective electromagnetic field is $E = mgh \sin\theta$, $B = I_s \omega$. Corresponding to the precession velocity (by $v = E/B$)

$$\Omega_p = \frac{v}{\sin\theta} = \frac{mgh}{L_0}.$$

Some may take v as Ω_p and miss the denominator $\sin\theta$.

(h) Nutation angle

$$\theta_N = \frac{2mghI_p}{I_s^2 \omega^2}.$$

Nutation period

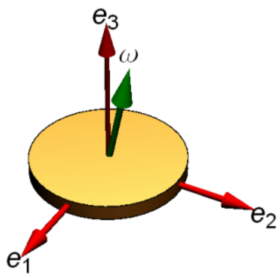
$$T_N = \frac{2\pi}{\omega} \frac{I_p}{I_s}.$$

Some miss the factor 2 in θ_N.

2. Torque-Free Precession of a Disk

Torque-free precession occurs when the axis of rotation differs slightly from an axis about which the object can rotate stably: a maximum or minimum principal axis. For example, when a plate is thrown, the plate may have some rotation around an axis that is not its axis of symmetry. This occurs because the angular momentum $\boldsymbol{L}$ (the loner arrow) is constant in absence of torques. Therefore it will have to be constant in the external reference frame, but the moment of inertia tensor $\mathbf{I}$ is non-constant in this frame because of the lack of symmetry. Therefore the spin angular velocity vector $\boldsymbol{\omega}_s$ (the shorter arrow) about the spin axis will have to evolve in time so that the matrix product $\boldsymbol{L} = \mathbf{I}\boldsymbol{\omega}_s$ remains constant.

Among the three principle axes of a disk, one ($\boldsymbol{e}_3$) is perpendicular to the disk, and the other two ($\boldsymbol{e}_1$ and $\boldsymbol{e}_2$) lies in the plane of the disk. Let I_s be the moment of inertia about the $\boldsymbol{e}_3$ axis, and I_p be moment of inertia about either of the other two principal axes. They should be the same, due to the symmetry of the disk.



At any instance, the three principle axes span a rotating frame of reference (in which the disk looks static for the moment) known as the frame of principle axes.

(a) If the angular velocity in the frame of principle axes is $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \omega_3\boldsymbol{e}_3$, what is the angular momentum $\boldsymbol{L} = L_1\boldsymbol{e}_1 + L_2\boldsymbol{e}_2 + L_3\boldsymbol{e}_3$? (Give the expression for L_1, L_2 and L_3)

Because the frame of principle axes is rotating with the angular velocity $\boldsymbol{\omega}$, its base vector will change with respect to time according to $\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i$ in the external inertial reference frame. Since there is no torque exerting on the disk, we should have

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{\tau} = 0$$

in the external frame.

(b) Show that the above equation leads to Euler's equation in the frame of principle axes,

$$\frac{\tilde{\mathrm{d}}\boldsymbol{L}}{\tilde{\mathrm{d}}t} + \boldsymbol{\omega} \times \boldsymbol{L} = 0,$$

where the derivative $\tilde{\mathrm{d}}/\tilde{\mathrm{d}}\mathrm{t}$ only applies to the components of the vector $\boldsymbol{L}$ in the frame of principle axes.

The torque-free precession will occur if the angular velocity $\boldsymbol{\omega}$ neither falls on the $\boldsymbol{e}_3$ axis nor lies in the plane of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

(c) Let α be the angle between $\boldsymbol{\omega}$ and $\boldsymbol{e}_3$, show that the angular velocity of torque-free precession is

$$\Omega = \omega\left(I_s/I_p - 1\right)\cos\alpha.$$

Solution:

In the frame of principle axes

(a) $\boldsymbol{L} = I_p\omega_1\boldsymbol{e}_1 + I_p\omega_2\boldsymbol{e}_2 + I_s\omega_3\boldsymbol{e}_3$.

(c) Thus

$$\boldsymbol{\omega} \times \boldsymbol{L} = \left(I_s - I_p\right)\omega_2\omega_3\boldsymbol{e}_1 + \left(I_p - I_s\right)\omega_1\omega_3\boldsymbol{e}_2.$$

Substitute into Euler's equation

$$I_p\dot{\omega}_1 + (I_s - I_p)\omega_2\omega_3 = 0,$$
$$I_p\dot{\omega}_2 + (I_p - I_s)\omega_1\omega_3 = 0,$$
$$I_p\dot{\omega}_3 = 0.$$

From the last equation, $\omega_3 = \omega\cos\alpha$ is a constant. Then the first two equations become

$$I_p\dot{\omega}_1 + \omega(I_s - I_p)\cos\alpha\,\omega_2 = 0,$$
$$I_p\dot{\omega}_2 - \omega(I_s - I_p)\cos\alpha\,\omega_1 = 0.$$

By introducing $\Omega = \omega(I_s/I_p - 1)\cos\alpha$, they can be decoupled into

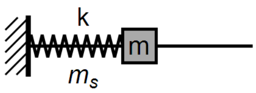
$$\ddot{\omega}_1 = -\Omega^2\omega_1,$$
$$\ddot{\omega}_2 = -\Omega^2\omega_2.$$

This shows that $\boldsymbol{\omega}$ (and hence $\boldsymbol{L}$) will precession around $\boldsymbol{e}_3$ in the frame of principle axes. Therefore in the external frame, it is the principle axes $\boldsymbol{e}_3$ that will precession around $\boldsymbol{L}$ with the same frequency Ω.

Chap. 21 The Harmonic Oscillator

1. Massive Spring (I): Effective Mass

If the mass of a spring m_s is not negligible but is small compared to the mass m of the object attached to it. Let k be the (static) force constant, what is the period T of motion for the mass m?



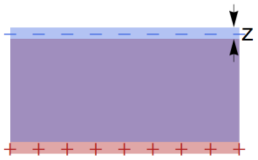
Note: Take into account the mass of the spring, the oscillation period is longer than $T_0 = 2\pi\sqrt{m/k}$. We can introduce the effective mass m^* from $T = 2\pi\sqrt{m^*/k}$. We can see the massive spring effectively increase the mass of the oscillator.

Solution:

$$T = 2\pi\sqrt{\frac{m + m_{s/3}}{k}}.$$

2. Plasma Oscillation (I): Plasma Frequency

Plasma oscillation means a kind of collective oscillation of charged particles in the material, typically metals. When an external field is switched on, all electrons in the metal will displace a distance z relative to the positive ions. There will be the positive and negative charge accumulations on the opposite sides of the metal, as shown in the following figure.



(a) Given the electron number density n in the metal, what is the surface charge density (charge per surface area) on both sides?

The electric field induced by the charged layers balances the external field such that the system is in equilibrium. When the external field is switched off, the balance is broken, and the electrons are to move under the influence of the electric field of the charged layers.

(b) How much electric force will be exerted on the electron inside the metal?

It turns out that the electric force is a restoring force that leads to the collective oscillation of electrons around their equilibrium positions.

(c) Given the effective mass m^* and electric charge e of the electron, what is its oscillating (circular) frequency ω_p?

Note: ω_p is known as the plasma frequency.

Solution:

(a) $\sigma = enz$.

(b) $E = \sigma/\epsilon_0 = enz/\epsilon_0$, and $F = eE = e^2nz/\epsilon_0$

(c) The electric force is a restoring force, which leads to oscillation

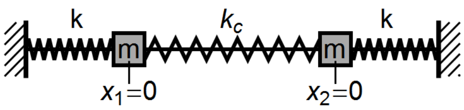
$$m^*\ddot{z} = -\frac{ne^2}{\epsilon_0}z,$$

so the plasma frequency

$$\omega_p = \left(\frac{ne^2}{\epsilon_0 m^*}\right)^{1/2}.$$

3. Coupled Oscillator

We have two spring and mass oscillators with stiffness constants k, and masses m coupled together by a third spring of stiffness k_c. It should be noted that $k_c \ll k$, so we can say that our system is weakly coupled. Let us ignore any damping forces, such as friction or air-resistance. And the motion of the masses is restricted to one dimension. We will make the same assumption that the springs obey Hooke's law and thus exert a linear restoring force on the masses.



Now we would like to find the general solution for the motion of the two masses. Their positions are given by $x_1(t)$ and $x_2(t)$.

(a) Write down the equations of motion for the two masses by applying Newton's second law.

There are two normal modes of vibration for this system. The symmetric normal mode occurs when both masses are displaced an equal amount in the same direction and released from rest. The anti-symmetric mode occurs when both masses are displaced an equal distance from their equilibrium positions but in opposite directions. These two normal modes oscillates with different frequency and the general behavior of the coupled oscillator is a superposition of the two modes.

(b) Decouple the equation of motions by transforming to normal coordinates q_1 and q_2:

$$\begin{cases} q_1 = x_1 + x_2 \\ q_2 = x_1 - x_2 \end{cases}.$$

q_1 gives the displacement of the symmetric mode, and q_2 is the displacement of the anti-symmetric mode. What are the equations of motion for q_1 and q_2? What are the frequencies of the two normal modes respectively?
(c) Let us assume that we displace the left mass by a distance $x_1(0) = A$, and the right mass by a distance $x_2(0) = B$, and release the masses from rest $x_1'(0) = 0$, $x_2'(0) = 0$. Give the solution of $x_1(t)$ and $x_2(t)$ by superpose the normal modes.
(d) With $A = 1$ and $B = 0$, plot $x_1(t)$ and $x_2(t)$ as functions of t.
We can now clearly see that both the masses display the beating phenomenon.
(e) Find the beating frequency.
Notice that the peaks of x_1 correspond to the troughs of x_2. The energy of the system is continuously transferred between the two masses, while the total energy of the system remains constant.

Solution:
(a) Equation of motion for x_1 and x_2

$$m\ddot{x}_1 = -kx_1 + k_c(x_2 - x_1),$$
$$m\ddot{x}_2 = -kx_2 + k_c(x_1 - x_2).$$

(b) Equation of motion for q_1 and q_2

$$m\ddot{q}_1 = -kq_1,$$
$$m\ddot{q}_2 = -(k + 2k_c)q_2.$$

Corresponds to the frequencies

$$\omega_1 = \sqrt{k/m}.$$
$$\omega_2 = \sqrt{(k + 2k_c)/m}.$$

(c) Solution

$$x_1 = \frac{A+B}{2}\cos\omega_1 t + \frac{A-B}{2}\cos\omega_2 t.$$
$$x_2 = \frac{A+B}{2}\cos\omega_1 t - \frac{A-B}{2}\cos\omega_2 t.$$

(d)

$$x_1 = \frac{1}{2}(\cos\omega_1 t + \cos\omega_2 t) \cong \cos\frac{k_c t}{2\sqrt{km}}\sin\sqrt{k/m}t.$$
$$x_2 = \frac{1}{2}(\cos\omega_1 t - \cos\omega_2 t) \cong \sin\frac{k_c t}{2\sqrt{km}}\sin\sqrt{k/m}t.$$

(e) Beating frequency

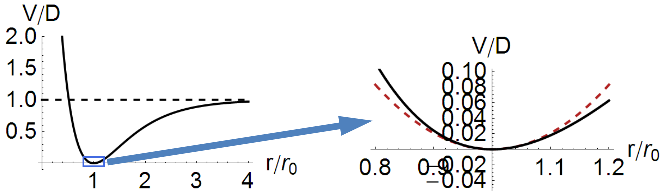
$$\frac{k_c}{2\sqrt{km}}.$$

4. Morse Potential

The Morse potential is a commonly used potential energy function for the potential energy of a diatomic molecule. It was proposed in 1929 by P. M. Morse[1] to be

$$V(r) = D\left(1 - e^{-\alpha(r-r_0)}\right)^2,$$

where r is the distance between the atoms, r_0 is the equilibrium bond distance, D is the well depth (defined relative to the dissociated atoms), and α controls the 'width' of the potential. For the hydrogen atoms in H_2 molecules, the parameters are[2] $r_0 = 0.7416\text{Å}$, $D = 457.8\ \text{kJ} \cdot \text{mol}^{-1}$, $\alpha = 1.9425\text{Å}^{-1}$. We can use the Morse potential to study the vibration of hydrogen molecules.



Around the equilibrium distance, $r = r_0 + x$ (assuming x to be small), the potential can be Taylor expanded in power series of x.

$$V(r_0 + x) = V_0 + V_1 x + \frac{1}{2} V_2 x^2 + O(x^3).$$

(a) Find the coefficients V_0, V_1 and V_2. Why does V_1 vanish?

In this way, almost any kind of potential can be approximated by the harmonic potential (the red dashed curve in the figure) near the bottom of the potential well. The coefficient V_2, which in general called the stiffness, may be identified as "the constant of the spring" in a oscillator analog. That is why harmonic oscillator is so frequently seen in physics.

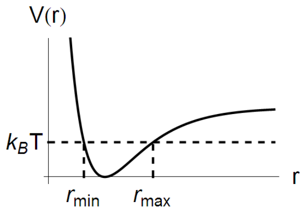
(b) Determine the frequency ω_0 of *small* vibration about equilibrium for two identical atoms of mass m bound to each other by the Morse potential.

(c) Given the mass of hydrogen atom $m_H = 1\ \text{g} \cdot \text{mol}^{-1}$, calculate the vibration frequency of hydrogen molecule. According to the dimension analysis based on quantum mechanics and thermodynamics, what is the characteristic temperature scale for such frequency?

Note: Below this temperature scale, the vibration mode of hydrogen molecule cannot be exited due to the energy quantization of microscopic oscillators, meaning that the energy must be absorbed or released quantum by quantum. This is why the specific heat of H_2 gas under room temperature is $C_V \simeq 5R/2$ roughly, a contribution from three translational degrees of freedom and two rotational degrees of freedom, with the vibrational degree of freedom being "frozen".

If the temperature is high, the vibration amplitude may become larger (for the atoms becoming more energetic), and the assumption of small vibration breaks down. The vibration of diatomic molecule is no longer harmonic at high temperature. The anharmonic vibration leads to the effect of *thermal expansion*, the tendency of material to increase in volume when heated.

In a gas of temperature T, the average total energy of the diatomic molecule is $E \simeq k_B T$, where k_B is the Boltzmann constant. If $k_B T < D$, the atoms are still bound together, but their relative distance varies from $r_{\min}$ to $r_{\max}$ in the vibration.



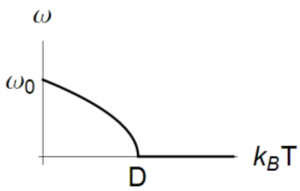
(d) Find $r_{\min}$ and $r_{\max}$. Let us roughly define the mean distance between atoms by $\bar{r} = (r_{\min} + r_{\max})/2$. What is the temperature dependence of $\bar{r}$?

Note: You will see the origin of thermo expansion, that the molecule grows larger when raising the temperature. This is an effect due to the x^3 and higher order terms (known as the anharmonic terms) in the expansion for the potential $V(r_0 + x)$.

(e) What will be the *relative* velocity v of the atoms when the distance between them is r? Hint: Use the conservation law of energy.

(f) Show that the frequency of the molecule vibration is red-shifted by heating, which is an effect that can be used to probe the mechanical properties of molecules by the vibrational spectroscopy in experiments.

$$\omega = \omega_0 \left(1 - \frac{k_B T}{D}\right)^{1/2}.$$



Hint: First, calculate the period of vibration from

$$\tau = 2 \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v(r)}.$$

The above formula for vibration period can be generalized for any kind of potential well. So *any* problem of oscillation frequency (not just for harmonic oscillation) is solvable, at least, in principle. The following integral will be useful

$$\int_{-a}^{a} \frac{\mathrm{d}x}{(1-x)\sqrt{a^2 - x^2}} = \int_0^{\pi} \frac{\mathrm{d}\theta}{1 - a\cos\theta} = \frac{\pi}{\sqrt{1-a^2}}.$$

[1] P. M. Morse, Phys. Rev. **34**, 57-64 (1929).

[2] O. Bayrak and I. Boztosun, J. Mol. Struct. – Theochem **802**, 17 (2007).

Solution:

(a) $V_0 = 0,\ V_1 = 0,\ V_2 = 2D\alpha^2$. V_1 vanishes because the expansion is around equilibrium (or the saddle point).

(b) Use the reduced mass $m^* = m/2$, then the frequency can be determined from the equation of motion

$$\omega_0 = \left(\frac{V_2}{m^*}\right)^{1/2} = 2\alpha \left(\frac{D}{m}\right)^{1/2}.$$

Some puts a N_A in front of m.

(c) For hydrogen molecule $\omega_0 = 8.30 \times 10^{14} \mathrm{rad \cdot s^{-1}}$ or 1.32×10^{14}Hz, corresponding temperature 6340K.

Some does not correctly use the equal partition theorem, and consider $\hbar\omega_0 = k_B T/2$.

(d) From $V(r) = k_B T$, the solution is

$$r_{\min} = r_0 - \alpha^{-1} \ln\left(1 + \xi^{1/2}\right),$$
$$r_{\max} = r_0 - \alpha^{-1} \ln\left(1 - \xi^{1/2}\right),$$

where $\xi \equiv k_B T/D$.

$$\bar{r} = r_0 - \frac{1}{2\alpha} \ln\left(1 - \frac{k_B T}{D}\right),$$

which increase with the temperature.

(e) $m^* v^2/2 + V(r) = k_B T$,

$$v = 2\left(\frac{D}{m}\right)^{1/2}\left(\frac{k_B T}{D} - \left(1 - e^{-\alpha(r-r_0)}\right)^2\right)^{1/2}.$$

(f) The period

$$\tau = 2\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{v(r)} = \frac{1}{\alpha}\left(\frac{m}{D}\right)^{1/2}\int_{\ln(1-\xi^{1/2})}^{\ln(1+\xi^{1/2})} (\xi - (1-e^x)^2)^{-1/2}\mathrm{d}x.$$

Let $y = 1 - e^x$, then $\mathrm{d}x = \mathrm{d}y/(y-1)$,

$$\tau = \frac{1}{\alpha}\left(\frac{m}{D}\right)^{1/2}\int_{-\xi^{1/2}}^{\xi^{1/2}} \frac{\mathrm{d}y}{(1-y)(\xi - y^2)^{1/2}} = \frac{1}{\alpha}\left(\frac{m}{D}\right)^{1/2}\frac{\pi}{\sqrt{1-\xi}}.$$

Therefore

$$\omega = \frac{2\pi}{\tau} = 2\alpha\left(\frac{D}{m}\right)^{1/2}\left(1 - \frac{k_B T}{D}\right)^{1/2} = \omega_0\left(1 - \frac{k_B T}{D}\right)^{1/2}.$$

Chap. 23 Resonance

1. Plasma Oscillation (II): Drude Conductivity

In the uniform electric field, a free electron will be accelerated constantly. However it is not the case for electrons in the metal, due to the presence of the lattice. The scattering of the electron with the lattice leads to two effects. First the mass of the electron will be renormalized to the effective mass m^* by elastic scattering. Secondly the motion of electron will be damped due to the inelastic scattering with the lattice vibrations (the electron-phonon scattering). To describe the effect of damping, we introduced the relaxation time τ, such that without a driving force the velocity of the electron will diminish within the time scale τ following the form of exponentially decay $e^{-t/\tau}$. This model is known as the *Drude model*.

(a) What is the form of the damping force so as to produce the exponentially decay $e^{-t/\tau}$ behavior of the velocity?

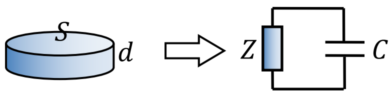
Driven by a static uniform electric field in the metal, the electron will drift with constant speed, so that the driving force will be balanced by the damping.

(b) Given the electron density n, what is the DC conductivity σ_0 of the metal according to the Drude model? Note: the DC conductivity is defined by $j = \sigma_0 E$, where j is the DC current density and E the static electric field.

If the metal is in a AC circuit, the electric field in the metal will be oscillating with some frequency ω.

(c) What is the frequency-dependent (or AC) conductivity $\sigma(\omega)$? Hint: use the complex number method you have learnt in the Lecture.

Consider such metal material is made into a thin cylinder with basal area S and height d. The metal cylinder is effectively a parallel circuit of a complex impedance Z and a capacitor C, because the body of the cylinder responses to the electric field like a impedance of the conductivity $\sigma(\omega)$, while the surface of the cylinder stores electric charges like a capacitor.



(d) Show that the capacitor of a thin cylinder is given by

$$C = \frac{\epsilon_0 S}{d}.$$

Note: You may think of why the dielectric constant we used here is the one of vacuum ϵ_0 but not the one of the metal. Because the capacitor C here only models the capacitance due to the effect of two surfaces. The capacitance due to the charge transfer in the metal lies in the impedance.

(e) Show that the Z-C circuit will oscillate will the plasma frequency ω_p in the long relaxation time limit ($\omega_p \tau \gg 1$). Note: If you have forgotten what is the plasma frequency ω_p, refer to the Problem 2 of Chap. 21.

(f) Show that if the relaxation time τ is shorter than $(1/2)\omega_p^{-1}$ the plasma oscillation will be quenched by the large damping.

Solution:

(a) $f = -m^* \tau^{-1} \dot{z}$.

(b) From the force balance $-eE - m^* \tau^{-1} \dot{z} = 0$,

$$\dot{z} = -\frac{eE\tau}{m^*}.$$

The DC conductivity can be obtained form

$$\sigma_0 = \frac{j}{E} = -\frac{ne\dot{z}}{E} = \frac{ne^2\tau}{m^*}$$

(c) $m^*(\ddot{z} + \tau^{-1}\dot{z}) = -eE$, therefore (according to Feynman's convention $e^{i\omega\tau}$)

$$\dot{z} = -\frac{eE\tau}{m^*(1 + i\omega\tau)}.$$

Then

$$\sigma(\omega) = \frac{\sigma_0}{1 + i\omega\tau}.$$

Some may take the module of σ.

(d) $\epsilon_0 S/d$.

(e) The total impedance in the circuit is

$$Z + \frac{1}{i\omega C} = \frac{d}{\sigma(\omega)S} + \frac{d}{i\omega\epsilon_0 S} = \frac{d\tau}{i\omega\sigma_0 S}\left(\frac{\sigma_0}{\epsilon_0\tau} - \omega^2 + i\tau^{-1}\omega\right).$$

By substitution of the DC Drude conductivity

$$\frac{\sigma_0}{\epsilon_0\tau} = \frac{ne^2}{\epsilon_0 m^*} = \omega_p^2,$$

we find the total impedance vanishes if

$$\omega_p^2 - \omega^2 + i\tau^{-1}\omega = 0,$$

which in the $\omega_p\tau \gg 1$ limit leads to $\omega = \omega_p$, meaning the current of frequency ω_p in the circuit will persists in the circuit for no impedance.

2. Nuclear Magnetic Resonance

Atomic nuclei have magnetic moment $\boldsymbol{\mu}$ proportional to their spin angular momentum $\boldsymbol{S}$,

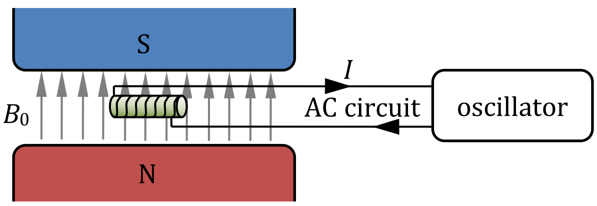
$$\boldsymbol{\mu} = \gamma \boldsymbol{S}.$$

The gyromagnetic ratio γ can be either positive or negative depending on the species of nucleus. As a model study, we can simply assume $\gamma > 0$ here. In the external magnetic field $\boldsymbol{B}$, the nuclear moment encounters torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}$, and will precess.

(a) What is the angular velocity Ω of precession for a nucleus in the magnetic field B?

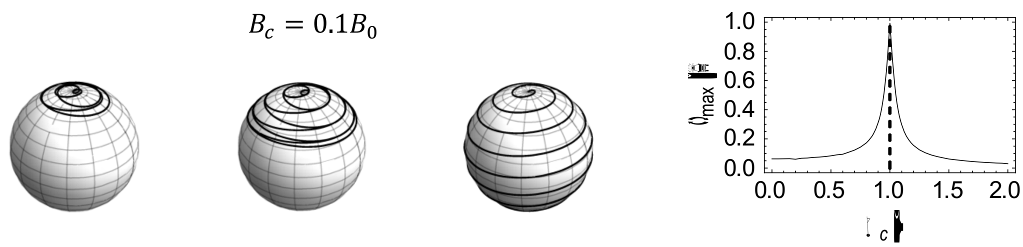
For example a proton in a 0.5T magnetic field will precess with 21MHz frequency, about the upper limit of radio short-wave II. If the electromagnetic wave of the same frequency shines on the precessing nucleus, it will resonate with the nucleus and hence being absorbed. This phenomena is known as the nuclear magnetic resonance. F. Bloch and E. M. Purcell won 1952 Nobel Prize for the development of new method measuring nuclear magnetic resonance. Today nuclear magnetic resonance is extensively used in physics, chemistry, biophysics, medicine etc. Samples are put in perpendicular uniform magnetic field B_0, with a weak alternating field $B_c \cos \omega_c t$ $(B_c \ll B_0)$ is added horizontally. The total magnetic field can be written as

$$\boldsymbol{B} = (B_c \cos \omega_c t, 0, B_0).$$



(b) Write down the equation of motion for the nuclear moment $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$.

(c) Design a computer program to solve the differential equation, under the condition that $B_c = 0.1 B_0$, for $\omega_c = 0.8\Omega$, 0.9Ω, 1.0Ω, 1.1Ω and 1.2Ω. The initial condition is given by $\boldsymbol{\mu}(t = 0) = (0,0,1)$, i.e. the moment points to the north pole initially. Plot your solutions on a sphere. Measure the maximum angle $\theta_{\max}$ that the moment can tilt away from the north pole for each ω_c. Hint: You are supposed to obtained the following figures for the first three cases.



$\omega_c = 0.8\Omega$ $\omega_c = 0.9\Omega$ $\omega_c = 1.0\Omega$ $\theta_{\max}$ peaks at resonance

From the computer simulation, we can see that the resonance happens when the alternating frequency ω_c of the horizontal field exactly matches the precession frequency Ω. Let us understand the physics. Since the horizontal field is oscillating, we may expect its torque on the nucleus will cancel on average. However if the circular frequency ω_c is equal to the precession frequency, the moment $\boldsymbol{\mu}$ will synchronize with the horizontal field, and gain energy from the field. The energy of the moment in the magnetic field is given by $E = -\boldsymbol{\mu} \cdot \boldsymbol{B}$, so as the moment becomes more and more energetic, it will flip to the opposite direction of the external field B_0. In this case, energy absorption of $2\mu B_0$ can be observed in the AC circuit.

(d) On resonance, what is the average power of the work done by the horizontal field, when the angle between the moment $\boldsymbol{\mu}$ and the magnetic field $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$ is θ?

(e) How much time it takes to flip the moment from the north pole to the south pole on resonance?

In the small B_c limit ($B_c/B_0 \ll 1$), the equation of motion can be solved by perturbation approach, i.e. to solve the equation with $B_c = 0$ first then to correct the result up to the first order of B_c/B_0.

(f) Use the perturbation approach to show that the angle θ between $\boldsymbol{\mu}$ and $\boldsymbol{B}_0$ follows a almost linear behavior on resonance

$$\theta = \frac{\gamma B_c}{2}\left(t + \frac{\sin 2\Omega\, t}{2\Omega}\right).$$

Solution:

(a) $\Omega = \gamma B$.

(b) From $\mathrm{d}\boldsymbol{\mu}/\mathrm{d}t = \gamma \boldsymbol{\mu} \times \boldsymbol{B}$,

$$\dot{\mu}_x = \gamma B_0 \mu_y,$$
$$\dot{\mu}_y = -\gamma B_0 \mu_x + \gamma B_c \cos \omega_c t\, \mu_z,$$
$$\dot{\mu}_z = -\gamma B_c \cos \omega_c t\, \mu_y.$$

(c) The maximum angle

ω_c/Ω	0.8	0.9	1.0	1.1	1.2
$\theta_{\max}$	0.543	0.974	3.116	0.931	0.491

(d) $P = \boldsymbol{\tau} \cdot \boldsymbol{\omega} = (\boldsymbol{\mu} \times \boldsymbol{B}) \cdot \boldsymbol{\omega}$ where $\boldsymbol{\omega} \cong (0,0,\Omega)$

$$\langle P \rangle = \mu B_c \Omega \sin\theta \,\langle \cos^2 \Omega t \rangle = \frac{1}{2} \mu B_c \Omega \sin\theta.$$

(e) $E = -\boldsymbol{\mu} \cdot \boldsymbol{B} = -\mu B_0 \cos\theta$, then $\langle P \rangle = \mathrm{d}E/\mathrm{d}t = \mu B_0 \sin\theta\, (\mathrm{d}\theta/\mathrm{d}t)$,

$$\frac{d\theta}{dt} = \frac{\gamma B_c}{2}.$$

θ goes from 0 to π, it will take the time $T = 2\pi/\gamma B_c$.

(f) Substitute $\mu_y = \mu \sin\theta \cos \Omega t$ into $\dot{\mu}_z = -\gamma B_c \cos \Omega t\, \mu_y$,

$$\frac{d\theta}{dt} = \frac{\gamma B_c}{2}(1 + \cos 2\Omega t).$$

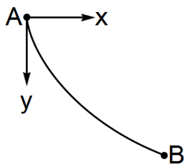
The solution is

$$\theta = \frac{\gamma B_c}{2}\left(t + \frac{\sin 2\Omega\, t}{2\Omega}\right).$$

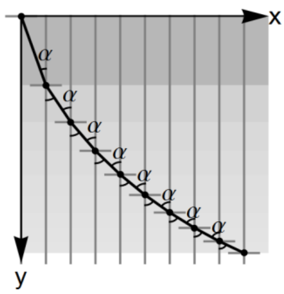
Chap. 26 Optics: The Principle of Least Time

1. The Curve of Fastest Descent (II): Optical Analog

The brachistochrone problem (or the problem of fastest descent) was posed by Johann Bernoulli in *Acta Eruditorum* in June 1696: given two points A and B in a vertical plane, what is the curve on which a particle, acted on only by gravity, starts at A and reaches B in the shortest time. To solved this problem, Johann Bernoulli appeals to Fermat's principle , namely that light always follows the shortest possible time of travel.
What he considered is not a particle descending along a curve, but a beam of light travelling in a non-uniform medium. The index of refraction is such *designed* that the speed of light increases *as if* following a constant vertical acceleration (that of gravity g). It is not saying that the speed of light increases when propagating downwards in a gravity field in the same manner as the mass particle, but just to analog the effect of gravitational acceleration on a mass by using the refraction index gradient of the medium, and thus there is no gravitational effect for light.



(a) What should be the index of refraction $n(y)$ as a function of vertical position y in order to allow the speed of light in the medium to be just the same as a mass in the gravity field g. (Note that the y axis has been flipped down.)
According to Fermat's principle, the propagation of light automatically choose the curve of the shortest travelling time, and hence the curve of fastest descent is simply the trajectory of light in the medium with index $n(y)$.



Johann Bernoulli noted that Fermat's principle is equivalent to the law of refraction. So he divides the plane into strips and he assumes that the particle follows a straight line in each strip. The path is then piecewise linear. The problem is to determine the angle α of the straight line segment in each strip as a function of y by the law of refraction.
(b) Show that the vertical position y and the angle α are related by $y = 2R\sin^2\alpha$, where R is a constant of the motion.
(c) Use trigonometry to express α in terms of $\mathrm{d}y/\mathrm{d}x$, so as to obtained the differential equation that determines the curve of fasted descent.
(d) Show that the inverted cycloid,

$$\begin{cases} x = R(\theta - \sin\theta) \\ y = R(\cos\theta - 1) \end{cases},$$

satisfies the differential equation, and is therefore the solution to the brachistochrone problem.

Solution:

(a) For the mass in the gravity field, its speed follows from the conservation of energy as $v = \sqrt{2gy}$. So to simulate the mass particle, the index of refraction should be designed as

$$n(y) = \frac{c}{v} = \frac{c}{\sqrt{2gy}}.$$

(b) $n \sin\alpha = \text{const}$, so

$$\frac{c}{\sqrt{2gy}} \sin\alpha = \text{const}.$$

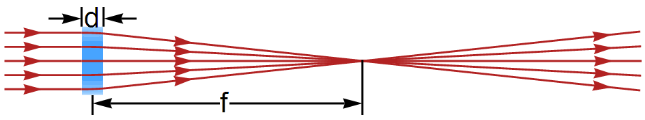
Therefore $y/\sin^2\alpha = 2R$ is a constant of motion.

(c) $y(1 + (\mathrm{d}y/\mathrm{d}x)^2) = 2R$. So

$$\alpha = \arcsin\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{1/2}.$$

2. Gradient Index (GRIN) Lenses

Gradient index (GRIN) materials are materials that have non-uniform index of refraction. It can be fabricated by the hierarchically structured polymer optical materials. By layering polycarbonate (PC) ($n = 1.58$) and polymethylmethacrylate (PMMA) ($n = 1.49$) films, the index of refraction can be tuned between 1.49 and 1.58 smoothly throughout the material. The GRIN material show promise for use in optical systems that have fewer, lighter lenses than traditional lens systems. The following figure shows how light is focused by a flat GRIN lens, which is simply a polymer disk of uniform thickness $d = 0.5$cm, but with non-uniform index.



The index of refraction at r distance away from the center of the lens is described by the function

$$n(r) = n_0 - \alpha r^2,$$

where $n_0 = 1.58$ and $\alpha = 0.167\text{cm}^{-2}$. Use Fermat's principle to determine the focus f of the GRIN lens.

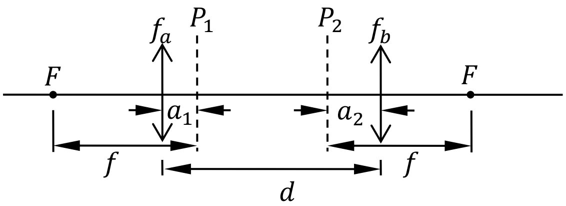
Solution:

$$f = \frac{1}{2d\alpha} = 6.0\text{cm}.$$

Chap. 27 Geometrical Optics

1. Principle Planes of Compound Lenses

In the Lecture, we have learnt that the optical system of compound lenses can be considered as a thin lens of focus f separated to the two principal planes that rays received at one principle plane is transferred to the corresponding point of the other. Here is an example for you. Two thin lenses with the focus lengths f_a and f_b are placed by the distance d. The principle planes P_1 and P_2 are located at a_1 behind the front lens and a_2 in front of the back lens respectively. Find the expressions for a_1, a_2 and f in terms of f_a, f_b and d.



Solution:

$$a_1 = \frac{f_a d}{f_a + f_b - d},$$

$$a_2 = \frac{f_b d}{f_a + f_b - d},$$

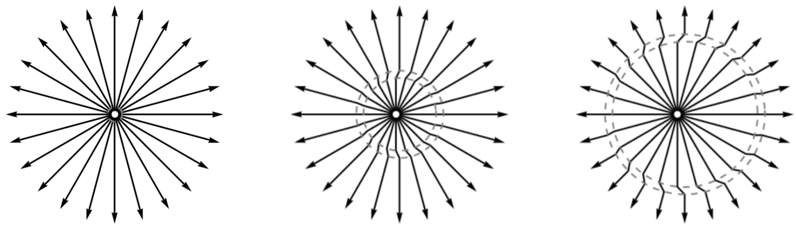
$$f = \frac{f_a f_b}{f_a + f_b - d}.$$

Chap. 28 Electromagnetic Radiation

1. Radiation From an Accelerated Charge

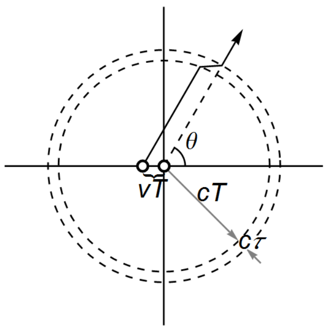
We have learnt from the Lecture, that an accelerating charge gives out electromagnetic radiations. J. J. Thomson provides a simple derivation for the intensity and angular distribution of the radiation from a point charge subject to an arbitrary but small acceleration.

We start by taking a point charge q at rest at time $t = 0$. Coulomb's law implies that its electric field lines will be purely radial as shown in the figure [a] below. We now accelerate the charge (to the left) by the acceleration a for a small time interval τ, and look at the field line configuration at a time T after this. Then we will see a thin shell of kinks of the field lines propagates outwards with the speed of light, as in the figure [b] and [c].



[a] [b] [c]

Why is there a kink in the field line? Because outside of a sphere of radius $R = cT$, the news about the acceleration has not arrived yet, so that the field line still points towards the original position of the charge. While inside this sphere, the electric field has been told that the charge is now moving with a constant speed $v = a\tau$, so that the field lines are now able to “stick to” and “move with” the charge, as what it should look like the charge’s frame. So there's an inner region with field lines pointing away from the charge, and an outer region with field lines pointing away from the original position of the charge. These two regions are separated by a transition region: a spherical shell of the thickness $c\tau$ that expands outward at the speed of light.



The next question is, what does the field look like in the transition region? The electric field lines have to be connected! Otherwise it will contradict to Gauss’s Law. Therefore in the transition region, the electric field has a non-radial component E_θ, and the electric field can be written as $\boldsymbol{E} = E_r\boldsymbol{e}_r + E_\theta\boldsymbol{e}_\theta$, where E_r is the radial component.

(a) What is the radial component E_r of the electric field at a distance R from the charge? (Assume $v \ll c$ in the non-relativistic limit.)

(b) Determine the tangent component E_θ in the transition region (expressed as a function of R). Hint: use the geometric relations between E_r and E_θ.

Note that E_θ falls off with distance as R^{-1} in contrast to $E_r \sim R^{-2}$. Thus far from the accelerated charge (large R), only the tangent component E_θ will contribute significantly to the radiation field. The energy per unit volume stored in this field, proportional to $|\boldsymbol{E}|^2$, therefore fall off as R^{-2}, so the total energy contained in the shell is unchanged as the shell expanded.

(c) According to Poynting’s Law, determine the energy flow density $\boldsymbol{S}$ (the Poynting vector) in each direction θ. Hint: analog to the propagation of plane-wave in the free space.

(d) What is the total power of radiation?

Solution:

(a) According to Gauss’s Law

$$E_r = \frac{q}{4\pi\epsilon_0 R^2} = \frac{q}{4\pi\epsilon_0 c^2 T^2}$$

(b) According to the geometry

$$\frac{E_\theta}{E_r} = \frac{vT\sin\theta}{c\tau} = \frac{aT\sin\theta}{c},$$

thus

$$E_\theta = \frac{qa\sin\theta}{4\pi\epsilon_0 c^2 R}.$$

(c) According to Poynting's Law $S = \epsilon_0 c E_\theta^2$,

$$S = \frac{q^2 a^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3 R^2}.$$

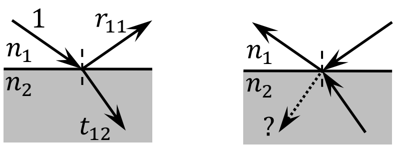
(d) The radiation power

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}.$$

Chap. 29 Interference

1. Interference on the Interface

Reflection and refraction happen on the interface of two mediums (1 and 2) with different indexes of refraction n_1 and n_2. Suppose the incident light in medium 1 is coming with the amplitude 1 (as in Fig. A). By saying that the reflection coefficient from medium 1 to 1 is r_{11}, and the transmission coefficient from medium 1 to 2 is t_{12}, we mean that the amplitude of the reflected and refracted light will be r_{11} and t_{12} respectively.



(A) (B)

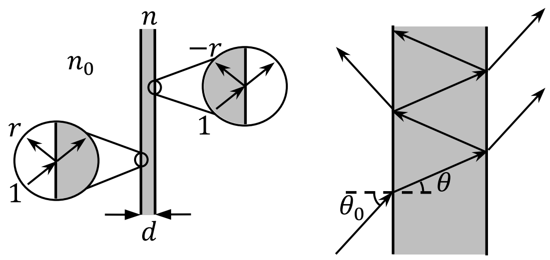
According to the principle of ray reversibility, if the reflected and refracted light proceed in the reverse direction along the same path, the incident light of the original amplitude should be generated (see Fig. B). One may doubt is an extra ray will be generated as the dashed ray in Fig. B. This extra ray contradict with the reversibility principle, and is fairly possible if we consider light as particles. The wave nature of light saves the reversibility principle, which indicates that the light will undergo completely destructive interference on the path of the extra ray, and thus the extra ray does not exists.

If the above statement is true, what should be the relations between r_{11}, r_{22}, t_{12} and t_{21}?

Solution: $r_{11} = -r_{22} = r$ and $t_{12}t_{21} + r^2 = 1$.

2. Film Interference

A thin film of the thickness d and refraction index n is sandwiched in the medium of the refraction index n_0. The coefficient of reflection on the interface is r. Light shined on the film will be reflected between both interfaces for many times. Each time the light reaches the back interface produces a transmission wave. The transmission waves interfere with each other leads to the phenomenon known as the film interference.



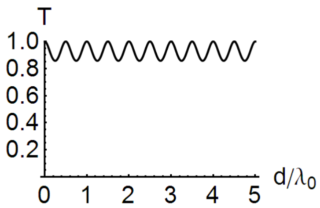
(a) Suppose the incident angle is θ_0, and the wave length in the medium n_0 is λ_0, show that the transmittance of the film takes the following form

$$T = \frac{1}{\alpha \sin^2 \delta + 1},$$

and determine the coefficient α and the angle δ.

The equal thickness interference is a special case of film interference, when the incident angle is fixed at zero.

(b) Fix $\theta_0 = 0$ and plot the transmittance as a function of film thickness d. Explain the oscillations you see.



Somehow by coating the surface, people are able to change the reflection coefficient r.

(c) How does the equal thickness interference affected by the coefficient r, especially when $r \to 1$? Find the standing wave modes in resonance with the incident light that leads to the peaks in transmittance.

The equal inclination interference is another case of film interference, in which the thickness is fixed but the incident angle varies. The equal inclination interference is more complicated because the coefficient of reflection also changes with the incident angle. Let us suppose hereon $n_0 = 1.5$ (glass) and $n = 1$ (air), and the incident light is polarized to S wave.

(d) For $d = 0.1 n_0 \lambda_0$, $d = n_0 \lambda_0$ and $d = 5 n_0 \lambda_0$, plot the transmittance as a function of the incident angle θ_0 for θ_0 ranging from 0 to $\pi/2$. Explain the oscillations in the transmittance. Hint: you may determine how r depends on θ_0 first.

(e) Why does the transmittance drops to zero at around a critical angle $\theta_c = 41°49'$? What is the physical meaning of this angle?

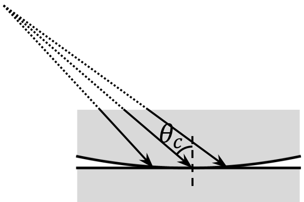
Finite transmittance remains when the incident angle exceeds the critical angle. This is a quantum mechanical effect of light, known as the optical tunneling, that can be demonstrated by simple experiments even in the undergraduate lab.

(f) Calculate the transmittance as a function of film thickness at critical angle $\theta_0 = \theta_c$.

Optical tunneling is prominent only for very thin films whose thickness should be controlled within a few wavelength of light. To realize a thin film like this, we evoke the Newton's ring device. Around the center of Newton's ring, there is a centimeter large region in which the air layer thickness varies smoothly within several wavelength according to

$$d = \frac{x^2 + y^2}{2R},$$

where $\boldsymbol{r} = (x, y)$ is the displacement from the center.



(g) What kind of transmission pattern do you expected to observe on the other side of Newton's ring device, if the light is emitted from a point source and falls on the center at the critical angle. Hint: design a computer program to help you calculate the transmitted light intensity at each point (x, y).

Solution:

(a) The phase difference due to translation through the film is

$$\delta = k_x d = kd \cos\theta,$$

where k is the wave number of light in the medium n, which is given by

$$k = \frac{2\pi nd}{n_0 \lambda_0}.$$

The angle θ can be determined from the incident angle θ_0 according to Snell's Law,

$$n \sin\theta = n_0 \sin\theta_0.$$

Then we have

$$\delta = \frac{2\pi d}{n_0 \lambda_0}(n^2 - n_0^2 \sin^2\theta_0)^{1/2}.$$

Suppose the transmission coefficient is t from the medium n_0 to n, and t' from n to n_0, then t and t' is related to r through $r^2 + tt' = 1$. Given the reflection and transmission coefficient, the amplitude of the transmitted light can be determined

$$t = \sum_{m=0}^{\infty} tt'(-r)^{2m} e^{i(2m+1)\delta} = \frac{tt' e^{i\delta}}{1 - r^2 e^{i2\delta}}.$$

Therefore the transmittance is

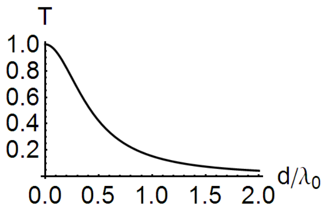
$$T = |t|^2 = \frac{1}{\alpha \sin^2\theta + 1},$$

where

$$\alpha = \frac{4r^2}{(1 - r^2)^2}.$$

(e) For S wave at critical angle

$$T = \left(\frac{\pi^2(n_0^2 - 1)}{n_0^2 \lambda_0^2} d^2 + 1\right)^{-1} = \left(\frac{5\pi^2}{9}\frac{d^2}{\lambda_0^2} + 1\right)^{-1} = \left(5.483\frac{d^2}{\lambda_0^2} + 1\right)^{-1}$$



Chap. 30 Diffraction

1. Diffraction by Opaque Screens

Solution:

The object plane x', and the screen x, separated by the distance z. The amplitude at x is given by

$$A(x) = \int_0^\infty \mathrm{d}x' \frac{e^{ikr}}{r^{1/2}},$$

where $r = \sqrt{(x'-x)^2 + z^2}$ is the distance to the source. The denominator takes into account the conservation of energy for cylindrical wave.

Let $y = x' - x$, $A(x) = B(+\infty) + B(x)$ (B is an odd function), by defining

$$B(y) = \int_0^y \mathrm{d}y \frac{e^{ik\sqrt{y^2+z^2}}}{(y^2+z^2)^{1/4}}.$$

In the large positive y limit,

$$B(+\infty) - B(y \to +\infty) \simeq \int_y^\infty \mathrm{d}y \frac{e^{iky}}{\sqrt{y}} \simeq -\frac{e^{iky}}{ik\sqrt{y}}.$$

In large z limit, we can use the saddle point approximation to evaluate $B(+\infty)$,

$$B(+\infty) = \left(\frac{\pi i}{2k}\right)^{1/2} e^{ikz}.$$

Therefore we obtain the asymptotic behavior,

$$B(y \to +\infty) \simeq \left(\frac{\pi i}{2k}\right)^{1/2} e^{ikz} + \frac{e^{iky}}{ik\sqrt{y}}.$$

Thus for large positive x, we have

$$A(x) = \left(\frac{2\pi i}{k}\right)^{1/2} e^{ikz} + \frac{e^{ikx}}{ik\sqrt{x}},$$

and the intensity $I = |A|^2$ oscillates between

$$I(x) = \left(\frac{2\pi}{k}\right)\left(1 - \frac{2}{\sqrt{2\pi kx}}\cos\left(k(x-z) + \frac{\pi}{4}\right)\right),$$

normalized by $I_0 = 2\pi/k$,

$$\frac{I(x)}{I_0} = 1 - \frac{2}{\sqrt{2\pi kx}}\cos\left(kx + \frac{\pi}{4}\right).$$

For large negative x, we make use of the fact that B is an odd function, so

$$A(x) = B(+\infty) + B(x) = B(+\infty) - B(|x|) = -\frac{e^{ik|x|}}{ik\sqrt{|x|}}.$$

Therefore the intensity decays by $I(x) = (k^2|x|)^{-1}$, and hence

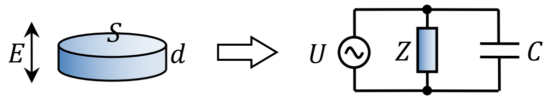
$$\frac{I(x)}{I_0} = \frac{1}{2\pi k|x|}.$$

2. Fraunhofer Diffraction Patterns of Lattice

Chap. 31 Origin of the Refractive Index

1. Plasma Oscillation (III): Absorption of Light

In the Problem 1 of Chap. 23 we have shown that a thin metal cylinder can be considered as a Z-C circuit effectively. When the electromagnetic wave is beamed on the metal, the oscillating electric field E applied across the metal will introduce a AC voltage source U in the effective circuit.



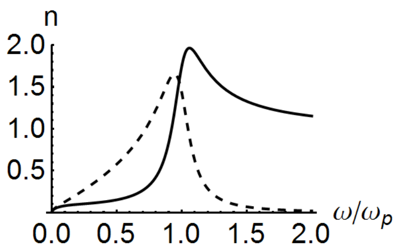
(a) Calculate the total current I in the metal (in terms of Z and C) in response to the electric field E.

According to Maxwell's theory of displacement current, the current inside the metal will not end up at the surfaces, but will be continued by the displacement current in electromagnetic field. The displacement current *density* (i.e. current per unit sectional area) by definition is the time derivative of electric displacement field D.

(b) Find the AC dielectric constant $\epsilon(\omega)$ as the ratio between D and E. Express your result in terms of the vacuum dielectric constant ϵ_0, the plasma frequency ω_p and the relaxation time τ. Hint: you should substitute the expression for Z and C. If you forget the result in the Problem 1 of Chap. 23, do it once more.

(c) Find the index of refraction for the metal. Hint: the index of refraction can be obtained as the ration of the phase velocity and the light speed. The phase velocity of light in the metal can be extracted from the wave equation, which can be deduced from the Maxwell equations ($\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$ and $\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}$).

The following figure shows the real (solid line) and imaginary (dashed line) part of the index of refraction as a function of frequency ω of light for the case of $\omega_p \tau = 5$.



(d) Explain why metal are usually opaque to light. Roughly upon which frequency does the metal become transparent?

It is known that light does not have a rest mass in vacuum. This is why light can propagate

with the speed c and does not contradict with special relativiey. However this is not true for the light in the metal. Due to the strong scattering with electrons in the metal, light will be slowed down as if the photon acquires a rest mass. This effect can be shown in the limit that the relaxation time tends to infinity, which is known as the *perfect metal* limit, when the resistivity drops to zero and the metal becomes a perfect conductor (not superconductor).
(e) Show the dispersion relation of light in the perfect metal, and find the rest mass of photon.
(f) As the photon acquires rest mass, it can be stopped. However the rest photon is never observed in experiments. How does the nature forbid us to see the rest photon?

Solution:
(a) $I = (Z^{-1} + i\omega C)U = (Z^{-1} + i\omega C)Ed$.
(b) The conduction current should be compensated by the displacement current

$$I = S\partial_t D = i\omega SD,$$

so

$$(Z^{-1} + i\omega C)Ed = i\omega SD,$$

and the AC dielectric constant is obtained

$$\epsilon(\omega) = \frac{D}{E} = \frac{(Z^{-1} + i\omega C)d}{i\omega S}.$$

Substitute the expression of Z and C,

$$\epsilon(\omega) = \epsilon_0 \left(1 + \frac{\omega_p^2 \tau}{i\omega(1 + i\omega\tau)}\right).$$

(c) Index of refraction can be calculated from $n(\omega) = \epsilon_0/\epsilon(\omega)$,

$$n(\omega) = \left(1 + \frac{\omega_p^2 \tau}{i\omega(1 + i\omega\tau)}\right)^{-1/2}.$$

(d) Upon plasma frequency the metal becomes transparent.
(e) In the good metal limit,

$$n = \left(1 - \omega_p^2/\omega^2\right)^{-1/2}.$$

From $\omega = nck$, the dispersion relation reads

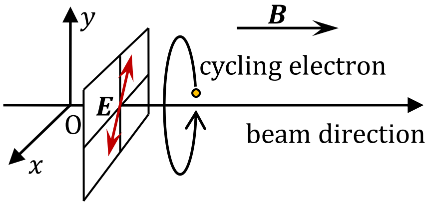
$$\omega^2 = c^2k^2 + \omega_p^2.$$

So the rest mass of photon is $\hbar\omega_p/c^2$.

Chap. 33 Polarization

1. Faraday Effect

The Faraday effect or Faraday rotation is a magneto-optical phenomenon, where the direction of polarization is rotated about the beam axis as the light is traveling through the material with applied magnetic field in the parallel direction.



The Faraday effect is a result of ferromagnetic resonance of light and electrons in the magnetic field. Without the light, the electron in the material will be cycling in the magnetic field B with frequency ω_c.

(a) Determine the cycling frequency ω_c given the electron mass m and charge e.

When a beam of light falls on the electron, the electron will be driven back and forth in the cycling plane, hence undergoing a forced oscillation, which may go into resonance if the light frequency ω matches the cycling frequency ω_c.

(b) Write down the equation of motion for the electron.

The polarization of the material is related to the electron displacement by $\boldsymbol{P} = -ne\boldsymbol{x}$, where n is the electron density. The linear response of polarization to the electron field defines the *polarizability* χ, which in general is a matrix.

$$\boldsymbol{P} = \epsilon_0 \chi \boldsymbol{E}.$$

(c) Use the complex number method to find the polarizability matrix χ.

(d) Diagonalize the polarizability matrix to find all eigen modes with their eigen values respectively. Explain the physical meaning of the eigen modes.

The coupling of light to the cycling electron causes the linearly polarized ray to be decomposed into two circularly polarized rays which propagate at different speeds, a property known as circular birefringence.

(e) Find the index of refraction n for the right-handed and left-handed circularly polarized light.

The rays can be considered to re-combine upon emergence from the medium, however owing to the difference in propagation speed they do so with a net phase offset, resulting in a rotation of the angle of linear polarization.

(f) Determine the angle of Faraday rotation after light propagating for the distance d, up to the first order of the component of the applied magnetic field in the direction of the beam of light.

Solution:

(a) $\omega_c = eB/m$.

(b) Equation of motion,

$$m\ddot{x} = eB\dot{y} - eE_x,$$
$$m\ddot{y} = -eB\dot{x} - eE_y.$$

(c) Cast the equation of motion into matrix form

$$\frac{m\omega}{e}\begin{bmatrix} \omega & i\omega_c \\ -i\omega_c & \omega \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} E_x \\ E_y \end{bmatrix}.$$

Thus the polarization responses according to

$$-\frac{m\omega}{ne^2}\begin{bmatrix} \omega & i\omega_c \\ -i\omega_c & \omega \end{bmatrix}\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} E_x \\ E_y \end{bmatrix},$$

form which the polarizability matrix can be found

$$\chi = \frac{ne^2}{\epsilon_0 m\omega(\omega_c^2 - \omega^2)}\begin{bmatrix} \omega & -i\omega_c \\ i\omega_c & \omega \end{bmatrix}.$$

(d) Diagonalization leads to two eigen modes:
The right-handed circular polarization $E_x = iE_y$,
The left-handed circular polarization $E_x = -iE_y$.
Corresponding to the eigen polarizabilities (for $E_x = \pm iE_y$),

$$\chi_\pm = -\frac{ne^2}{\epsilon_0 m\omega(\omega \pm \omega_c)}$$

(e) Index of refraction,

$$n_\pm = \sqrt{1+\chi_\pm} = \sqrt{1 - \frac{ne^2}{\epsilon_0 m\omega(\omega \pm \omega_c)}}.$$

2. Coherency Matrix of Light Polarization

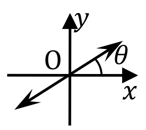
In the Lecture, we have learn that the polarization state of light may be expressed by a complex vector

$$|E\rangle = \begin{bmatrix} E_x \\ E_y \end{bmatrix},$$

which contains two complex numbers E_x and E_y that provides both the amplitudes and the phases of electric field vibrations in both x and y directions. For example, the polarization state pictured in [Fig. 33-2] are

picture	a	b	c	…	g	…
state$\lvert E\rangle$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ e^{i\pi/4} \end{bmatrix}$	$\begin{bmatrix} 1 \\ i \end{bmatrix}$	…	$\begin{bmatrix} 1 \\ -i \end{bmatrix}$	…
polarization	linear	elliptic	circular (LH)	…	circular (RH)	…

(a) According to this notation, find the vector $|\theta\rangle$ that represents a linearly polarized state, with the polarization direction at angle θ with the x-axis.



In quantum mechanics, such a vector $|E\rangle$ is known as the state vector. Every state vector $|E\rangle$ has a duel, obtained by transposing and taking complex conjugate, known as the its adjoint vector, denoted by

$$\langle E| = [E_x^* \quad E_y^*].$$

The principle of quantum mechanics is that each sate can be represented by the state vector. Physical observables are represented by operators (which means matrixes here). Each measurement can be taken by applying the corresponding matrix of that physical observable to the state vector, and then project to its adjoint vector, the resulting number is the expectation value of the physical observable.
For example, the light intensity is represented by the light intensity operator I, which is a

2×2 matrix

$$I = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix}.$$

Then to measure the light intensity of the polarization state $|E\rangle$, we take the matrix I, apply to vector $|E\rangle$, and then project to the vector $\langle E|$.

$$\langle I \rangle = \langle E|I|E\rangle = [E_x^* \quad E_y^*] \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}.$$

(b) Given that the light intensity is given by $\langle I \rangle = |E_x|^2 + |E_y|^2$, what should the matrix I be so as to produce this result.

(c) Find the matrix $I(\theta)$ that corresponds to the measurement of light intensity through a polarizer whose passing direction is at angle θ with the x axis.

Hints: You can try the measurement on different states, e.g. the x-polarized, y-polarized and θ-direction linearly polarized states, or circularly polarized states as well. From these special examples, you will be able to determine the four matrix elements of $I(\theta)$.

We have considered linearly, circularly and elliptically polarized light, which only covers a very small portion of the polarization states known as the *fully polarized* states. There is the case of *unpolarized* light, also called the *natural light*. Sun light is a typical example of natural light. There are even more cases of *partially polarized* light, which are mixtures of fully polarized and unpolarized light.

Natural light does not have a certain polarization. It is a statistically combining of all polarization states with equal probability. Naively one may wish to describe the natural light by a superposition of all fully polarized states $|E\rangle$.

$$|N\rangle = \sum_E |E\rangle = 0.$$

But unfortunately the result is 0 due to the phase cancelation effect.

(d) Show that at least the superposition of all linearly polarized states results in zero. Hints: use integral instead of summation $\int \mathrm{d}\theta\, |\theta\rangle$.

In quantum mechanics, the fully polarized states are called pure states, and all the other partially polarized and unpolarized states are called mixed states. Only pure states can be described by state vectors. Superposition of any pure states is still a pure state. Because the superposition of state vectors is a quantum mechanical mixing that has interference effect. To obtain the mixed state, statistical (thermal) mixing is required, which means that the states are mixed by probability but not by probability amplitude. Therefore there is no hope to represent mixed states by state vectors.

Nevertheless, both pure and mixed states can be represented by matrixes, known as the *coherency matrix*. The coherency matrix of light polarization is a 2×2 matrix. For fully polarized state $|E\rangle$, the coherency matrix ρ is defined as

$$\rho = |E\rangle\langle E|.$$

(e) Find the coherency matrix $\rho(\theta)$ representing the θ-direction linearly polarized light.

(f) Find the coherency matrix $\rho(R)$ and $\rho(L)$ representing the right-handed and left-handed circular light.

The coherency matrix of natural light can be obtained by combining the coherency matrix of all θ-direction linearly polarized light with equal probability.

(g) Find the coherency matrix of the natural light.

In the coherency matrix formulism, the measurement of a physical observable represented by matrix I is given by taking trace

$$\langle I \rangle = \mathrm{Tr}\, \rho I.$$

(h) Calculate the angular distribution of light intensity for the θ-direction linearly polarized state. Hints: Use $I(\alpha)$ to implement the measurement.

(i) Calculate the angular distribution of light intensity for the circularly polarized light and the natural light. Is there any difference between them?

(j) Find the determinant $\det \rho$ for both the circularly polarized light and the natural light. Can you see the difference?

A partially polarized light is in general described by the coherency matrix

$$\rho = \begin{bmatrix} a & c^* \\ c & b \end{bmatrix}.$$

Any partially polarized light can be described as a statistical mixing of a natural light with a fully polarized light. The degree of polarization (DOP) is a quantity used to describe the portion of light which is polarized, defined as the ratio of the light intensity of polarization component to the total light intensity. The fully polarized light has a DOP equal to 1, while the DOP for natural light is 0.

(k) Determine the DOP given the coherency matrix ρ. Under what condition does the polarization lose coherence completely and become natural light?

Hints: decompose ρ into the summation of fully polarized and unpolarized parts. The fully polarized coherency matrix has zero determinant.

The effect of a polarizer is a projection operator $P(\theta)$ which is also a 2×2 matrix. With the initial polarization described by the coherency matrix ρ, the light is sent into a polarizer whose pass axis is at angle θ. The polarization of the out-coming light will be described by $P(\theta)\rho P^\dagger(\theta)$.

(l) Find the matrix $P(\theta)$.

The birefringence effect of cellophane can be formulated in similar way. Each cellophane is represented by an unitary matrix U. Light passing through a cellophane changes the polarization state by SU(2) rotation of the coherency matrix $\rho \to U\rho U^\dagger$. For example, the matrix representation of a quarter-wave plate is

$$U = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

It is easy to show that the quarter-wave plate changes the linearly polarized light in [Fig. 33-2 a] to circular polarization.

If light passes through a series of polarizers P and cellophanes U, the effect is to multiply the P or U matrixes together one by one in right order. Then perform the transform on the coherency matrix of incident light results in the coherency matrix of the transmitted light.

(m) Use the matrix representation of polarizer to demonstrate the paradox that some light will be transmitted if a third polarizer with pass axis at $45°$ in plugged in between two polarizers of perpendicular pass axes.

Solution:

(b) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $I(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$.

Chap. 40 The Principle of Statistical Mechanics

1. Effusion of Gas

The escape of gas through a tiny pore or pinhole in its container is called *effusion*. Effusion occurs if the diameter of the hole is considerably smaller than the mean free path of the gas particles. Things that effuse can be either *molecules* or *photons* depending on the constituent of the gas. The effusion of light from the hole of a cavity is a typical kind of black-body radiation.

The effusion rate Γ is defined as the number of particles escaped from the hole per unit of time and per hole area. Let n be the number density of gas particles. You might expect the effusion rate being proportional to the product of n and the mean speed of particle $\bar{v}$.

$$\Gamma \propto n\bar{v}.$$

Because if the particle is to escape from the container, it must first touch the hole area A. Within the time interval $\mathrm{d}t$, the particles can fly for a distance roughly $\bar{v}\mathrm{d}t$. So in each time slice $\mathrm{d}t$, the particles in the volume $A\bar{v}\mathrm{d}t$ will be flying out of the hole, thus the number of particles that escape is $\mathrm{d}N \sim nA\bar{v}\mathrm{d}t$, which gives you a rough estimate of the effusion rate $\Gamma = \mathrm{d}N/(A\mathrm{d}t)$.

But this estimate cannot give us the correct coefficient. To obtain the formula for effusion rate, let us assume the speed of the particles follows a probability distribution described by the function $f(v)$ such that the probability finding a particle with speed between $v \sim v + \mathrm{d}v$ is $f(v)\mathrm{d}v$. Then the mean speed of the gas particle is well defined as

$$\bar{v} \equiv \int_0^\infty \mathrm{d}v\, v\, f(v).$$

(a) Calculate the effusion rate Γ in terms of n and $\bar{v}$. Hints: Consider particles are flying towards all directions with equal probability. Only those that flies towards the hole will possibly get escape.

Let us consider the effusion of molecules first.

(b) Suppose the mass of the molecule is m, what is the speed distribution function $f(v)$ given the temperature T of the gas? Note: The result is known as Maxwell's speed distribution.

(c) Use the function $f(v)$ to calculate $\bar{v}$, and give the effusion rate as a function of n, m and T. Note: The effusion rate is inversely proportional to the mass of the molecule, thus provide us a method for isotope separation.

Now let us turn to the photon gas. As all photons moves with the same speed, the mean speed for photon is simply the speed of light $\bar{v} = c$. However the density of photon depends on temperature of the cavity.

(d) Find the effusion rate for photon as a function of temperature. Hint: You may find the temperature dependence of the number density of photons in a 3D cavity simply by thermodynamic arguments.

Solution:
(a) The particles of a particular velocity $\boldsymbol{v}$ will escape in $\mathrm{d}t$ if it is included in the volume $Av\cos\theta\,\mathrm{d}t$, where θ is the angle between $\boldsymbol{v}$ and the normal direction of the hole. The probability to find a particular with such velocity is $f(v)\sin\theta\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}\phi/(4\pi)$. Therefore the effusion rate is
$$\Gamma = \frac{n}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}v\, v\cos\theta\, f(v)\sin\theta = \frac{1}{4}n\bar{v}.$$
(b) Maxwell's speed distribution
$$f(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-\frac{mv^2}{2k_B T}}.$$
(c) The mean speed of molecules
$$\bar{v} = \left(\frac{8k_B T}{\pi m}\right)^{1/2}.$$
Therefore
$$\Gamma = n\left(\frac{k_B T}{2\pi m}\right)^{1/2}.$$

Chap. 42 Applications of Kinetic Theory

1. Polar Dielectrics

There are three kinds of dielectrics: polar, non-polar dielectrics, and ferroelectrics. The molecules of polar dielectric materials have permanent dipole moments, with fixed magnitudes and random orientations. Thus the polar dielectric can be considered as a set of free electric dipoles responding to the external electric field. A free dipole $\boldsymbol{p}$ in the electric field $\boldsymbol{E}$ has the potential energy
$$V = -\boldsymbol{p}\cdot\boldsymbol{E}.$$
(a) If dipoles obey Boltzmann statistics, what is the thermal average of dipole moment $\langle\boldsymbol{p}\rangle$ given the temperature T under external electric field $\boldsymbol{E}$?
If we consider unit volume of substance, we have dipole moment density, i.e. the polarization
$$\boldsymbol{P} = n\langle\boldsymbol{p}\rangle,$$
where n is the number density of dipole. In the weak field limit ($E \to 0$), the polarization of polar dielectric will linearly respond to E, hence
$$\boldsymbol{P} = \chi\epsilon_0\boldsymbol{E}$$
defining the polarizability χ.
(b) Find the polarizability χ in the weak field limit.
The thermal motion randomize the dipole orientations, thus suppressing the polarizability.

Solution:
(a) Partition function
$$Z = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\, e^{\beta pE\cos\theta} = \frac{4\pi}{\beta pE}\sinh\beta pE,$$
where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{E}$. The average moment $\langle\boldsymbol{p}\rangle$ lies along the direction

of $\boldsymbol{E}$, and its magnitude should be given by

$$\frac{1}{\beta}\partial_E \ln Z = p\left(\coth\beta pE - \frac{1}{\beta pE}\right).$$

$\langle\boldsymbol{p}\rangle = p(\coth\eta - \eta^{-1})\boldsymbol{e}_z$, where $\boldsymbol{e}_z$ is the unit vector along the direction of $\boldsymbol{E}$, and the parameter $\eta = pE/k_BT$.

Some students may consider it as the Ising model, if their answer is correct, that is also ok. For Ising model $\langle\boldsymbol{p}\rangle = p\tanh\eta\,\boldsymbol{e}_z$.

(b) The polarization in the weak field limit,

$$P = np = \frac{np^2E}{3k_BT}.$$

So the polar dielectric polarizability

$$\chi = \frac{P}{\epsilon_0 E} = \frac{np^2}{3\epsilon_0 k_BT}.$$

Note: for Ising model, $\chi = np^2/\epsilon_0 k_BT$ which is 3 times greater.

2. Thermodynamics Origin of Intermolecular Force

The dipole-dipole interaction, also called Keesom interaction, is a kind of intermolecular force caused by the attraction between permanent dipoles in molecules. Let us suppose two molecules with electric dipole moments $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ respectively separating by the relative displacement $\boldsymbol{r}$. The magnitude of the dipole moments are the same, $|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = p$, but the orientation can be different due to thermal fluctuation.

However, to simplify the calculation, let us project our problem in one dimension. Assuming each dipole only aligns in parallel or anti-parallel to their intermolecular displacement $\boldsymbol{r}$, which means if we define $p_1 r = \boldsymbol{p}_1\cdot\boldsymbol{r}$, $p_2 r = \boldsymbol{p}_2\cdot\boldsymbol{r}$, the scalars p_1 and p_2 only takes the value of $\pm p$.

(a) Show that the electrostatic potential energy between two dipoles along the same line is given by

$$V = \frac{p_1 p_2}{2\pi\epsilon_0 r^3}.$$

Due to thermal fluctuations, one cannot tell for sure weather p_1 and p_2 are of the same sign or not. However, different configurations may have different Boltzmann weight, and thus physical quantities should be averaged accordingly.

(b) Find the average potential energy $\bar{V}$ of the dipole-dipole system. Is it an attractive interaction or a repulsive one?

(c) Show that $\bar{V} \propto r^{-3}$ at low temperature, and $\bar{V} \propto r^{-6}$ when the temperature is high.

Note: The temperature being considered high or low is by comparison to the energy scale of static dipole-dipole interaction. The intermolecular interaction becomes stronger at low temperature. With strong attractions, molecules will condensed into liquid or solid. As the temperature rises, the interaction becomes weaker, and molecules evaporate into gas. For molecules in gas, the dipole-dipole interaction no longer follows the r^{-3} law, but shows the r^{-6} behavior. This is a thermodynamic effect.

Solution:

(b) The interaction is attractive,

$$\bar{V} = -\frac{p^2}{2\pi\epsilon_0 r^3}\tanh\frac{\beta p^2}{2\pi\epsilon_0 r^3}.$$

(c) At low temperature

$$\bar{V} = -\frac{p^2}{2\pi\epsilon_0 r^3}.$$

At high temperature

$$\bar{V} = -\frac{p^4}{4\pi^2\epsilon_0^2 r^6}\frac{1}{k_B T}.$$

Chap. 44 The Law of Thermodynamics

1. Finding the Entropy

For equilibrium system, the entropy is a function of state and can be determined up to a constant, from the equation of state and internal energy law. In this problem, we will determine the entropy of gas, solid and light.

All ideal gas follows the equation of state, $pV = Nk_BT$. For gas of molecules, the internal energy is $U = C_V T$, where the constant volume specific heat C_V is considered a constant independent of temperature. For photon gas (light), $U = aVT^4$, where a is also a constant.

(a) Determine the entropy $S(T,V)$ of molecule gas as a function of temperature and volume.

(b) Determine the entropy $S(V,p)$ of molecule gas as a function of volume and pressure.

(c) Determine the entropy $S(T,p)$ of molecule gas as a function of temperature and pressure.

(d) Determine the entropy $S(T,V)$ of photon gas as a function of temperature and volume.

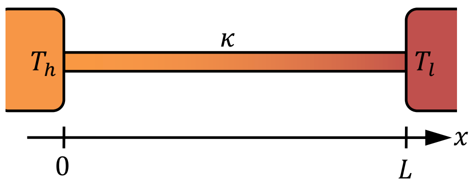
(e) Determine the entropy $S(V,p)$ of photon gas as a function of volume and pressure.

The equation of state for the incompressible idea solid simply states that the volume is a constant $V = \text{const}$. The internal energy can be given by $U = mcT$, where m is the mass of the solid, and c is the specific heat capacity.

(f) Determine the entropy $S(T)$ of incompressible idea solid as a function of temperature.

2. Entropy Production in Heat Conduction

A bar of length L, sectional area A and thermal conductivity κ is in contact with two heat reservoirs of different temperatures T_h and T_l (suppose $T_h > T_l$).



Assume the temperature gradient is uniform throughout the bar.

(a) Write down the temperature $T(x)$ as a function of position x.

(b) What is the amount of heat $\mathrm{d}Q$ flowing through the sectional area A of the bar in time

$\mathrm{d}t$?

(c) How much entropy flows into the bar from the high temperature reservoir, and how much entropy flows out of the bar to the low temperature reservoir, during the time $\mathrm{d}t$?

You will see that there is more entropy flowing out of the bar. The bar gains negative entropy of the heat reservoirs, but its thermodynamic state does not change. The where does this part of entropy go? This is a paradox. Because there is no conservation law for entropy. Even if the state of the system does not change, entropy can be produced by non-equilibrium process, such as heat transport. The entropy production rate $\sigma = \mathrm{d}S_{\mathrm{prod}}/(\mathrm{d}t\, V)$ measures the entropy production per unit of time per unit volume.

(d) Find the entropy production rate $\sigma(x)$ at each point x along the bar. On which side is the entropy production faster, the hotter side or the colder side?

(e) Show that the entropy production along the whole bar exactly balanced the negative entropy from the reservoirs, so that the state of the bar is not changed.

Solution:

(a) $T(x) = (T_l - T_h)x + T_h$.

(b) $\mathrm{d}Q = A\kappa(T_h - T_l)\mathrm{d}t/L$.

(c) The net entropy flowing into the reservoirs

$$\Delta S = \frac{\mathrm{d}Q}{T_l} - \frac{\mathrm{d}Q}{T_h} = A\kappa dt \frac{(T_h - T_l)^2}{L T_h T_l}.$$

(d) Entropy production rate

$$\sigma(x) = \frac{\kappa}{T^2}(\partial_x T)^2 = \frac{\kappa}{\big((T_l - T_h)x + T_h\big)^2}\left(\frac{T_h - T_l}{L}\right)^2.$$

Entropy produces faster on the colder side.

Chap. 45 Illustration of Thermodynamics

1. Thermodynamics of Solid

A often used phenomenological internal energy for solids is

$$U = A e^{b(V-V_0)^2} S^{\frac{4}{3}} e^{\frac{S}{3R}},$$

where A, b and V_0 are positive constants and R is the ideal gas constant; U is the internal energy, S the entropy and V the volume. We will see that many thermodynamic properties of solid can be deduced from this internal energy equation.

(a) Show that this internal energy satisfies the third law of thermodynamics. Hint: You may first find the temperature from the first law of thermodynamics.

(b) Show that the constant volume specific heat C_V is proportional to T^3 at low temperature (Debye's Law), and that it approaches to $3R$ at high temperature (Dulong-Petit's Law).

(c) Calculate the pressure p. What is the physical interpretation of V_0?

The thermal expansion coefficient α_p at constant pressure is defined as

$$\alpha_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p.$$

(d) Find the behavior of α_p for $p \to 0$. Is this behavior reasonable?

(e) Show that

$$\left(\frac{\partial p}{\partial T}\right)_V = -\frac{6bRS(4R+S)(V-V_0)}{4R^2+8RS+S^2}.$$

Note: The equation of state can be solved from this differential equation, however the result may be complicated.

Solution:

(a) The temperature can be found from

$$T = \left(\frac{\partial U}{\partial S}\right)_V = Ae^{b(V-V_0)^2}S^{\frac{1}{3}}e^{\frac{S}{3R}}\left(\frac{4}{3}+\frac{S}{3R}\right).$$

T is a monatomic function of S. As $T \to 0$, $S \to 0$, thus satisfying the third law of thermodynamics.

(b) The constant volume specific heat is

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial S}\right)_V\left(\frac{\partial^2 U}{\partial S^2}\right)_V^{-1} = \frac{3RS(4R+S)}{4R^2+8RS+S^2}.$$

At low temperature $T \propto S^{1/3}$, and $C_V = 3S$, so $C_V \propto T^3$. At high temperature $S \to \infty$, and $C_V = 3R$.

(e) Let $f(V) = Ae^{b(V-V_0)^2}$ and $g(S) = S^{4/3}e^{S/3R}$. Then $U = f(V)g(S)$, and $T = f(V)g'(S)$. Differential on both sides, to keep T as a constant

$$0 = \mathrm{d}T = f'(V)g'(S)\mathrm{d}V + f(V)g''(S)\mathrm{d}S,$$

thus

$$\left(\frac{\partial S}{\partial V}\right)_T = -\frac{f'(V)g'(S)}{f(V)g''(S)} = -\frac{6bRS(4R+S)(V-V_0)}{4R^2+8RS+S^2}.$$

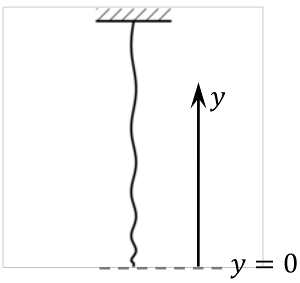
According to the Maxwell relation

$$\left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T = -\frac{6bRS(4R+S)(V-V_0)}{4R^2+8RS+S^2}.$$

Chap. 47 Sound. The Wave Equation

1. Transverse Wave in Hanging Rope

A uniform rope of length L hangs from the ceiling.



(a) Find the speed of a transverse wave in the rope is a function of y (the distance from the lower end).
(b) Find the time it takes a transverse wave to travel the length of the rope. Note: This time is independent of the actual mass of the rope. Therefore you can determine the length a long rope hanging down from the ceiling by striking its lower end and measuring the time intervals between echoes.

Solution:
(a) Tension in the rope: $T = \lambda g y$, where λ is the mass density of the rope. The velocity of transverse wave in the rope can be calculated from the formula

$$v = \sqrt{T/\lambda} = \sqrt{gy}.$$

(b) From velocity

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v = \sqrt{gy}.$$

By integral

$$t = 2g^{-1/2}y^{1/2}\Big|_0^L = 2\sqrt{L/g}.$$

2. Massive Spring (II): Sound Wave in Elastic Medium

In the Lecture, we have learnt the wave equation of sound in the air. In this problem we will deduce the sound wave equation in the elastic medium (solid). Our simple model is a massive spring of the original length L. Its total mass is m (uniformly distributed throughout the spring), and the stiffness (or the force constant) is k.

You can imagine the longitudinal sound wave propagating in the massive spring. Let us label each segment of the spring by its position x at equilibrium, and use $\chi(x,t)$ to denote the displacement of that piece of spring at time t.
(a) Deduce the wave equation for $\chi(x,t)$. Hint: analyze the motion of a little segment of the spring.
(b) Write down the dispersion relation $\omega(q)$ (circular frequency as a function of wave number) for the sound wave. Find the sound velocity c.
Assume a sound wave propagating in the massive spring,

$$\chi(x,t) = \chi_0 \cos q(x - ct).$$

(c) Find the kinetic energy density and potential energy density. Are they always the same everywhere? Note: the energy density here denotes the energy in the spring segment of unit *original* length.
The kinetic energy density and the potential energy density reach their maximum and minimum at the same time and at the same position. It is one of the most important difference between waves and harmonic oscillators, that the kinetic and the potential energy do not mutually transform in waves, but to grow together and to fade together. Does it contradict with the conservation of energy? No, because what is varying is not the total energy but the energy density. There is no conservation law for energy density, since it can

be transported by the energy flow.

(d) What is the energy flow on the spring? Find the instantaneous energy flow as well as the energy flow averaged over a wave period. Note: the energy flow is the amount of energy transferred per unit of time.

Solution:

(a) $(m/L)\partial_t^2\chi - (kL)\partial_x^2\chi = 0$.

(b) $\omega = cq$, where $c = L\sqrt{k/m}$.

(c) The energy density

$$\mathcal{T} = \frac{1}{2}\left(\frac{m}{L}\right)(\partial_t\chi)^2 = \frac{1}{2}kL\chi_0^2 q^2 \sin^2 q(x - ct),$$

$$\mathcal{V} = \frac{1}{2}(kL)(\partial_x\chi)^2 = \frac{1}{2}kL\chi_0^2 q^2 \sin^2 q(x - ct).$$

They are always the same everywhere.

(d) The total energy density

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = kL\chi_0^2 q^2 \sin^2 q(x - ct).$$

The energy flow

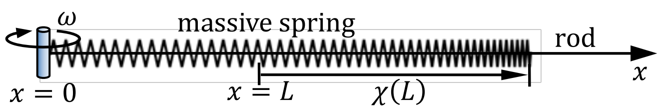
$$\mathcal{S} = c\mathcal{E} = ckL\chi_0^2 q^2 \sin^2 q(x - ct).$$

The average energy flow

$$\langle\mathcal{S}\rangle = \frac{1}{2}ckL\chi_0^2 q^2.$$

3. Massive Spring (III): Dispersive Sound Wave

Let us still look at the uniform massive spring of the original length L, the mass m and the stiffness k. But this time it is threaded on a straight rod that rotates in the plane with constant angular velocity ω about the axis at the end of the rod. The spring is also tied to the axis so that it will not get thrown during the rotation. However the centrifugal force will stretch the spring, so that there could be non-uniform deformation as you can see in the following figure.



We will label each segment of the spring by its equilibrium position x *when there is no rotation* ($\omega = 0$). Then the relative displacement (deformation) is described by $\chi(x, t)$.

(a) In the rotating frame, what is the additional potential energy that the spring will experience due to the action of centrifugal force?

(b) Deduce the wave equation for the deformation field $\chi(x, t)$. Hint: It might be easier to consider from the energy viewpoint. The total energy of the spring comes from three parts: the kinetic energy, the elastic potential energy and the centrifugal potential energy. Compare with the energy of the non-rotating spring, you might be able to see what extra term should be added to the wave equation.

The wave equation is not a linear equation of χ because $\chi = 0$ is not the equilibrium deformation. So we need to find the static configuration of χ first, upon which we can study the sound wave dynamics.

(c) Under the static condition $\partial_t \chi_0(x,t) = 0$ what is the solution of $\chi_0(x,t)$ that satisfies the wave equation?

(d) How long the spring will be upon rotation with the angular velocity ω?

(e) What is the critical angular velocity ω_c on which the spring will be ruptured? Note: The rupture strength tends to infinity in our ideal model.

Now we should consider the propagation of the little perturbation $\delta\chi$ on the equilibrium deformation χ_0.

(f) Substitute $\chi = \chi_0 + \delta\chi$ into the wave equation for χ to obtain the wave equation for $\delta\chi$, from which determine the dispersion relation $\Omega(q)$ of the sound wave in the rotating spring. Note: Let Ω be the circular frequency of the sound wave, and q be the corresponding wave number.

(g) Find the phase velocity and group velocity of the sound. Compare these velocities with the sound speed in the non-rotating massive spring.

In quantum mechanics, the quantized sound wave is called a *phonon*. The dispersion relation of the sound wave gives the energy-momentum relation of the phonon.

(h) What is the "rest mass" of the phonon in the rotating spring? Note: The so called "rest mass" here is by the analog of sound to light.

Upon rotation, the phonon in the spring acquires rest mass. This is because the rotation breaks the translational symmetry of the spring, which indicates that even a global displacement of the spring will experience the restoring force, and hence the phonon mode will be gapped. The broken symmetry is one possible origin of mass.

Solution:

(a) The centrifugal potential energy

$$V_{\text{cent}} = \int_0^L \mathrm{d}x \frac{m\omega^2}{2L}(x+\chi)^2.$$

(b) Wave equation for χ

$$\left(\frac{m}{L}\right)\partial_t^2\chi - (kL)\partial_x^2\chi - \left(\frac{m\omega^2}{L}\right)(x+\chi) = 0.$$

(c) $\partial_t \chi_0(x,t) = 0$, then

$$\partial_x^2\chi_0 = -(\omega/c)^2(x+\chi_0),$$

where we have defined $c^2 = kL^2/m$. Let $u = x + \chi_0$ the equation becomes

$$\partial_x^2 u = -(\omega/c)^2 u.$$

The solution is

$$u = A\cos(\omega x/c) + B\sin(\omega x/c).$$

Therefore with the boundary condition $\chi_0|_{x=0} = 0$ and $\partial_x\chi_0|_{x=L} = 0$,

$$\chi_0(x,t) = -x + \frac{\sin(\omega x/c)}{(\omega/c)\cos(\omega L/c)}.$$

(d) Length under rotation

$$L(\omega) = \frac{\tan(\omega L/c)}{(\omega/c)}.$$

(e) $\omega_c = (\pi/2)\sqrt{k/m}$.

(f) Wave equation for $\delta\chi$

$$\left(\frac{m}{L}\right)\partial_t^2\delta\chi - (kL)\partial_x^2\delta\chi - \left(\frac{m\omega^2}{L}\right)\delta\chi = 0.$$

Dispersion relations

$$\Omega^2 = c^2q^2 - \omega^2,$$

where $c^2 = kL^2/m$.

(g) Phase velocity

$$v_p = c\left(1 - \left(\frac{\omega}{cq}\right)^2\right)^{1/2}.$$

Group velocity

$$v_g = c\left(1 - \left(\frac{\omega}{cq}\right)^2\right)^{-1/2}.$$

We have $v_g < c < v_p$.

(h) $m_0 = i\hbar\omega/c^2$.

Chap. 48 Beats

Chap. 49 Modes

1. Pulsation Period of White Dwarf

The period of a pulsating variable star may be estimated by considering the star to be executing radial longitudinal pulsations in the fundamental standing wave mode, that is, the radius varies periodically with time, with a displacement antinode at the surface.

(a) Would you expect the center of the star to be a displacement node or antinode?

(b) By analogy with the open organ pipe, show that the period T of pulsation is given by

$$T = \frac{4R}{v}.$$

where R is the equilibrium radius of the star and v is the average sound speed.

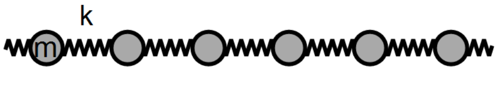
(c) Typical white dwarf starts have pressure of $10^{22}\mathrm{Pa}$, density of $10^{10}\mathrm{kg\cdot m^{-3}}$, ratio of specific heats of 4/3, and radius approximately the same as the radius of the Earth. Estimate the pulsation period of a white dwarf.

Solution:

(c) 22s.

2. Wave in Solids

The dispersion of wave in solids can be demonstrated by the simple model of coupled oscillations in one-dimensional crystal. Suppose atoms of mass m are aligned in a line and interact with each other. The interacting force between neighboring atoms can be modeled by a spring of the stiffness k. At equilibrium, these atoms form the one-dimensional crystal with lattice constant a (the spacing between neighboring atoms).



Now if these atoms are exited, sound wave will propagate in the crystal as a collective mode. Let u_l be the displacement of the lth atom from its equilibrium position.

(a) Write down the equation of motion for u_l.

It is known that sound waves are normal modes of these coupled oscillators.

(b) Write down your trial wave $u_l(t)$, and use it to find the frequency of all normal mode. Show that the sound wave in the crystal is dispersive

$$\omega(q) = 2\sqrt{k/m}\sin|qa/2|,$$

where q is the wave number and ω the frequency.

(c) Find the velocity of sound in the long wave length limit.

(d) Calculate the group velocity. Show that the highest frequency model cannot propagate in the crystal. Can you figure out why?

3. Modes on Regular Triangular Drum

We have learnt in the Lecture that the circular frequency ω of vibration mode on a square drum of the side length a is given by

$$\omega^2 = \left(\frac{\pi c}{a}\right)^2 (m^2 + n^2),$$

where c is the sound velocity in the drumhead, and m, n are positive integers labeling the modes.

(a) Find the density of states (the number of modes per unit circular frequency per unit area) of the square drum around a given ω, in the continuum limit (i.e. $\omega \gg \pi c/a$).

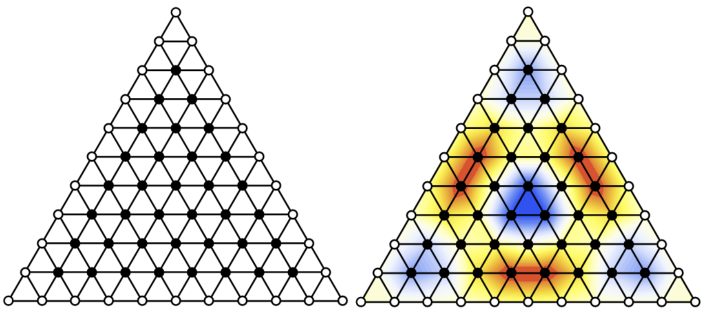
Now let us consider a regular triangular drum with side length a. The mode pattern of the displacement ϕ of the triangular drumhead is determined by the eigen problem

$$-c^2\nabla^2\phi = \omega^2\phi,$$

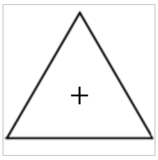
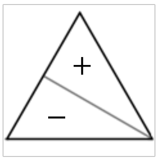
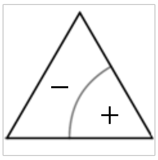
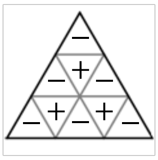
subject to the boundary condition that ϕ must vanish on the triangular edge.

(b) Design a computer program to solve the eigen problem. Draw the patterns of the first 11 modes and find their corresponding frequencies.

Hint: You may discretize the drumhead into a grid of little triangles as shown below. Label the grid points, and collect the displacement field ϕ on them into a column vector.



Then discretize the operator ∇^2 on the triangular lattice to its matrix representation, such that it can be applied to the vector of displacement field, just as applying ∇^2 operator to ϕ. Thus the eigen problem of a partial differential equation is reduced to the diagonalization of the matrix representation of ∇^2, which can be easily solved by a mathematical software in few seconds. The boundary points (hollow circles in the figure) should be treated specially to ensure the open boundary condition. The trick is to assign very large diagonal elements for the boundary points, such that the boundary states will be rised to a subspace of extremely high momentum, hence being effectively projected out in the low momentum subspace which we concern. You are expected to obtain the following result:

Mode shape	m	n	$(\omega/\omega_0)^2$	ω/ω_0
	1	1	5.33	2.31
	1	2	12.44	3.53
	2	1	12.44	3.53
...?	...?	...?	...?	...?
	3	3	48.00	6.93

Here the circular frequency is measured in the unit $\omega_0 \equiv \pi c/a$.

(c) Use your numerical result to check that the mode frequencies of the triangular drum are given by the following formula:

$$\omega^2 = \left(\frac{4\pi c}{3a}\right)^2 (m^2 + mn + n^2),$$

with m and n both positive integers.

Hint: For those who wish to deduce this formula analytically, just generalize Feynman's plane wave reflection arguments in the Lecture. Note that on the triangular drum, we have six waves to be added together instead of four for the square drum.

(d) Find the density of states of the triangular drum in the continuum limit. Compare it with the square drum case.

Although the frequency spectrums are totally different for different shape of drums, their densities of states somehow remain the same. This is an important property for the density of sate. It has a notable consequence in thermodynamic, that the thermal behaviors of gas (e.g. the ideal gas law $pV = nRT$) are totally independent of the shape of the container. No matter square or round, only the volume matters.

Solution:

(a) The number of states below the $m^2 + n^2 = r^2$ is

$$N = \frac{1}{4}\pi r^2.$$

While the circular frequency on $m^2 + n^2 = r^2$ follows $(\omega a/\pi c)^2 = r^2$, so

$$N = \frac{a^2}{4\pi c^2}\omega^2.$$

Thus from $g = A^{-1}\mathrm{d}N/\mathrm{d}\omega$,

$$g = \frac{1}{2\pi c^2}\omega,$$

where $A = a^2$ is the area of the drumhead.

(d) Under the curve $m^2 + mn + n^2 = r^2$, the area is given by the integral

$$N = \int_0^r \mathrm{d}n \left(-\frac{n}{2} + \frac{1}{2}\sqrt{4r^2 - 3n^2}\right) = \frac{\pi}{3\sqrt{3}} r^2,$$

while $r^2 = (3a\omega/4\pi c)^2$, so

$$N = \frac{\pi}{3\sqrt{3}}\left(\frac{3a\omega}{4\pi c}\right)^2.$$

Hence the same density of state

$$g = \frac{1}{2\pi c^2}\omega,$$

where the surface area is given by $A = \sqrt{3}a^2/4$ for the regular triangle.

Chap. 51 Waves

1. Water Wave

Surface wave in liquid has different phase and group velocity. The general expression for phase velocity is

$$v_p = \left(\left(\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}\right)\tanh\frac{2\pi h}{\lambda}\right)^{1/2},$$

where h is the depth, ρ is the density, and σ is the surface tension (for water, $\sigma = 73 \times 10^{-3}\mathrm{N}\cdot\mathrm{m}^{-1}$).

(a) For deep water, find the phase velocity.

(b) Show that the phase velocity has a minimum when the wave length is $\lambda_0 = 1.7\mathrm{cm}$.

According to the wavelength, the surface wave can be classified as gravity wave and capillary wave. The capillary waves are ripples.

(c) Find the phase velocity and group velocity for gravity wave. What is the ratio between group velocity and phase velocity?

(d) Find the phase velocity and group velocity for capillary wave. What is the ratio between group velocity and phase velocity?

(e) If the water is shallow, find the phase velocity and group velocity for gravity wave.

Solution:

(a) If $h \gg \lambda$

$$v_p = \left(\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}\right)^{1/2},$$

(b) $\lambda_0 = 2\sqrt{\sigma/(\rho g)} = 1.7\mathrm{cm}$.

(c) For gravity wave,

$$v_p = \sqrt{\frac{g\lambda}{2\pi}},$$

$$v_g = \frac{1}{2} v_p.$$

(d) For capillary wave,

$$v_p = \sqrt{\frac{2\pi\sigma}{\rho\lambda}},$$

$$v_g = \frac{3}{2} v_p.$$

(e) For shallow water

$$v_g = v_p = \sqrt{gh}.$$