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Quantum Mechanics B (Physics 212B) Winter 2019
Worksheet 5 – Solutions

Problems

1. **Spin-1 Pair**

   Consider a pair of spin-1 particles with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ where each factor is spanned by S^z eigenstates $\mathcal{H}_i = \text{span}\{|1\rangle_i, |0\rangle_i, |-1\rangle_i\}$

   (a) Calculate the dimension of $\mathcal{H}$

   $\text{dim}\mathcal{H} = 3 \times 3 = 9$

   (b) Write the fusion rule for two spin-1's and confirm the dimension of $\mathcal{H}$ agrees with its representation as a direct sum

   $1 \times 1 = 0 + 1 + 2$ so therefore $\mathcal{H} = \mathcal{H}_{\ell=0} \oplus \mathcal{H}_{\ell=1} \oplus \mathcal{H}_{\ell=2}$

   $\text{dim}\mathcal{H}_\ell = 2\ell + 1$ so $\text{dim}\mathcal{H} = 1 + 3 + 5 = 9$ which agrees with the above

   (c) Define $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ as the spin vector for the i-th spin. The spin generators in the $\ell = 1$ representation are given as:

   $$S_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \begin{pmatrix} 0 & -\mathbf{i} & 0 \\ \mathbf{i} & 0 & -\mathbf{i} \\ 0 & \mathbf{i} & 0 \end{pmatrix} \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

   Confirm that $\vec{S}_i^2 = \ell(\ell+1)\mathbb{1}$ for $\ell = 1$

   Just matrix multiplication. $\vec{S}_i^2 = 2\mathbb{1}$

   (d) The 'total spin' is $\vec{S} = \sum_i \vec{S}_i = \vec{S}_1 + \vec{S}_2$. Compute the Casimir operator $\vec{S}^2$ using the relation above.

   $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1 \cdot \vec{S}_2 = 4\mathbb{1} + 2\vec{S}_1 \cdot \vec{S}_2$

   (e) By the fusion rule, $\vec{S}^2$ has 3 distinct eigenvalues. Write it as a matrix in the direct sum basis $|j, m_j\rangle$

   The values of j are $\{0, 1, 2\}$ so it is a 9×9 diagonal matrix with eigenvalues: $\vec{S}^2 = \text{diag}\{0, 2, 2, 2, 6, 6, 6, 6, 6\}$

   (f) This way of writing it is $\vec{S}^2 = \sum_j j(j+1)P^{(j)}$ where j are the allowed spins in the direct sum and $P^{(j)}$ is the projector onto the j-spin subspace.

   The projectors satisfy: $P^{(j)}|j', m_{j'}\rangle = 0$ for $j' \neq j$ otherwise $P^{(j)}|j, m_j\rangle = |j, m_j\rangle$

   Express the projectors $P^{(0)}, P^{(1)}, P^{(2)}$ for the system above as polynomials in $\vec{S}^2$

   Consider $P^{(0)}$ first. It must vanish when $\vec{S}^2$ acts on a state of eigenvalue 2 or 6. This implies: $P^{(0)} \propto (\vec{S}^2 - 2\mathbb{1})(\vec{S}^2 - 6\mathbb{1})$

The proportionality constant is determined by $P^{(0)}$ giving 1 when $\vec{S}^2 = 0$. This gives: $P^{(0)} = \frac{1}{12}(\vec{S}^2 - 2\mathbb{1})(\vec{S}^2 - 6\mathbb{1})$
We can use the same trick for the other projectors;
multiply factors of $(\vec{S}^2 - \ell(\ell+1)\mathbb{1})$ where ℓ are the spins being projected out.
$P^{(1)} = -\frac{1}{8}\vec{S}^2(\vec{S}^2 - 6\mathbb{1})$ and $P^{(2)} = \frac{1}{24}\vec{S}^2(\vec{S}^2 - 2\mathbb{1})$

(g) Use the above result to write the spin-2 projector as:

$$P^{(2)} = \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{2}(\vec{S}_1 \cdot \vec{S}_2) + \frac{1}{3}\mathbb{1} \tag{2}$$

This is just plugging in $\vec{S}^2 = 4\mathbb{1} + 2\vec{S}_1 \cdot \vec{S}_2$ into the above polynomial expression.

2. **Two Becomes Four**

Consider a pair of spin-$\frac{1}{2}$ particles. We'd like to represent each spin-1 particle above as a subspace of the pair.

(a) Given just one pair of spin-$\frac{1}{2}$, what state must be projected out from the Hilbert space to make a spin-1 particle?
The fusion rule is $\frac{1}{2} \times \frac{1}{2} = 0 + 1$ so one needs to project out the singlet state with $j = 0$.
The triplet states make a spin-1 and are symmetric wavefunctions:
$|1\rangle = |\uparrow\uparrow\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |-1\rangle = |\downarrow\downarrow\rangle$

(b) Let's represent our pair of spin-1 particles as four spin-$\frac{1}{2}$ particles. Write the fusion rule for the four spin-$\frac{1}{2}$'s
We can fuse each pair first to get: $(0+1) \times (0+1)$
This then gives: $0 + 1 + 1 + (0 + 1 + 2)$
One can check the dimensions add up correctly: $1 + 3 + 3 + 1 + 3 + 5 = 16 = 2^4$

(c) You should notice that the spin-2 appears only from one term. The upshot of this is that the projector $P^{(2)}$ will annihilate any state of the four qubits where *any* two are in a singlet.
In particular, $P^{(2)}$ will annihilate the four qubit state:

$$|\psi\rangle = |\uparrow\rangle_1(|\uparrow\rangle_2|\downarrow\rangle_3 - |\downarrow\rangle_2|\uparrow\rangle_3)|\uparrow\rangle_4 = |\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle \tag{3}$$

To recover a state in the Hilbert space of a pair of spin-1 particles this must be symmetrized.
Write this wavefunction explicitly in both the four qubit representation and the two spin-1 representation.
The symmetrized factor one should add is $|\uparrow\rangle_2(|\uparrow\rangle_1|\downarrow\rangle_4 - |\downarrow\rangle_1|\uparrow\rangle_4)|\uparrow\rangle_3$
This gives a four qubit state: $|\Psi\rangle = |\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\uparrow\rangle$
This can be factorized as: $|\Psi\rangle = |\uparrow\uparrow\rangle \otimes (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) - (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \otimes |\uparrow\uparrow\rangle$
Writing it in spin-1 language: $|\Psi\rangle = |1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle$

3. **Many Spins**

Consider a circle of L many spin-1 particles with the Hamiltonian:

$$H = \sum_{i=1}^{L} \frac{1}{6}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{2}(\vec{S}_i \cdot \vec{S}_{i+1}) + \frac{1}{3}\mathbb{1} = \sum_{i=1}^{L} P^{(2)}[i, i+1] \tag{4}$$

While this is a fine-tuned model, the coefficients of the terms are very specific, it describes a spin-chain with nearest-neighbor interactions and spin-rotation symmetry. It is known as the 'AKLT model'.

(a) Conclude that if you find a wavefunction $|\Psi\rangle$ which is annihilated by each of the $P^{(2)}[i, i+1]$ that it is a ground-state of the model.

The Hamiltonian is a sum of projects which are all manifestly non-negative. Because there is no possibility for cancellations, any state where each projector gives 0 must be the lowest energy state of the model. Such a Hamiltonian is 'frustration free'.

(b) Consider each spin-1 particle to be broken up as a pair of spin-$\frac{1}{2}$ particles. Find the generalization of the state above on 4 qubits that is annihilated by all the projectors $P^{(2)}[i, i+1]$.

Let $|\alpha_i\beta_i\rangle = \frac{1}{\sqrt{2}}(|\alpha_i\beta_i\rangle + |\beta_i\alpha_i\rangle)$ be the symmetrized 2-qubit wavefunction associated with the original spin-1 at site i. Each α and β can be either $\uparrow$ or $\downarrow$; denoted by 0 or 1 respectively.

In this notation, the wavefunction for a single pair of spin-1 particles above (written as 4 qubits) was given as $|\psi\rangle = \epsilon^{\beta_1\alpha_2}\epsilon^{\beta_2\alpha_1}|\alpha_1\beta_1\rangle|\alpha_2\beta_2\rangle = |00\rangle|10\rangle - |01\rangle|00\rangle$

The generalization to many sites is then $|\Psi\rangle = \epsilon^{\beta_1\alpha_2}\epsilon^{\beta_2\alpha_3}\cdots\epsilon^{\beta_L\alpha_1}|\alpha_1\beta_1\rangle|\alpha_2\beta_2\rangle\cdots|\alpha_L\beta_L\rangle$

The picture of this state is that each bond, written as 4 qubits, has a singlet state between the middle two.