University of California at San Diego – Department of Physics – TA: Shauna Kravec

Quantum Mechanics A (Physics 212A) Fall 2018
Worksheet 4 – Solutions

Problems

1. **Interferometry**

   We can consider the path taken of a photon as (approximately) a two-state quantum system spanned by $|u\rangle$, $|d\rangle$ for whether it went up or down respectively.

   Consider the following two interferometers:

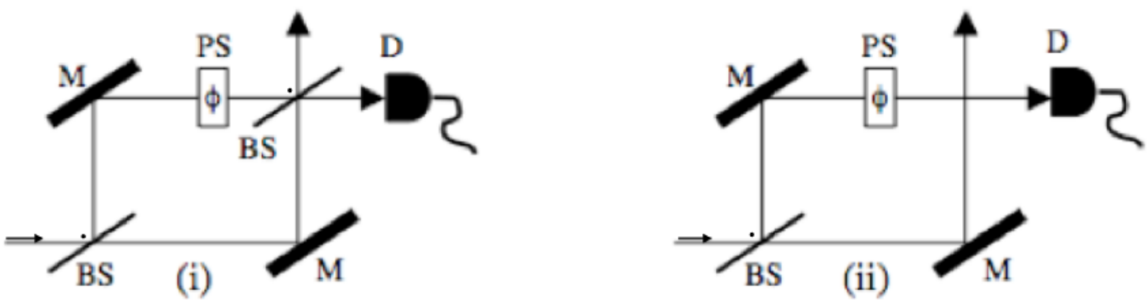

Figure 1: A balanced Mach-Zender interferometer (i) and another with the final beam splitter removed (ii)

   The elements BS are beam splitters which implement the Hadamard gate on the incoming beam: $H = \frac{1}{\sqrt{2}}[(|u\rangle + |d\rangle)\langle u| + (|u\rangle - |d\rangle)\langle d|]$

   The elements M are mirrors which transform $|\text{In}\rangle \rightarrow |\text{Out}\rangle = -|\text{In}\rangle$

   The element PS is a phase shifter which transform $|\text{In}\rangle \rightarrow |\text{Out}\rangle = e^{\mathbf{i}\phi}|\text{In}\rangle$

   The element D is a detector which measures photons going in.

   For each device, determine the probability for detecting a photon as a function of ϕ.

   Let's do setup (ii) first.

   After the first beam splitter the state is $|1\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$

   The pair of mirrors introduces an overall minus sign $|2\rangle = -|1\rangle$ which is irrelevant

   The phase shifter then acts on the up travelling beam $|3\rangle = \frac{1}{\sqrt{2}}(e^{\mathbf{i}\phi}|u\rangle + |d\rangle)$

   But then the only part of the beam entering the detector D is the up component. Therefore $P(\phi) = |\langle u|3\rangle|^2 = \frac{1}{2}$. Pretty boring.

   For the true MZ interferometer (i) we apply the Hadamard:

   $|4\rangle = H|3\rangle = \frac{1}{2}(e^{\mathbf{i}\phi} + 1)|u\rangle + \frac{1}{2}(e^{\mathbf{i}\phi} - 1)|d\rangle$

   Then again it's measuring 'up' component so $P(\phi) = |\langle u|4\rangle|^2 = \frac{1}{4}|e^{\mathbf{i}\phi}+1|^2 = \frac{1}{2}(1+\cos\phi)$

2. **Time-energy uncertainty**

Consider an observable $\mathbf{A}$ which has no explicit dependence on time. Define the variance of $\mathbf{A}$ in a state ψ to be

$$\Delta A \equiv \sqrt{\langle(\mathbf{A} - \langle\mathbf{A}\rangle_\psi)^2\rangle_\psi}$$

as usual, with $\langle\mathbf{A}\rangle_\psi \equiv \langle\psi|\mathbf{A}|\psi\rangle$. Let

$$\Delta T \equiv \frac{\Delta A}{|\partial_t \langle\mathbf{A}\rangle_\psi|}.$$

This is a measure of the time required for $\langle\mathbf{A}\rangle$ to change significantly, *i.e.* by an amount comparable to its variance ΔA. Show that

$$\Delta E \Delta T \geq \frac{\hbar}{2},$$

where ΔE is the variance of $\mathbf{H}$, the Hamiltonian.

Let $H_0 \equiv \mathbf{H} - \langle\mathbf{H}\rangle_\psi \equiv \mathbf{H} - E$ and $A_0 \equiv \mathbf{A} - \langle\mathbf{A}\rangle_\psi$. Apply Cauchy-Schwarz to $|\alpha\rangle \equiv A_0|\psi\rangle$ and $|\beta\rangle \equiv H_0|\psi\rangle$. This gives

$$(\Delta A)^2(\Delta E)^2 \geq \frac{1}{4}|\langle\psi|[\mathbf{A}, \mathbf{H}]|\psi\rangle|^2 = \frac{1}{4}|\mathbf{i}\hbar\partial_t\langle\mathbf{A}\rangle_\psi|^2.$$

where in the last step we've used the Heisenberg equation of motion.

3. **Entropy and thermodynamics**

Consider a quantum system with hamiltonian $\mathbf{H}$ and Hilbert space $\mathcal{H} = \text{span}\{|n\rangle, n = 0, 1, 2...\}$. Its behavior at finite temperature can be described using the *thermal density matrix*

$$\boldsymbol{\rho}_\beta \equiv \frac{1}{Z}e^{-\beta\mathbf{H}}$$

where $\beta \equiv \frac{1}{T}$ specifies the temperature and Z is a normalization factor. (We can think about this as the reduced density matrix resulting from coupling the system to a heat bath and tracing out the Hilbert space of the (inaccessible) heat bath.)

(a) Find a formal expression for Z by demanding that $\boldsymbol{\rho}_\beta$ is normalized appropriately. (Z is called the *partition function.*)
Recall that $\text{Tr}\ [\rho_\beta] = 1 = \frac{1}{Z}\text{Tr}\ [e^{-\beta\mathbf{H}}] \rightarrow Z = \text{Tr}\ [e^{-\beta\mathbf{H}}] = \sum_i e^{-\beta E_i}$

(b) Show that the von Neumman entropy of $\boldsymbol{\rho}_\beta$ can be written as

$$S_\beta = E/T + \log Z \tag{1}$$

where $E \equiv \langle\hat{H}\rangle_\rho = \text{Tr}\ \rho\hat{H}$ is the expectation value for the energy. The expression above for S_β is the thermal entropy.
Recall the definition of the von Neumman entropy: $S[\rho_\beta] = -\text{Tr}\ [\rho_\beta \log \rho_\beta]$
$\log \rho_\beta = \log \frac{e^{-\beta\mathbf{H}}}{Z} = \log e^{-\beta\mathbf{H}} - \log Z = -(\beta\mathbf{H} + \log Z)$
$\rightarrow S = \text{Tr}\ [\frac{e^{-\beta\mathbf{H}}}{Z}(\beta\mathbf{H} + \log Z)] = \text{Tr}\ [\rho_\beta\beta\mathbf{H}] + \text{Tr}\ [\rho_\beta \log Z] = \beta\langle\mathbf{H}\rangle + \log Z \text{Tr}\ [\rho_\beta]$
Therefore $S = \frac{E}{T} + \log Z$ by $\text{Tr}\ [\rho_\beta] = 1$ and $\beta = \frac{1}{T}$

(c) Evaluate Z and E for the case where the system is a simple harmonic oscillator

$$\mathbf{H} = \hbar\omega\left(\mathbf{n} + \frac{1}{2}\right)$$

with $\mathbf{n}|n\rangle = n|n\rangle$.

$Z = \sum_n e^{-\beta E_n} = \sum_i e^{-\omega\beta(n+\frac{1}{2})} = e^{-\frac{\omega\beta}{2}} \sum_{n=0}^{\infty} e^{-\omega\beta n}$

This sum is a geometric series: $\sum_n x^{-n} = \frac{x}{x-1}$ which implies: $Z = e^{-\frac{\beta\omega}{2}} \frac{e^{\beta\omega}}{e^{\beta\omega}-1}$

One way to evaluate E is to put our statistical mechanics knowledge to work: $\langle E \rangle = -\frac{\partial \log Z}{\partial \beta} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{\omega}{2}(\frac{1+e^{\beta\omega}}{e^{\beta\omega}-1})$