Quantum Mechanics

Second Quantization

Second Quantization

■ Quantum Many-Body States

■ Identical Particles

Second quantization is a formalism used to describe quantum many-body systems of **identical particles**.

- Classical mechanics: each particle is labeled by a distinct position $\boldsymbol{r}_i \Rightarrow$ any different configuration of $\{\boldsymbol{r}_i\}$ correspond to a different classical many-body state.
- Quantum mechanics: particles are *identical*, such that *exchanging* two particles ($\boldsymbol{r}_i \leftrightarrow \boldsymbol{r}_j$) does *not* lead to a different quantum many-body state.

Permutation symmetry of identical particles ⇒ **joint probability distribution** must be *invariant* under permutation:

$$p(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots) = p(\ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_i, \ldots), \tag{1}$$

where the probability distribution p is related to the **many-body wave function** Ψ by

$$p(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots) = |\Psi(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots)|^2. \tag{2}$$

The wave function can only change up to an *overall phase factor*.

$$\Psi(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots) = e^{i\varphi}\,\Psi(\ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_i, \ldots). \tag{3}$$

It forms as a **one-dimensional representation** of the **permutation group**. Mathematical fact: there are only *two* 1-dim representations for any permutation group,

- **trivial** representation ⇒ **bosons**

$$\Psi_B(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots) = +\Psi_B(\ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_i, \ldots) \tag{4}$$

- **sign** representation ⇒ **fermions**

$$\Psi_F(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots) = -\Psi_F(\ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_i, \ldots) \tag{5}$$

■ Dirac Notations

Let us rephrase this using *Dirac ket-state notation* (more concise). Consider a complete set of **single-particle states** $|\alpha\rangle$ (labeled by α)

$$|\alpha\rangle = \int d^d \boldsymbol{r}\, \psi_\alpha(\boldsymbol{r})\, |\boldsymbol{r}\rangle, \tag{6}$$

where $\psi_\alpha(\boldsymbol{r})$ is the wave function representing the state.

- A two-particle state with the 1st particle in $|\alpha_1\rangle$ and the 2nd particle in $|\alpha_2\rangle$ will be described by

$$\begin{aligned}|\alpha_1\rangle \otimes |\alpha_2\rangle &= \int d^d \boldsymbol{r}_1 \int d^d \boldsymbol{r}_2\, \psi_{\alpha_1}(\boldsymbol{r}_1)\, \psi_{\alpha_2}(\boldsymbol{r}_2)\, |\boldsymbol{r}_1\rangle \otimes |\boldsymbol{r}_2\rangle \\ &= \int d^d \boldsymbol{r}_1 \int d^d \boldsymbol{r}_2\, \Psi(\boldsymbol{r}_1, \boldsymbol{r}_2)\, |\boldsymbol{r}_1\rangle \otimes |\boldsymbol{r}_2\rangle.\end{aligned} \tag{7}$$

  $\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \psi_{\alpha_1}(\boldsymbol{r}_1)\, \psi_{\alpha_2}(\boldsymbol{r}_2)$ is identified as the two-body wave function.

- Exchanging $\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2$ in the wave function $\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ leads to a new wave function $\Psi'(\boldsymbol{r}_1, \boldsymbol{r}_2)$

$$\Psi'(\boldsymbol{r}_1, \boldsymbol{r}_2) = \Psi(\boldsymbol{r}_2, \boldsymbol{r}_1) = \psi_{\alpha_1}(\boldsymbol{r}_2)\, \psi_{\alpha_2}(\boldsymbol{r}_1) = \psi_{\alpha_2}(\boldsymbol{r}_1)\, \psi_{\alpha_1}(\boldsymbol{r}_2), \tag{8}$$

  which corresponds to a new state

$$\begin{aligned}&\int d^d \boldsymbol{r}_1 \int d^d \boldsymbol{r}_2\, \Psi'(\boldsymbol{r}_1, \boldsymbol{r}_2)\, |\boldsymbol{r}_1\rangle \otimes |\boldsymbol{r}_2\rangle \\ &= \int d^d \boldsymbol{r}_1 \int d^d \boldsymbol{r}_2\, \psi_{\alpha_2}(\boldsymbol{r}_1)\, \psi_{\alpha_1}(\boldsymbol{r}_2)\, |\boldsymbol{r}_1\rangle \otimes |\boldsymbol{r}_2\rangle \\ &= |\alpha_2\rangle \otimes |\alpha_1\rangle,\end{aligned} \tag{9}$$

  describing a two-particle state with the 1st particle in $|\alpha_2\rangle$ and the 2nd particle in $|\alpha_1\rangle$.

Conclusion: exchanging the positions of two particles ($\boldsymbol{r}_1 \leftrightarrow \boldsymbol{r}_2$) $\Leftrightarrow$ exchanging the labels of the single-particle state ($\alpha_1 \leftrightarrow \alpha_2$).

First-Quantized States

First-quantization approach:

- Suppose the **single-particle Hilbert space** is D dimensional, spanned by the **single-particle basis states** $|\alpha\rangle$ ($\alpha = 1, 2, \ldots, D$).

- The **many-body Hilbert space** of N particles will be D^N dimensional, spanned by the **many-body basis states**

$$\boxed{|[\alpha]\rangle \equiv |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots \otimes |\alpha_N\rangle,} \tag{10}$$

  where $\alpha_i = 1, 2, \ldots, D$ labels the state of the ith particle.

- A generic **first-quantized state** is a linear superposition of these basis states

$$|\Psi\rangle = \sum_{[\alpha]} \Psi[\alpha]\, |[\alpha]\rangle, \tag{11}$$

  where the coefficient $\Psi[\alpha] \in \mathbb{C}$ is also called the **many-body wave function** (as a more general function of labels α_i not positions $\boldsymbol{r}_i$).

Most of the first-quantized states are *not* qualified to describe systems of *identical particles*.

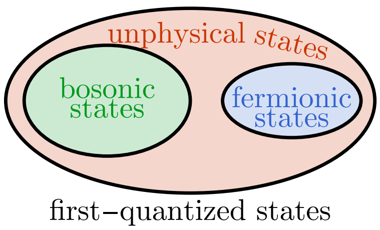
- For identical **bosons**, $\Psi[\alpha]$ must be **symmetric**

$$\Psi_B(\ldots, \alpha_i, \ldots, \alpha_j, \ldots) = +\Psi_B(\ldots, \alpha_j, \ldots, \alpha_i, \ldots) \tag{12}$$

- For identical **fermions**, $\Psi[\alpha]$ must be **antisymmetric**

$$\Psi_F(\ldots, \alpha_i, \ldots, \alpha_j, \ldots) = -\Psi_F(\ldots, \alpha_j, \ldots, \alpha_i, \ldots) \tag{13}$$

These states only span a *subspace* of the *first-quantized* Hilbert space.



We would like to pick out (or construct) the **basis states** for the **bosonic** and **fermionic** subspaces. Starting from a generic basis state $|[\alpha]\rangle$, we can construct

- **bosonic** states by **symmetrization**

$$\boxed{\begin{aligned}\mathcal{S}\,|[\alpha]\rangle &= \mathcal{S}\,|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots \otimes |\alpha_N\rangle \\ &\equiv \sum_{\pi \in S_N} |\alpha_{\pi(1)}\rangle \otimes |\alpha_{\pi(2)}\rangle \otimes \ldots \otimes |\alpha_{\pi(N)}\rangle,\end{aligned}} \tag{14}$$

- **fermionic** states by **antisymmetrization**

$$\boxed{\begin{aligned}\mathcal{A}\,|[\alpha]\rangle &= \mathcal{A}\,|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots \otimes |\alpha_N\rangle \\ &\equiv \sum_{\pi \in S_N} (-)^\pi\, |\alpha_{\pi(1)}\rangle \otimes |\alpha_{\pi(2)}\rangle \otimes \ldots \otimes |\alpha_{\pi(N)}\rangle,\end{aligned}} \tag{15}$$

π denotes an S_N group element and $(-)^\pi$ is the **permutation sign** of π.

$$(-)^\pi = \begin{cases} +1 & \text{if } \pi \text{ has even number of inversions} \\ -1 & \text{if } \pi \text{ has odd number of inversions} \end{cases} \tag{16}$$

An inversion is a pair (x, y) such that $x < y$ and $\pi(x) > \pi(y)$. Take the S_3 group for example:

$$\begin{array}{ccccccc} \pi(123) & 123 & 231 & 312 & 321 & 213 & 132 \\ (-)^\pi & +1 & +1 & +1 & -1 & -1 & -1 \end{array} \tag{17}$$

- Examples of **bosonic** states (unnormalized):

$$\begin{aligned} &\mathcal{S}\,|\alpha\rangle \otimes |\beta\rangle = |\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle, \ (\text{assuming } \alpha \neq \beta) \\ &\mathcal{S}\,|\alpha\rangle \otimes |\alpha\rangle = |\alpha\rangle \otimes |\alpha\rangle. \end{aligned} \tag{18}$$

- Examples of **fermionic** states (unnormalized):

$$\begin{aligned} &\mathcal{A}\,|\alpha\rangle \otimes |\beta\rangle = |\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle, \ (\text{assuming } \alpha \neq \beta) \\ &\mathcal{A}\,|\alpha\rangle \otimes |\alpha\rangle = 0 \ \Rightarrow \ \text{no such fermionic state}. \end{aligned} \tag{19}$$

Pauli exclusion principle: two (or more) identical fermions can not occupy the same state simultaneously.

Originally $|\alpha\rangle \otimes |\beta\rangle$ and $|\beta\rangle \otimes |\alpha\rangle$ (for $\alpha \neq \beta$) are two orthogonal first-quantized states, under either *symmetrization* or *antisymmetrization*, they correspond to the same state (up to ±1 overall phase)

$$\begin{aligned} \mathcal{S}\,|\alpha\rangle \otimes |\beta\rangle &= \mathcal{S}\,|\beta\rangle \otimes |\alpha\rangle, \\ \mathcal{A}\,|\alpha\rangle \otimes |\beta\rangle &= -\mathcal{A}\,|\beta\rangle \otimes |\alpha\rangle. \end{aligned} \tag{20}$$

- The first-quantized Hilbert space is *redundant* ⇒ there are *fewer* basis states in the bosonic and fermionic subspaces.
- Consider N particles, each can take one of D different single-particle states,
  - the **dimension** of **bosonic subspace**:

$$\mathcal{D}_B = \frac{(N+D-1)!}{N!\,(D-1)!}. \tag{21}$$

  - the **dimension** of **fermionic subspace**:

$$\mathcal{D}_F = \frac{D!}{N!\,(D-N)!}. \tag{22}$$

  It turns out that $\mathcal{D}_B + \mathcal{D}_F \leq D^N$ as long as $N > 1$ ⇒ the remaining basis states in the first-quantized Hilbert space are *unphysical* (for identical particles).

These unphysical states are annoying: we can not combine the states in the Hilbert space freely. We must always remember to symmetrized/antisymmetrized the state. ⇒ Is there a better way to organize the many-body Hilbert space, such that all states in the space are physical?

Second-Quantized States (Fock States)

Sometimes *difficulties* in physics arise from the *inappropriate language* we used. There are two different ways to describe many-body states:

- In **first-quantization**, we ask: *Which particle is in which state*?
- In **second-quantization**, we ask: *How many particles are there in every state*?

The question we ask in first-quantization is inappropriate: if the particles are *identical*, it will be impossible to tell which particle is which in the first place. We need to switch to a new language

$\lvert\alpha\rangle \otimes \lvert\beta\rangle$	$\lvert\beta\rangle \otimes \lvert\alpha\rangle$
the 1st particle on $\lvert\alpha\rangle$ the 2nd particle in $\lvert\beta\rangle$	the 1st particle on $\lvert\beta\rangle$ the 2nd particle in $\lvert\alpha\rangle$
↘	↙
there is one particle in $\lvert\alpha\rangle$, another particle in $\lvert\beta\rangle$	

The new description does not require the labeling of particles. ⇒ It contains no redundant information. ⇒ It leads to a more precise and succinct description.

In the second-quantization approach,

- Each **basis state** in the many-body Hilbert space is labeled by a set of **occupation numbers** n_α (for $\alpha = 1, 2, \ldots, D$)

$$\boxed{|[n]\rangle \equiv |n_1, n_2, \ldots, n_\alpha, \ldots, n_D\rangle,} \tag{23}$$

meaning that there are n_α particles in the state $|\alpha\rangle$.

$$n_\alpha = \begin{cases} 0, 1, 2, 3, \ldots & \text{bosons,} \\ 0, 1 & \text{fermions.} \end{cases} \tag{24}$$

- For **bosons**, n_α can be any non-negative integer.
- For **fermions**, n_α can only take 0 or 1, due to the *Pauli exclusion principle*.
- The occupation numbers n_α sum up to the total number of particles, i.e. $\sum_\alpha n_\alpha = N$.

- The states $|[n]\rangle$ are also known as **Fock states**.
- All Fock states form a complete set of basis for the many-body Hilbert space, or the **Fock space**.
- Any generic **second-quantized** many-body state is a linear combination of *Fock states*,

$$|\Psi\rangle = \sum_{[n]} \Psi[n]\, |[n]\rangle. \tag{25}$$

Representation of Fock States

The **first**- and the **second-quantization** formalisms can both provide *legitimate* description of identical particles. (The first-quantization is just awkward to use, but it is still valid.)

Every *Fock state* has a **first-quantized representation**.

- The Fock state with all occupation numbers to be zero is called the **vacuum state**, denoted as

$$|0\rangle \equiv |\ldots, 0, \ldots\rangle \tag{26}$$

It corresponds to the *tensor product unit* in the first-quantization, which can be written as

$$|0\rangle_B = |0\rangle_F = 1. \tag{27}$$

We use a subscript B/F to indicate whether the Fock state is **bosonic** (B) or **fermionic** (F). For vacuum state, there is no difference between them.

- The Fock state with only one *non-zero* occupation number is a **single-mode Fock state**, denoted as

$$|n_\alpha\rangle = |\ldots, 0, n_\alpha, 0, \ldots\rangle \tag{28}$$

In terms of the first-quantized states

$$\begin{aligned} |1_\alpha\rangle_B &= |1_\alpha\rangle_F = |\alpha\rangle, \\ |2_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle, \\ |3_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle, \\ |n_\alpha\rangle_B &= \underbrace{|\alpha\rangle \otimes |\alpha\rangle \otimes \ldots \otimes |\alpha\rangle}_{n_\alpha \text{ factors}} \equiv |\alpha\rangle^{\otimes n_\alpha}. \end{aligned} \tag{29}$$

- For **multi-mode Fock states** (meaning more than one single-particle state $|\alpha\rangle$ is involved), the first-quantized state will involve appropriate symmetrization depending on the particle statistics. For example,

$$\begin{aligned} |1_\alpha, 1_\beta\rangle_B &= \frac{1}{\sqrt{2}}(|\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle), \\ |1_\alpha, 1_\beta\rangle_F &= \frac{1}{\sqrt{2}}(|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle). \end{aligned} \tag{30}$$

Note the difference between bosonic and fermionic Fock states (even if their occupation numbers are the same). Here are more examples

$$\begin{aligned} |2_\alpha, 1_\beta\rangle_B &= \frac{1}{\sqrt{3}}(|\alpha\rangle \otimes |\alpha\rangle \otimes |\beta\rangle + |\alpha\rangle \otimes |\beta\rangle \otimes |\alpha\rangle + |\beta\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle), \\ |1_\alpha, 1_\beta, 1_\gamma\rangle_F &= \frac{1}{\sqrt{6}}(|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle + |\beta\rangle \otimes |\gamma\rangle \otimes |\alpha\rangle + \\ &\qquad |\gamma\rangle \otimes |\alpha\rangle \otimes |\beta\rangle - |\gamma\rangle \otimes |\beta\rangle \otimes |\alpha\rangle - |\beta\rangle \otimes |\alpha\rangle \otimes |\gamma\rangle - |\alpha\rangle \otimes |\gamma\rangle \otimes |\beta\rangle). \end{aligned} \tag{31}$$

Ok, you get the idea. In general, the Fock state can be represented as

- for **bosons**,

$$\boxed{|[n]\rangle_B = \left(\frac{\prod_\alpha n_\alpha!}{N!}\right)^{1/2} \mathcal{S} \bigotimes_\alpha |\alpha\rangle^{\otimes n_\alpha}.} \tag{32}$$

- for **fermions**,

$$\boxed{|[n]\rangle_F = \frac{1}{\sqrt{N!}} \mathcal{A} \bigotimes_\alpha |\alpha\rangle^{\otimes n_\alpha}.} \tag{33}$$

$\mathcal{S}$ and $\mathcal{A}$ are symmetrization and antisymmetrization operators defined in Eq. (14) and Eq. (15).

Creation and Annihilation Operators

State Insertion and Deletion

The **creation** and **annihilation operators** are introduced to *create* and *annihilate* particles in the quantum many-body system, as indicated by their names. The first step towards defining them is to understand how to *insert* and *delete* a single-particle state from the first-quantized state in a *symmetric* (or *antisymmetric*) manner.

Let us first declare some notations:

- Let $|\alpha\rangle$, $|\beta\rangle$ be single-particle states.
- Let $\mathit{1}$ be the *tensor identity* (meaning that $|\alpha\rangle \otimes \mathit{1} = \mathit{1} \otimes |\alpha\rangle = |\alpha\rangle$).
- Let $|\Psi\rangle$, $|\Phi\rangle$ be generic *first-quantized states* as in Eq. (11).

Now we define the **insertion operator** $\triangleright_\pm$ and **deletion operator** $\triangleleft_\pm$ by the following rules:

- Linearity (for $a, b \in \mathbb{C}$)

$$\begin{aligned} |\alpha\rangle \triangleright_\pm (a\,|\Psi\rangle + b\,|\Phi\rangle) &= a\,|\alpha\rangle \triangleright_\pm |\Psi\rangle + b\,|\alpha\rangle \triangleright_\pm |\Phi\rangle, \\ |\alpha\rangle \triangleleft_\pm (a\,|\Psi\rangle + b\,|\Phi\rangle) &= a\,|\alpha\rangle \triangleleft_\pm |\Psi\rangle + b\,|\alpha\rangle \triangleleft_\pm |\Phi\rangle. \end{aligned} \tag{34}$$

- Vacuum action

$$\begin{aligned} |\alpha\rangle \triangleright_\pm \mathit{1} &= |\alpha\rangle, \\ |\alpha\rangle \triangleleft_\pm \mathit{1} &= 0. \end{aligned} \tag{35}$$

- Recursive relation

$$\begin{aligned} |\alpha\rangle \triangleright_\pm |\beta\rangle \otimes |\Psi\rangle &= |\alpha\rangle \otimes |\beta\rangle \otimes |\Psi\rangle \pm |\beta\rangle \otimes (|\alpha\rangle \triangleright_\pm |\Psi\rangle), \\ |\alpha\rangle \triangleleft_\pm |\beta\rangle \otimes |\Psi\rangle &= \langle \alpha \mid \beta\rangle\, |\Psi\rangle \pm |\beta\rangle \otimes (|\alpha\rangle \triangleleft_\pm |\Psi\rangle). \end{aligned} \tag{36}$$

$\langle \alpha \mid \beta\rangle = \delta_{\alpha\beta}$ if $|\alpha\rangle$ and $|\beta\rangle$ are orthonormal basis states. The subscript $\pm$ of the insertion or deletion operators indicates whether symmetrization $(+)$ or antisymmetrization $(-)$ is implemented.

Boson Creation and Annihilation

- The **boson creation operator** $b_\alpha^\dagger$ *adds* a boson to the single-particle state $|\alpha\rangle$, *increasing* the occupation number by one $n_\alpha \to n_\alpha + 1$. It acts on a N-particle first-quantized state $|\Psi\rangle$ as

$$\boxed{b_\alpha^\dagger\,|\Psi\rangle = \frac{1}{\sqrt{N+1}}\,|\alpha\rangle \triangleright_+ |\Psi\rangle,} \tag{37}$$

  where $|\alpha\rangle \triangleright_+$ *inserts* the single-particle state $|\alpha\rangle$ to $N+1$ possible insertion positions *symmetrically.*

- The **boson annihilation operator** b_α *removes* a boson from the single-particle state $|\alpha\rangle$, *reducing* the occupation number by one $n_\alpha \to n_\alpha - 1$ (while $n_\alpha > 0$). It acts on a N-particle first-quantized state $|\Psi\rangle$ as

$$\boxed{b_\alpha\,|\Psi\rangle = \frac{1}{\sqrt{N}}\,|\alpha\rangle \triangleleft_+ |\Psi\rangle,} \tag{38}$$

  where $|\alpha\rangle \triangleleft_+$ *removes* the single-particle state $|\alpha\rangle$ from N possible deletion positions *symmetrically.*

Single-Mode Fock States

Based on these definitions, we can show that the creation and annihilation operators acting on single-mode Fock states as

$$b_\alpha^\dagger\,|n_\alpha\rangle = \frac{1}{\sqrt{n_\alpha+1}}\,|\alpha\rangle \triangleright_+ |\alpha\rangle^{\otimes n_\alpha}$$

$$
\begin{aligned}
&= \frac{n_\alpha + 1}{\sqrt{n_\alpha + 1}} \, |\alpha\rangle^{\otimes(n_\alpha+1)} \\
&= \sqrt{n_\alpha + 1} \, |n_\alpha + 1\rangle. \\
b_\alpha |n_\alpha\rangle &= \frac{1}{\sqrt{n_\alpha}} \, |\alpha\rangle \lhd_+ |\alpha\rangle^{\otimes n_\alpha} \\
&= \frac{n_\alpha}{\sqrt{n_\alpha}} \, |\alpha\rangle^{\otimes(n_\alpha-1)} \\
&= \sqrt{n_\alpha} \, |n_\alpha - 1\rangle.
\end{aligned}
\tag{40}
$$

Thus we conclude

$$
\boxed{
\begin{aligned}
b_\alpha^\dagger |n_\alpha\rangle &= \sqrt{n_\alpha + 1} \, |n_\alpha + 1\rangle, \\
b_\alpha |n_\alpha\rangle &= \sqrt{n_\alpha} \, |n_\alpha - 1\rangle.
\end{aligned}
}
\tag{41}
$$

- Especially, when acting on the vacuum state

$$
\begin{aligned}
b_\alpha^\dagger |0_\alpha\rangle &= |1_\alpha\rangle, \\
b_\alpha |0_\alpha\rangle &= 0.
\end{aligned}
\tag{42}
$$

- Using Eq. (41), we can show that

$$
\boxed{b_\alpha^\dagger \, b_\alpha |n_\alpha\rangle = n_\alpha |n_\alpha\rangle,}
\tag{43}
$$

  meaning that $b_\alpha^\dagger \, b_\alpha$ is the **boson number operator** of the $|\alpha\rangle$ state.

All the single-mode Fock state can be constructed by the boson creation operator from the vacuum state

$$
\boxed{|n_\alpha\rangle = \frac{1}{\sqrt{n_\alpha!}} \left(b_\alpha^\dagger\right)^{n_\alpha} |0_\alpha\rangle.}
\tag{44}
$$

Generic Fock States

The above result can be generalized to any Fock state of bosons

$$
\boxed{
\begin{aligned}
b_\alpha^\dagger |\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle_B &= \sqrt{n_\alpha + 1} \, |\ldots, n_\beta, n_\alpha + 1, n_\gamma, \ldots\rangle_B, \\
b_\alpha |\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle_B &= \sqrt{n_\alpha} \, |\ldots, n_\beta, n_\alpha - 1, n_\gamma, \ldots\rangle_B.
\end{aligned}
}
\tag{45}
$$

These two equations can be considered as the **defining properties** of boson creation and annihilation operators.

Operator Identities

Eq. (45) implies the following operator identities

$$\boxed{\left[b_\alpha^\dagger, b_\beta^\dagger\right] = [b_\alpha, b_\beta] = 0, \left[b_\alpha, b_\beta^\dagger\right] = \delta_{\alpha\beta}.} \quad (46)$$

These relations can be considered as the **algebraic definition** of boson creation and annihilation operators.

Fermion Creation and Annihilation

- The **fermion creation operator** $c_\alpha^\dagger$ *adds* a fermion to the single-particle state $|\alpha\rangle$, *increasing* the occupation number by one $n_\alpha \to n_\alpha + 1$ (while $n_\alpha = 0$). It acts on a N-particle first-quantized state $|\Psi\rangle$ as

$$\boxed{c_\alpha^\dagger |\Psi\rangle = \frac{1}{\sqrt{N+1}} |\alpha\rangle \triangleright_- |\Psi\rangle,} \quad (47)$$

  where $|\alpha\rangle \triangleright_-$ *inserts* the single-particle state $|\alpha\rangle$ to $N+1$ possible insertion positions *anti-symmetrically.*

- The **fermion annihilation operator** c_α *removes* a fermion from the single-particle state $|\alpha\rangle$, *reducing* the occupation number by one $n_\alpha \to n_\alpha - 1$ (while $n_\alpha = 1$). It acts on a N-particle first-quantized state $|\Psi\rangle$ as

$$\boxed{c_\alpha |\Psi\rangle = \frac{1}{\sqrt{N}} |\alpha\rangle \triangleleft_- |\Psi\rangle,} \quad (48)$$

  where $|\alpha\rangle \triangleleft_-$ *removes* the single-particle state $|\alpha\rangle$ from N possible deletion positions *anti-symmetrically.*

Single-Mode Fock States

Based on these definitions, we can show that the creation and annihilation operators acting on single-mode Fock states as

$$\begin{aligned} c_\alpha^\dagger |0_\alpha\rangle &= |\alpha\rangle \triangleright_- \mathit{1} = |\alpha\rangle = |1_\alpha\rangle \\ c_\alpha^\dagger |1_\alpha\rangle &= \frac{1}{\sqrt{2}} |\alpha\rangle \triangleright_- |\alpha\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\alpha\rangle - |\alpha\rangle \otimes |\alpha\rangle) = 0 \end{aligned} \quad (49)$$

$$\begin{aligned} c_\alpha |0_\alpha\rangle &= 0 \\ c_\alpha |1_\alpha\rangle &= |\alpha\rangle \triangleleft_- |\alpha\rangle = \mathit{1} = |0_\alpha\rangle. \end{aligned} \quad (50)$$

Thus we conclude (note that $n_\alpha = 0, 1$ only take two values)

$$\boxed{\begin{aligned} c_\alpha^\dagger |n_\alpha\rangle &= \sqrt{1 - n_\alpha}\, |1 - n_\alpha\rangle, \\ c_\alpha |n_\alpha\rangle &= \sqrt{n_\alpha}\, |1 - n_\alpha\rangle. \end{aligned}} \quad (51)$$

- Using Eq. (51), we can show that

$$c_\alpha^\dagger \, c_\alpha \, |n_\alpha\rangle = n_\alpha \, |n_\alpha\rangle, \tag{52}$$

meaning that $c_\alpha^\dagger \, c_\alpha$ is the **fermion number operator** of the $|\alpha\rangle$ state.

All the single-mode Fock state can be constructed by the boson creation operator from the vacuum state

$$|n_\alpha\rangle = \left(c_\alpha^\dagger\right)^{n_\alpha} |0_\alpha\rangle. \tag{53}$$

Generic Fock States

The above result can be generalized to any Fock state of bosons

$$\begin{aligned} c_\alpha^\dagger \, |\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle_F &= (-)^{\sum_{\beta<\alpha} n_\beta} \sqrt{1 - n_\alpha} \; |\ldots, n_\beta, 1 - n_\alpha, n_\gamma, \ldots\rangle_F, \\ c_\alpha \, |\ldots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle_F &= (-)^{\sum_{\beta<\alpha} n_\beta} \sqrt{n_\alpha} \; |\ldots, n_\beta, 1 - n_\alpha, n_\gamma, \ldots\rangle_F. \end{aligned} \tag{54}$$

These two equations can be considered as the **defining properties** of fermion creation and annihilation operators.

Operator Identities

Eq. (54) implies the following operator identities

$$\left\{c_\alpha^\dagger, c_\beta^\dagger\right\} = \{c_\alpha, c_\beta\} = 0, \; \left\{c_\alpha, c_\beta^\dagger\right\} = \delta_{\alpha\beta}. \tag{55}$$

These relations can be considered as the **algebraic definition** of fermion creation and annihilation operators.