Classical Shadow Tomography with Locally Scrambled Quantum Dynamics

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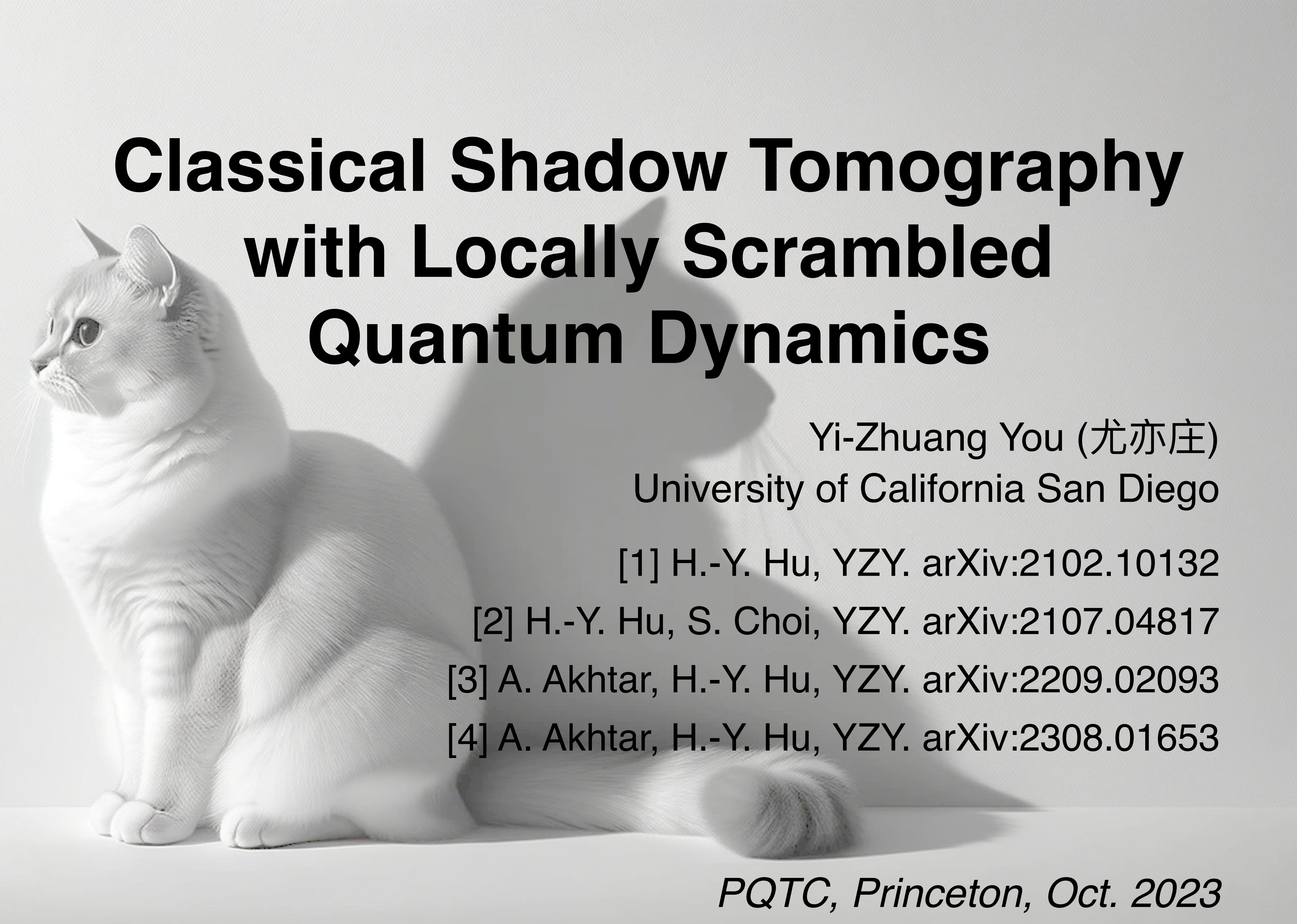
[1] H.-Y. Hu, YZY. arXiv:2102.10132

[2] H.-Y. Hu, S. Choi, YZY. arXiv:2107.04817

[3] A. Akhtar, H.-Y. Hu, YZY. arXiv:2209.02093

[4] A. Akhtar, H.-Y. Hu, YZY. arXiv:2308.01653

PQTC, Princeton, Oct. 2023



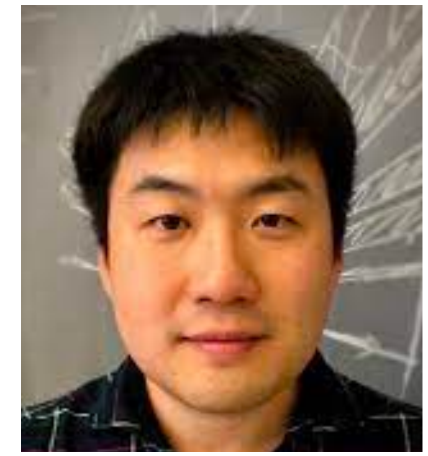
“The Shadow of a Ket” by OpenAI DALL·E 3

Acknowledgement

- Collaborators

Ahmed Akhtar
(UCSD)

Hong-Ye Hu
(Harvard)



Soonwon Choi
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- UC Hellman Fellow (2021)

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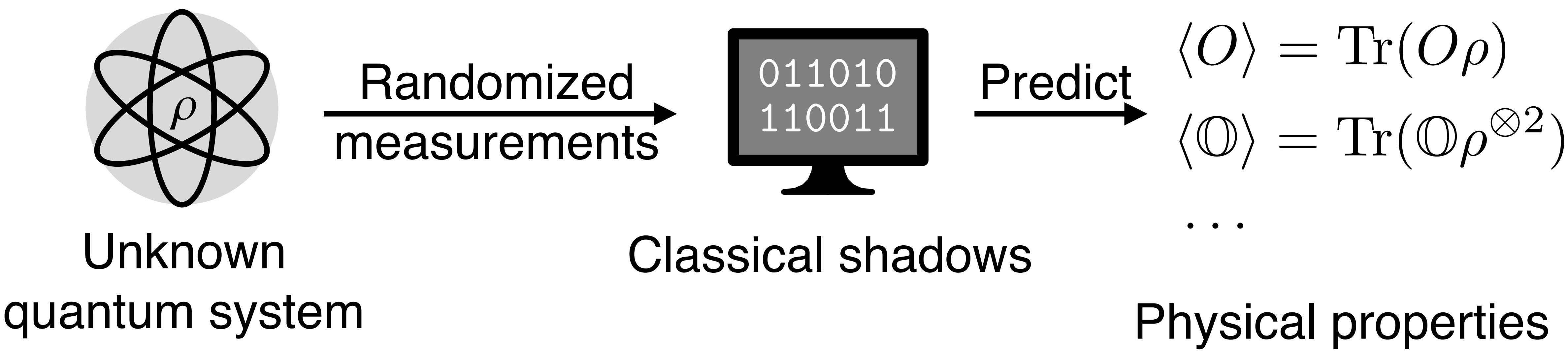
Quantum State Tomography

- Quantum state tomography: using repeated measurements to extract information about a quantum system.

- **Classical shadow tomography**: a general-purpose tomography scheme with superior sample efficiency

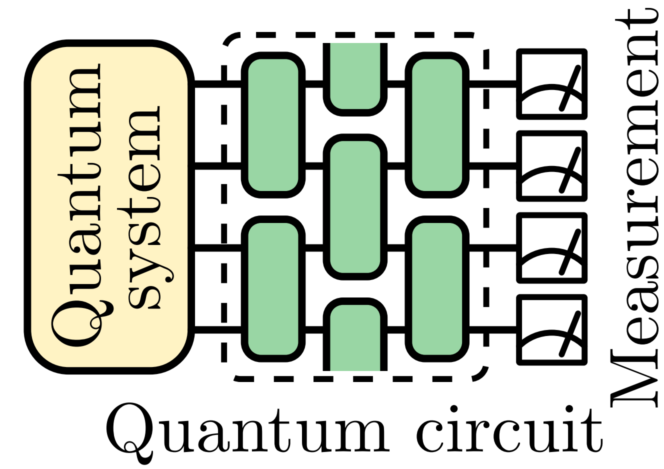


- Key idea: use randomized measurement to sample classical shadows $\hat{\sigma}$, without reconstructing ρ explicitly.

Huang, Kueng, Preskill, Nat. Phys. arXiv:2002.08953 (2020)

Elben et.al. *The randomized measurement toolbox*, Nat. Rev. Phys. (2022)

Randomized Measurement



- Randomized measurement protocol
  - Prepare $\rho_{\text{full}} = \rho \otimes (|0\rangle\langle 0|)^{\otimes N_{\text{anc}}}$
  - Pick $U \sim p(U)$, perform
  $$\rho_{\text{full}} \to \rho'_{\text{full}} = U \rho_{\text{full}} U^\dagger$$
  - Measure a subset of qubits in the computational (Z) basis
  $$\rho'_{\text{full}} \to \frac{\Pi_{\boldsymbol{b}} \rho'_{\text{full}} \Pi_{\boldsymbol{b}}}{\mathrm{Tr}(\Pi_{\boldsymbol{b}} \rho'_{\text{full}} \Pi_{\boldsymbol{b}})}$$
  with measurement outcome $\boldsymbol{b} \in \{0,1\}^{\times N_{\text{msr}}}$
  - Record the measurement event $(U, \boldsymbol{b})$, then repeat
- Goal: predict properties of ρ from $\{(U, \boldsymbol{b})\}$

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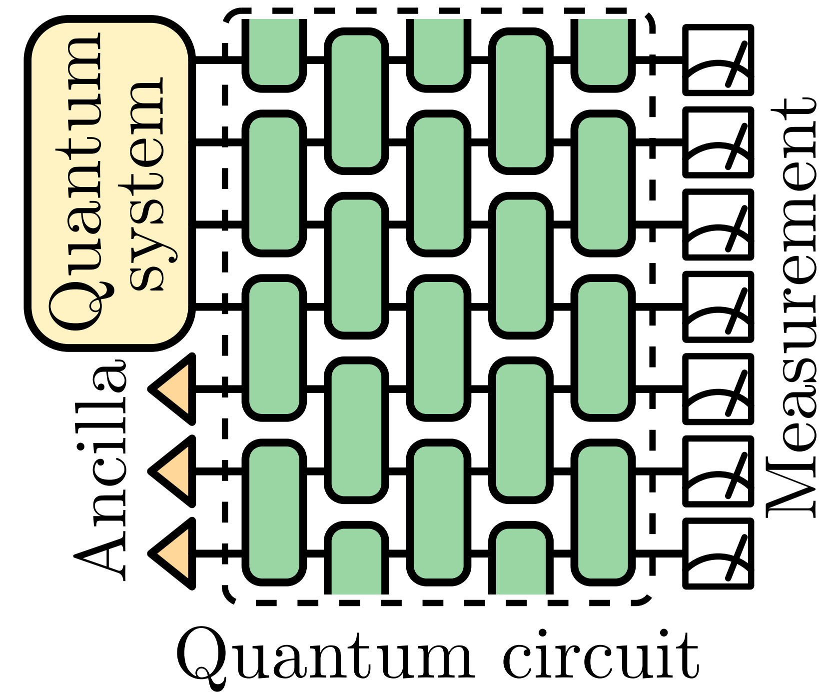
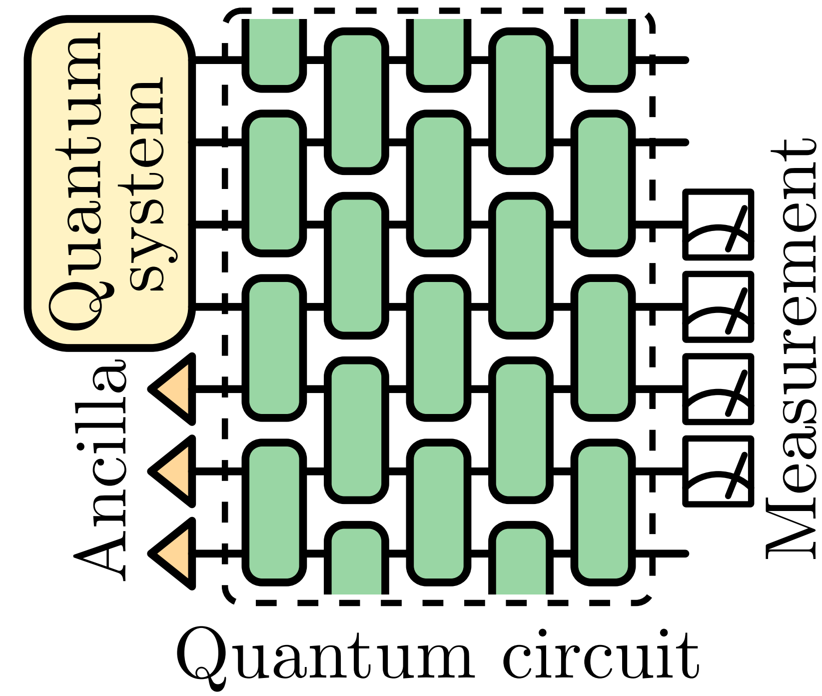


Randomized Measurement



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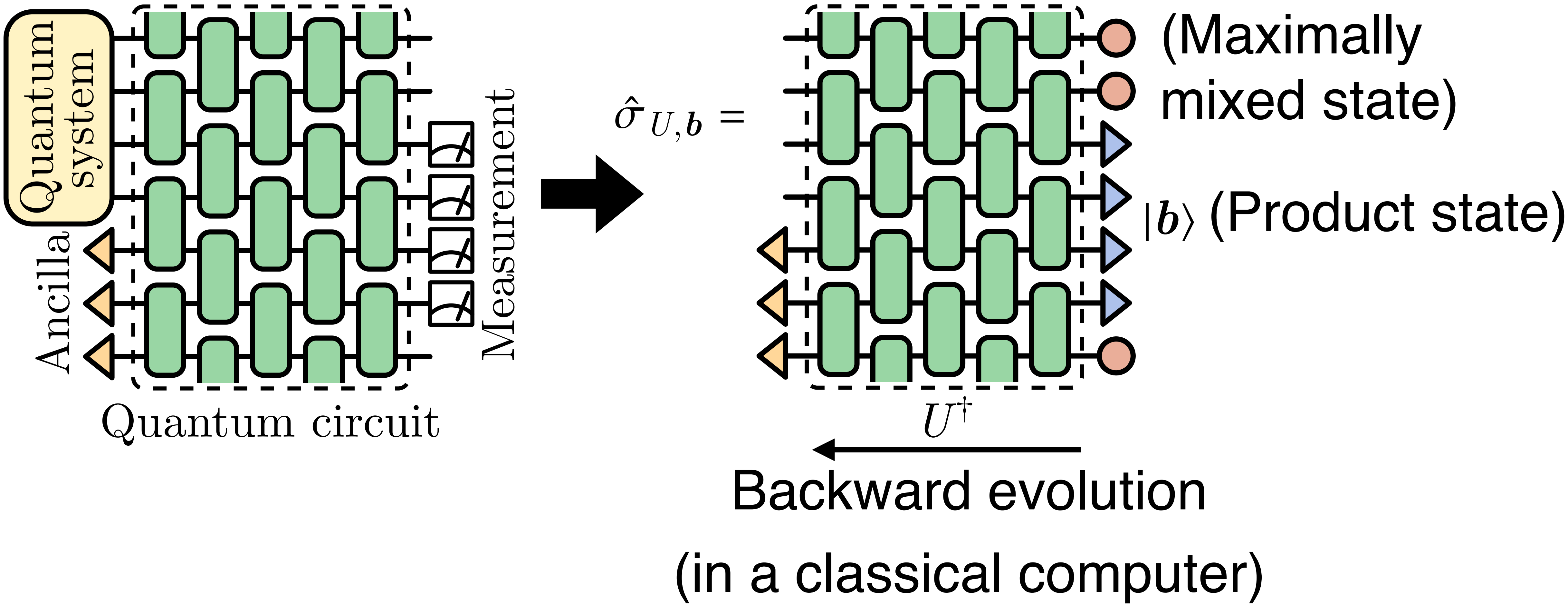
Randomized Measurement

- Each randomized measurement event $(U, \boldsymbol{b})$ is characterized by a classical shadow state $\hat{\sigma}_{U,\boldsymbol{b}}$, such that

$$p(\hat{\sigma}_{U,\boldsymbol{b}}|\rho) \propto \mathrm{Tr}(\hat{\sigma}_{U,\boldsymbol{b}}\rho)$$



Measurement Channel

- **Measurement channel**: measure ρ prepare $\hat{\sigma}$

$$\rho \to \sigma := \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma}|\rho)} \hat{\sigma} = \mathcal{M}[\rho]$$

  - $\mathcal{M}$ is invertible, if the scheme is tomographically complete

- **Reconstruction map**: reconstruct ρ from $\hat{\sigma}$

$$\rho = \mathcal{M}^{-1}[\sigma] = \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma}|\rho)} \mathcal{M}^{-1}[\hat{\sigma}]$$

  - $\mathcal{M}^{-1}$ is *not* a physical quantum process, and can only be implemented by classical computing.
  - It specifies how to post-process the classical data to make predictions, e.g.

$$\langle O\rangle_\rho = \operatorname{Tr} O\rho = \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma}|\rho)} \operatorname{Tr}(O\mathcal{M}^{-1}[\hat{\sigma}])$$

Operator Shadow Norm

- **Sample complexity**: the number M of samples needed to control the estimation error within the level of ϵ,

$$M \sim \frac{1}{\epsilon^2} \|O\|^2_{\mathcal{E}_\sigma}$$

Huang, Kueng, Preskill (2020)

  - The scaling $\epsilon \sim 1/\sqrt{M}$ follows the large number theorem
  - The coefficient is set by the operator shadow norm

$$\|O\|^2_{\mathcal{E}_\sigma} := \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma})} \big(\operatorname{Tr}(O\mathcal{M}^{-1}[\hat{\sigma}])\big)^2$$

$$= \frac{\operatorname{Tr}(O\mathcal{M}^{-1}[O])}{\operatorname{Tr}\mathbb{1}}$$

  which depends on

    - The observable O of interest
    - The randomized measurement channel $\mathcal{M}$

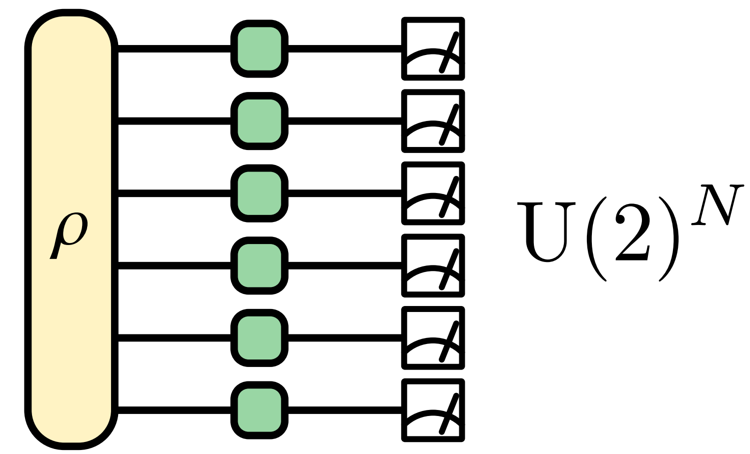
H.-Y. Hu, S. Choi, Y.-Z.You. arXiv:2107.04817

Pauli and Clifford Measurements

- Everything boils down to computing $\mathcal{M}^{-1}$.
- Know results
  - Randomized Pauli measurement

$$\mathcal{M}^{-1}[\sigma] = \bigotimes_i (3\sigma_i - \mathbb{1}_i)$$

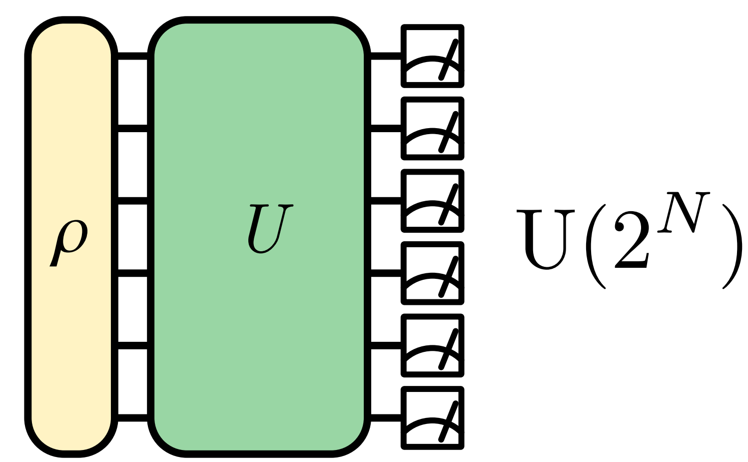
$$\|O\|^2_{\mathcal{E}_\sigma} = 3^{|\operatorname{supp} O|}$$



  - Randomized Clifford measurement

$$\mathcal{M}^{-1}[\sigma] = (2^N + 1)\sigma - \mathbb{1}$$

$$\|O\|^2_{\mathcal{E}_\sigma} \simeq \operatorname{Tr} O^2$$



- What about other more general randomized unitary ensemble (beyond Haar measure of unitary groups)?

Locally-Scrambled Shadow Tomography

- Much progress can be made by a mild assumption: the prior distribution of classical shadows is local basis independent — the random unitary is locally scrambled (twirled).

$$\forall V = \bigotimes_i V_i \text{ with } V_i \in \mathrm{U}(2):$$

$$p(\hat{\sigma}) = p(V^\dagger \hat{\sigma} V)$$

H.-Y. Hu, S. Choi, Y.-Z.You.
arXiv:2107.04817

- The reconstruction map is given by

$$\boxed{\mathcal{M}^{-1}[O] = \sum_{P \in \mathcal{P}} \frac{\operatorname{Tr} OP}{w_{\mathcal{E}_\sigma}(P) \operatorname{Tr} \mathbb{1}} P}$$

  - Only knowledge about Pauli weights is needed ($P \in \mathcal{P}$)

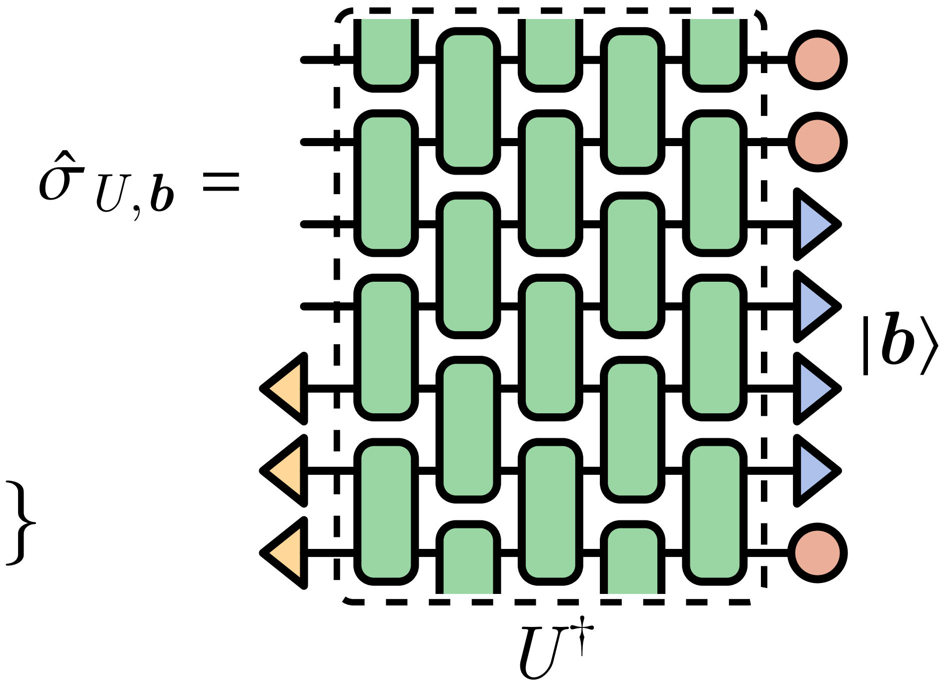
$$w_{\mathcal{E}_\sigma}(P) = \mathop{\mathbb{E}}_{\sigma \in \mathcal{E}_\sigma} (\operatorname{Tr} P\sigma)^2$$

K. Bu, D. Enshan Koh, R. J. Gracia, A. Jaffe. arXiv: 2202.03272

Locally-Scrambled Quantum Dynamics

- Classical shadows are constructed from backward quantum dynamics
  - Physical dynamics $\sigma \to \mathcal{C}[\sigma]$
  - Ensemble dynamics

$$\mathcal{E}_\sigma \to \mathcal{E}_{\mathcal{C}[\sigma]} = \{\mathcal{C}[\sigma] | \sigma \in \mathcal{E}_\sigma, \mathcal{C} \in \mathcal{E}_\mathcal{C}\}$$



  - Pauli weight dynamics

$$\boxed{w_{\mathcal{E}_{\mathcal{C}[\sigma]}}(P) = \sum_{P'} w_{\mathcal{E}_\mathcal{C}}(P, P') w_{\mathcal{E}_\sigma}(P')}$$

Definitions:

$$w_{\mathcal{E}_\sigma}(P) = \mathop{\mathbb{E}}_{\sigma \in \mathcal{E}_\sigma} (\operatorname{Tr} P\sigma)^2$$

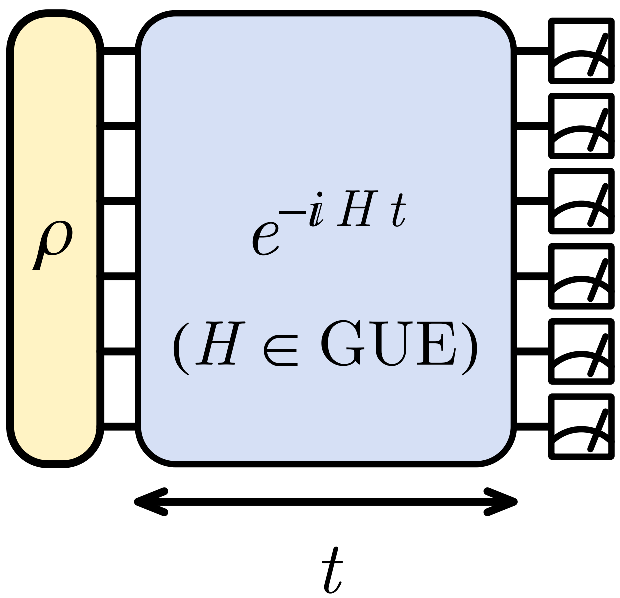
(State Pauli weight)

$$w_{\mathcal{E}_\mathcal{C}}(P, P') = \mathop{\mathbb{E}}_{\mathcal{C} \in \mathcal{E}_\mathcal{C}} \left(\frac{\operatorname{Tr}(P\mathcal{C}[P'])}{\operatorname{Tr} \mathbb{1}} \right)^2$$

(Channel Pauli weight)

Locally-Scrambled Tomography Schemes

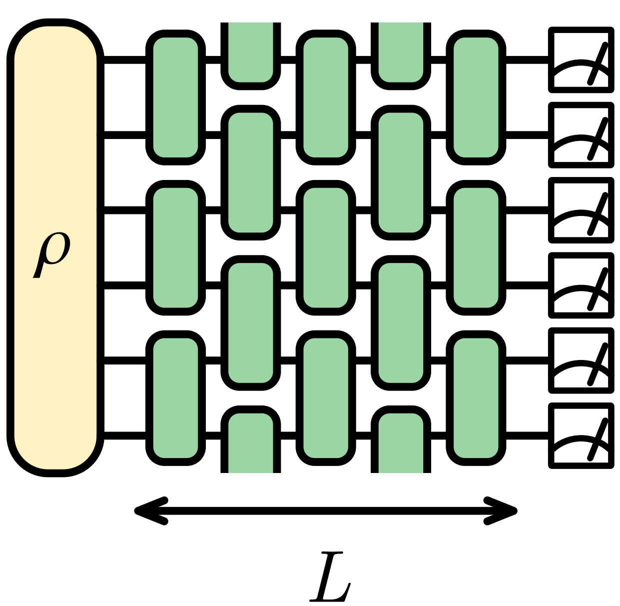
- Hamiltonian-driven shadow tomography



Hamiltonian H can be:
GUE, GOE, GSE …
SYK, ETH, Quantum simulators …

H.-Y. Hu, Y.-Z. You. arXiv:2102.10132

- Shallow shadow tomography



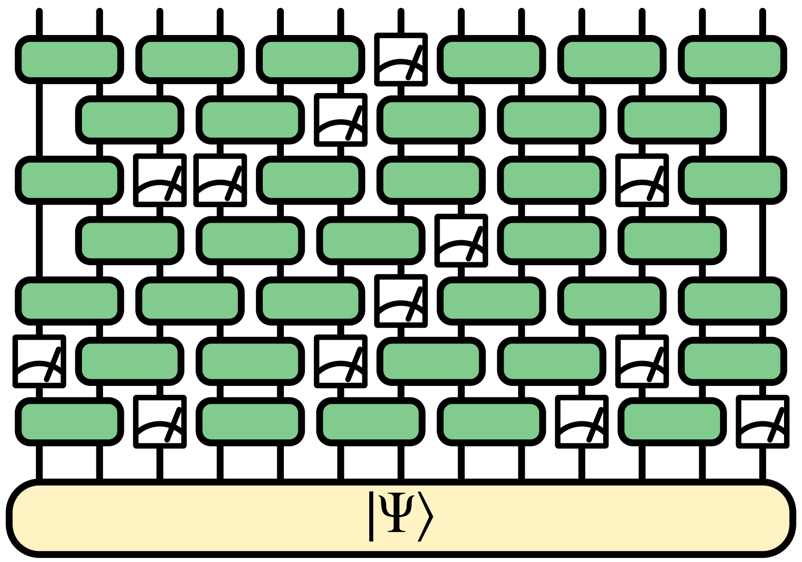
Finite-depth local random unitary
Gate: Haar, Clifford …
Friendly to NISQ devices

H.-Y. Hu, S. Choi, Y.-Z.You. arXiv:2107.04817

Locally-Scrambled Tomography Schemes

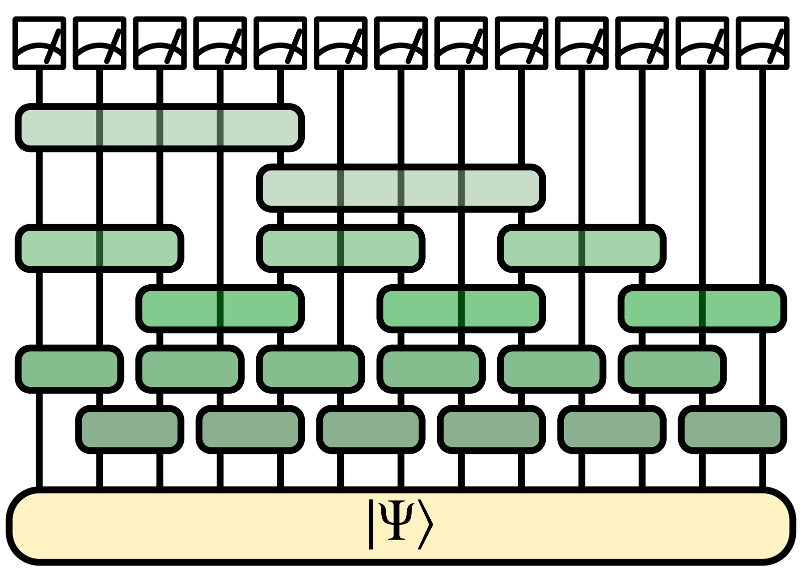
- Hybrid shadow tomography

A.Akhtar, H.-Y. Hu, YZY. arXiv:2308.01653



Intermediate measurements with a measurement rate p
— Optimal sample efficiency achieved at the measurement-induced phase transition.

- Holographic shadow tomography



Local measurements in the holographic bulk map to scale-free measurements on the holographic boundary.

X. Feng, S. Zhang, M. Ippoliti, YZY. (To appear)

Scalable Tomography on Clifford Circuits

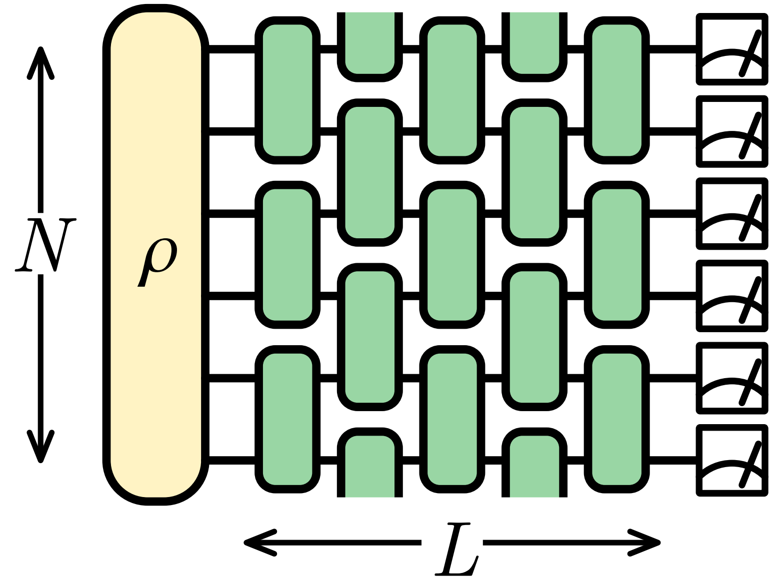
- Test states:

$$\rho_{\mathrm{GHZ}} = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| \qquad |\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|00\cdots\rangle + |11\cdots\rangle)$$

$$\rho_{\mathrm{ZXZ}} = |\mathrm{ZXZ}\rangle\langle\mathrm{ZXZ}| \qquad Z_{i-1}X_iZ_{i+1}|\mathrm{ZXZ}\rangle = |\mathrm{ZXZ}\rangle$$

(Cluster state)

- Randomized measurement scheme:



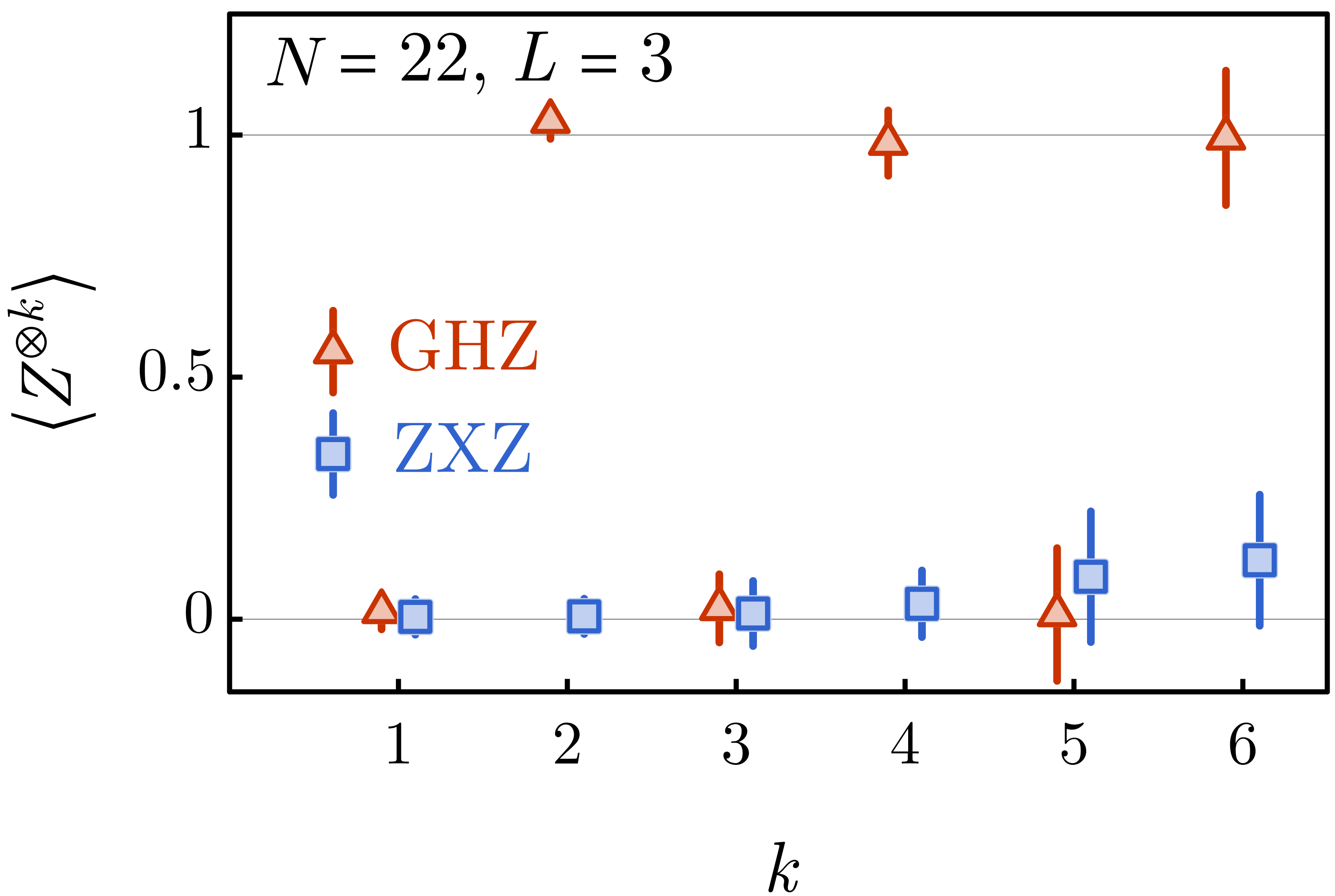
  - Brick-wall arrangement of 2-qubit random Clifford gates
  - Width (number of qubits): N
  - Depth (number of layers): L

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

Pauli Operator Estimation

- Task 2: Estimate the expectation value of a Pauli string operator

$$Z^{\otimes k} := \prod_{i=1}^{k} Z_i$$



- Estimations are unbiased (converge to the ground truth)
- Variance (shadow norm) increases with the weight (size) of the Pauli string
  - Note: For Pauli measurements ($L = 0$)

$$\|Z^{\otimes k}\|_{\mathcal{E}_\sigma}^2 \sim 3^k$$

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

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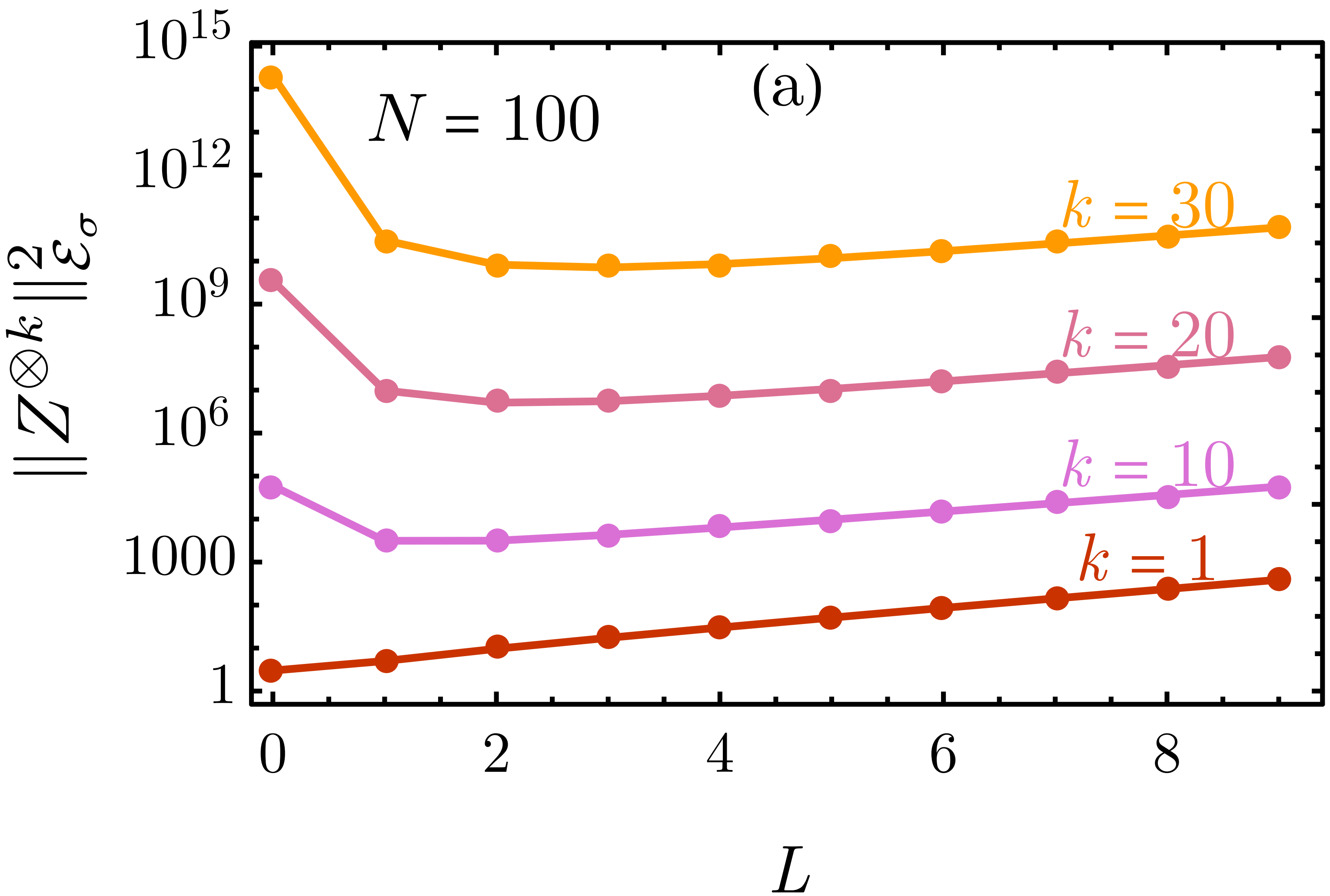
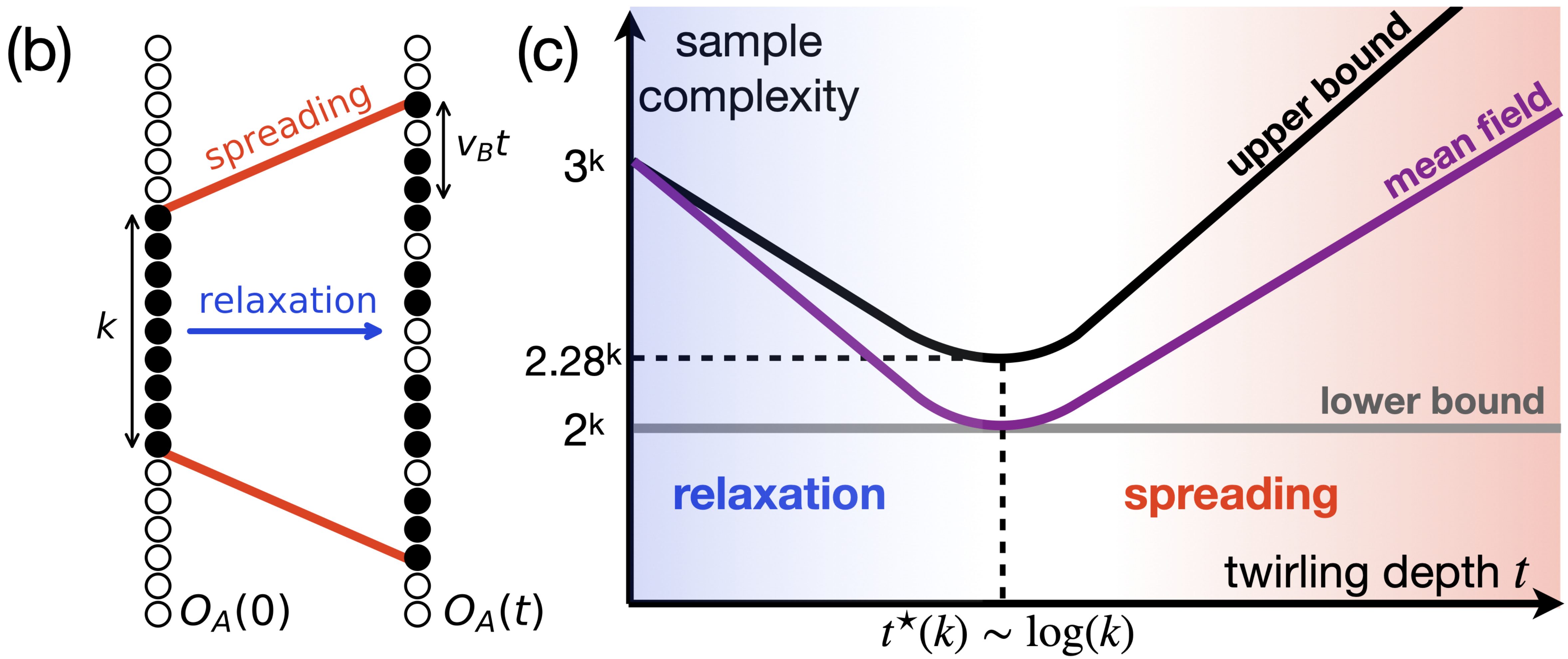


- Given the size k of the Pauli string, there is an optimal circuit depth L^* minimizing the shadow norm (sample complexity)

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

Pauli Operator Estimation

- Considering continuous time limit, operator dynamics $O_A \to O_A(t) = U O_A U^\dagger$ (with $k = |A|$)



- Shadow norm ~ $3^{k_{\text{eff}}}$

$$k_{\text{eff}} \sim \left(\frac{3}{4} + \frac{1}{4}\mathrm{e}^{-\gamma t}\right)(k + 2v_B t) \xrightarrow{\text{Min}} t^* \sim \gamma^{-1} \log k$$

M. Ippoliti, Y. Li, T. Rakovszky, V. Khemani arXiv:2212.11963

Pauli Operator Estimation

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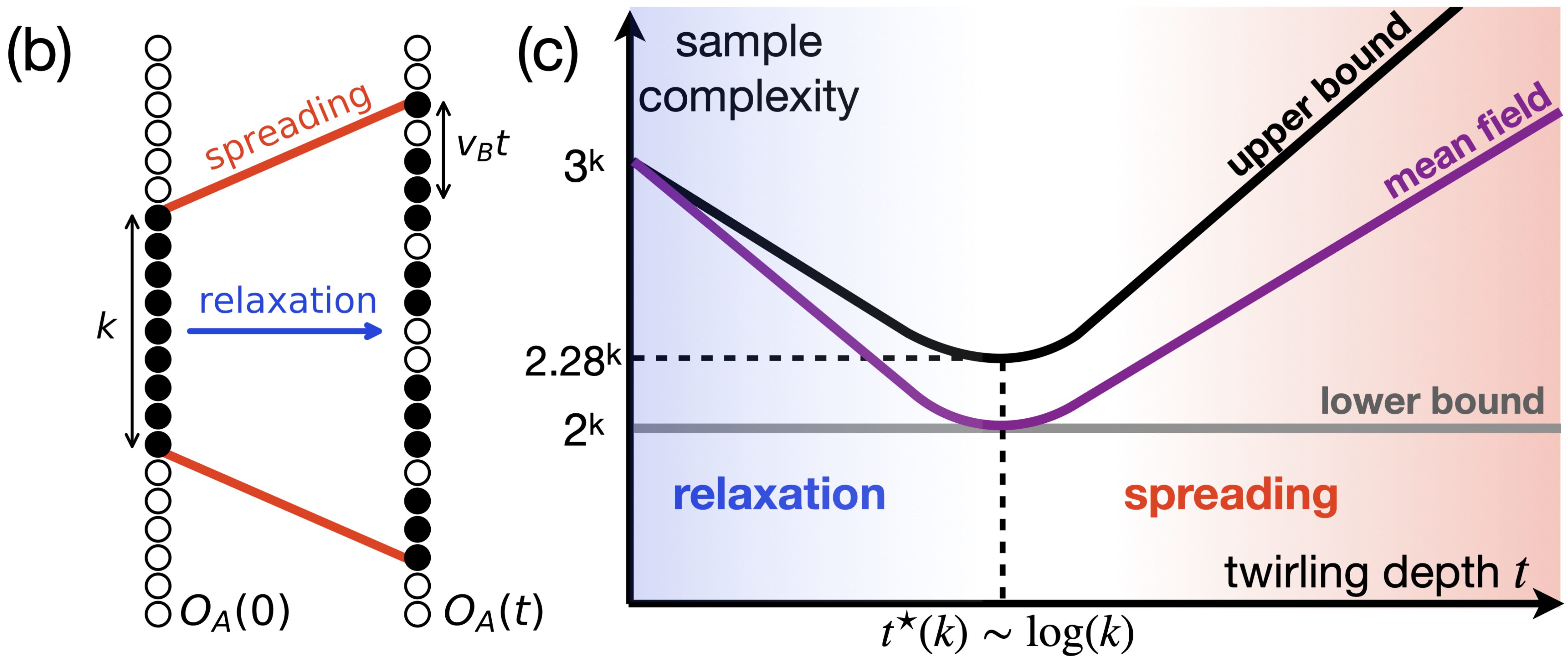


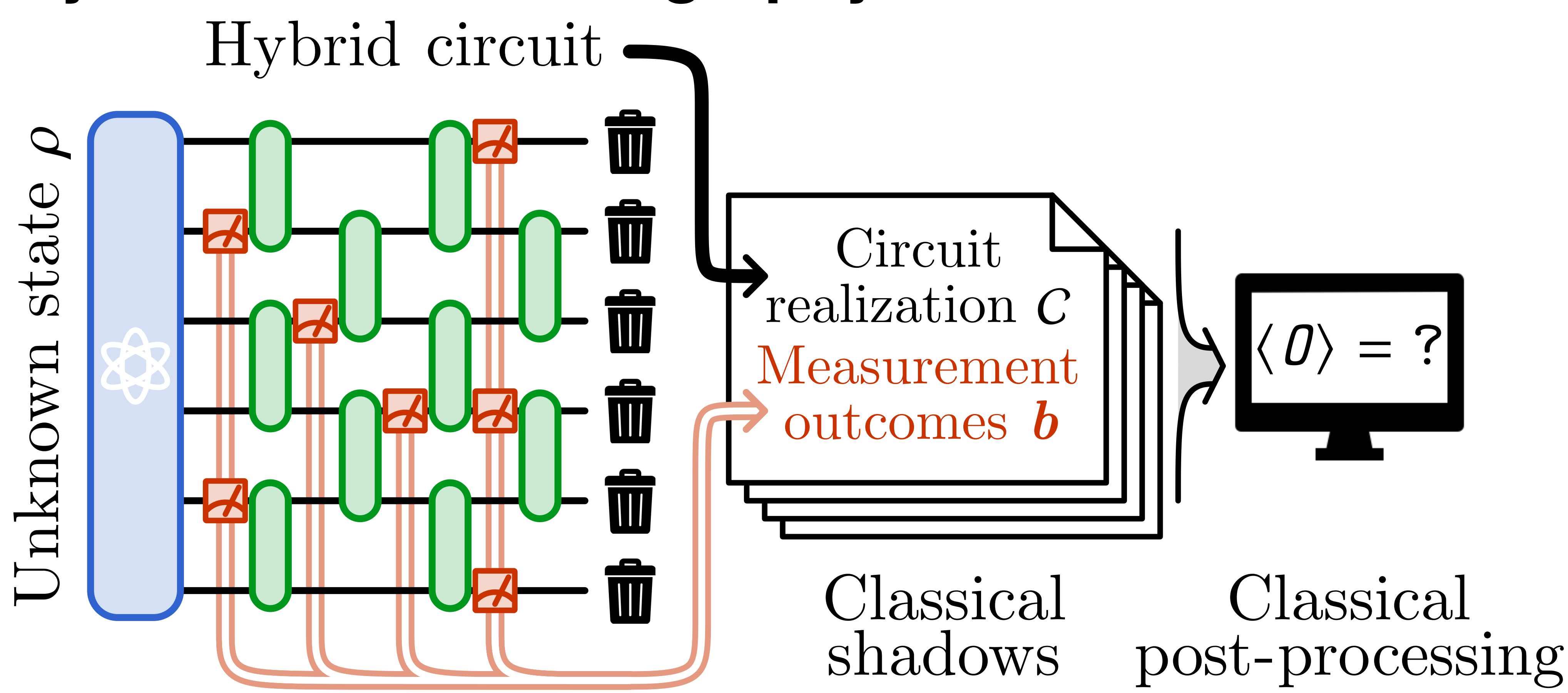
- At optimal circuit depth t^* shadow norm scales with k with a smaller base

$$2^k \lesssim \|O_A(t)\|^2_{\mathcal{E}_\sigma} \lesssim 3^{\frac{3}{4}k} \approx 2.28^k$$

M. Ippoliti, Y. Li, T. Rakovszky, V. Khemani arXiv:2212.11963

Hybrid Shadow Tomography

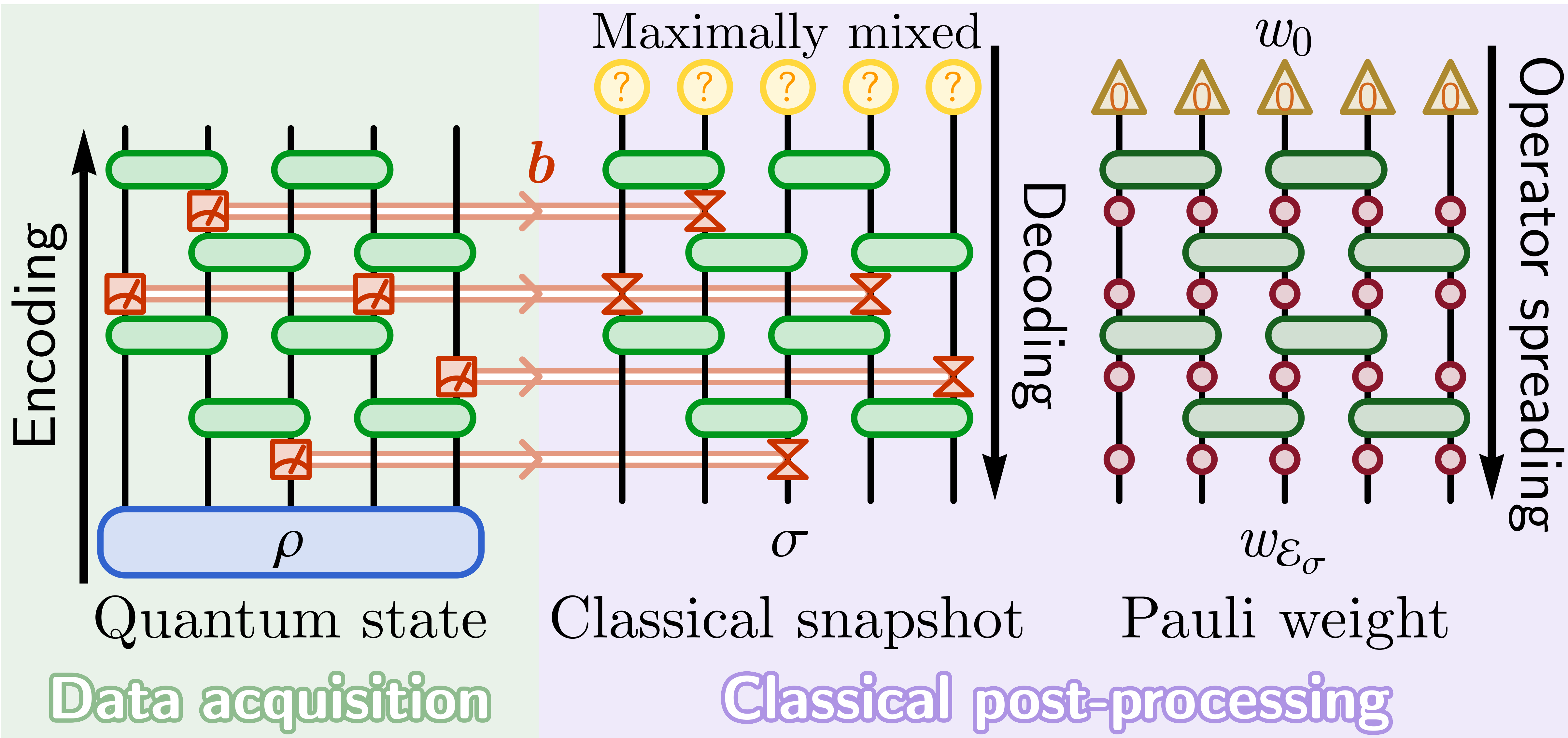
- **Shallow shadow tomography** has an advantage in sample complexity scaling, but only achievable if the circuit depth is adjusted with the size of the observable.
- Can we perform the measurement on one circuit and make predictions for observables of all sizes optimally?
  — **Hybrid shadow tomography**



A.Akhtar, H.-Y. Hu, E. Altman, YZY. arXiv:2307.xxxxx

Hybrid Shadow Tomography

- Post-processing scheme

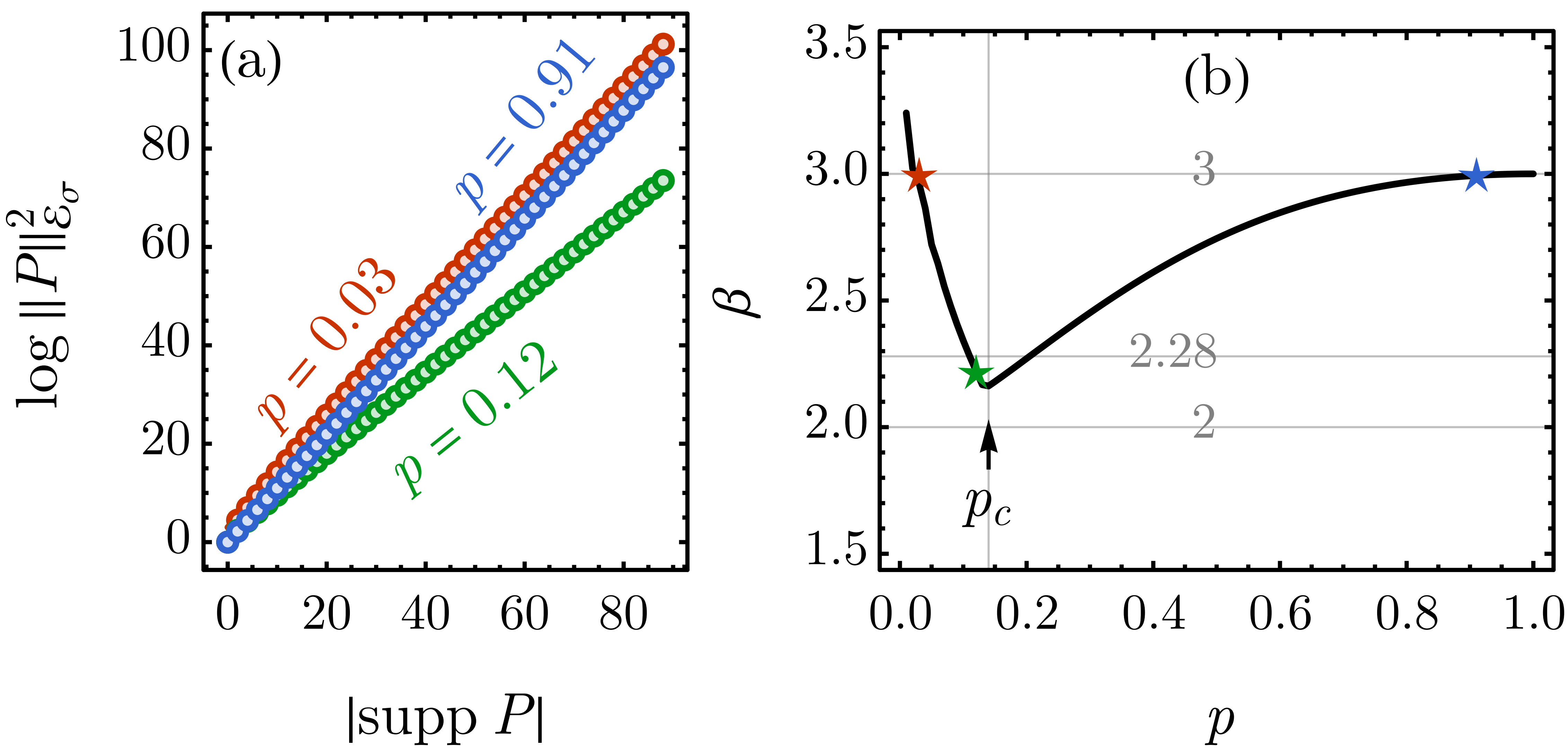


A.Akhtar, H.-Y. Hu, E. Altman, YZY. arXiv:2307.xxxxx

Hybrid Shadow Tomography

- Shadow norm scaling

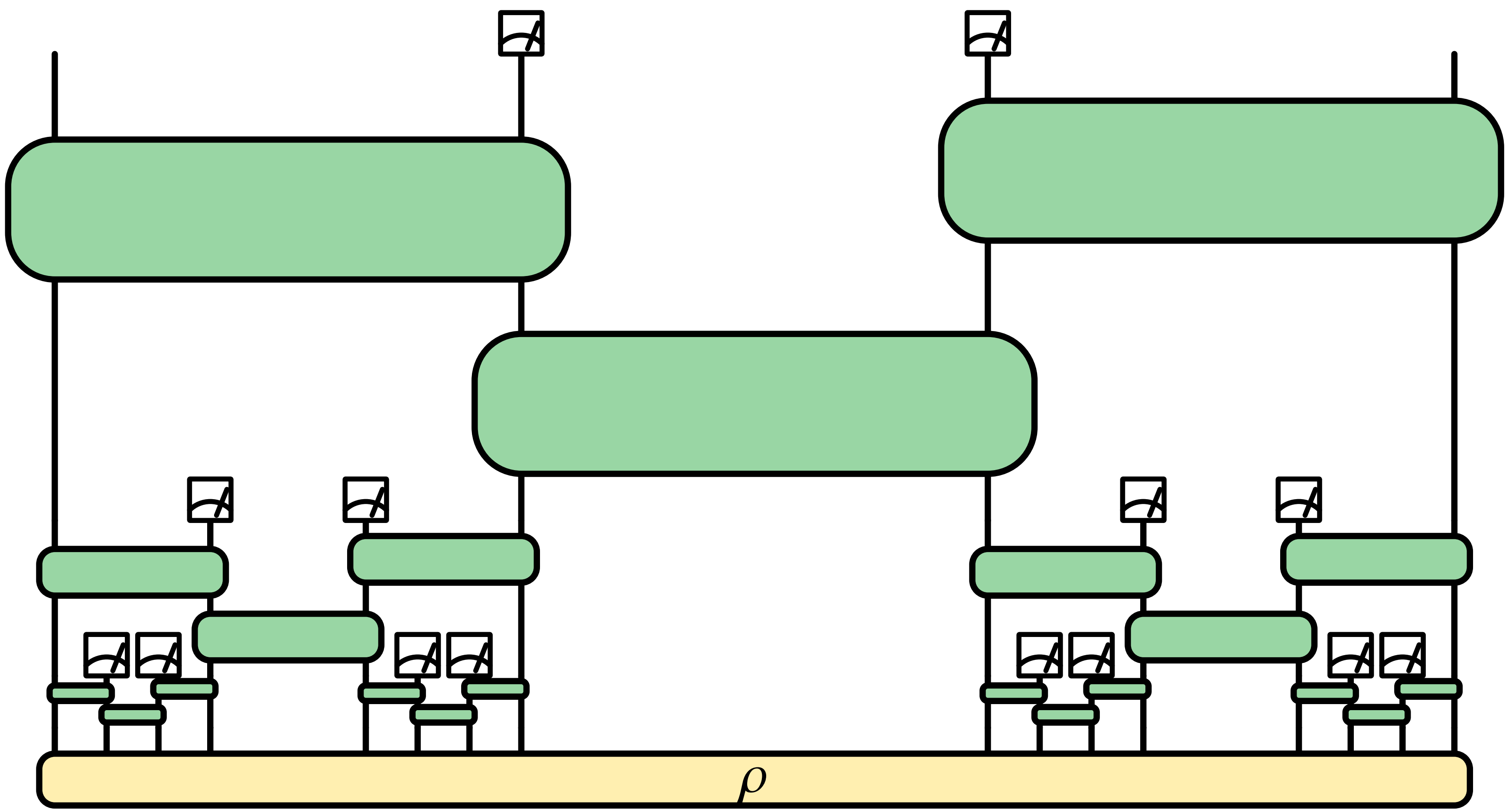
$$\|P\|^2_{\mathcal{E}_\sigma} \simeq \beta^k \mathrm{poly}(k) \qquad k = |\,\mathrm{supp}\, P|$$



A.Akhtar, H.-Y. Hu, YZY. arXiv:2308.01653

Holographic Shadow Tomography

- However, **hybrid shadow tomography** requires fine-tuning the measurement-induced criticality.
- **Holographic shadow tomography** is automatically critical.



H.-Y. Hu, S. Choi, YZY. arXiv:2107.04817
X. Feng, S. Zhang, M. Ippoliti, YZY. (To appear)

Summary

- With locally-scrambled quantum dynamics, we extend the classical shadow tomography to a large class of quantum circuits, which is
    - Scalable (efficient classical post-processing)
    - Flexible (arbitrary circuit structure / quantum dynamics)
    - NISQ friendly (shallow circuits, simple gates, available devices)
- We expect our approach to have broad applications in many quantum information processing tasks (e.g. quantum error mitigation)

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Thanks for your attention!