Problem Set

Section I (Oct. 4)

1. Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Let $|i\rangle$ be a complete set of basis ket states. Suppose $\langle i \mid \alpha\rangle$ and $\langle i \mid \beta\rangle$ are known for all basis states $|i\rangle$. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

2. We now consider a qubit system and let $|\alpha\rangle = |\sigma^z = +1\rangle$ and $|\beta\rangle = |\sigma^x = +1\rangle$. Write down the explicit square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the σ^z basis.

3. Construct the state $|\boldsymbol{n}\cdot\boldsymbol{\sigma} = +1\rangle$ such that

$$\boldsymbol{n}\cdot\boldsymbol{\sigma}\,|\boldsymbol{n}\cdot\boldsymbol{\sigma} = +1\rangle = (+1)\,|\boldsymbol{n}\cdot\boldsymbol{\sigma} = +1\rangle, \tag{1}$$

where $\boldsymbol{n} = (n_x, n_y, n_z)$ is a unit vector.

- $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ is an *operator*

$$\boldsymbol{n}\cdot\boldsymbol{\sigma} = n_x\,\sigma^x + n_y\,\sigma^y + n_z\,\sigma^z. \tag{2}$$

- $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is a *vector* of *operators*, i.e. each component of the vector $\boldsymbol{\sigma}$ is an operator.

If we treat the qubit as a spin, the spin operators are related by

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}. \tag{3}$$

4. A beam of electrons goes through a series of Stern-Gerlach measurements as follows: (a) the first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms; (b) the second measurement accepts $s_{\boldsymbol{n}} = \hbar/2$ atoms and rejects $s_{\boldsymbol{n}} = -\hbar/2$ atoms, where $s_{\boldsymbol{n}}$ is the eigenvalue of the operator $\boldsymbol{n}\cdot\boldsymbol{S}$; (c) the third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms. What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximizing the intensity of the final $s_z = -\hbar/2$ beam?

Section II (Oct. 11)

1. An operator (or matrix) $\hat{A}$is normal if is satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

(a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal

(b) Show that any operator can be written as $\hat{A} = \hat{H} + i\,\hat{G}$, where $\hat{H}$ and $\hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger$, $\hat{A} - \hat{A}^\dagger$]. Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

(c) Show that a normal operator $\hat{A}$ admits a spectral representation

$$\hat{A} = \sum_{i=1}^{N} \lambda_i \, \hat{P}_i \tag{4}$$

for a set of projectors $\hat{P}_i$ and complex numbers λ_i.

2. Recall the trace of an operator $\mathrm{Tr}[A] = \sum_m \langle m| \, A \, |m\rangle$ for some basis $\{ |\, \mathtt{m} \rangle \}$

(a) Prove that this definition is independent of basis. This implies if A is diagonalizable with eigenvalues λ_i that $\mathrm{Tr}[A] = \sum_i \lambda_i$.

(b) Prove the cycle property: Tr[ABC]=Tr[BCA]=Tr[CAB]

(c)Consider an operator A. Show the following identity

$$\det e^{A} = e^{\mathrm{Tr}[A]} \tag{5}$$

3. Clock and shift operators

Consider an N-dimensional Hilbert space, with orthonormal basis $\{|n\rangle, n = 0, \; .., N-1\}$. Consider operators T and U which act on this N-state system by

$$T \, |n\rangle = |n+1\rangle, \;\; U \, |n\rangle = e^{\frac{2 \pi i n}{N}} \, |n\rangle. \tag{6}$$

In the definition of T, the label on the ket should be understood as its value modulo N.

(a) Find the matrix representations of T and U in the basis $\{|n\rangle\}$.

(b) What are the eigenvalues of U? What are the eigenvalues of its adjoint $U^\dagger$?

(c) Show that

$$U \, T \; = \; e^{\frac{2 \pi i}{N}} \; T \; U \tag{7}$$

(d) From the definition of adjoint, how does $T^\dagger$act?

(e) Show that the clock operator T is normal.

(f) Find the eigenvalues and eigenvectors of T. [Hint: consider states of the form $|\theta\rangle \equiv \sum_n e^{i \, n \theta} \, |n\rangle$].