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Quantum Mechanics B (Physics 212B) Winter 2019
Worksheet 1 – Solutions

Problems

1. **Bogliubov Transformation**

   In our last discussion we solved a new Hamiltonian by defining a set of transformed creation/annihilation operators which satisfy the same algebra $[A, A^\dagger] = \mathbb{1}$

   More generally consider $\hat{b} = \hat{a}\cosh\eta + \hat{a}^\dagger \sinh\eta$

   (a) Show that $[\hat{b}, \hat{b}^\dagger] = \mathbb{1}$
   $[\hat{b}, \hat{b}^\dagger] = \cosh^2\eta[a, a^\dagger] - \sinh^2\eta[a, a^\dagger] = 1[a, a^\dagger] = \mathbb{1}$

   (b) Show that $\hat{b} = U\hat{a}U^\dagger$ for $U = e^{\frac{\eta}{2}(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger)}$
   Time to bust out BCH: $U\hat{a}U^\dagger = \hat{a} + [A, a] + \frac{1}{2}[A, [A, a]] + \cdots$
   Where $A \equiv \frac{\eta}{2}(\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger)$. Note $[a^\dagger a^\dagger, a] = -2a^\dagger$ and $[aa, a^\dagger] = 2a$
   So $[A, a] = \eta a^\dagger$ and $[A, [A, a]] = \eta[A, a^\dagger] = \eta^2 a$
   So everything decomposes into odd terms with $a^\dagger$ and even terms a. All plus signs. This gives the form of b

   Now consider the Hamiltonian

   $$\hat{H} = \omega\hat{a}^\dagger\hat{a} + \frac{V}{2}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger) \tag{1}$$

   (c) Diagonalize the Hamiltonian (1) using the $\hat{b}$ operators for suitably chosen η
   We should look for an operator of the form $H = \Omega b^\dagger b + F$ for some constant F
   $b^\dagger b = (\cosh^2\eta + \sinh^2\eta)a^\dagger a + \sinh^2\eta + \cosh\eta\sinh\eta(aa + a^\dagger a^\dagger)$
   So $\Omega\cosh(2\eta) = \omega$ and $\Omega\sinh(2\eta) = V \implies V = \omega\tanh(2\eta)$
   It also must be that $\Omega\sinh^2\eta + F = 0$; solving for Ω and F independently is just algebra. $\Omega = \omega\cosh(2\eta) - V\sinh(2\eta)$
   The spectrum is simple now though! $E_n = \Omega n + F$

   (d) Show there is a limit on V for which this Hamiltonian makes physical sense
   Ω must be positive. $\omega\cosh(2\eta) > V\sinh(2\eta) \implies V < \frac{\omega}{\tanh(2\eta)} = \frac{\omega^2}{V}$
   Thus $V < \omega$

2. **Fermionic Harmonic Oscillator**

   Let's discuss a system with no simple classical analog. Consider "fermionic" operators c and $c^\dagger$ which obey the following anti-commutation relation:

   $$\hat{c}^2 = 0 = (\hat{c}^\dagger)^2 \qquad \{\hat{c}, \hat{c}^\dagger\} \equiv cc^\dagger + c^\dagger c = \mathbb{1} \tag{2}$$

(a) Assuming the Hilbert space cannot be empty, show there exists a state $|0\rangle$ for which $c|0\rangle = 0$. Similarly, show there must exist a non-vanishing state $|1\rangle \equiv c^\dagger|0\rangle$

By assumption there must exist a state $|\psi\rangle$. Either $c|\psi\rangle = 0$, in which case $|\psi\rangle \equiv |0\rangle$ and we're done, or $c|\psi\rangle \neq 0$

In the latter case $|0\rangle \propto c|\psi\rangle$ as $c^2|\psi\rangle = 0$ by fermionic algebra.

To show $c^\dagger|0\rangle \neq 0$ we consider acting c on it.

$cc^\dagger|0\rangle = (1 - c^\dagger c)|0\rangle = |0\rangle$ from algebra. This cannot happen if $c^\dagger|0\rangle$ vanished.

(b) Write explicit matrix representations of the operators c and $c^\dagger$ in the $|0\rangle, |1\rangle$ basis.

$c = \sigma^- = X - \mathbf{i}Y$ and $c^\dagger = \sigma^+ = X + \mathbf{i}Y$ where X and Y are Pauli matrices. From Pauli algebra they obey the fermionic statistics.

(c) Show that $\hat{d} = \hat{c}\cos\theta + \hat{c}^\dagger \sin\theta$ is the analogous Bougliobuv transform

Similar but the anticommutation means $\{d, d^\dagger\} = \cos^2\theta\{\hat{c}, \hat{c}^\dagger\} + \sin^2\theta\{\hat{c}, \hat{c}^\dagger\} = \mathbb{1}$

(d) Define the Hamiltonian in analogy to the bosonic SHO as:

$$H = \frac{\omega}{2}(c^\dagger c - cc^\dagger) \tag{3}$$

What are it's eigenstates and eigenvalues?

It's easy to see that it's diagonal in the $|0\rangle, |1\rangle$ basis which eigenvalues $E_\pm = \pm\frac{\omega}{2}$ with $|0\rangle$ as the groundstate.

The operator $c^\dagger c$ is again a number operator and labels $0, 1$ are fermion number.