Université Paris Diderot - Paris 7
Master première année
31 UKMT 42

UFR de Mathématiques
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Francis Comets et Justin Salez

Probability and stochastic Processes

No books, no notes, no phones, no electronic equipment of any kind are allowed. Problems are independent. Do not waste too much time on a single question: it is possible to assume the result stated in one of the questions and use it later on. Please use a separate sheet for each problem.

Problem I (approx. 11 points)

In a fair game of heads and tails, a gambler bets on the current leader. In this problem we study this strategy, and compute the probability of ruin.

Let $Z_n = +1$ [resp. -1] if the nth throw is heads [resp. tails]. Thus, $Z_n, n \geq 1$ are i.i.d. Bernoulli variables with $\mathbf{P}(Z_i = -1) = \mathbf{P}(Z_i = +1) = 1/2$, and $S_n = Z_1 + \ldots + Z_n$ is the number of heads minus the number of tails by time n ($S_0 = 0$). Define the sign function as

$$\operatorname{sign}(x) = \begin{cases} +1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

If the gambler uses the strategy "follow the leader", his fortune M_n at time n is

$$M_n = M_{n-1} + \operatorname{sign}(S_{n-1}) \times Z_n = \sum_{i=1}^{n} \operatorname{sign}(S_{i-1}) \times Z_i.$$

Hence, $M_0 = M_1 = 0$. Define $L_n = \sum_{i=1}^n \mathbf{1}_{\{S_{i-1}=0\}}$ the number of visits to 0 ($L_0 = 0$).

1. Show that $(S_n)_n$ and $(M_n)_n$ are martingales. Are $(S_n^2)_n$ and $(M_n^2)_n$ submartingales ? Compute the brackets $\langle S \rangle_n$ and $\langle M \rangle_n$ in terms on n, L_n.

2. a) Prove by induction that for every $n \geq 0$, $|S_n| = L_n + M_n$.
   b) Deduce that $M_n \in \sigma(|S_i|, i \leq n)$. What is the Doob's decomposition of $|S_n|$?
   c) Prove that $|S_n|$ is a Markov chain. Is it irreducible? Recurrent ? Ergodic ?

3. In this question, we fix two integers $a, b \geq 1$, and let
$$\tau = \inf\big\{n \geq 1 : M_n \in \{-a, b\}\big\}. \qquad (1)$$
   Set $\delta = 2(a+b)$. By considering the events $A_k = \{Z_{(k-1)\delta+1} = \cdots = Z_{k\delta} = 1\}$ show that
$$\mathbf{P}\big(\tau > k\delta\big) \leq \left(1 - \left(\frac{1}{2}\right)^{\delta}\right)^{k}, \qquad \forall k \geq 1.$$
   Conclude that τ is square-integrable. Is τ a stopping time ?

4. Is $(M_{n\wedge\tau})_n$ is a martingale ? Compute the values of $\mathbf{P}(M_\tau = -a)$ and $\mathbf{P}(M_\tau = b)$.

5. We now take $a = b \geq 1$. We use the notation $\tau(a) = \tau$ for (1). Prove that

$$\mathbf{E}M^2_{\tau(a)} = \mathbf{E}\tau(a) - \mathbf{E}L_{\tau(a)}, \qquad \mathbf{E}|S_{\tau(a)}| = \mathbf{E}L_{\tau(a)}, \qquad \mathbf{E}S^2_{\tau(a)} = \mathbf{E}\tau(a).$$

   Deduce an equivalent of $\mathbf{E}\tau(a)$ as $a \to \infty$.

Problem II (approx. 9 points)

Let $\mathcal{A}$ be a finite or countable set, and let $p = (p(a))_{a\in\mathcal{A}}$ be a collection of numbers satisfying

$$\forall a \in \mathcal{A},\ p(a) > 0 \quad \text{and} \quad \sum_{a\in\mathcal{A}} p(a) = 1.$$

Consider a sequence of i.i.d. random symbols $(Z_n)_{n\geq 1}$ distributed according to p. We are interested in the stochastic process $(X_n)_{n\geq 0}$ defined by $X_n := (Z_{n+1}, Z_{n+2}, Z_{n+3})$. Thus, $X_0 = (Z_1, Z_2, Z_3)$, $X_1 = (Z_2, Z_3, Z_4)$, $X_2 = (Z_3, Z_4, Z_5)$, and so on.

Part A. In this question we consider the special case where $\mathcal{A} = \{0, 1\}$ and $p(0) = p(1) = \frac{1}{2}$.

1. Show that $(X_n)_{n\geq 1}$ is a Markov chain and draw the associated diagram.
2. Justify that the chain admits a unique invariant law, and find it. Is it reversible ?
3. Let $T = \inf\{n \geq 1 \colon X_n = X_0\}$ be the time of first return to the initial state (which is itself random). What is the value of $\mathbf{E}[T]$?
4. What is the probability that the state $(1, 1, 1)$ is visited before the state $(0, 0, 1)$?

Part B. We now come back to the general case. Recall that $\mathcal{A}$ is not necessarily finite!

1. Show that $(X_n)_{n\geq 1}$ is a Markov chain and specify its transition matrix.
2. Justify that the chain is irreducible and aperiodic.
3. What is the initial law X_0? What about X_1? What can we deduce from this regarding transience/null-recurrence/positive-recurrence?
4. Let a, b, c be three distinct symbols in $\mathcal{A}$. Let T_1, T_2 denote the first and second time at which the state (a, b, c) gets visited, i.e.

$$T_1 := \inf\{n \geq 0 \colon X_n = (a, b, c)\} \quad \text{and} \quad T_2 := \inf\{n > T_1 \colon X_n = (a, b, c)\}$$

   Justify that T_1, T_2 are almost-surely finite, and then compute the expected number of visits to the state (c, b, a) between times T_1 and T_2.

5. A random text is produced by hitting n times a 26-letter keyboard, uniformly at random and independently. How many times does the word CLT appear, for n large? What about a word with repeated letters, such as LLN? Justify precisely.