Université Paris Diderot - Paris 7 | UFR de Mathématiques
Master première année | March 9th, 2017
UKMT 42 | Francis Comets et Benoît Laslier

Probability and stochastic Processes

Midterm exam (duration 2h30)

No books, no notes, no phones, no electronic equipment of any kind are allowed.

All problems are independent. Do not waste too much time on a single question: it is possible to assume the result stated in one of the questions and use it later on.

Compose Problems I and II on one set of copies, and Problems III on another one.

Problem I (approximatively 7.5 points)

The birth-and-death process is the Markov chain $(X_n)_{n\geq 0}$ on the integers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with transition

$$Q(x, x+1) = p_x, \quad Q(x, x-1) = q_x, \quad Q(x, x) = r_x \quad (x \geq 0),$$

where the non-negative numbers p_x, r_x, q_x add up to 1, with $q_0 = 0 < p_0$ and $p_x > 0, q_x > 0$ for all $x \geq 1$.

1. Show that the chain is irreducible on $\mathbb{N}$.
2. Given integers $0 \leq a < b$, we define the probabity to exit to the left

$$u(x) = P_x(T_a < T_b) \qquad (a \leq x \leq b)$$

with $T_y = \inf\{t \geq 0 : X_t = y\} \in [0, \infty]$ the hitting time of $y \in \mathbb{N}$.

(a) Show that for $a < x < b$,

$$u(x) = p_x u(x+1) + r_x u(x) + q_x u(x-1),$$

with boundary conditions $u(a) = 1, u(b) = 1$.

(b) What is the equation satisfied by $v(x) = u(x) - u(x+1)$? Solve this equation, and, denoting

$$\gamma_x = \prod_{y=1}^{x} \frac{p_y}{q_y}\ , \qquad \gamma_0 = 1,$$

deduce that, for $x = a, a+1, \ldots b-1$,

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}\ .$$

3. Derive that $P_x(T_a < \infty) = \lim_{b\to\infty} P_x(T_a < T_b)$ and compute its value in terms of the γ_x's. What is the condition for recurrence [resp., transience] of the chain ?

4. Determine the reversible measures fo the chain. When are there reversible probability measures ?

5. Determine the condition on the sequence γ_x's for ergodicity of the chain. In the sequel, we assume this condition holds.

   (a) In that case, in what sense does the limit $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \mathbf{1}_{X_i \leq 13}$ exist ? What is its value ?

   (b) Compute the value of $w(x) = E_x T_0$.

Problem II

We consider in this question a random urn process. At time t we will have an urn with t balls in it and add a single ball to it to obtain the urn at time $t+1$. Balls have three types called "rock", "paper" and "scissors" and we will call $p_R(t), p_P(t), p_S(t)$ the proportion of balls in the urn and $n_R(t), n_P(t), n_S(t)$ the number of balls. We say that rock beats scissors, scissors beats paper and paper beats rock.

To find the type of the ball we add from t to $t+1$ we sample tree balls from the urn uniformly at random (putting them back after looking at their type so each is sampled according to $p(t)$). If they all share a type, we add a ball of the same type. If they have exactly two types, we add a ball of the winning type. If we sampled all three types we chose a type at random and add a ball of that type.

We start the process with 3 balls in the urn, one of each type (note that this means that we start at time $t = 3$). To simplify notations, we will allow ourself to write $p_R p_S + p_S p_P + p_P p_R = \sum_i p_i p_{i+1}$ and similar expressions.

We let $M_t = \sum_{i\in\{R,P,S\}} \sum_{k=n_i(t)}^{t-1} \frac{1}{k}$ and we recall Euler's formula $\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \gamma + o(1)$ for some constant γ.

1. Let $q(t) = (q_R(t), q_P(t), q_S(t))$ be the law of the ball added at time t. Write $q(t)$ in terms of $p(t)$.

2. Show that $\mathbf{E}[M_{t+1} - M_t | \mathcal{F}_t] = \frac{-1}{t} \sum_i (p_i - p_{i+1})^2$. Is M a super-martingale ? In which sense can we say that it converges ? We let M_∞ denote its limit if it exists.

3. Show that almost surely $n_R(t), n_P(t)$ and $n_S(t)$ tend to infinity.

4. Show that almost surely, if p is any limit point ("valeure d'adhérence") of the sequence p_t then $M_\infty = -\sum_i \log p_i$. Show that $p_i \in (0,1)$.

5. Show that any limit point p of the sequence p_t has to satisfy $\sum_i (p_i - p_{i+1})^2 = 0$ and conclude about the almost sure convergence of p_t.