Université Paris Diderot - Paris 7 | UFR de Mathématiques
Master première année | May 17th, 2017
UKMT 42 | Francis Comets et Benoît Laslier

Probability and stochastic Processes

Exam (duration 3h)

No books, no notes, no phones, no electronic equipment of any kind are allowed.

All problems are independent. Do not waste too much time on a single question: it is possible to assume the result stated in one of the questions and use it later on.

Compose Problems I and II on one set of copies, and Problems III on another one.

Exercise (approximatively 5 points)

Let $\xi_n, n \geq 1$, an i.i.d. integrable sequence, $\phi(t) = Ee^{it\xi_1}$ and $m = E\xi_1$. Let $\alpha \in (-1, 1)$. We consider

$$X_{n+1} = \alpha X_n + \xi_{n+1}\ , \qquad X_0 = 0. \tag{1}$$

1. We know that a characteristic function is always continuous and that $\phi(0) = 1$, therefore $\phi(t)$ stays in a neighbourhood of 1 (hence in $\mathbb{C} \setminus \mathbb{R}_-$) for t in a neighbourhood of 0. ξ is integrable so ϕ is differentiable (one can reprove it with the dominated convergence theorem) and $\phi'(0) = im$ so $\log \phi(t) = imt + o(t)$ as a first order Taylor expansion.

2. From 1 and applying the formula giving the characteristic functions of sums of independent variables and scaling, we see that $\Phi_0(t) = 1$ and $\Phi_{n+1}(t) = \Phi_n(\alpha t)\phi(t)$. Therefore $\Phi_n = \prod_{k=0}^{n-1} \phi(\alpha^k t)$.

3. First if t is small enough so that $\phi(t) \in \mathbb{C} \setminus \mathbb{R}$, $\log \Phi_n$ is well defined. From the question 1, $\log \phi(\alpha^k t) \sim im\alpha^k t$ which is summable so by comparison of series $\log \Phi_n$ converges. To see why the convergence is uniform, let t_0 be such that for $|t| \leq t_0$, $|\log \phi(t)| \leq 2mt$ (which exists by question 1). Now fix $\epsilon > 0$ and let n_0 be such that $\sum_{k \geq n_0} 2m\alpha^k t_0 \leq \epsilon$. For any compact set K, there is a n_1 such that $\alpha^{n_1} t \leq t_0$ uniformly over $t \in K$ and hence for any $N \geq n_0 + n_1$, $|\Phi_N(t) - \lim \Phi_n(t)| \leq \epsilon$ uniformly over K.

4. $(\log \Phi_n)_{n \geq 0}$ is a sequence of continuous functions in a neighbourhood of 0 that converges uniformly over compacts so its limit is continuous in a neighbourhood of 0. By composition $\Phi_n = \exp \log \Phi_n$ converges to a function continuous in a neighbourhood of 0 and in particular at 0.

5. By Levy's theorem the convergence of question 3 implies the convergence in law of X_n.

Problem I (approximatively 8 points)

The birth-and-death process is the Markov chain $(X_n)_{n\geq 0}$ on the integers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with transition

$$Q(x, x+1) = p_x, \quad Q(x, x-1) = q_x, \quad Q(x, x) = r_x \quad (x \geq 0),$$

where the non-negative numbers p_x, r_x, q_x add up to 1, with $q_0 = 0 < p_0$ and $p_x > 0, q_x > 0$ for all $x \geq 1$.

1. Since $p_x > 0$ for all x and $q_x > 0$ for all $x \geq 1$, the chain is irreducible. If there is at least one x such that $r_x > 0$ then the chain is aperiodic. On the other hand if $r_x = 0$ for all x, then the chain is periodic of period 2.

   Given integers $0 \leq a < b$, define the probability to exit to the right of the interval (a, b),

   $$u(x) = P_x(T_a > T_b) \qquad (a \leq x \leq b)$$

   with $T_y = \inf\{t \geq 0 : X_t = y\} \in [0, \infty]$ the hitting time of $y \in \mathbb{N}$.

2. Conditioning on the first step we have

   $$u(x) = P_x(\{T_a > T_b\}|\{X_1 = x-1\})q_x + P_x(\{T_a > T_b\}|\{X_1 = x\})r_x + P_x(\{T_a > T_b\}|\{X_1 = x+1\})p_x.$$

   By the Markov property, the conditional expectations are equal to the $u(y)$ which concludes. The boundary conditions are obvious since starting in a, $T_a = 0$ and reciprocally for b.

3. We have

   $$\begin{aligned} p_x u(x+1) &= u(x) - r_x u(x) - q_x u(x-1) \\ &= (p_x + q_x)u(x) - q_x u(x-1), \end{aligned}$$

   so $p_x v(x) = q_x v(x-1)$. The boundary condition is $v(a) > 0$ and $\sum v(x) = 1$. We can solve the equation as $v(x) = v(a) \prod_{y=1}^{x} q_y/p_y = \gamma_x v(a)$.

4. From any point in $[a, b]$, there is a uniformly positive probability to have $\min\{T_a, T_b\} \leq b - a$. We conclude by the Markov property and Borel-Cantelli.

5. Reasoning as in question 4, it is easy to see that for all $x \leq b$, $\mathbf{P}(T_b < \infty) = 1$, therefore the event $\{T_0 = \infty\}$ is included in any event of the form $\{T_0 > T_b\}$ with $b \geq x$ and $P_x(T_0 = \infty) \leq \lim P_x(T_0 > T_b)$. On the other hand $P_x(T_0 = \infty) = \lim P_x(T_0 \geq n) \geq \lim P_x(T_0 \geq T_{n+x})$ because X needs at least n steps to reach b.

   If $\sum \gamma_y = \infty$ then $P_x(T_0 = \infty) = 0$ and the chain is recurrent. If $\sum \gamma_y < \infty$ then $P_x(T_0 = \infty) = \frac{\sum_0^{x-1} \gamma_y}{\sum_0^{\infty} \gamma_y} > 0$ and the chain is transient.

6. The condition for μ to be a reversible measure is $\mu(x)p_x = \mu(x+1)q_{x+1}$ so if there is a reversible measure it is of the form $\mu(x) = \mu(0)\prod_{y=0}^{x-1} p_y/q_{y+1} = \frac{p_0}{\gamma_x q_x}\mu_0$. Therefore an invariant reversible probability measure exists if and only if this is summable.

7. If the chain is recurrent, then $\lim \frac{1}{n}\sum \mathbf{1}_{X_i \leq 13}$ converges almost surely by the ergodic theorem. The limit is $\mu([0,13])$ if there exists an invariant measure μ (and in particular under the condition of question 6) and 0 otherwise.

   If the chain is transient, then the number of visits to $0, 1, \dots, 13$ are almost surely finite and in particular $\lim \frac{1}{n}\sum \mathbf{1}_{X_i \leq 13}$ converges almost surely to 0.

8. Similarly to question 3, w satisfies the equation $w(x) = 1+p_x w(x+1)+r_x w(x)+q_x w(x-1)$ for all $x \geq 1$ with $w(0) = 0$.

$$w(x+1) - w(x) = -\frac{1}{p_x} + \frac{q_x}{p_x}(w(x) - w(x-1)).$$

   By induction this can be solved to give :

$$w(x+1) - w(x) = -(\sum_{y=1}^{x} \frac{1}{p_y} \prod_{z=y+1}^{x} \frac{q_z}{p_z}) + (\prod_{z=1}^{x} \frac{q_z}{p_z})(w(1) - w(0)).$$

   Note that the induction step does not use the equation for $x = 0$ so it is valid.

   The only remaining unknown is $w(1)$ which we find by using the relation between the return time to 0 and the invariant measure. Indeed if we let T_0^+ be the first return time to 0 we have $1/\mu(x) = \mathbf{E}_0(T_0^+) = 1.r_0 + (1 + w(1)).p_0$ by conditioning on the first step and applying the Markov property. Plugging it back into the formula we find

$$w(x+1) - w(x) = \sum_{y=1}^{x} -\frac{1}{p_y} \prod_{z=y+1}^{x} \frac{q_z}{p_z} + \prod_{z=1}^{x} \frac{q_z}{p_z} \frac{1/\mu(0) - 1}{p_0}.$$

   In the case where there exists a reversible invariant measure, $\mu(0)$ can be written in terms of the p_x, q_x, r_x.

   We will conclude by proving separately that all invariant measures are reversible for this chain, hence proving that the conditions of question 6 are optimal. Let μ be an invariant measure, we have for all $x \geq 1$

$$\mu(x) = p_{x-1}\mu(x-1) + r_x\mu(x) + q_{x+1}\mu(x+1),$$

   or equivalently,

$$p_x\mu(x) + q_x\mu(x) = p_{x-1}\mu(x-1) + q_{x+1}\mu(x+1).$$

   At 0 the equation is similar but the is no term coming from -1 so it becomes $p_0\mu(0) = q_1\mu(1)$ which is exactly the reversibility condition for 0 and 1 ! Plugging it into $p_1\mu(1) + q_1\mu(1) = p_0\mu(0) + q_2\mu(2)$ yields $p_1\mu(1) = q_2\mu_2$ and by induction $\mu(x)p_x = \mu(x+1)q_{x+1}$ for all x so the measure is reversible.

Problem II (approximatively 7 points)

We consider in this question a random urn process. At time t we will have an urn with t balls in it and add a single ball to it to obtain the urn at time $t+1$. Balls have three types called "rock", "paper" and "scissors" and we will call $p_R(t), p_P(t), p_S(t)$ the proportion of balls in the urn and $n_R(t), n_P(t), n_S(t)$ the number of balls. We say that rock beats scissors, scissors beats paper and paper beats rock.

To find the type of the ball we add from t to $t+1$ we sample three balls from the urn uniformly at random (putting them back after looking at their type so each is sampled according to $p(t) = (p_R(t), p_P(t), p_S(t)))$. If they all share a type, we add a ball of the same type. If they have exactly two types, we add a ball of the winning type. If we sampled all three types we chose a type at random and add a ball of that type.

We start the process with 3 balls in the urn, one of each type (note that this means that we start at time $t = 3$). To simplify notations, we will allow ourself to write $p_Rp_S+p_Sp_P+p_Pp_R = \sum_i p_ip_{i+1}$, $(p_R-p_S)^2 + (p_S-p_P)^2 + (p_P-p_R)^2 = \sum_i(p_i-p_{i+1})^2$ and similar expressions.

Let $\mathcal{F}_t$ be the σ-field of events observable before time t, and denote

$$M_t = \sum_{i\in\{R,P,S\}} \sum_{k=n_i(t)}^{t-1} \frac{1}{k}.$$

1. To add a rock, either we sampled only rocks (probability p_R^3), only rock and scissors (probability $p_R^2p_S$ or $p_Sp_R^2$ and three permutations each) or all three and chose rock (probability $\frac{1}{3}p_Rp_Pp_S$ and 6 permutations). Overall $q_R = p_R^3 + 3p_R^2p_S + 3p_Rp_S^2 + 2p_Rp_Sp_P$ and similarly for the others.

2. Note that if the ball added at time t is of type i then $M_{t+1} - M_t = \frac{3}{t} - \frac{1}{n_i}$. Therefore

$$\begin{aligned}
\mathbf{E}[M_{t+1} - M_t|\mathcal{F}_t] &= \frac{3}{t} - \sum_i \frac{q_i}{n_i} \\
&= \frac{3}{t} - \frac{1}{t}\sum_i \frac{p_i^3 + 3p_i^2p_{i+1} + 3p_ip_{i+1}^2 + 2p_ip_{i+1}p_{i+2}}{p_i} \\
&= \frac{3}{t} - \frac{1}{t}\sum_i p_i^2 + 3p_{i+1}^2 + 3p_ip_{i+1} + 2p_{i+1}p_{i+2} \\
&= \frac{3}{t} - \frac{1}{t}\sum_i 4p_i^2 + 5p_ip_{i+1}.
\end{aligned}$$

   Where in the last line we relabelled some of the terms.

   Now we note that $\sum_i p_i^2 + 2p_ip_{i+1} = \sum_i p_i^2 + p_{i-1}p_i + p_ip_{i+1} = \sum_i p_i(p_i + p_{i+1} + p_{i-1}) = 1$. Hence

$$\begin{aligned}
\mathbf{E}[M_{t+1} - M_t|\mathcal{F}_t] &= -\frac{1}{t}\sum_i p_i^2 - p_ip_{i+1} \\
&= -\frac{1}{2t}\sum_i p_i^2 + p_{i+1}^2 - 2p_ip_{i+1}
\end{aligned}$$

as desired.

3. M is an adapted process integrable process and $\mathbf{E}[M_{t+1} - M_t | \mathcal{F}_t] < 0$ so it is a submartingale. Since M is positive, we can apply the martingale convergence theorem to deduce that M converges almost surely to some limit M_∞.

   Remark : It is actually not known whether $(M_t)_{t \geq 3}$ is uniformly integrable.

4. Let us consider a fixed realisation and assume by contradiction that one of the n_i does not converges to infinity. Without loss of generality we can assume that n_R stays bounded by C. Then we have $M_t \geq \sum_{k=C}^{t} \frac{1}{k} \to_{t \to +\infty} +\infty$. By the previous question we know that M_t converges almost surely to a finite limit so in particular all n_i tend to infinity almost surely. Overall the event that one of the n_i stays bounded is included in the event that M_t tends to infinity so it has probability 0.

5. We write Euler's formula: $\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \gamma + \epsilon(n)$ for some constant γ and writing the $o(1)$ explicitly.

   Replacing in the definition of M, we get $M_t = \sum_i \ln(t) - \ln(n_i(t)) + \epsilon(t) - \epsilon(n_i(t)) = \sum_i \ln p_i(t) + o(1)$ because the n_i converge to infinity. As a consequence if p is any limit point, then taking a subsequence such that $p(t)$ converge to p and replacing we get $M_\infty = \sum_i \log p_i$.

6. Suppose by contradiction that p is a limit point such that $\sum_i (p_i - p_{i+1})^2 = \epsilon > 0$. Let t_k be a subsequence on which $p(t)$ converges to p. We can assume without loss of generality by dropping the first terms that for all k we have $\sum_i (p_i(t_k) - p_{i+1}(t_k))^2 \geq \epsilon/2$.

   Now note that $p(t)$ can be seen as a point in a triangle and that deterministically we have $||p(t+1) - p(t)||_\infty \leq 3/t$. Note also that the function $q \to \sum_i (q_i - q_{i+1})^2$ is uniformly continuous on that triangle. Hence we can find $\delta > 0$ such that for all $t \in [t_k, t_k(1+\delta)]$, we have $\sum_i (p_i(t) - p_{i+1}(t))^2 \geq \epsilon/4$.

   Without loss of generality by taking a sub-sequence of the t_k, we can assume that for all k, $t_{k+1} \geq t_k(1 + 2\delta)$. Hence we see that on the event that $p(t)$ has p as a limit point,

$$
\begin{aligned}
-\sum_t \mathbf{E}[M_{t+1} - M_t | \mathcal{F}_t] &\geq \sum_k \sum_{t=t_k}^{(1+\delta)t_k} \mathbf{E}[M_{t+1} - M_t | \mathcal{F}_t] \\
&\geq \sum_k \sum_{t=t_k}^{(1+\delta)t_k} \frac{\epsilon}{8t} \\
&\geq \sum_k \frac{\epsilon \delta}{8(1+\delta)} = +\infty.
\end{aligned}
$$

   On the other hand by the monotonous convergence theorem, $\mathbf{E} \sum_t \mathbf{E}[M_{t+1} - M_t | \mathcal{F}_t] = \sum_t \mathbf{E}[M_{t+1} - M_t] = \lim \mathbf{E} M_t - \mathbf{E} M_0$. By Fatou's theorem this is finite and therefore $\mathbf{P}\left(\sum_t \mathbf{E}[M_{t+1} - M_t | \mathcal{F}_t] = -\infty\right) = 0$.

   Since $(1/3, 1/3, 1/3)$ is the only probability measure such that $\sum_i (p_i - p_{i+1})^2 = 0$, it is the only limit point of the sequence and therefore $p(t)$ converges almost surely to $(1/3, 1/3, 1/3)$.