A New General Construction of Complementary Sequence Pairs over 4^q-QAM

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Abstract

The previous constructions of quadrature amplitude modulation (QAM) Golay complementary sequences (GCSs) were generalized as 4^q-QAM GCSs of length 2^m by Li (Generalised Cases I-III for $q \geq 2$) in 2010 and Liu (Generalised Cases IV-V for $q \geq 3$) in 2013 respectively. The results can be given by weighted sum of q quadrature phase shift keying (QPSK) standard GCSs, which is represented as q-dimensional vectorial generalized Boolean functions (V-GBFs) in this paper. In line with their work, an extended construction of 4^q-QAM GCSs of length 2^m is proposed, including Generalized Gases I-V as special cases. If q is a composite number, a great number of new GCSs will arise. For example, in the case $q = 4$, the number of new GCSs is more than seven times than those in generalized cases IV-V. In the case $q = 6$, the ratio of the number of new GCSs and the Generalized Cases IV-V is greater than six and will increase in proportion with m. Moreover, based on the theory of array and paraunitary matrices, the proof implies all the mentioned GCSs are actually projections of Golay complementary arrays of size $2 \times 2 \times \cdots \times 2$.

1 Introduction

A pair of sequences is called a Golay complementary pair (GCP) if their aperiodic autocorrelation sums for any nonzero shift is equal to zero [11]. Each sequence in the GCP is called a Golay complementary sequence (GCS). The concept of binary GCP was extended later to polyphase case [22] and complementary sequence set [23]. These sequences have found numerous applications in various fields of science and engineering, especially in orthogonal frequency-division multiplexing (OFDM) systems. One of the major impediments to deploying OFDM is the high peak-to-mean envelope power ratio (PMEPR) of uncoded OFDM signals. A convenient approach [1][19] to PMEPR reduction in OFDM transmission is to use codes constructed from the sequences in complementary set, especially GCSs.

GCPs were initially constructed by recursive method [11][3]. An extensive study on this topic was made by Davis and Jedwab in [7] by a direct construction of polyphase complementary sequences based

on generalized Boolean functions (GBFs), which have been subsequently referred to as the standard GCSs. Non-standard GCPs were studied in [16],[8],[9] and complementary sequence sets were constructed in [18][21] later on.

All of the aforementioned sequences are over the phase-shift keying (PSK) constellations. Since quadrature amplitude modulation (QAM) are widely employed in high rate OFDM transmissions, 16-QAM sequences based on weighted quaternary PSK (QPSK) GCSs were studied by Rößing and Tarokh [20] in 2001. Chong *et al.* [5] then proposed a construction of 16-QAM GCSs based on standard GCSs over QPSK and first-order offsets in three cases. It is pointed out in [5] that an OFDM system with 16-QAM GCSs has a higher code rate than that with binary or quaternary standard GCSs, given the same PMEPR constraint. In 2006, Lee and Golomb [13] proposed a construction of 64-QAM GCSs with weighted-sum of three standard GCSs over QPSK and first-order offsets in five cases. And it was improved by Li *et al.* [14] and Chang *et al.* [6] later on. These results were generalized for the construction of general 4^q-QAM GCSs by Li *et al.* [15] in 2001 and Liu *et al.* [17] in 2013, respectively. Depending on the algebraic structure of the compatible offsets for standard GCSs over QPSK, the GCSs proposed in [15] and [17] are called cases I-III and cases IV-V, respectively.

In 2018, Budišin *et al.* [2] introduced a new algorithm in multiplicative form to generate GCPs over QAM by para-unitary (PU) matrices. The element of a sequence is generated by indexing the entries of unitary matrices with the binary representation of the discrete time index (which is easily implemented as a binary counter). The 1Qum (based on one QAM unitary matrix) sequences are identical to the sequences in generalized Cases I-III[15]. The 2Qum (based on two QAM unitary matrices) constructions produce not only all the known 4^q-QAM GCSs in Generalized Cases IV,V[17], but also many new sequences. (For instance, for the case $q = 3$ and 4, the ratio between 2Qum sequences and Generalized Cases IV,V is about 10 and 100 respectively.) As the constellation size increase ($q \geq 4$), MQum (based on $M(\geq 3)$ QAM unitary matrices) sequences can also derived from the construction.

However, there are also some drawbacks in the algorithm. For given sequence length 2^m and given q, acceptable unitary matrices can be obtained only by exhaustive search. For example, in 2Qum case, the adjacent QAM unitary matrices must be determined computationally by choosing Gaussian integers that satisfy 8 inequalities. (Remark 12 in[2].) In the MQum case with higher M, the situation is even more complicated. We have to deal with the great complexity of duplication in constructing the algorithm. And the lack of explicit algebraic expression makes the use of GCSs difficult.

In this paper, we propose two new general constructions of GCSs over 4^q-QAM of length 2^m by explicit GBFs based on standard GCSs over QPSK and compatible offsets. The known generalized cases I-V in [15, 17] are special cases of our new constructions. Moreover, if q is a composite number, the proposed constructions significantly increase the number of the GCSs given in [15, 17]. From Table 1, we can see the number of new GCSs is more than seven times than those in generalized cases IV-V

[17] if $q = 4$. If $q = 6$, notice that the number of new GCSs is cubic expression of m, the ratio with the generalized cases IV-V is greater than six and will increase in proportion with m.

The approach to construct GCSs over QAM in this paper is different from all the aforementioned methods. Golay array pairs (GAPs) over PSK and their relationship with GCPs was introduced by F. Fiedler *et al.* [10] in 2008. We extend this idea to GAPs and QAM, and introduce a mapping from a GAP to a large number of GCPs, which in some sense answers the open problem posed in [10]. Then we can construct GCPs by GAPs. Inspired by the idea in [2], we make a connection between GAP and specified PU matrix with multi-variables over 4^q-QAM. Finally, by the technique to derive GBFs from PU matrices introduced in our recent work [24, 25], two new general constructions of GCSs over 4^q-QAM based on explicit GBFs by standard GCSs over QPSK and compatible offsets are given.

The reminder of this paper is organized as follows. In the next section, we introduce the definitions of GCP and GCS, and the known constructions of the 4^q-QAM GCPs. In Section 3, we present two new general constructions of 4^q-QAM GCSs including generalized Cases I-V as special cases. An enumeration of the new GCSs other than generalized Cases I-V for $q = 4$ and $q = 6$ is given in Section 4. The proof of our main results is presented step by step in the following two sections. In Section 5, we explain our main theory on GAPs, PU matrices and corresponding GBFs. What's more, we propose the array form of our result, namely 4^q-QAM GAPs based on PU matrices, from which the main result on GCSs can be derived directly. The proof of the array form results is shown in Section 6. We conclude the paper in Section 7.

2 Preliminary

The following notations will be used throughout the paper.

- q, p, m, L are all positive integers, where $0 \leq p < q$.
- For any positive integer N, $\mathbb{Z}_N = \{0, 1, \cdots N-1\}$ is the residue class ring modulo N.
- For positive integer n and N, define $n \cdot \mathcal{Z}_N$ as set $\{0, n, \cdots, n(N-1)\}$ and $n \cdot \mathcal{Z}_1 = \{0\}$.
- $\mathbb{F}_2$ is the finite field with two elements, and $\mathbb{F}_2^m$ is m-dimension vector space on $\mathbb{F}_2$.
- $\boldsymbol{x} = (x_1, x_2, \cdots x_m) \in \mathbb{F}_2^m$, where every x_i is a indeterminate for $1 \leq i \leq m$.
- $\mathbb{C}$ is the complex field, and $\mathbb{R}$ is the real number field.
- For any $\alpha \in \mathbb{C}$, $\overline{\alpha}$ is the conjugation of α.
- π is a permutation of symbols $\{1, 2, \cdots, m\}$, and σ is a permutation of symbols $\{0, 1, \cdots, m\}$.

2.1 Golay Complementary Pair

A complex value sequence of length L can be expressed by a function $F(y) : \mathbb{Z}_L \to \mathbb{C}$, i.e.,

$$F(y) = (F(0), F(1), \cdots, F(L-1)).$$

The *aperiodic auto-correlation* of $F(y)$ at shift τ $(1 - L \leq \tau \leq L - 1)$ is defined by

$$C_F(\tau) = \sum_y F(y+\tau) \cdot \overline{F(y)},$$

where $F(y+\tau) \cdot \overline{F(y)} = 0$ if $F(y+\tau)$ or $F(y)$ is not defined.

A pair of sequences $\{F(y), G(y)\}$ of length L is said to be *Golay complementary pair* (GCP) if

$$C_F(\tau) + C_G(\tau) = 0, \quad (\forall \tau \neq 0). \tag{1}$$

And either sequence in a GCP is called a *Golay complementary sequence* (GCS) [11].

2.2 GCPs over QPSK

In this subsection, we give the concepts and some results about sequence and GCP over QPSK based on generalized Boolean function.

A *generalized Boolean function* (GBF) $f(\boldsymbol{x})$ (or $f(x_1, x_2, \cdots, x_m)$) over $\mathbb{Z}_4$ is a function from $\mathbb{F}_2^m$ to $\mathbb{Z}_4$. Such a function can be uniquely expressed as a linear combination over $\mathbb{Z}_4$ of the monomials

$$1, x_1, x_2, \cdots, x_m, x_1x_2, x_1x_3, \cdots, x_{m-1}x_m, \cdots, x_1x_2x_3\cdots x_m,$$

where the coefficient of each monomial belongs to $\mathbb{Z}_4$.

For $0 \leq y < 2^m$, y can be written uniquely in a binary expansion as $y = \sum_{k=1}^m x_k \cdot 2^{k-1}$ where $x_k \in \mathbb{Z}_2$. Then a sequence $F(y)$ of length $L = 2^m$ over QPSK can be associated with a GBF $f(\boldsymbol{x})$ over $\mathbb{Z}_4$ by

$$F(y) = \xi^{f(\boldsymbol{x})}. \tag{2}$$

where $\xi = \sqrt{-1}$ is a 4-th primitive root of unity.

There are several constructions of GCPs over QPSK based on GBFs, such as [7], [16],[8] and [9]. The most typical GCPs are so called standard GCPs given in [7], which are associated with GBFs over $\mathbb{Z}_4$ given below.

Fact 1 *[7] Let*

$$f(\boldsymbol{x}) = 2 \cdot \sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} c_k \cdot x_k + c_0, \tag{3}$$

where $c_k \in \mathbb{Z}_4 (0 \le k \le m)$. For any $c' \in \mathbb{Z}_4$, the sequence pair associated with the GBFs over $\mathbb{Z}_4$

$$\begin{cases} f(\boldsymbol{x}), \\ f(\boldsymbol{x}) + 2x_{\pi(1)} + c', \end{cases} \quad or \quad \begin{cases} f(\boldsymbol{x}), \\ f(\boldsymbol{x}) + 2x_{\pi(m)} + c'. \end{cases}$$

form a GCP over QPSK.

2.3 GCPs over QAM

In this subsection, we show some of the known results about sequence and GCP over QAM based on vectorial generalized Boolean function.

A *vectorial generalized Boolean function (V-GBF)* is a function from $\mathbb{F}_2^m$ to $\mathbb{Z}_4^q$, denoted by

$$\vec{f}(\boldsymbol{x}) = (f^{(0)}(\boldsymbol{x}), f^{(1)}(\boldsymbol{x}), \cdots, f^{(q-1)}(\boldsymbol{x})),$$

where each component function $f^{(p)}(\boldsymbol{x}) (0 \le p < q)$ is a GBF over $\mathbb{Z}_4$.

A sequence over 4^q-QAM can be viewed as the weighted sums of q sequences over QPSK, with weights in the ratio of $2^{q-1} : 2^{q-2} : \cdots : 1$. A sequence over 4^q-QAM of length 2^m can be associated with a V-GBF $\vec{f}(\boldsymbol{x}) = (f^{(0)}(\boldsymbol{x}), f^{(1)}(\boldsymbol{x}), \cdots, f^{(q-1)}(\boldsymbol{x}))$ over $\mathbb{Z}_4$ by

$$F(y) = \sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(y)} \tag{4}$$

where $y = \sum_{k=1}^{m} x_k \cdot 2^{k-1}$ and $f^{(p)}(y) = f^{(p)}(\boldsymbol{x}) (0 \le p < q)$. Obviously, the sequence over QPSK can be seen as a special case of QAM sequence when $q = 1$.

The GCPs $\{F(y), G(y)\}$ of length 2^m over 4^q-QAM were well studied by their associated V-GBFs $\{\vec{f}(\boldsymbol{x}), \vec{g}(\boldsymbol{x})\}$ in the literature. Such V-GBFs $\{\vec{f}(\boldsymbol{x}), \vec{g}(\boldsymbol{x})\}$ are usually given by

- standard GCSs $f(\boldsymbol{x})$ in form (3),
- *offset* V-GBF $\vec{s}(\boldsymbol{x}) = \big(s^{(0)}(\boldsymbol{x}) = 0, s^{(1)}(\boldsymbol{x}), \cdots, s^{(q-1)}(\boldsymbol{x})\big)$,
- *paring difference* V-GBF $\vec{\mu}(\boldsymbol{x}) = \big(\mu^{(0)}(\boldsymbol{x}), \mu^{(1)}(\boldsymbol{x}), \cdots, \mu^{(q-1)}(\boldsymbol{x})\big)$,

or more clearly,

$$\begin{cases} \vec{f}(\boldsymbol{x}) = \vec{1} \cdot f(\boldsymbol{x}) + \vec{s}(\boldsymbol{x}) \\ \vec{g}(\boldsymbol{x}) = \vec{f}(\boldsymbol{x}) + \vec{\mu}(\boldsymbol{x}) \end{cases} \tag{5}$$

where $\vec{1}$ denotes the q-dimension vector $(1, 1, \cdots, 1)$. The offset V-GBFs $\vec{s}(\boldsymbol{x})$ and paring difference V-GBFs $\vec{\mu}(\boldsymbol{x})$ proposed in generalized constructions for Cases I, II, III given by Li *et al.* [15] and Cases IV, V given by Liu *et al.* [17] are shown below respectively.

Fact 2 *(Generalized Cases I-III [15]) $\{F(y), G(y)\}$ form a 4^q-QAM GCP of length 2^m if the offset V-GBF $\vec{s}(\boldsymbol{x})$ and paring difference V-GBF $\vec{\mu}(\boldsymbol{x})$ in their associated V-GBFs $\{\vec{f}(\boldsymbol{x}), \vec{g}(\boldsymbol{x})\}$ satisfy one of the following cases:*

(1) Generalized Case I:

$$s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(1)}, 1 \le p \le q-1, \vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(m)},$$

(2) Generalized Case II:

$$s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(m)}, 1 \le p \le q-1, \vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(1)},$$

(3) Generalized Case III:

$$s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(\omega)} + d_2^{(p)} x_{\pi(\omega+1)}, 1 \le p \le q-1,$$

with $\vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(1)}$ or $\vec{2} \cdot x_{\pi(m)}$, $1 \le \omega \le m-1$, and $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$.

Definition 1 *([17]) A complex number is called a Gaussian integer if its real and imaginary part are both integers. Define*

$$Q(b_1, b_2, \ldots, b_{q-1}) = 2^{q-1} + \sum_{p=1}^{q-1} 2^{q-1-p} \xi^{b_p}, \quad b_p \in \mathbb{Z}_4 \tag{6}$$

as a one-to-one mapping from $\mathbb{Z}_4^{q-1}$ to $\mathcal{Q}_q$, which is a set including 4^{q-1} Gaussian integers.

Suppose that $Q_0 = Q(b_1, b_2, \ldots, b_{q-1}) \in \mathcal{Q}_q$ and $Q_1 = Q(b'_1, b'_2, \ldots, b'_{q-1}) \in \mathcal{Q}_q$ are a pair of distinct Gaussian integers with identical magnitude, and which are not conjugate with each other, namely:

$$|Q_0| = |Q_1|, \ Q_0 \ne Q_1, \ \text{and } Q_0 \ne \overline{Q_1}, \tag{7}$$

then (Q_0, Q_1) is called a non-symmetrical Gaussian integer pair (NSGIP).

Fact 3 *(Generalized Cases IV-V [17]) Given a non-symmetrical Gaussian integer pair (Q_0, Q_1), $\{F(y), G(y)\}$ form a 4^q-QAM GCP of length 2^m if the offset V-GBFs $\vec{s}(\boldsymbol{x})$ and paring difference V-GBFs $\vec{\mu}(\boldsymbol{x})$ in their associated V-GBFs $\{\vec{f}(\boldsymbol{x}), \vec{g}(\boldsymbol{x})\}$ satisfy one of the following cases:*

(4) Generalized Case IV:

$$s^{(p)}(\boldsymbol{x}) = b_p + (b'_p - b_p)x_{\pi(\omega)}, 1 \le p \le q-1,$$

with $\vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(1)}$ *or* $\vec{2} \cdot x_{\pi(m)}$, $2 \le \omega \le m-1$.

(5) Generalized Case V:

$$s^{(p)}(\boldsymbol{x}) = b_p + (b'_p - b_p)x_{\pi(\omega)} + (-b'_p - b_p)x_{\pi(\upsilon)}, 1 \le p \le q-1,$$

with $\vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(1)}$ *or* $\vec{2} \cdot x_{\pi(m)}$, $1 \le \omega \le m-2$, $\omega+2 \le \upsilon \le m$.

3 Main Results

In this section, we will propose two new constructions of GCPs over 4^q-QAM. The aforementioned generalized constructions for cases I, II, III [15] are special cases of our first construction, and the constructions for cases IV, V [17] are included in our second construction. Different from the offset V-GBFs $\vec{s}(\boldsymbol{x})$ and paring difference V-GBFs $\vec{\mu}(\boldsymbol{x})$ shown in Fact 2 and 3, the new proposed $\vec{s}(\boldsymbol{x})$ and $\vec{\mu}(\boldsymbol{x})$ relate to the factorization of the integer q.

Definition 2 *Let* $q = q_0 \cdot q_1 \cdots q_m$ *be a factorization of* q, *where* $q_k (0 \le k \le m)$ *are positive integers. For* $1 \le k \le m$, *define* $\mathcal{T}_k$ *as sets of integers by*

$$\mathcal{T}_k = (q_0 \cdot q_1 \cdots q_{k-1}) \cdot \mathcal{Z}_{q_k} = \left(\prod_{i=0}^{k-1} q_i \right) \cdot \mathcal{Z}_{q_k}. \tag{8}$$

For the case $k = 0$, *the definition is extended to* $\mathcal{T}_0 = \mathcal{Z}_{q_0}$. *For any given permutation* σ *of symbols* $\{0, 1, \cdots, m\}$, *define mappings* $\rho_k : \mathcal{Z}_q \to \mathcal{T}_{\sigma(k)} (0 \le k \le m)$, *such that* $p = \rho_0(p) + \rho_1(p) + \cdots + \rho_m(p)$.

We now illustrate the definition of $\rho_k : \mathcal{Z}_q \to \mathcal{T}_{\sigma(k)} (0 \le k \le m)$ is reasonable.

Remark 1 *Any* $p \in \mathcal{Z}_q$ *can be uniquely decomposed as* $p = p_0 + p_1 + \cdots + p_m$, *where* $p_k \in \mathcal{T}_k$. *In fact,* p_k $(0 \le k \le m)$ *can be uniquely determined by the following recursive formula*

$$\begin{cases} p_0 \equiv p \ (mod\, q_0), \\ \\ p_k \equiv p - \sum_{i=0}^{k-1} p_i \ (mod \prod_{i=0}^{k} q_k), 1 \le k \le m. \end{cases}$$

It is easy to verify $\rho_k(p) = p_{\sigma(k)}$.

Example 1 *Let $q = 6$ and $m = 3$. For factorization $q = q_0 \cdot q_1 \cdot q_2 \cdot q_3 = 2 \times 1 \times 3 \times 1 = 6$ and permutation $\sigma(0,1,2,3) = (1,0,3,2)$, we have*

k	q_k	$\prod_{i=0}^{k-1} q_i \cdot \mathcal{Z}_{q_k}$	$\mathcal{T}_k$	$\mathcal{T}_{\sigma(k)}$
0	2	$\mathcal{Z}_2$	$\{0,1\}$	$\{0\}$
1	1	$2 \cdot \mathcal{Z}_1$	$\{0\}$	$\{0,1\}$
2	3	$2 \times 1 \cdot \mathcal{Z}_3$	$\{0,2,4\}$	$\{0\}$
3	1	$2 \times 1 \times 3 \cdot \mathcal{Z}_1$	$\{0\}$	$\{0,2,4\}$

For $p \in \mathcal{Z}_6$, the decomposition of $p = p_0 + p_1 + p_2 + p_3 (p_k \in \mathcal{T}_{\sigma(k)})$ is given below:

p	$\mathcal{T}_{\sigma(k)}$	0	1	2	3	4	5
$\rho_0(p)$	$\{0\}$	0	0	0	0	0	0
$\rho_1(p)$	$\{0,1\}$	0	1	0	1	0	1
$\rho_2(p)$	$\{0\}$	0	0	0	0	0	0
$\rho_3(p)$	$\{0,2,4\}$	0	0	2	2	4	4

Definition 3 *For $0 \le p \le q-1$, define $d_0^{(p)}, d_1^{(p)}, d_2^{(p)} \in \mathbb{Z}_4$ such that $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$. In particular we always define $d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0$.*

For any $1 \le p \le q-1$, There are 16 possible values of $(d_0^{(p)}, d_1^{(p)}, d_2^{(p)})$, which respectively are

$$(0,0,0),(0,1,3),(0,2,2),(0,3,1),(1,0,2),(1,1,1),(1,2,0),(1,3,3),$$
$$(2,0,0),(2,1,3),(2,2,2),(2,3,1)(3,0,2),(3,1,1),(3,2,0),(3,3,3).$$

3.1 The First Construction

Our first construction extent the results in generalized Cases I-III in [15].

Theorem 1 *For any factorization $q = q_0 \cdot q_1 \cdots q_m$ and permutation σ, let $\rho_k : \mathcal{Z}_q \to \mathcal{T}_{\sigma(k)}$ $(1 \le k \le m)$ be mappings given in Definition 2, and $d_0^{(p)}, d_1^{(p)}, d_2^{(p)}$ $(0 \le p \le q-1)$ elements over $\mathbb{Z}_4$ given in Definition 3. Denote the vectors $\vec{d_1}$ and $\vec{d_2}$ respectively by*

$$\vec{d_1} = \left(d_1^{(\rho_0(0))}, d_1^{(\rho_0(1))}, \cdots, d_1^{(\rho_0(q-1))}\right) \quad \text{and} \quad \vec{d_2} = \left(d_2^{(\rho_m(0))}, d_2^{(\rho_m(1))}, \cdots, d_2^{(\rho_m(q-1))}\right). \tag{9}$$

Sequences over 4^q-QAM of length 2^m associated with V-GBFs in (5) form a GCP if the offset V-GBFs $\vec{s}(\boldsymbol{x})$ and paring difference V-GBFs $\vec{\mu}(\boldsymbol{x})$ satisfy the following conditions:

$$\begin{cases} s^{(p)}(\boldsymbol{x}) = \sum_{k=1}^{m} \left(d_1^{(\rho_k(p))} + d_2^{(\rho_{k-1}(p))} \right) x_{\pi(k)} + \sum_{k=0}^{m} d_0^{(\rho_k(p))}, \\ \vec{\mu}(\boldsymbol{x}) = 2x_{\pi(1)} \cdot \vec{1} + \vec{d_1} \quad or \quad 2x_{\pi(m)} \cdot \vec{1} + \vec{d_2}. \end{cases} \tag{10}$$

We explain Theorem 1 in details by the following examples.

Let the factorization of q be trivial, i.e., $q = q_0 \cdot q_1 \cdots q_m$ such that $q_k = q$ for $k = \omega$ and $q_k = 1$ for $k \neq \omega$, and $\sigma(p) = p$. We have

$$\mathcal{T}_k = \begin{cases} \mathcal{Z}_q, & \text{if} \quad k = \omega; \\ \{0\}, & \text{otherwise}, \end{cases} \qquad \text{and} \quad \rho_k(p) = \begin{cases} p, & \text{if} \quad k = \omega; \\ 0, & \text{otherwise}. \end{cases}$$

From Definition 3, we obtain $\left(d_0^{(\rho_k(p))}, d_1^{(\rho_k(p))}, d_2^{(\rho_k(p))} \right)$ equals $\left(d_0^{(p)}, d_1^{(p)}, d_2^{(p)} \right)$ if $k = \omega$, and equals $\{0, 0, 0\}$ otherwise. Then the offset and pairing difference set V-GBFs for different ω can be obtained immediately by Theorem 1.

(1) If $\omega = 0$, i.e. $q = q_0 \cdot q_1 \cdots q_m = q \times 1 \times \cdots \times 1$, then $\vec{d_1} = (d_1^{(0)}, d_1^{(1)}, \cdots, d_1^{(q-1)})$ and $\vec{d_2} = \vec{0}$.

$$\begin{cases} s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_2^{(p)} x_{\pi(1)}, \\ \vec{\mu}(\boldsymbol{x}) = \vec{2} x_{\pi(m)}; \end{cases} \tag{11}$$

or

$$\begin{cases} s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_2^{(p)} x_{\pi(1)}, \\ \vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(1)} + \vec{d_1}. \end{cases} \tag{12}$$

(2) If $\omega = m$, i.e. $q = q_0 \cdot q_1 \cdots q_m = 1 \times 1 \times \cdots \times q$, then $\vec{d_2} = (d_2^{(0)}, d_2^{(1)}, \cdots, d_2^{(q-1)})$ and $\vec{d_1} = \vec{0}$.

$$\begin{cases} s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(m)}, \\ \vec{\mu}(\boldsymbol{x}) = \vec{2} x_{\pi(1)}; \end{cases} \tag{13}$$

or

$$\begin{cases} s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_1^{(p)} x_{\pi(m)}, \\ \vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(m)} + \vec{d_2}. \end{cases} \tag{14}$$

(3) If $1 \le \omega \le m-1$, i.e. $q = q_0 \cdot q_1 \cdots q_m = 1 \times \cdots q \cdots \times 1$, then $\vec{d}_1 = \vec{d}_2 = \vec{0}$.

$$\begin{cases} s^{(p)}(\boldsymbol{x}) = d_0^{(p)} + d_1^{(p)} \cdot x_{\pi(\omega)} + d_2^{(p)} \cdot x_{\pi(\omega+1)}, \\ \vec{\mu}(\boldsymbol{x}) = \vec{2} x_{\pi(1)} \quad \text{or} \quad \vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(m)}. \end{cases} \tag{15}$$

It is obvious that the offset V-GBFs $\vec{s}(\boldsymbol{x})$ and paring difference V-GBFs $\vec{\mu}(\boldsymbol{x})$ shown in (11), (13), (15) agree with the generalized Cases I-III in [15]. New paring difference V-GBFs shown in (12), (14) lead to new GCPs over 4^q-QAM, although they do not produce new GCSs over 4^q-QAM.

Moreover, New GCPs and GCSs over QAM can be identified from Theorem 1 if q is a composite number and $q = q_0 \cdot q_1 \cdots q_m$ is a non-trivial factorization. Two examples are given below to illustrate it.

Example 2 *Suppose* $m = 3$, $q = q_0 \cdot q_1 \cdot q_2 \cdot q_3 = 2 \times 1 \times 1 \times 2 = 4$, $\sigma(0,1,2,3) = (0,1,2,3)$. *Then*

k	q_k	$\prod_{i=0}^{k-1} q_i \cdot \mathcal{Z}_{q_k}$	$\mathcal{T}_{\sigma(k)}$
0	2	$\mathcal{Z}_2$	$\{0,1\}$
1	1	$2 \cdot \mathcal{Z}_1$	$\{0\}$
2	1	$2 \times 1 \cdot \mathcal{Z}_1$	$\{0\}$
3	2	$2 \times 1 \times 1 \cdot \mathcal{Z}_2$	$\{0,2\}$

From Theorem 1, the 256*-QAM sequence pair* $\{F(y), G(y)\}$ *associated with the V-GBFs*

$$\begin{cases} \vec{f}(\boldsymbol{x}) = \vec{1} \cdot f(\boldsymbol{x}) + \left(s^{(0)}(\boldsymbol{x}), s^{(1)}(\boldsymbol{x}), s^{(2)}(\boldsymbol{x}), s^{(3)}(\boldsymbol{x})\right), \\ \vec{g}(\boldsymbol{x}) = \vec{f}(\boldsymbol{x}) + \vec{2} \cdot x_{\pi(1)} + \vec{d}_1 \quad \textit{or} \quad \vec{f}(\boldsymbol{x}) + \vec{2} x_{\pi(3)} + \vec{d}_2 \end{cases}$$

form a GCP of length $L = 8$, *where* $f(\boldsymbol{x})$ *is given in (3). Notice that* $\rho_1(p) = \rho_2(p) \equiv 0$, *the offset* $s^{(p)}(\boldsymbol{x})$ *and vectors* $\vec{d}_1, \vec{d}_2$ *are given in the table respectively:*

p	$\rho_0(p)$	$\rho_3(p)$	$d_1^{(\rho_0(p))}$	$d_2^{(\rho_3(p))}$	*offset :* $s^{(p)}(\boldsymbol{x})$
0	0	0	$d_1^{(0)}$	$d_2^{(0)}$	0
1	1	0	$d_1^{(1)}$	$d_2^{(0)}$	$d_2^{(1)} x_{\pi(1)} + d_0^{(1)}$
2	0	2	$d_1^{(0)}$	$d_2^{(2)}$	$d_1^{(2)} x_{\pi(3)} + d_0^{(2)}$
3	1	2	$d_2^{(1)}$	$d_1^{(2)}$	$d_2^{(1)} x_{\pi(1)} + d_1^{(2)} x_{\pi(3)} + d_0^{(1)} + d_0^{(2)}$

where $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$ *for* $1 \le p \le q-1$ *and* $d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0$.

Example 3 *Suppose* $m = 3$, $q = q_0 \cdot q_1 \cdot q_2 \cdot q_3 = 2 \times 1 \times 3 \times 1 = 6$, $\sigma(0,1,2,3) = (1,0,3,2)$. *Then* $\mathcal{T}_k$ $(k \in \{0,1,2,3\})$ *and the decomposition of* $p = \rho_0(p) + \rho_1(p) + \rho_2(p) + \rho_3(p)(\rho_k(p) \in \mathcal{T}_k)$ *are shown in Example 1. Thus the* 4^6*-QAM sequence pair* $\{F(y), G(y)\}$ *associated with the V-GBFs*

$$\begin{cases} \vec{f}(\boldsymbol{x}) = \vec{1} \cdot f(\boldsymbol{x}) + \left(s^{(0)}(\boldsymbol{x}), s^{(1)}(\boldsymbol{x}), \cdots, s^{(5)}(\boldsymbol{x})\right), \\ \vec{g}(\boldsymbol{x}) = \vec{f}(\boldsymbol{x}) + \vec{2} \cdot x_{\pi(1)} \quad or \quad \vec{f}(\boldsymbol{x}) + \vec{2} x_{\pi(3)} + \vec{d}_2 \end{cases}$$

form a GCP of length $L = 8$, *where* $f(\boldsymbol{x})$ *is given in (3). Notice that* $\rho_0(p) = \rho_2(p) \equiv 0$, *the offset* $s^{(p)}(\boldsymbol{x})$ *and vectors* $\vec{d}_1, \vec{d}_2$ *are given in the table respectively:*

p	$\rho_1(p)$	$\rho_3(p)$	$d_1^{(\rho_0(p))}$	$d_2^{(\rho_3(p))}$	*offset :* $s^{(p)}(\boldsymbol{x})$
0	0	0	0	$d_2^{(0)}$	0
1	1	0	0	$d_2^{(0)}$	$d_1^{(1)} x_{\pi(1)} + d_2^{(1)} x_{\pi(2)} + d_0^{(1)}$
2	0	2	0	$d_2^{(2)}$	$d_1^{(2)} x_{\pi(3)} + d_0^{(2)}$
3	1	2	0	$d_2^{(2)}$	$d_1^{(1)} x_{\pi(1)} + d_2^{(1)} x_{\pi(2)} + d_1^{(2)} x_{\pi(3)} + d_0^{(1)} + d_0^{(2)}$
4	0	4	0	$d_2^{(4)}$	$d_1^{(4)} x_{\pi(3)} + d_0^{(4)}$
5	1	4	0	$d_2^{(4)}$	$d_1^{(1)} x_{\pi(1)} + d_2^{(1)} x_{\pi(2)} + d_1^{(4)} x_{\pi(3)} + d_0^{(1)} + d_0^{(4)}$

where $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$ *for* $1 \le p \le q-1$ *and* $d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0$.

To the best of our knowledge, the expressions of offset V-GBFs shown in the above examples have never been reported before.

3.2 The Second Construction

In this subsection, we slightly modify the conditions in our first construction, and obtain our second construction which include the generalized cases IV-V in [17].

Definition 4 *Let* $q = q' \cdot q_0 \cdot q_1 \cdots q_m$ *be a factorization of* q, *where* $q' \ge 3$ *and* $q_k (0 \le k \le m)$ *are positive integers. For* $1 \le k \le m$, *define* $\mathcal{T}'_k$ *as sets of integers by*

$$\mathcal{T}'_k = (q' \cdot q_0 \cdot q_1 \cdots q_{k-1}) \cdot \mathcal{Z}_{q_k} = \left(q' \cdot \prod_{i=0}^{k-1} q_i\right) \cdot \mathcal{Z}_{q_k}. \tag{16}$$

For the case $k = 0$, *we extend the definition to* $\mathcal{T}'_0 = q' \cdot \mathcal{Z}_{q_0}$. *For any given permutation* σ *of symbols* $\{0, 1, \cdots, m\}$, *define mappings* $\rho'_k : \mathcal{Z}_q \to \mathcal{T}'_{\sigma(k)} (0 \le k \le m)$ *and* $\rho' : \mathcal{Z}_q \to \mathcal{Z}_{q'}$, *such that* $p = \rho'(p) + \rho'_0(p) + \rho'_1(p) + \cdots + \rho'_m(p)$.

Similar to the mappings ρ_k in the first construction, $p \in \mathcal{Z}_q$ can be uniquely decomposed as $p = p' + p'_0 + p'_1 + \cdots + p'_m$ such that $p' \in \mathcal{Z}_{q'}$ and $p'_k \in \mathcal{T}'_{\sigma(k)}$ here, so the above definition is reasonable.

Theorem 2 *For any factorization $q = q' \cdot q_0 \cdot q_1 \cdots q_m$ and permutation σ, let ρ'_k and ρ' be mappings given in Definition 4, and $d_0^{(p)}, d_1^{(p)}, d_2^{(p)}$ $(0 \le p \le q-1)$ elements over $\mathbb{Z}_4$ given in Definition 3. Suppose $G_0 = G(b_1, b_2, \ldots, b_{q'-1})$ and $G_1 = G(b'_1, b'_2, \ldots, b'_{q'-1})$ are NSGIP introduced in Definition 1. Denote the vectors $\vec{d}_1$ and $\vec{d}_2$ here respectively by*

$$\vec{d}_1 = \left(d_1^{(\rho'_0(0))}, d_1^{(\rho'_0(1))}, \cdots, d_1^{(\rho'_0(q-1))}\right) \quad \textit{and} \quad \vec{d}_2 = \left(d_2^{(\rho'_m(0))}, d_2^{(\rho'_m(1))}, \cdots, d_2^{(\rho'_m(q-1))}\right). \tag{17}$$

Sequences over 4^q-QAM of length 2^m associated with V-GBFs in (5) form a GCP if the offset V-GBFs $\vec{s}(\boldsymbol{x})$ satisfy
case (a): for $2 \le \omega \le m-1$,

$$s^{(p)}(\boldsymbol{x}) = \sum_{k=1}^{m} \left(d_1^{(\rho'^{\,k}(p))} + d_2^{(\rho'^{\,k-1}(p))}\right) x_{\pi(k)} + \sum_{k=0}^{m} d_0^{(\rho'^{\,k}(p))} + \left((b'_{\rho'(p)} - b_{\rho'(p)}) x_{\pi(\omega)} + b_{\rho'(p)}\right), \tag{18}$$

case (b): for $1 \le \omega \le m-2$, $\omega + 2 \le \upsilon \le m$,

$$\begin{aligned} s^{(p)}(\boldsymbol{x}) = &\sum_{k=1}^{m} \left(d_1^{(\rho'^{\,k}(p))} + d_2^{(\rho'^{\,k-1}(p))}\right) x_{\pi(k)} + \sum_{k=0}^{m} d_0^{(\rho'^{\,k}(p))} \\ &+ \left((b'_{\rho'(p)} - b_{\rho'(p)}) x_{\pi(\omega)} + (-b'_{\rho'(p)} - b_{\rho'(p)}) x_{\pi(\upsilon)} + b_{\rho'(p)}\right), \end{aligned} \tag{19}$$

and paring difference V-GBFs are given by

$$\vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(1)} + \vec{d}_1 \quad or \quad \vec{\mu}(\boldsymbol{x}) = \vec{2} \cdot x_{\pi(m)} + \vec{d}_2.$$

We explain Theorem 2 in details by the following examples.

Let the factorization $q = q' \cdot q_0 \cdot q_1 \cdots q_m$ be restricted as $q' = q$ and $q_k = 1$ for $0 \le k \le m$, and $\sigma(p) = p$. We have $\rho'(p) = p$ and $\rho'^{\,k}(p) = 0$. Notice $d_1^{(0)} = d_2^{(0)} = d_0^{(0)} = 0$, we have $d_0^{(\rho_k(p))} = d_1^{(\rho_k(p))} = d_2^{(\rho_k(p))} = 0$ and $\vec{d}_1 = \vec{d}_2 = \vec{0}$. Then the offset V-GBFs in formula (18) and (19), can be simplified to

$$s^{(p)}(\boldsymbol{x}) = b_p + (b'_p - b_p) x_{\pi(\omega)}$$

and

$$s^{(p)}(\boldsymbol{x}) = b_p + (b'_p - b_p) x_{\pi(\omega)} + (-b'_p - b_p) x_{\pi(\upsilon)},$$

respectively, which coincide with the generalized Case IV, V in [17].

What's more, New GCPs and GCSs over QAM can be derived from Theorem 2 if $q \neq q'$. We give an example of case (b) to illustrate it. To the best of our knowledge, the expressions of offset V-GBFs shown below have never been reported before.

Example 4 *Suppose* $m = 3$, $q = q' \cdot q_0 \cdot q_1 \cdot q_2 \cdot q_3 = 3 \times 1 \times 2 \times 1 \times 1 = 6$, $\sigma(0,1,2,3) = (0,1,2,3)$. *Then we have* $\mathcal{Z}_{q'} = \{0,1,2\}$, *and*

k	q_k	$q' \prod_{i=0}^{k-1} q_i \cdot \mathcal{Z}_{q_k}$	$\mathcal{T}_k$
0	1	$3 \cdot \mathcal{Z}_1$	$\{0\}$
1	2	$3 \times 1 \cdot \mathcal{Z}_2$	$\{0,3\}$
2	1	$3 \times 1 \times 2 \cdot \mathcal{Z}_1$	$\{0\}$
3	1	$3 \times 1 \times 2 \times 1 \cdot \mathcal{Z}_1$	$\{0\}$

Notice that $\rho'_0(p) = \rho'_2(p) = \rho'_3(p) \equiv 0$, *then* $\vec{d}_1 = \vec{d}_2 = \vec{0}$. *Thus the* 4^6*-QAM sequence pair* $\{F(y), G(y)\}$ *associated with the V-GBFs*

$$\begin{cases} \vec{f}(\boldsymbol{x}) = \vec{1} \cdot f(\boldsymbol{x}) + \left(s^{(0)}(\boldsymbol{x}), s^{(1)}(\boldsymbol{x}), \cdots, s^{(5)}(\boldsymbol{x})\right), \\ \vec{g}(\boldsymbol{x}) = \vec{f}(\boldsymbol{x}) + \vec{2} \cdot x_{\pi(1)} \quad or \quad \vec{f}(\boldsymbol{x}) + \vec{2} x_{\pi(3)}, \end{cases}$$

form a GCP of length $L = 8$, *where* $f(\boldsymbol{x})$ *is given in (3). For* $\omega = 1$ *and* $\upsilon = 3$ *in case (b) of Theorem 2, offset* $s^{(p)}(\boldsymbol{x}) = f^{(p)}(\boldsymbol{x}) - f(\boldsymbol{x})$ *are given as followings.*

p	$\rho'_1(p)$	$\rho'(p)$	*offset :* $s^{(p)}(\boldsymbol{x})$
0	0	0	0
1	0	1	$(b'_1 - b_1)x_{\pi(1)} + (-b'_1 - b_1)x_{\pi(3)} + b_1$
2	0	2	$(b'_2 - b_2)x_{\pi(1)} + (-b'_2 - b_2)x_{\pi(3)} + b_2$
3	3	0	$d_1^{(3)}x_{\pi(1)} + d_2^{(3)}x_{\pi(2)} + d_0^{(3)}$
4	3	1	$d_1^{(3)}x_{\pi(1)} + d_2^{(3)}x_{\pi(2)} + (b'_1 - b_1)x_{\pi(1)} + (-b'_1 - b_1)x_{\pi(3)} + d_0^{(3)} + b_1$
5	3	2	$d_1^{(3)}x_{\pi(1)} + d_2^{(3)}x_{\pi(2)} + (b'_2 - b_2)x_{\pi(1)} + (-b'_2 - b_2)x_{\pi(3)} + d_0^{(3)} + b_2$

where $2d_0^{(p)} + d_1^{(p)} + d_2^{(p)} = 0$ *for* $1 \leq p \leq q-1$, *and* $G(b_1, b_2)$,$G(b'_1, b'_2)$ *form an NSGIP.*

4 Enumeration

The number of the GCSs over 4^q-QAM of length 2^m constructed in Theorem 1 and 2 equals the product of the number of the standard GCSs $f(\boldsymbol{x})$ over QPSK and the number of the compatible offset $\overrightarrow{s}(\boldsymbol{x})$, i.e.,

$$\#\{\overrightarrow{s}(\boldsymbol{x})\} \times \#\{f(\boldsymbol{x})\}.$$

Table 1: Comparison of the number of the compatible offsets

	$q=4$	$q=6$
Generalized Cases I-III [15]	$4032m+4040$	$1047552m+1047584$
Generalized Case IV V [17]	$14(m^2-m-2)$	$584(m^2-m-2)$
New in this paper	$\geq 100(m^2-m-2)$	$\geq (3700+20m)(m^2-m-2)$

It is well known the number of the standard GCSs over QPSK is given by $\#\{f(\boldsymbol{x})\} = (m!/2)4^{(m+1)}$, so the enumeration of the GCSs is determined by the number of the compatible offset.

It was shown in [15] that the number of the compatible offsets in Generalized Cases I-III is

$$\mathcal{N}_{m,q}^{123} = (m+1)4^{2(q-1)} - (m+1)4^{(q-1)} + 2^{(q-1)},\ m \geq 2.$$

The enumeration of the compatible offsets in Generalized Cases IV-V $\mathcal{N}_{m,q}^{45}$ was given in [17] for $q \geq 3, m \geq 3$, especially

$$\mathcal{N}_{m,4}^{45} = 14 \cdot (m-2)(m+1) \text{ and } \mathcal{N}_{m,6}^{45} = 584 \cdot (m-2)(m+1).$$

If q is a prime, the construction in this paper is identical to generalized Cases I-V. If q is a composite number, new GCSs over QAM arise. The offset $\vec{s}(\boldsymbol{x})$ in generalized Cases I-V has no more than two Boolean variables x_k with non-zero coefficients. By studying the offset $\vec{s}(\boldsymbol{x})$ with three and four Boolean variables with non-zero coefficients, A lower bound of the new GCSs other than Generalized Cases I-V for $q=4$ and $q=6$ is given in this section. A comparison between the number of the compatible offsets in generalized case I-III and new in this paper for $q=4$ and $q=6$ is given in Table 1.

Before we list the new compatible offsets for $q=4$ and $q=6$, recall the values of $\overrightarrow{d}^{(p)} = (d_0^{(p)}, d_1^{(p)}, d_2^{(p)})$ in Definition 3. According to whether $d_1^{(p)}$ and $d_2^{(p)}$ equal to 0 or not, all the 16 possible values of $\overrightarrow{d}^{(p)}$ can be classified into four classes:

$$\begin{aligned}
\mathcal{C}_1 &= \{(1,1,1),(3,1,1),(0,1,3),(2,1,3),(0,2,2),(2,2,2),(0,3,1),(2,3,1),(1,3,3),(3,3,3)\},\\
\mathcal{C}_2 &= \{(1,0,2),(3,0,2)\},\\
\mathcal{C}_3 &= \{(1,2,0),(3,2,0)\},\\
\mathcal{C}_4 &= \{(0,0,0),(2,0,0)\}.
\end{aligned}$$

4.1 Enumeration for $q=4$

In this and the next subsections, the permutation π in the superscripts of Boolean variables are ignored, and define x_0 and x_{m+1} as "fake" variables with fixed value 0 for the convenience of expression.

Table 2: Coefficients of offsets for $q = 4$

	$s^{(p)}(\boldsymbol{x})$			
p	x_ω	$x_{\omega+1}$	x_υ	$x_{\upsilon+1}$
0	0	0	0	0
1	$d_1^{(1)}$	$d_2^{(1)}$	0	0
2	0	0	$d_1^{(2)}$	$d_2^{(2)}$
3	$d_1^{(1)}$	$d_2^{(1)}$	$d_1^{(2)}$	$d_2^{(2)}$

Since $q = 4 = 2 \cdot 2$ and $\mathcal{Z}_4 = \mathcal{Z}_2 + 2\mathcal{Z}_2$, the offset $\vec{s}(\boldsymbol{x})$ in Theorem 1 can be expressed by

$$s^{(p)}(\boldsymbol{x}) = \left(d_1^{(\rho_\omega(p))}x_\omega + d_2^{(\rho_\omega(p))}x_{\omega+1} + d_0^{(\rho_\omega(p))}\right) + \left(d_1^{(\rho_\upsilon(p))}x_\upsilon + d_2^{(\rho_\upsilon(p))}x_{\upsilon+1} + d_0^{(\rho_\upsilon(p))}\right), (0 \le p < q), \tag{20}$$

where $p = \rho_\omega(p) + \rho_\upsilon(p)$ $(0 \le \omega \ne \upsilon \le m)$ are given below.

p	0	1	2	3
$\rho_\omega(p)$	0	1	0	1
$\rho_\upsilon(p)$	0	0	2	2

Proposition 1 *For $m \ge 3$, if the coefficients are chosen such that $\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)} \in \mathcal{C}_1$ and the ordered pairs (ω, υ) are chosen such that $(\omega, \upsilon) \ne (0, m), (m, 0)$ and $|\omega - \upsilon| \ge 2$, different choice of $\left(\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)}, (\omega, \upsilon)\right)$ determines different offset in the form of (20) with at least three Boolean variables with non-zero coefficients.*

Proposition 1 can be verified immediately from the observation of the Table 2.

Proposition 2 *For $q = 4$ and $m \ge 3$, Proposition 1 identifies $100(m+1)(m-2)$ distinct compatible offsets other than Cases I-V.*

Proof If $\omega = 0$, we can select υ such that $2 \le \upsilon \le m - 1$. If $\omega = m$, we can choose υ such that $1 \le \upsilon \le m - 2$. If $\omega \ne 0, m$, we can select υ such that $1 \le \upsilon \le m$ and $\upsilon \ne m - 1, m, m + 1$. So there are totally $(m + 1)(m - 2)$ ordered pairs (ω, υ). For each ordered pair, there are $10 \times 10 = 100$ choices of $\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)}$ such that $\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)} \in \mathcal{C}_1$. Thus, Proposition 1 identifies $100(m + 1)(m - 2)$ distinct compatible offsets with at least three Boolean variables with non-zero coefficients. □

4.2 Enumeration for $q = 6$

Since $q = 6 = 3 \cdot 2$ and $\mathcal{Z}_6 = \mathcal{Z}_3 + 2\mathcal{Z}_2$, the offset $\vec{s}(\boldsymbol{x})$ in Theorem 2 can be expressed by

$$s^{(p)}(\boldsymbol{x}) = \left(d_1^{(\rho'_\kappa(p))}x_\kappa + d_2^{(\rho'_\kappa(p))}x_{\kappa+1} + d_0^{(\rho'_\kappa(p))}\right) + \left((b'_{\rho'(p)} - b_{\rho'(p)})x_\omega + b_{\rho'(p)}\right), (0 \le p < q), \tag{21}$$

where $0 \le \kappa \le m$, $2 \le \omega \le m-1$ for case (a) and

$$s^{(p)}(\boldsymbol{x}) = \left(d_1^{(\rho'_\kappa(p))} x_\kappa + d_2^{(\rho'_\kappa(p))} x_{\kappa+1} + d_0^{(\rho'_\kappa(p))}\right) + \left((b'_{\rho'(p)} - b_{\rho'(p)}) x_\omega + (-b'_{\rho'(p)} - b_{\rho'(p)}) x_\upsilon + b_{\rho'(p)}\right), (0 \le p < q), \tag{22}$$

where $0 \le \kappa \le m$, $1 \le \omega \le m-2$, $\omega+2 \le \upsilon \le m$ for case (b). The decomposition $p = \rho'(p) + \rho'_\kappa(p)$ is given below.

p	0	1	2	3	4	5
$\rho'_\kappa(p)$	0	0	0	3	3	3
$\rho'(p)$	0	1	2	0	1	2

We first show four cases of offsets for $q = 6$ in Theorem 1 and Theorem 2.

Case (1): For $q = 2 \cdot 3$ and $\mathcal{Z}_6 = \mathcal{Z}_2 + 3\mathcal{Z}_2$, the offset $\vec{s}(\boldsymbol{x})$ in Theorem 1 can be expressed in the form of (20), where $p = \rho_\omega(p) + \rho_\upsilon(p)$ $(0 \le \omega \ne \upsilon \le m)$ are given below.

p	0	1	2	3	4	5
$\rho_\omega(p)$	0	1	0	1	0	1
$\rho_\upsilon(p)$	0	0	2	2	4	4

Let the coefficients $\overrightarrow{d}^{(p)}$ $(p = 1, 2, 4)$ and the ordered pairs (ω, υ) satisfy the following conditions:

(1) $\overrightarrow{d}^{(1)} \in \mathcal{C}_1$, $\overrightarrow{d}^{(2)}, \overrightarrow{d}^{(4)} \notin \mathcal{C}_4$, $(\overrightarrow{d}^{(2)}, \overrightarrow{d}^{(4)}) \notin (\mathcal{C}_2, \mathcal{C}_2) \bigcup (\mathcal{C}_3, \mathcal{C}_3)$;

(2) $(\omega, \upsilon) \ne (0, m), (m, 0)$ and $|\omega - \upsilon| \ge 2$.

Case (2): For $q = 3 \cdot 2$ and $\mathcal{Z}_6 = \mathcal{Z}_3 + 2\mathcal{Z}_2$, the offset $\vec{s}(\boldsymbol{x})$ in Theorem 1 can be expressed in the form of (20), where $p = \rho_\omega(p) + \rho_\upsilon(p)$ $(0 \le \omega \ne \upsilon \le m)$ are given below.

p	0	1	2	3	4	5
$\rho_\omega(p)$	0	0	0	3	3	3
$\rho_\upsilon(p)$	0	1	2	0	1	2

Let the coefficients $\overrightarrow{d}^{(p)}$ $(p = 1, 2, 3)$ and the ordered pairs (ω, υ) satisfy the following conditions:

(1) $\overrightarrow{d}^{(3)} \in \mathcal{C}_1$, $\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)} \notin \mathcal{C}_4$, $(\overrightarrow{d}^{(1)}, \overrightarrow{d}^{(2)}) \notin (\mathcal{C}_2, \mathcal{C}_2) \bigcup (\mathcal{C}_3, \mathcal{C}_3)$;

(2) $(\omega, \upsilon) \ne (0, m), (m, 0)$ and $|\omega - \upsilon| \ge 2$.

Case (3): For $q = 3 \cdot 2$ and $\mathcal{Z}_6 = \mathcal{Z}_3 + 2\mathcal{Z}_2$, the offset $\vec{s}(\boldsymbol{x})$ in case (a) of Theorem 2 can be expressed in the form of (21). Let the coefficients $\overrightarrow{d}^{(p)}$ $(p = 1, 2, 3)$ and the ordered pairs (κ, ω) satisfy the following conditions:

Table 3: Coefficients of offsets for $q = 6$

	$s^{(p)}(\boldsymbol{x})$ in Case (1)				$s^{(p)}(\boldsymbol{x})$ in Case (2)				$s^{(p)}(\boldsymbol{x})$ in Case (3) and (4)			
p	x_ω	$x_{\omega+1}$	x_υ	$x_{\upsilon+1}$	$x_{\omega'}$	$x_{\omega'+1}$	$x_{\upsilon'}$	$x_{\upsilon'+1}$	x_κ	$x_{\kappa+1}$	x_ω	x_υ
0	0	0	0	0	0	0	0	0	0	0	0	0
1	$d_1^{(1)}$	$d_2^{(1)}$	0	0	0	0	$d_1^{(1)}$	$d_2^{(1)}$	0	0	$b_1' - b_1$	$-b_1' - b_1$
2	0	0	$d_1^{(2)}$	$d_2^{(2)}$	0	0	$d_1^{(2)}$	$d_2^{(2)}$	0	0	$b_2' - b_2$	$-b_2' - b_2$
3	$d_1^{(1)}$	$d_2^{(1)}$	$d_1^{(2)}$	$d_2^{(2)}$	$d_1^{(3)}$	$d_2^{(3)}$	0	0	$d_1^{(3)}$	$d_2^{(3)}$	0	0
4	0	0	$d_1^{(4)}$	$d_2^{(4)}$	$d_1^{(3)}$	$d_2^{(3)}$	$d_1^{(1)}$	$d_2^{(1)}$	$d_1^{(3)}$	$d_2^{(3)}$	$b_1' - b_1$	$-b_1' - b_1$
5	$d_1^{(1)}$	$d_2^{(1)}$	$d_1^{(4)}$	$d_2^{(4)}$	$d_1^{(3)}$	$d_2^{(3)}$	$d_1^{(2)}$	$d_2^{(2)}$	$d_1^{(3)}$	$d_2^{(3)}$	$b_2' - b_2$	$-b_2' - b_2$

(1) $\overrightarrow{d}^{(3)} \in \mathcal{C}_1$;

(2) $2 \le \omega \le m-1$, $1 \le \kappa \le m$, and $\kappa \ne \omega, \omega - 1, m$.

Case (4): For $q = 3 \cdot 2$ and $\mathcal{Z}_6 = \mathcal{Z}_3 + 2\mathcal{Z}_2$, the offset $\vec{s}(\boldsymbol{x})$ in case (b) of Theorem 2 can be expressed in the form of (22). Let the coefficients $\overrightarrow{d}^{(p)}$ $(p = 1, 2, 3)$ and the ordered triples $(\kappa, \omega, \upsilon)$ satisfy the following conditions:

(1) $\overrightarrow{d}^{(3)} \in \mathcal{C}_1$;

(2) $1 \le \omega \le m-2$, $\omega + 2 \le \upsilon \le m$, and $0 \le \kappa \le m$, $\kappa \ne \omega, \omega - 1, \upsilon, \upsilon - 1$.

Proposition 3 *For $q = 6$ and $m \ge 3$, the above Case (1)-(4) identifies $(3700 + 20m)(m+1)(m-2)$ distinct compatible offsets other than Cases I-V.*

Proof We verify this enumeration from the observation of the Table 3. Denote the collection of the offsets in Case (i) by $\mathcal{S}_i$ for $i = 1, 2, 3, 4$.

First of all, every offset in $\mathcal{S}_i$ $(i = 1, 2, 3, 4)$ must have at least three Boolean variables with non-zero coefficients, so these offsets must be different form those in Cases I-V.

Secondly, we prove $\mathcal{S}_i (i = 1, 2, 3, 4)$ has no intersection with each other. From the positions of the non-zero coefficients in $s^{(p)}(\boldsymbol{x})$ in Table 3, it is obviously $\mathcal{S}_1 \bigcap \mathcal{S}_2 = \varnothing$. From the positions of the non-zero coefficients of x_κ and $x_{\kappa+1}$, we have $\mathcal{S}_1 \bigcap (\mathcal{S}_3 \bigcup \mathcal{S}_4) = \varnothing$. From the definition of the NSGIP, we have both $(b_1' - b_1, b_2' - b_2) \ne (0, 0)$ and $(-b_1' - b_1, -b_2' - b_2) \ne (0, 0)$. Combine with the positions of the non-zero coefficients of x_κ and $x_{\kappa+1}$, we obatin $\mathcal{S}_3 \bigcap \mathcal{S}_4 = \varnothing$. If the offset $s^{(p)}(\boldsymbol{x})$ belongs to both $\mathcal{S}_2$ and $\mathcal{S}_3$, we have $\upsilon' = \omega$ and $\upsilon' + 1 = m + 1$ in Case (2) and (3) in Table 3, which contradicts to $\omega \ne m$ in Case (3). If the offset $s^{(p)}(\boldsymbol{x})$ belongs to both $\mathcal{S}_2$ and $\mathcal{S}_4$, we have $\upsilon' = \omega$ and $\upsilon' + 1 = \upsilon$ in Case (2) and (4) in Table 3, which contradicts to $\omega + 2 \le \upsilon$ in Case (4). Thus we obtain $\mathcal{S}_2 \bigcap (\mathcal{S}_3 \bigcup \mathcal{S}_4) = \varnothing$.

Thirdly, it is easy to prove that different parameters in each case determine different offset.

With the same arguments in Proposition 3, we can prove that there are totally $(m+1)(m-2)$ ordered pairs (ω, υ) in case (1). For each ordered pair, there are 10 choices of $\overrightarrow{d}^{(1)}$ such that $\overrightarrow{d}^{(1)} \in \mathcal{C}_1$, and there are $(14^2 - 2 \times 2^2)$ choices of $\overrightarrow{d}^{(2)}$ and $\overrightarrow{d}^{(4)}$ such that $\overrightarrow{d}^{(2)}, \overrightarrow{d}^{(4)} \notin \mathcal{C}_4, (\overrightarrow{d}^{(2)}, \overrightarrow{d}^{(4)}) \notin (\mathcal{C}_2, \mathcal{C}_2) \bigcup (\mathcal{C}_3, \mathcal{C}_3)$. Thus we have $\#\{\mathcal{S}_1\} = 1880 \times (m+1)(m-2)$. Similar to Case (1), we also have $\#\{\mathcal{S}_2\} = 1880 \times (m+1)(m-2)$. We consider the Case (3) and (4) together. It was proved in [17] that different choice of the subscript ω, υ and NSGIPs are $2 \cdot (m-2)(m+1)$. Moreover, we have 10 choices of $\overrightarrow{d}^{(3)}$ such that $\overrightarrow{d}^{(3)} \in \mathcal{C}_1$, and $(m-3)$ choices of κ satisfying condition in Case (3) and (4). Thus we have $\#\{\mathcal{S}_3 \bigcup \mathcal{S}_4\} = 20 \times (m-3)(m+1)(m-2)$.

From the discussion above, we obtain

$$\#\{\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4\} = \#\{\mathcal{S}_1\} + \#\{\mathcal{S}_2\} + \#\{\mathcal{S}_3\} + \#\{\mathcal{S}_4\} = (3700 + 20m)(m^2 - m - 2),$$

which completes the proof. □

5 Main Theory

In this section, we will introduce our main theory on how to construct GCPs over 4^q-QAM.

Firstly, inspired by the idea in [12, 10], we generalize the concept of Golay array pair (GAP) from PSK [12, 10] to 4^q-QAM, argue that a Golay complementary sequence (GCS) is naturally viewed as a projection of a Golay complementary array, and demonstrate that a large number of GCPs can be constructed from a GAP over 4^q-QAM.

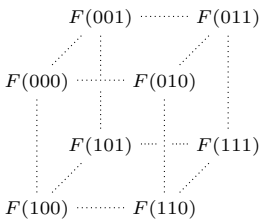
Secondly, inspired by the idea in [2, 24, 25], we introduce the generating function of array, and make a connection between GAP and specified para-unitary (PU) matrix over 4^q-QAM.

Finally, we introduce our method to construct the desired PU matrix over 4^q-QAM. Thus, a greatly simplified and completely elementary process for constructing GAPs and GCSs over 4^q-QAM is given.

5.1 Golay Array and Sequence Pair over QAM

A complex value array of size $2 \times 2 \times \cdots \times 2$ can be expressed by a function $F(x_1, x_2, \cdots, x_m)$ (or $F(\boldsymbol{x})$ for short) from $\mathbb{Z}_2^m$ to $\mathbb{C}$. An array is in a sense a multidimensional sequence. For example, the array for $m = 3$ can be viewed as the following cube:

$F(x_1, x_2, x_3) =$



Definition 5 *Let $F(\boldsymbol{x})$ be an array of size $2 \times 2 \times \cdots \times 2$. The* aperiodic auto-correlation *of $F(\boldsymbol{x})$ at shift $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots \tau_m)$ ($\tau_k = -1, 0$ or 1) is defined by*

$$C_F(\boldsymbol{\tau}) = \sum_{\boldsymbol{x}} F(\boldsymbol{x} + \boldsymbol{\tau}) \cdot \overline{F(\boldsymbol{x})}, \tag{23}$$

where $F(\boldsymbol{x} + \boldsymbol{\tau}) \cdot \overline{F(\boldsymbol{x})} = 0$ if $F(\boldsymbol{x} + \boldsymbol{\tau})$ or $F(\boldsymbol{x})$ is not defined.

Definition 6 *A pair of arrays $\{F(\boldsymbol{x}), G(\boldsymbol{x})\}$ of size $2 \times 2 \times \cdots \times 2$ is said to be a* Golay array pair *(GAP) if*

$$C_F(\boldsymbol{\tau}) + C_G(\boldsymbol{\tau}) = 0, \forall \boldsymbol{\tau} \neq \mathbf{0}. \tag{24}$$

For more details on the concepts and results for GAPs, see [10, 24, 25]. An array over QPSK of size $2 \times 2 \times \cdots \times 2$ can be described by a GBF over $\mathbb{Z}_4$ [24, 25]. Similarly, an array of size $2 \times 2 \times \cdots \times 2$ over 4^q-QAM can be described by a V-GBF $\vec{f}(\boldsymbol{x}) = (f^{(0)}(\boldsymbol{x}), f^{(1)}(\boldsymbol{x}), \cdots, f^{(q-1)}(\boldsymbol{x}))$ over $\mathbb{Z}_4$ by

$$F(\boldsymbol{x}) = \sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(\boldsymbol{x})}. \tag{25}$$

Arrays over QPSK can be obviously regarded as arrays over 4^q-QAM for $q = 1$.

Note that a sequence $F(y)$ of length 2^m can be connected with an array $F(\boldsymbol{x})$ by setting $y = \sum_{k=1}^{m} x_k \cdot 2^{k-1}$. The aperiodic auto-correlation $C_F(\tau)$ of sequence $F(y)$ can be derived from the sum of aperiodic auto-correlation $C_F(\boldsymbol{\tau})$ of array $F(\boldsymbol{x})$ by restricting $\tau = \sum_{k=1}^{m} 2^{k-1}\tau_k$, i.e.,

$$C_F(\tau) = \sum_{\boldsymbol{\tau} \in \mathcal{D}(\tau)} C_F(\boldsymbol{\tau}),$$

where $\mathcal{D}(\tau) = \{\boldsymbol{\tau} | \tau = \sum_{k=1}^{m} 2^{k-1}\tau_k\}$. Thus, if the arrays $F(\boldsymbol{x})$ and $G(\boldsymbol{x})$ over QAM described by V-GBFs $\vec{f}(\boldsymbol{x})$ and $\vec{g}(\boldsymbol{x})$ form a GAP, the sequences associated with V-GBFs $\vec{f}(\boldsymbol{x})$ and $\vec{g}(\boldsymbol{x})$ must form a GCP too. Furthermore, we have the following results.

Theorem 3 *Let $f'(\boldsymbol{x}) = \sum_{k=1}^{m} c_k x_k + c_0$ be an affine vectorial function from $\mathbb{Z}_2^m$ to $\mathbb{Z}_4^q$ for $c_k \in \mathbb{Z}_4$. Denote $\pi \cdot f(\boldsymbol{x}) = f(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(m)})$ and $\pi \cdot \vec{f}(\boldsymbol{x}) = \vec{f}(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(m)})$. If a pair of arrays over QAM described by V-GBFs $\{\vec{f}(\boldsymbol{x}), \vec{g}(\boldsymbol{x})\}$ form a GAP, the sequences associated with V-GBFs*

$$\left\{\pi \cdot \vec{f}(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \vec{1},\ \pi \cdot \vec{g}(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \vec{1}\right\}$$

is also a GCP.

Proof We first prove the sequences associated with V-GBFs $\{\pi \cdot \vec{f}(\boldsymbol{x}), \pi \cdot \vec{g}(\boldsymbol{x})\}$ form a GCP. The aperiodic autocorrelation of array described by $\vec{f}(\boldsymbol{x})$ is given by

$$C_{\vec{f}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{x}} F(\boldsymbol{x} + \boldsymbol{\tau}) \cdot \overline{F(\boldsymbol{x})}.$$

Then the aperiodic autocorrelation of array described by $\pi \cdot \vec{f}(\boldsymbol{x})$ is given by

$$C_{\pi \cdot \vec{f}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{x}} F\left(\pi \cdot \boldsymbol{x} + \boldsymbol{\tau}\right) \cdot \overline{F(\pi \cdot \boldsymbol{x})} = \sum_{\boldsymbol{x}} F\left(\pi \cdot (\boldsymbol{x} + \pi^{-1} \cdot \boldsymbol{\tau})\right) \cdot \overline{F(\pi \cdot \boldsymbol{x})}.$$

As $\boldsymbol{x}$ goes over $\mathbb{Z}_2^m$, $\pi \cdot \boldsymbol{x}$ also runs over $\mathbb{Z}_2^m$. So $C_{\pi \cdot \vec{f}}(\boldsymbol{\tau}) = C_{\vec{f}}(\pi^{-1} \cdot \boldsymbol{\tau})$. Thus for $\boldsymbol{\tau} \neq \mathbf{0}$, we have

$$C_{\pi \cdot \vec{f}}(\boldsymbol{\tau}) + C_{\pi \cdot \vec{g}}(\boldsymbol{\tau}) = C_{\vec{f}}(\pi^{-1} \cdot \boldsymbol{\tau}) + C_{\vec{g}}(\pi^{-1} \cdot \boldsymbol{\tau}) = 0.$$

Next, we prove the sequences associated with V-GBFs $\left\{\vec{f}(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \vec{1}, \vec{g}(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \vec{1}\right\}$ form a GCP. The aperiodic autocorrelation of array described by $\vec{f}(\boldsymbol{x})$ is given by

$$C_{\vec{f}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{x}} \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(\boldsymbol{x}+\boldsymbol{\tau})} \right) \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{-f^{(p)}(\boldsymbol{x})} \right).$$

Then the aperiodic autocorrelation of array described by $\vec{f}(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \vec{1}$ is given by

$$\begin{aligned}
C_{\vec{f}+\vec{f'}}(\boldsymbol{\tau}) &= \sum_{\boldsymbol{x}} \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(\boldsymbol{x}+\boldsymbol{\tau}) + f'(\boldsymbol{x}+\boldsymbol{\tau})} \right) \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{-f^{(p)}(\boldsymbol{x}) - f'(\boldsymbol{x})} \right) \\
&= \sum_{\boldsymbol{x}} \xi^{f'(\boldsymbol{x}+\boldsymbol{\tau}) - f'(\boldsymbol{x})} \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(\boldsymbol{x}+\boldsymbol{\tau})} \right) \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{-f^{(p)}(\boldsymbol{x})} \right) \\
&= \xi^{\sum_{k=1}^m c_k \tau_k} \sum_{\boldsymbol{x}} \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{f^{(p)}(\boldsymbol{x}+\boldsymbol{\tau})} \right) \left(\sum_{p=0}^{q-1} 2^p \cdot \xi^{-f^{(p)}(\boldsymbol{x})} \right) \\
&= \xi^{\sum_{k=1}^m c_k \tau_k} \cdot C_{\vec{f}}(\boldsymbol{\tau}).
\end{aligned}$$

Similarly, the aperiodic autocorrelation of array described by $\vec{g}(\boldsymbol{x}) + f'(\boldsymbol{x}) \cdot \vec{1}$ is given by

$$C_{\vec{g}+\vec{f'}}(\boldsymbol{\tau}) = \xi^{\sum_{k=1}^m c_k \tau_k} \cdot C_{\vec{g}}(\boldsymbol{\tau}).$$

We have that $C_{\vec{f}+\vec{f'}}(\boldsymbol{\tau}) + C_{\vec{g}+\vec{f'}}(\boldsymbol{\tau}) = \xi^{\sum_{k=1}^m c_k \tau_k} \cdot (C_{\vec{g}}(\boldsymbol{\tau}) + C_{\vec{g}}(\boldsymbol{\tau})) = 0 (\boldsymbol{\tau} \neq \mathbf{0})$, which completes the proof. □

From Theorem 3, we can construct GAPs to produce a large number of GCPs over QAM.

5.2 Golay Array Pair and PU Matrix over QAM

For a complex value array of size $2 \times 2 \times \cdots \times 2$, denoted by $F(\boldsymbol{x})$, we can define its *generating function* by

$$F(z_1, z_2, \cdots, z_m) = \sum_{x_1, x_2, \cdots, x_m} F(x_1, x_2, \cdots, x_m) z_1^{x_1} z_2^{x_2} \cdots z_m^{x_m}, \tag{26}$$

or $F(\boldsymbol{z}) = \sum_{\boldsymbol{x}} F(\boldsymbol{x}) \cdot \boldsymbol{z}^{\boldsymbol{x}}$ for short. It is easy to verify

$$F(\boldsymbol{z}) \cdot \overline{F}(\boldsymbol{z}^{-1}) = \sum_{\boldsymbol{\tau}} C_F(\boldsymbol{\tau}) z_1^{\tau_1} z_2^{\tau_2} \cdots z_m^{\tau_m} \tag{27}$$

where $\boldsymbol{z}^{-1} = (z_1^{-1}, z_2^{-1}, \cdots, z_m^{-1})$. So a pair of arrays $\{F(\boldsymbol{x}), G(\boldsymbol{x})\}$ of size $2 \times 2 \times \cdots \times 2$ forms a GAP if and only if their generating functions $\{F(\boldsymbol{z}), G(\boldsymbol{z})\}$ satisfy

$$F(\boldsymbol{z}) \cdot \overline{F}(\boldsymbol{z}^{-1}) + G(\boldsymbol{z}) \cdot \overline{G}(\boldsymbol{z}^{-1}) = c, \tag{28}$$

where c is a real constant.

Notice that the array described by $\vec{f}(\boldsymbol{x})$ over QAM (or $f(\boldsymbol{x})$ over QPSK) is uniquely determined by the generating function $F(\boldsymbol{z})$, and vice versa.

Let $F_{i,j}(\boldsymbol{x})$ $(0 \le i, j \le 1)$ be m-dimensional arrays over QPSK of size $2 \times 2 \times \cdots \times 2$ corresponding to GBF $f_{i,j}(\boldsymbol{x})$ and generating function $F_{i,j}(\boldsymbol{z})$. These arrays can be expressed by a formalized array matrix $\boldsymbol{M}(\boldsymbol{x})$, where each entry is $F_{i,j}(\boldsymbol{x})$, i.e.,

$$\boldsymbol{M}(\boldsymbol{x}) = \begin{bmatrix} F_{0,0}(\boldsymbol{x}) & F_{0,1}(\boldsymbol{x}) \\ F_{1,0}(\boldsymbol{x}) & F_{1,1}(\boldsymbol{x}) \end{bmatrix}. \tag{29}$$

Also, these arrays can be described by a formalized matrix with the GBF entry, i.e.,

$$\widetilde{\boldsymbol{M}}(\boldsymbol{x}) = \begin{bmatrix} f_{0,0}(\boldsymbol{x}) & f_{0,1}(\boldsymbol{x}) \\ f_{1,0}(\boldsymbol{x}) & f_{1,1}(\boldsymbol{x}) \end{bmatrix}, \tag{30}$$

and described by a formalized matrix with the generating-function entry, i,e.,

$$\boldsymbol{M}(\boldsymbol{z}) = \begin{bmatrix} F_{0,0}(\boldsymbol{z}) & F_{0,1}(\boldsymbol{z}) \\ F_{1,0}(\boldsymbol{z}) & F_{1,1}(\boldsymbol{z}) \end{bmatrix}. \tag{31}$$

$\boldsymbol{M}(\boldsymbol{z})$ is called the generating-function matrix of $\boldsymbol{M}(\boldsymbol{x})$ and $\widetilde{\boldsymbol{M}}(\boldsymbol{x})$.

Furthermore, suppose that $F_{i,j}(\boldsymbol{x})$ $(0 \le i, j \le 1)$ are m-dimensional arrays over 4^q-QAM of size $2 \times 2 \times \cdots \times 2$ corresponding to V-GBF $\vec{f}_{i,j}(\boldsymbol{x}) = (f_{i,j}^{(0)}(\boldsymbol{x}), f_{i,j}^{(1)}(\boldsymbol{x}), \cdots, f_{i,j}^{(q-1)}(\boldsymbol{x}))$ and generating

function $F_{i,j}(\boldsymbol{z})$. We can correspondingly define the formalized array matrix

$$\mathbb{M}(\boldsymbol{x}) = \begin{bmatrix} F_{0,0}(\boldsymbol{x}) & F_{0,1}(\boldsymbol{x}) \\ F_{1,0}(\boldsymbol{x}) & F_{1,1}(\boldsymbol{x}) \end{bmatrix}, \tag{32}$$

the formalized V-GBF matrix

$$\widetilde{\mathbb{M}}(\boldsymbol{x}) = \begin{bmatrix} \vec{f}_{0,0}(\boldsymbol{x}) & \vec{f}_{0,1}(\boldsymbol{x}) \\ \vec{f}_{1,0}(\boldsymbol{x}) & \vec{f}_{1,1}(\boldsymbol{x}) \end{bmatrix}, \tag{33}$$

and the generating-function matrix

$$\mathbb{M}(\boldsymbol{z}) = \begin{bmatrix} F_{0,0}(\boldsymbol{z}) & F_{0,1}(\boldsymbol{z}) \\ F_{1,0}(\boldsymbol{z}) & F_{1,1}(\boldsymbol{z}) \end{bmatrix}. \tag{34}$$

Denote the component GBF matrix of the V-GBF matrix $\widetilde{\mathbb{M}}(\boldsymbol{x})$ by

$$\widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x}) = \begin{bmatrix} f_{0,0}^{(p)}(\boldsymbol{x}) & f_{0,1}^{(p)}(\boldsymbol{x}) \\ f_{1,0}^{(p)}(\boldsymbol{x}) & f_{1,1}^{(p)}(\boldsymbol{x}) \end{bmatrix}$$

in the form of (30) for $0 \leq p < q$. Let $\boldsymbol{M}^{(p)}(\boldsymbol{z})$ be the corresponding generating-function matrix of $\widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x})$. Since an array over 4^q-QAM is the weighted sums of q arrays over QPSK, the generating-function matrix $\mathbb{M}(\boldsymbol{z})$ is the weighted sums of $\boldsymbol{M}^{(p)}(\boldsymbol{z})$, i.e.,

$$\mathbb{M}(\boldsymbol{z}) = \sum_{p=0}^{q-1} 2^p \cdot \boldsymbol{M}^{(p)}(\boldsymbol{z}). \tag{35}$$

Theorem 4 *Let $\mathbb{M}(\boldsymbol{z})$ in the form (34) be the generating-function matrix of a V-GBF matrix $\widetilde{\mathbb{M}}(\boldsymbol{x})$ in the form (33). If $\mathbb{M}(\boldsymbol{z})$ is a* para-unitary *(PU) matrix, i.e.,*

$$\mathbb{M}(\boldsymbol{z})\mathbb{M}^{\dagger}(\boldsymbol{z}^{-1}) = c \cdot \boldsymbol{I}, \tag{36}$$

where where c is a real number, $(\cdot)^{\dagger}$ denotes the Hermitian transpose and $\boldsymbol{I}$ is an identity matrix of order 2, the arrays over QAM described by every row (or column) of $\widetilde{\mathbb{M}}(\boldsymbol{x})$ form a GAP.

Proof It is straightforward from an alternative definition of the GAP in formula (28). □

From Theorem 4, GAPs can be constructed by studying the array matrix $\mathbb{M}(\boldsymbol{z})$ over QAM satisfying PU condition.

5.3 Construction of the desired PU matrices

The following notations of matrices of order 2 will be used in the rest of the paper.

- $\boldsymbol{D}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$, $\boldsymbol{D}(x) = \begin{bmatrix} 1-x & 0 \\ 0 & x \end{bmatrix}$.
- $\boldsymbol{J} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

The most simple PU matrix in the form of (31) over QPSK is the *Butson-type* [4] Hadamard matrix of order 2 with entries being fourth roots of unity, which corresponds to the GAP of dimension 1 and length 1 [24, 25]. Such a Butson-type Hadamard matrix $\boldsymbol{H}$ can be expressed by

$$\boldsymbol{H}(d_0, d_1, d_2) = \xi^{d_0} \cdot \begin{bmatrix} 1 & \\ & \xi^{d_1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & \xi^{d_2} \end{bmatrix} = \begin{bmatrix} \xi^{d_0} & \xi^{d_0+d_2} \\ \xi^{d_0+d_1} & -\xi^{d_0+d_1+d_2} \end{bmatrix}. \tag{37}$$

Then the Hadamard matrix $\boldsymbol{H}(d_0, d_1, d_2)$ is uniquely determined by the vector (d_0, d_1, d_2) over $\mathbb{Z}_4$.

Furthermore, denote GBF matrix of $\boldsymbol{H}$ by $\widetilde{\boldsymbol{H}}$, we have

$$\widetilde{\boldsymbol{H}}(d_0, d_1, d_2) = \begin{bmatrix} d_0 & d_0+d_2 \\ d_0+d_1 & d_0+d_1+d_2+2 \end{bmatrix} = d_0 \cdot \boldsymbol{J} + d_1 \cdot \boldsymbol{A} + d_2 \cdot \boldsymbol{B} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}. \tag{38}$$

In this paper, we are only interested in the following Hadamard matrices.

Definition 7 *For $0 \le p < q$, define Hadamard matrices*

$$\boldsymbol{H}_p = \boldsymbol{H}(d_0^{(p)}, d_1^{(p)}, d_2^{(p)}) \tag{39}$$

where $d_1^{(p)}, d_2^{(p)}, d_0^{(p)} \in \mathbb{Z}_4$ are given in definition 3.

For $p = 0$, from Definition 3, we have

$$\boldsymbol{H}_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{H}}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Lemma 1 *For $0 \le p < q$, let $\boldsymbol{H}_p$ be Hadamard matrices given in Definition 7, and $\mathcal{T}$ an arbitrary nonempty subset of $\mathcal{Z}_q$. Then*

$$\mathbb{H} = \sum_{p \in \mathcal{T}} 2^p \cdot \boldsymbol{H}_p \tag{40}$$

is a unitary matrix, i.e.,

$$\mathbb{H}\mathbb{H}^{\dagger} = c \cdot \boldsymbol{I},$$

where c is a real number.

Proof Since $2d_0 + d_1 + d_2 = 0$ over $\mathbb{Z}_4$, we have

$$\boldsymbol{H}(d_0, d_1, d_2) = \begin{bmatrix} \xi^{d_0} & \xi^{d_0+d_2} \\ \xi^{d_0+d_1} & -\xi^{d_0+d_1+d_2} \end{bmatrix} = \begin{bmatrix} \xi^{d_0} & \overline{\xi^{d_0+d_1}} \\ \xi^{d_0+d_1} & -\overline{\xi^{d_0}} \end{bmatrix}.$$

Then $\mathbb{H}$ can be re-expressed by

$$\mathbb{H} = \begin{bmatrix} \alpha & \overline{\beta} \\ \beta & -\overline{\alpha} \end{bmatrix},$$

where $\alpha = \sum_{p\in\mathcal{T}} 2^p \cdot \xi^{d_0^{(p)}}$ and $\beta = \sum_{p\in\mathcal{T}} 2^p \cdot \xi^{d_0^{(p)}+d_1^{(p)}}$. Its easy to verify that

$$\mathbb{H} \cdot \mathbb{H}^{\dagger} = (|\alpha|^2 + |\beta|^2) \cdot \boldsymbol{I},$$

which completes the proof. □

Theorem 5 *Suppose that the sets $\mathcal{T}_k$, the permutation σ, the mappings ρ_k, the elements $d_0^{(p)}, d_1^{(p)}, d_2^{(p)}$, and the vectors $\vec{d_1}, \vec{d_2}$ are the same as those in Theorem 1. Let*

$$\mathbb{H}^{\{k\}} = \sum_{p\in\mathcal{T}_{\sigma(k)}} 2^{\rho_k(q-1)-p} \cdot \boldsymbol{H}_p,$$

where $\boldsymbol{H}_p$ is given in Definition 7. Then

$$\mathbb{M}_{Thm\text{-}1}(\boldsymbol{z}) = \mathbb{H}^{\{0\}} \cdot \prod_{k=1}^{m} \left(\boldsymbol{D}(z_k) \cdot \mathbb{H}^{\{k\}} \right) \tag{41}$$

is a PU matrix over 4^q-QAM. Moreover, it is the generating matrix of the V-GBF matrix

$$\widetilde{\mathbb{M}}_{Thm\text{-}1}(\boldsymbol{x}) = \left(\vec{1} \cdot f(\boldsymbol{x}) + \vec{s}(\boldsymbol{x}) \right) \cdot \boldsymbol{J} + \left(\vec{2} \cdot x_1 + \vec{d_1} \right) \cdot \boldsymbol{A} + \left(\vec{2} \cdot x_m + \vec{d_2} \right) \cdot \boldsymbol{B} \tag{42}$$

where $f(\boldsymbol{x}) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}$, $\vec{s}(\boldsymbol{x}) = \left(0, s^{(1)}(\boldsymbol{x}), \cdots, s^{(q-1)}(\boldsymbol{x})\right)$ with

$$s^{(p)}(\boldsymbol{x}) = \sum_{k=1}^{m} \left(d_1^{(\rho_k(p))} + d_2^{(\rho_{k-1}(p))} \right) x_k + \sum_{k=0}^{m} d_0^{(\rho_k(p))}, \ (0 \le p \le q-1).$$

Notice that the V-GBFs in first row and first column of $\widetilde{\mathbb{M}}_{\text{Thm-1}}(\boldsymbol{x})$ are

$$\begin{cases} \vec{f}(\boldsymbol{x}) = \vec{1} \cdot f(\boldsymbol{x}) + \vec{s}(\boldsymbol{x}), \\ \vec{g}(\boldsymbol{x}) = \vec{f}(\boldsymbol{x}) + \vec{2} \cdot x_1 + \vec{d_1}, \end{cases} \quad \text{and} \quad \begin{cases} \vec{f}(\boldsymbol{x}) = \vec{1} \cdot f(\boldsymbol{x}) + \vec{s}(\boldsymbol{x}), \\ \vec{g}(\boldsymbol{x}) = \vec{f}(\boldsymbol{x}) + \vec{2} \cdot x_m + \vec{d_2}, \end{cases}$$

which are both GAPs. By applying Theorem 3, Theorem 1 is proved immediately if Theorem 5 is valid.

We introduce another construction of PU matrices over 4^q-QAM involving NSGIP as following.

For NSGIP $Q_0 = Q(b_1, b_2, \ldots, b_{q'-1})$ and $Q_1 = Q(b'_1, b'_2, \ldots, b'_{q'-1})$, define two matrices

$$diag\{Q_0, Q_1\} = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_1 \end{bmatrix} \text{ and } \mathbb{Q} = \begin{bmatrix} Q_0 & Q_1 \\ \overline{Q_1} & \overline{Q_0} \end{bmatrix}.$$

Then $diag\{Q_0, Q_1\}$ is a unitary matrix. Moreover, if $\boldsymbol{M}(\boldsymbol{z}) = \begin{bmatrix} F_{0,0}(\boldsymbol{z}) & F_{0,1}(\boldsymbol{z}) \\ F_{1,0}(\boldsymbol{z}) & F_{1,1}(\boldsymbol{z}) \end{bmatrix}$ is PU matrix, then

$$\mathbb{Q} \odot \boldsymbol{M}(\boldsymbol{z}) = \begin{bmatrix} Q_0 \cdot F_{0,0}(\boldsymbol{z}) & Q_1 \cdot F_{0,1}(\boldsymbol{z}) \\ \overline{Q_1} \cdot F_{1,0}(\boldsymbol{z}) & \overline{Q_0} \cdot F_{1,1}(\boldsymbol{z}) \end{bmatrix}$$

is also a PU matrix, where the symbol $\odot$ means the element-wise product of matrices.

Theorem 6 *Suppose that the sets $\mathcal{T}'_k$, the permutation σ, the mappings ρ'_k and ρ', the elements $d_0^{(p)}, d_1^{(p)}, d_2^{(p)}$, and the vectors $\vec{d}_1, \vec{d}_2$ are the same as those in Theorem 2. Let*

$$\mathbb{H}^{\{k\}} = \sum_{p \in \mathcal{T}'_{\sigma(k)}} 2^{\rho'_k(q-1)-p} \cdot \boldsymbol{H}_p,$$

where $\boldsymbol{H}_p$ is given in Definition 7. Then both
case (a):

$$\mathbb{M}_{Thm\text{-}2}(\boldsymbol{z}) = \mathbb{H}^{\{0\}} \cdot \prod_{k=1}^{\omega-1} \left(\boldsymbol{D}(z_k) \cdot \mathbb{H}^{\{k\}} \right) \cdot diag\{Q_0, Q_1\} \cdot \prod_{k=\omega}^{m} \left(\boldsymbol{D}(z_k) \cdot \mathbb{H}^{\{k\}} \right), \tag{43}$$

where $2 \le \omega \le m-1$,
case (b):

$$\mathbb{M}_{Thm\text{-}2}(\boldsymbol{z}) = \prod_{k=0}^{\omega-1} \left(\mathbb{H}^{\{k\}} \cdot \boldsymbol{D}(z_{k+1}) \right) \cdot \left(\mathbb{Q} \odot \left(\mathbb{H}^{\{\omega\}} \cdot \prod_{k=\omega+1}^{\upsilon-1} \left(\boldsymbol{D}(z_k) \cdot \mathbb{H}^{\{k\}} \right) \right) \right) \cdot \prod_{k=\upsilon}^{m} \left(\boldsymbol{D}(z_k) \cdot \mathbb{H}^{\{k\}} \right), \tag{44}$$

where $1 \le \omega \le m-2$, $\omega+2 \le \upsilon \le m$, *are PU matrices over* 4^q*-QAM. Moreover, it is the generating matrix of the V-GBF matrix*

$$\widetilde{\mathbb{M}}_{Thm\text{-}2}(\boldsymbol{x}) = \left(\vec{1}\cdot f(\boldsymbol{x}) + \vec{s}(\boldsymbol{x})\right)\cdot \boldsymbol{J} + \left(\vec{2}\cdot x_1 + \vec{d}_1\right)\cdot \boldsymbol{A} + \left(\vec{2}\cdot x_m + \vec{d}_2\right)\cdot \boldsymbol{B} \tag{45}$$

where $f(\boldsymbol{x}) = 2\cdot\sum_{k=1}^{m-1} x_k x_{k+1}$ *and* $s^{(p)}(\boldsymbol{x}) = s_0^{(p)}(\boldsymbol{x}) + s'^{(p)}(\boldsymbol{x})$ *with*

$$s_0^{(p)}(\boldsymbol{x}) = \sum_{k=1}^{m}\left(d_1^{(\rho'_k(p))} + d_2^{(\rho'_{k-1}(p))}\right)x_k + \sum_{k=0}^{m} d_0^{(\rho'_k(p))}$$

and

$$\begin{cases} s'^{(p)}(\boldsymbol{x}) = (b'_{\rho'(p)} - b_{\rho'(p)})x_\omega + b_{\rho'(p)}, & case(a); \\ s'^{(p)}(\boldsymbol{x}) = (b'_{\rho'(p)} - b_{\rho'(p)})x_\omega + (-b'_{\rho'(p)} - b_{\rho'(p)})x_\upsilon + b_{\rho'(p)} & case(b) \end{cases}$$

for $0 \le p \le q-1$.

Similar to Theorem 5, by applying Theorem 3 and 6, Theorem 2 is proved immediately.

It is easy to verify both $\mathbb{M}_{\text{Thm-1}}(\boldsymbol{z})$ and $\mathbb{M}_{\text{Thm-2}}(\boldsymbol{z})$ in Theorem 5 and 6 are PU matrices. We will prove $\widetilde{\mathbb{M}}_{\text{Thm-1}}(\boldsymbol{x})$ and $\widetilde{\mathbb{M}}_{\text{Thm-2}}(\boldsymbol{x})$ are their respectively corresponding V-GBF matrices in next section.

6 Proof of the Main Theorems

From the discussion in Subsection 5.3, it is sufficient to prove Theorem 1 and 2 if Theorem 5 and 6 are valid. To prove Theorem 5 and 6, we should develop a method to derive the corresponding V-GBFs from the desired PU matrices over QAM. Such a method has been deeply studied in [24, 25] for PU matrices over PSK. We introduce this method for QPSK case and prove Theorem 5 and 6 in this section.

6.1 GBF Matrix and Its Generating Matrix

In this subsection, we introduce some basic results on how to derive GBF matrix in the form of (29) from its generating matrix over QPSK in the form of (31).

Lemma 2 *Let the matrices* $\boldsymbol{D}(x)$, $\boldsymbol{J}$, $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be given in Subsection 5.3, we have*

(1) $\boldsymbol{J}\cdot\boldsymbol{D}(x)\cdot\boldsymbol{J} = \boldsymbol{J}$;

(2) $\boldsymbol{A}\cdot\boldsymbol{D}(x)\cdot\boldsymbol{J} = \boldsymbol{A}$;

(3) $\boldsymbol{B}\cdot\boldsymbol{D}(x)\cdot\boldsymbol{J} = x\cdot\boldsymbol{J}$;

(4) $\boldsymbol{J}\cdot\boldsymbol{D}(x)\cdot\boldsymbol{A} = x\cdot\boldsymbol{J}$;

(5) $\boldsymbol{J}\cdot\boldsymbol{D}(x)\cdot\boldsymbol{B}=\boldsymbol{B}$.

Suppose that $\{\boldsymbol{z}_0,\boldsymbol{z}_1,\ldots,\boldsymbol{z}_m,z_1,\ldots,z_m\}$ are multivariate variables which do not intersect with each other, and $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m,x_1,\ldots,x_m\}$ are their corresponding Boolean variables respectively. Let

$$\begin{cases}\boldsymbol{z}=(\boldsymbol{z}_0,\boldsymbol{z}_1,\ldots,\boldsymbol{z}_m,z_1,\ldots,z_m),\\ \boldsymbol{x}=(\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m,x_1,\ldots,x_m).\end{cases}\tag{46}$$

Lemma 3 *For $m=1$ in (46), let $\boldsymbol{M}^{\{0\}}(\boldsymbol{z}_0)$ and $\boldsymbol{M}^{\{1\}}(\boldsymbol{z}_1)$ be generating matrices of GBF matrices $\widetilde{\boldsymbol{M}}^{\{0\}}(\boldsymbol{x}_0)$ and $\widetilde{\boldsymbol{M}}^{\{1\}}(\boldsymbol{x}_1)$ over QPSK respectively. Then*

$$\boldsymbol{M}(\boldsymbol{z})=\boldsymbol{M}^{\{0\}}(\boldsymbol{z}_0)\cdot\boldsymbol{D}(z)\cdot\boldsymbol{M}^{\{1\}}(\boldsymbol{z}_1)\tag{47}$$

is a PU matrix and its corresponding GBF matrix is given by

$$\widetilde{\boldsymbol{M}}(\boldsymbol{x})=\widetilde{\boldsymbol{M}}^{\{0\}}(\boldsymbol{x}_0)\cdot\boldsymbol{D}(x)\cdot\boldsymbol{J}+\boldsymbol{J}\cdot\boldsymbol{D}(x)\cdot\widetilde{\boldsymbol{M}}^{\{1\}}(\boldsymbol{x}_1).\tag{48}$$

Proof It is obvious that $\boldsymbol{M}(\boldsymbol{z})$ is PU matrix. Let $\boldsymbol{M}(\boldsymbol{x})$, $\boldsymbol{M}^{\{0\}}(\boldsymbol{x}_0)$, and $\boldsymbol{M}^{\{1\}}(\boldsymbol{x}_1)$ be the array matrices of the generating matrices $\boldsymbol{M}(\boldsymbol{z})$, $\boldsymbol{M}^{\{0\}}(\boldsymbol{z}_0)$ and $\boldsymbol{M}^{\{1\}}(\boldsymbol{z}_1)$ respectively. Notice that $\boldsymbol{D}(z)=\sum_{x=0}^{1}\boldsymbol{D}(x)\cdot z^x$, we have

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{z}) &= \boldsymbol{M}^{\{0\}}(\boldsymbol{z}_0)\cdot\boldsymbol{D}(z)\cdot\boldsymbol{M}^{\{1\}}(\boldsymbol{z}_1)\\ &= \sum_{\boldsymbol{x}_0}\left(\boldsymbol{M}^{\{0\}}(\boldsymbol{x}_0)\cdot\boldsymbol{z}_0^{\boldsymbol{x}_0}\right)\cdot\sum_{x}\left(\boldsymbol{D}(x)\cdot z^x\right)\cdot\sum_{\boldsymbol{x}_1}\left(\boldsymbol{M}^{\{1\}}(\boldsymbol{x}_1)\cdot\boldsymbol{z}_1^{\boldsymbol{x}_1}\right)\\ &= \sum_{\boldsymbol{x}_0}\sum_{x}\sum_{\boldsymbol{x}_1}\boldsymbol{M}^{\{0\}}(\boldsymbol{x}_0)\cdot\boldsymbol{D}(x)\cdot\boldsymbol{M}^{\{1\}}(\boldsymbol{x}_1)\cdot\boldsymbol{z}_0^{\boldsymbol{x}_0}\cdot z^x\cdot\boldsymbol{z}_1^{\boldsymbol{x}_1}\\ &= \sum_{\boldsymbol{x}}\boldsymbol{M}^{\{0\}}(\boldsymbol{x}_0)\cdot\boldsymbol{D}(x)\cdot\boldsymbol{M}^{\{1\}}(\boldsymbol{x}_1)\cdot\boldsymbol{z}^{\boldsymbol{x}},\end{aligned}$$

On the other hand, we known that

$$\boldsymbol{M}(\boldsymbol{z})=\sum_{\boldsymbol{x}}\boldsymbol{M}(\boldsymbol{x})\cdot\boldsymbol{z}^{\boldsymbol{x}},$$

then

$$\boldsymbol{M}(\boldsymbol{x})=\boldsymbol{M}^{\{0\}}(\boldsymbol{x}_0)\cdot\boldsymbol{D}(x)\cdot\boldsymbol{M}^{\{1\}}(\boldsymbol{x}_1)$$

for $\forall\boldsymbol{x}$ follows. Then we have

$$M_{i,j}(\boldsymbol{x})=M_{i,x}^{\{0\}}(\boldsymbol{x}_0)\cdot M_{x,j}^{\{1\}}(\boldsymbol{x}_1),$$

which leads to

$$\widetilde{M}_{i,j}(\boldsymbol{x}) = \widetilde{M}^{\{0\}}_{i,x}(\boldsymbol{x}_0) + \widetilde{M}^{\{1\}}_{x,j}(\boldsymbol{x}_1) \tag{49}$$

for $i, j, x = 0, 1$. One can check that formula (48) and (49) are equivalent. □

By applying item (1) in Lemma 2 and Lemma 3 iteratively, we get a powerful result.

Theorem 7 *For $0 \leq k \neq m$, let $\boldsymbol{M}^{\{k\}}(\boldsymbol{z}_k)$ be generating matrices of GBF matrices $\widetilde{\boldsymbol{M}}^{\{k\}}(\boldsymbol{x}_k)$ over QPSK. Then*

$$\boldsymbol{M}(\boldsymbol{z}) = \boldsymbol{M}^{\{0\}}(\boldsymbol{z}_0) \cdot \left(\prod_{k=1}^{m} \left(\boldsymbol{D}(z_k) \cdot \boldsymbol{M}^{\{k\}}(\boldsymbol{z}_k) \right) \right) \tag{50}$$

is a PU matrix and its corresponding GBF matrix is given by

$$\widetilde{\boldsymbol{M}}(\boldsymbol{x}) = \widetilde{\boldsymbol{M}}^{\{0\}}(\boldsymbol{x}_0) \cdot \boldsymbol{D}(x_1) \cdot \boldsymbol{J} + \sum_{k=1}^{m-1} \boldsymbol{J} \cdot \boldsymbol{D}(x_k) \cdot \widetilde{\boldsymbol{M}}^{\{k\}}(\boldsymbol{x}_k) \cdot \boldsymbol{D}(x_{k+1}) \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{D}(x_m) \cdot \widetilde{\boldsymbol{M}}^{\{m\}}(\boldsymbol{x}_m). \tag{51}$$

Corollary 1 *[2, 24] Let $\boldsymbol{H}_0$ be shown in Definition 7. The corresponding GBF matrix of PU matrix*

$$\boldsymbol{U}(\boldsymbol{z}) = \boldsymbol{H}_0 \cdot \left(\prod_{k=1}^{m} (\boldsymbol{D}(z_k) \cdot \boldsymbol{H}_0) \right) \tag{52}$$

is given by

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot \boldsymbol{J} + 2x_1 \cdot \boldsymbol{A} + 2x_m \cdot \boldsymbol{B}, \tag{53}$$

where $f(\boldsymbol{x}) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}$.

Proof For $0 \leq k \leq m$, let $\boldsymbol{M}^{\{k\}}(\boldsymbol{z}_k) = \boldsymbol{H}_0$ in Theorem 7, we have

$$\begin{aligned} \widetilde{\boldsymbol{H}}_0 \cdot \boldsymbol{D}(x_1) \cdot \boldsymbol{J} &= 2x_1 \cdot \boldsymbol{A}, \\ \boldsymbol{J} \cdot \boldsymbol{D}(x_k) \cdot \widetilde{\boldsymbol{H}}_0 \cdot \boldsymbol{D}(x_{k+1}) \cdot \boldsymbol{J} &= 2x_k x_{k+1} \cdot \boldsymbol{J}, \\ \boldsymbol{J} \cdot \boldsymbol{D}(x_m) \cdot \widetilde{\boldsymbol{H}}_0 &= 2x_m \cdot \boldsymbol{B}. \end{aligned}$$

The proof is completed by formula (51). □

6.2 Proof of Theorem 5

According to the definition of ρ_k in Definition 2 and $\mathbb{H}^{\{k\}}$ in Theorem 5, we have

$$\mathbb{M}_{\text{Thm-1}}(\boldsymbol{z}) = \sum_{p=0}^{q-1} 2^{q-1-p} \cdot \boldsymbol{M}^{(p)}_{\text{Thm-1}}(\boldsymbol{z})$$

where

$$\boldsymbol{M}^{(p)}_{\text{Thm-1}}(\boldsymbol{z}) = \boldsymbol{H}_{\rho_0(p)} \cdot \prod_{k=1}^{m} \left(\boldsymbol{D}(z_k) \cdot \boldsymbol{H}_{\rho_k(p)} \right).$$

According to Theorem 7, their corresponding GBF matrices are given by

$$\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-1}}(\boldsymbol{x}) = \widetilde{\boldsymbol{H}}_{\rho_0(p)} \cdot \boldsymbol{D}(x_1) \cdot \boldsymbol{J} + \sum_{k=1}^{m-1} \boldsymbol{J} \cdot \boldsymbol{D}(x_k) \cdot \widetilde{\boldsymbol{H}}_{\rho_k(p)} \cdot \boldsymbol{D}(x_{k+1}) \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{D}(x_m) \cdot \widetilde{\boldsymbol{H}}_{\rho_m(p)}$$

In particular, $\boldsymbol{M}^{(0)}_{\text{Thm-1}}(\boldsymbol{z}) = \boldsymbol{U}(\boldsymbol{z})$ and $\widetilde{\boldsymbol{M}}^{(0)}_{\text{Thm-1}}(\boldsymbol{x}) = \widetilde{\boldsymbol{U}}(\boldsymbol{x})$ are shown in Corollary 1. The difference between $\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-1}}(\boldsymbol{x})$ and $\widetilde{\boldsymbol{M}}^{(0)}_{\text{Thm-1}}(\boldsymbol{x})$ can be calculated by

$$(\widetilde{\boldsymbol{H}}_{\rho_0(p)} - \widetilde{\boldsymbol{H}}_0) \cdot \boldsymbol{D}(x_1) \cdot \boldsymbol{J} + \sum_{k=1}^{m-1} \boldsymbol{J} \cdot \boldsymbol{D}(x_k) \cdot (\widetilde{\boldsymbol{H}}_{\rho_k(p)} - \widetilde{\boldsymbol{H}}_0) \cdot \boldsymbol{D}(x_{k+1}) \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{D}(x_m) \cdot (\widetilde{\boldsymbol{H}}_{\rho_m(p)} - \widetilde{\boldsymbol{H}}_0)$$

According to (38), we have

$$(\widetilde{\boldsymbol{H}}_p - \widetilde{\boldsymbol{H}}_0) = d_0^{(p)} \cdot \boldsymbol{J} + d_1^{(p)} \cdot \boldsymbol{A} + d_2^{(p)} \cdot \boldsymbol{B}.$$

According to Lemma 2, the items in above difference can be simplified as

$$\begin{aligned}
(\widetilde{\boldsymbol{H}}_{\rho_0(p)} - \widetilde{\boldsymbol{H}}_0) \cdot \boldsymbol{D}(x_1) \cdot \boldsymbol{J} &= (d_2^{(\rho_0(p))} x_1 + d_0^{(\rho_0(p))}) \cdot \boldsymbol{J} + d_1^{(\rho_0(p))} \cdot \boldsymbol{A}, \\
\boldsymbol{J} \cdot \boldsymbol{D}(x_k) \cdot (\widetilde{\boldsymbol{H}}_{\rho_k(p)} - \widetilde{\boldsymbol{H}}_0) \cdot \boldsymbol{D}(x_{k+1}) \cdot \boldsymbol{J} &= (d_1^{(\rho_k(p))} x_k + d_2^{(\rho_k(p))} x_{k+1} + d_0^{(\rho_k(p))}) \boldsymbol{J}, \\
\boldsymbol{J} \cdot \boldsymbol{D}(x_m) \cdot (\widetilde{\boldsymbol{H}}_{\rho_m(p)} - \widetilde{\boldsymbol{H}}_0) &= (d_1^{(\rho_m(p))} x_m + d_0^{(\rho_m(p))}) \cdot \boldsymbol{J} + d_2^{(\rho_m(p))} \cdot \boldsymbol{B}.
\end{aligned}$$

Then we have

$$\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-1}}(\boldsymbol{x}) - \widetilde{\boldsymbol{M}}^{(0)}_{\text{Thm-1}}(\boldsymbol{x}) = s^{(p)}(\boldsymbol{x}) \cdot \boldsymbol{J} + d_1^{(\rho_0(p))} \cdot \boldsymbol{A} + d_2^{(\rho_m(p))} \cdot \boldsymbol{B}$$

where

$$s^{(p)}(\boldsymbol{x}) = \sum_{k=1}^{m} \left(d_1^{(\rho_k(p))} + d_2^{(\rho_{k-1}(p))} \right) x_k + \sum_{k=0}^{m} d_0^{(\rho_k(p))}.$$

By applying Corollary 1, we obtain

$$\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-1}}(\boldsymbol{x}) = \left(f(\boldsymbol{x}) + s^{(p)}(\boldsymbol{x}) \right) \cdot \boldsymbol{J} + \left(2x_1 + d_1^{(\rho_0(p))} \right) \cdot \boldsymbol{A} + \left(2x_m + d_2^{(\rho_m(p))} \right) \cdot \boldsymbol{B} \tag{54}$$

where $f(\boldsymbol{x}) = 2 \cdot \sum_{k=1}^{m-1} x_k x_{k+1}$, which complete the proof.

6.3 Proof of Theorem 6

Lemma 4 *Let $\boldsymbol{M}(\boldsymbol{z})$ be the generating matrix of GBF matrix $\widetilde{\boldsymbol{M}}(\boldsymbol{x})$ over QPSK. Suppose $\alpha, \beta, c_{i,j} (0 \leq i, j \leq 1) \in \mathbb{Z}_4$. We have*

(1) $\boldsymbol{M}(\boldsymbol{z}) \cdot diag\{\xi^{\alpha}, \xi^{\beta}\}$ is the generating matrix of GBF matrix $\widetilde{\boldsymbol{M}}(\boldsymbol{x}) + \boldsymbol{J} \cdot diag\{\alpha, \beta\}$;

(2) $\boldsymbol{M}(\boldsymbol{z}) \odot \boldsymbol{C}$ (or $\boldsymbol{C} \odot \boldsymbol{M}(\boldsymbol{z})$)is the generating matrix of GBF matrix $\widetilde{\boldsymbol{M}}(\boldsymbol{x}) + \widetilde{\boldsymbol{C}}$, where $\boldsymbol{C} = \begin{bmatrix} \xi^{c_{0,0}} & \xi^{c_{0,1}} \\ \xi^{c_{1,0}} & \xi^{c_{1,1}} \end{bmatrix}$ and $\widetilde{\boldsymbol{C}} = \begin{bmatrix} c_{0,0} & c_{0,1} \\ c_{1,0} & c_{1,1} \end{bmatrix}$.

According to the definition of ρ'_k and ρ' in Definition 4 and $\mathbb{H}^{\{k\}}$ in Theorem 6, we have

$$\mathbb{M}_{\text{Thm-2}}(\boldsymbol{z}) = \sum_{p=0}^{q-1} 2^{q-1-p} \cdot \boldsymbol{M}^{(p)}_{\text{Thm-2}}(\boldsymbol{z})$$

where

$$\boldsymbol{M}^{(p)}_{\text{Thm-2}}(\boldsymbol{z}) = \boldsymbol{H}_{\rho'_0(p)} \cdot \prod_{k=1}^{\omega-1} \left(\boldsymbol{D}(z_k) \cdot \boldsymbol{H}_{\rho'_k(p)} \right) \cdot diag\{\xi^{b_{\rho'(p)}}, \xi^{b'_{\rho'(p)}}\} \cdot \prod_{k=\omega}^{m} \left(\boldsymbol{D}(z_k) \cdot \boldsymbol{H}_{\rho'_k(p)} \right)$$

where $2 \leq \omega \leq m-1$, for case (a), and

$$\begin{aligned} \boldsymbol{M}^{(p)}_{\text{Thm-2}}(\boldsymbol{z}) \quad = \quad & \prod_{k=0}^{\omega-1} \left(\boldsymbol{H}_{\rho'_k(p)} \cdot \boldsymbol{D}(z_{k+1}) \right) \cdot \left(\begin{bmatrix} \xi^{b_{\rho'(p)}} & \xi^{b'_{\rho'(p)}} \\ \xi^{-b'_{\rho'(p)}} & \xi^{-b_{\rho'(p)}} \end{bmatrix} \odot \left(\boldsymbol{H}_{\rho'_{\omega}(p)} \cdot \prod_{k=\omega+1}^{\upsilon-1} \left(\boldsymbol{D}(z_k) \cdot \boldsymbol{H}_{\rho'_k(p)} \right) \right) \right) \cdot \\ & \prod_{k=\upsilon}^{m} \left(\boldsymbol{D}(z_k) \cdot \boldsymbol{H}_{\rho'_k(p)} \right) \end{aligned}$$

where $1 \leq \omega \leq m-2$, $\omega + 2 \leq \upsilon \leq m$, for case (b).

According to Lemma 4, by iteratively using Theorem 7, their corresponding GBF matrices are given by

$$\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-2}}(\boldsymbol{x}) = \widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x}) + \boldsymbol{J} \cdot \boldsymbol{D}(x_{\omega-1}) \cdot \boldsymbol{J} \cdot diag\{b_{\rho'(p)}, b'_{\rho'(p)}\} \cdot \boldsymbol{D}(x_{\omega}) \cdot \boldsymbol{J}$$

for case (a), and

$$\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-2}}(\boldsymbol{x}) = \widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x}) + \boldsymbol{J} \cdot \boldsymbol{D}(x_{\omega}) \begin{bmatrix} b_{\rho'(p)} & b'_{\rho'(p)} \\ -b'_{\rho'(p)} & -b_{\rho'(p)} \end{bmatrix} \cdot \boldsymbol{D}(x_{\upsilon}) \cdot \boldsymbol{J}$$

for case (b), where

$$\widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x}) = \widetilde{\boldsymbol{H}}_{\rho'_0(p)} \cdot \boldsymbol{D}(x_1) \cdot \boldsymbol{J} + \sum_{k=1}^{m-1} \boldsymbol{J} \cdot \boldsymbol{D}(x_k) \cdot \widetilde{\boldsymbol{H}}_{\rho'_k(p)} \cdot \boldsymbol{D}(x_{k+1}) \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{D}(x_m) \cdot \widetilde{\boldsymbol{H}}_{\rho'_m(p)}.$$

The last term in $\widetilde{\boldsymbol{M}}^{(p)}_{\text{Thm-2}}(\boldsymbol{x})$ can be calculated by

$$\boldsymbol{J}\cdot\boldsymbol{D}(x_{\omega-1})\cdot\boldsymbol{J}\cdot diag\{b_{\rho'(p)},b'_{\rho'(p)}\}\cdot\boldsymbol{D}(x_{\omega})\cdot\boldsymbol{J}=((b'_{\rho'(p)}-b_{\rho'(p)})x_{\omega}+b_{\rho'(p)})\cdot\boldsymbol{J} \tag{55}$$

for case (a), and

$$\boldsymbol{J}\cdot\boldsymbol{D}(x_{\omega})\cdot\begin{bmatrix} b_{\rho'(p)} & b'_{\rho'(p)} \\ -b'_{\rho'(p)} & -b_{\rho'(p)}\end{bmatrix}\cdot\boldsymbol{D}(x_{\upsilon})\cdot\boldsymbol{J}=((b'_{\rho'(p)}-b_{\rho'(p)})x_{\omega}+(-b'_{\rho'(p)}-b_{\rho'(p)})x_{\upsilon}+b_{\rho'(p)})\cdot\boldsymbol{J} \tag{56}$$

for case (b). The term $\widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x})$ is well studied in the proof of Theorem 5. By replacing the subscript $\rho(p)$ by $\rho'(p)$ in formula (54), we have

$$\widetilde{\boldsymbol{M}}^{(p)}(\boldsymbol{x})=\left(f(\boldsymbol{x})+s_0^{(p)}(\boldsymbol{x})\right)\cdot\boldsymbol{J}+\left(2x_1+d_1^{(\rho'_0(p))}\right)\cdot\boldsymbol{A}+\left(2x_m+d_2^{(\rho'_m(p))}\right)\cdot\boldsymbol{B} \tag{57}$$

where $f(\boldsymbol{x})=2\cdot\sum_{k=1}^{m-1}x_kx_{k+1}$, and $s_0^{(p)}(\boldsymbol{x})=\sum_{k=1}^{m}\left(d_1^{(\rho'_k(p))}+d_2^{(\rho'_{k-1}(p))}\right)x_k+\sum_{k=0}^{m}d_0^{(\rho'_k(p))}$.

Combining the formulae (55), (56) and (57), we complete the proof.

7 Conclusion

This paper is devoted to the constructions of GCPs and GCSs over 4^q-QAM.

The first contribution of this paper is that we demonstrate a large number of GCPs can be constructed from a GAP over 4^q-QAM in Theorem 3. This argument greatly simplifies the process for constructing GCPs and GCSs over 4^q-QAM. And it answers the open problem posed in [10]: how can the three-stage construction process of the paper [10] be used to simplify or extend known results on the construction of GCSs in QAM modulation? Although the three-stage process in [10] are not involved here, Theorem 3 in this paper has the same power as the three-stage process in [10]. By Theorem 3, the proposed GAPs of size $2\times2\times\cdots\times2$ over 4^q-QAM can not only construct GCPs in generalized cases I-V[15, 17], but also produce a large number of new GCSs over 4^q-QAM.

The second contribution of this paper is that we make a connection between GAP and specified PU matrix with multi-variables over 4^q-QAM in Theorem 4. Our observation generalized the idea in [2] which make a connection between GCP and specified PU matrix with a single variable over 4^q-QAM. This generalization greatly simplifies the process for constructing the desired PU matrices. It should be pointed out that the constructions of PU matrices and GCSs given in [2] do not include the constructions in this paper, and vice versa.

The most important contribution of this paper is that we develop a method to derive the corresponding V-GBFs from the desired PU matrices in Section 5. Note that many new GCSs over QAM different from generalized cases I-V[15, 17] were also found in [2] by PU method and exhaustive search,

but these GCSs were realized by the algorithm in [2] instead of explicit V-GBFs. The new proposed method overcomes the disadvantage in [2]. A large number of the GCSs over 4^q-QAM with explicit V-GBFs, including generalized cases I-V, are given in Theorem 1 and 2 in this paper. If q is a composite number, new GCSs arise in our construction. For example, if $q = 4$, the number of new GCSs is more than seven times as much as those in generalized cases IV-V[17], and if $q = 6$, the ratio is greater than six and will increase in proportion with m.

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