ECE 470/ ME 445: Introduction to Robotics- Homework 03

Question 1.

A robot arm is designed as illustrated by the following figure. It can be assumed that the mass distributions of the links are insignificant and can be treated as lumped equivalent masses m_1 and m_2.

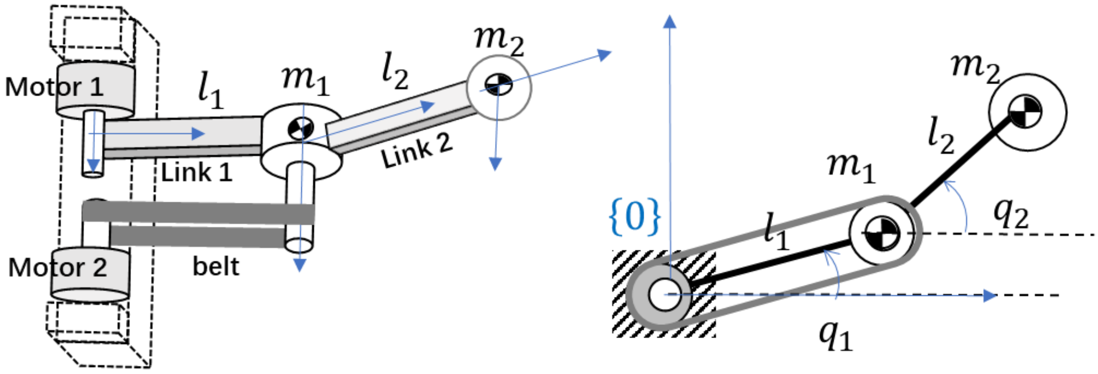

Figure 1

a) Write down the position of masses m_1 and m_2 in terms of q_1 and q_2 referenced from the given frame {0}. ***(2 marks)***

c) Show that the total kinetic energy of the system, K can be written as

$$K = \frac{1}{2}\left(m_1 {l_1}^2 + m_2 {l_1}^2\right){\dot{q}_1}^2 + m_2 l_1 l_2 \cos(q_2 - q_1)\,\dot{q}_1\dot{q}_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{q}_2}^2$$ ***(4 marks)***

d) Obtain the total potential energy of the system. ***(2 marks)***

e) Write down the Lagrangian L. ***(2 marks)***

f) Obtain the dynamic equations relating the torque output (n_1, n_2) of Motor 1 & 2 with the motion of the masses in $q_{1,2}, \dot{q}_{1,2}, \ddot{q}_{1,2}$ ***(5 marks)***

Solution

a)

$$\begin{cases} x_1 = l_1 \cos q_1 \\ y_1 = l_1 \sin q_1 \end{cases} \qquad \begin{cases} x_2 = l_1 \cos q_1 + l_2 \cos q_2 \\ y_2 = l_1 \sin q_1 + l_2 \sin q_2 \end{cases}$$

c)

$$\begin{cases} \dot{x}_1 = -l_1 \sin q_1 \, \dot{q}_1 \\ \dot{y}_1 = l_1 \cos q_1 \dot{q}_1 \end{cases} \quad \begin{cases} \dot{x}_2 = -(l_1 \sin q_1 \dot{q}_1 + l_2 \sin q_2 \dot{q}_2) \\ \dot{y}_2 = l_1 \cos q_1 \dot{q}_1 + l_2 \cos q_2 \dot{q}_2 \end{cases}$$

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{q}_1^2$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{q}_1^2 + 2 l_1 l_2 \cos(q_2 - q_1) \dot{q}_1 \dot{q}_2 + l_2^2 \dot{q}_2^2$$

$$K = \frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2$$

$$= \frac{1}{2}(m_1 {l_1}^2 + m_2 {l_1}^2){\dot{q}_1}^2 + m_2 l_1 l_2 \cos(q_2 - q_1) \, \dot{q}_1 \dot{q}_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{q}_2}^2$$

d)

$$U = [m_1 g l_1 \sin q_1 + m_2 g (l_1 \sin q_1 + l_2 \sin q_2)]$$

e)

Lagrangian $L = K - U$

$$L = \left[\frac{1}{2}(m_1 {l_1}^2 + m_2 {l_1}^2){\dot{q}_1}^2 + m_2 l_1 l_2 \cos(q_2 - q_1) \, \dot{q}_1 \dot{q}_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{q}_2}^2\right] \\ - [m_1 g l_1 \sin q_1 + m_2 g (l_1 \sin q_1 + l_2 \sin q_2)]$$

f)

$$n_1 = (m_1 + m_2) {l_1}^2 \ddot{q}_1 + m_2 l_1 l_2 \cos(q_2 - q_1) \ddot{q}_2 - m_2 l_1 l_2 \sin(q_2 - q_1) \, {\dot{q}_2}^2 \\ + (m_1 + m_2) g l_1 \cos(q_1)$$

$$n_2 = m_2 l_1 l_2 \cos(q_2 - q_1) \, \ddot{q}_1 + m_2 {l_2}^2 \ddot{q}_2 + m_2 l_1 l_2 \sin(q_2 - q_1) \, {\dot{q}_1}^2 + m_2 g l_2 \cos(q_2)$$

In matrix form:

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2) {l_1}^2 & m_2 l_1 l_2 \cos(q_2 - q_1) \\ m_2 l_1 l_2 \cos(q_2 - q_1) & m_2 {l_2}^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 \sin(q_2 - q_1) \, {\dot{q}_2}^2 \\ m_2 l_1 l_2 \sin(q_2 - q_1) \, {\dot{q}_1}^2 \end{bmatrix} + \\ \begin{bmatrix} (m_1 + m_2) g l_1 \cos q_1 \\ m_2 g l_2 \cos q_2 \end{bmatrix}$$

Question 2.

Compare your answer in Question 1 with that of the example discussed in class (Example 5.2 or Section 6.7, Equation (6.58), Reference Textbook J. Craig 3rd Ed.). shown in Figure 2.
State and comment on the differences. ***(5 Points)***

$$\tau_1 = m_2 l_2^2(\ddot\theta_1+\ddot\theta_2) + m_2 l_1 l_2 \cos\theta_2 (2\ddot\theta_1+\ddot\theta_2) + (m_1+m_2) l_1^2 \ddot\theta_1 - m_2 l_1 l_2 \sin\theta_2 \dot\theta_2^2 - 2 m_2 l_1 l_2 \sin\theta_2 \dot\theta_1 \dot\theta_2 + m_2 g l_2 \cos\theta_{12} + (m_1+m_2) g l_1 \cos\theta_1$$

$$\tau_2 = m_2 l_1 l_2 \cos\theta_2 \ddot\theta_1 + m_2 l_1 l_2 \sin\theta_2 \dot\theta_1^2 + m_2 g l_2 \cos\theta_{12} + m_2 l_2^2(\ddot\theta_1+\ddot\theta_2)$$

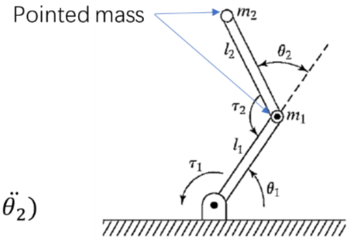

Figure 2

Solution

Example 5.2 discussed in class expresses the generalized forces (τ_1, τ_2) with configuration of coordinates (θ_1, θ_2) as:

$$\tau_1 = m_2 l_2^2(\ddot\theta_1+\ddot\theta_2) + m_2 l_1 l_2 \cos\theta_2 (2\ddot\theta_1+\ddot\theta_2) + (m_1+m_2) l_1^2 \ddot\theta_1 - m_2 l_1 l_2 \sin\theta_2 \dot\theta_2^2 - 2 m_2 l_1 l_2 \sin\theta_2 \dot\theta_1 \dot\theta_2 + m_2 g l_2 \cos\theta_{12} + (m_1+m_2) g l_1 \cos\theta_1$$

$$\tau_2 = m_2 l_1 l_2 \cos\theta_2 \ddot\theta_1 + m_2 l_1 l_2 \sin\theta_2 \dot\theta_1^2 + m_2 g l_2 \cos\theta_{12} + m_2 l_2^2(\ddot\theta_1+\ddot\theta_2)$$

Rewriting the expression

$$\tau_1 = (m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \cos\theta_2 \ddot\theta_2 + m_2 l_2^2(\ddot\theta_1+\ddot\theta_2) + 2 m_2 l_1 l_2 \cos\theta_2 \ddot\theta_1 - m_2 l_1 l_2 \sin\theta_2 \dot\theta_2^2 - 2 m_2 l_1 l_2 \sin\theta_2 \dot\theta_1 \dot\theta_2 + (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos(\theta_1+\theta_2)$$

$$\tau_2 = m_2 l_1 l_2 \cos\theta_2 \ddot\theta_1 + m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \sin\theta_2 \dot\theta_1^2 + m_2 g l_2 \cos(\theta_1+\theta_2) + m_2 l_2^2 \ddot\theta_1$$

By inspection, we see that mapping the coordinates (q_1, q_2) to (θ_1, θ_2) based on the relationship $(\theta_1 \ \ \theta_2)^{\mathrm{T}} = (q_1 \quad q_2 - q_1)^{\mathrm{T}}$, the respective generalized force coordinates of $(n_1 \ \ n_2)$ maps to $(\tau_1 \ \ \tau_2)$ accordingly.

$$(\tau_1 \ \ \tau_2)^{\mathrm{T}} = (n_1 + f_1 \quad n_2 + f_2)^{\mathrm{T}}$$

where

$$f_1 = m_2[l_2^2(\ddot\theta_1+\ddot\theta_2) + 2 l_1 l_2 \cos\theta_2 \ddot\theta_1 - 2 l_1 l_2 \sin\theta_2 \dot\theta_1 \dot\theta_2 + g l_2 \cos(\theta_1+\theta_2)]$$

$$f_2 = m_2(l_2^2 \ddot\theta_1)$$

** Presenting observation by inspection is sufficient for full mark. Analysis of the structures of f_1 and f_2 is not required.

In matrix form:

Configuration in coordinates (θ_1, θ_2)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} =$$

$$\begin{bmatrix} (m_1 + m_2){l_1}^2 + m_2{l_2}^2 & m_2 l_1 l_2 \cos\theta_2 + m_2{l_2}^2 \\ m_2 l_1 l_2 \cos\theta_2 + m_2{l_2}^2 & m_2{l_2}^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} +$$

$$\begin{bmatrix} -m_2 l_1 l_2 \sin\theta_2 \, {\dot{\theta}_2}^2 - 2m_2 l_1 l_2 \sin\theta_2 \, \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 \sin\theta_2 \, {\dot{\theta}_1}^2 \end{bmatrix} +$$

$$\begin{bmatrix} (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos(\theta_1 + \theta_2) \\ m_2 g l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$