Quantum Teleportation

An Application of Quantum Entanglement

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Quantum Teleportation

- An Application of Quantum Entanglement
- "Teleports" or Transfers Quantum Information from one location to another
- Basis is the Sharing of Entangled (EPR) Pairs
- Assumes Presence of two Communication Channels
  - Classical
  - Quantum

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Quantum Teleportation: The Scenario

- Alice wants to send Bob Quantum Information in the Form of a Qubit
$$|\Omega\rangle = \alpha|0\rangle + \beta|1\rangle$$
- Alice does not want to Send Bob $|\Omega\rangle$ over a Quantum Channel for Security Reasons
- Impossible to use Classical Channel since it would Require an Infinite Number (∞) of Classical Bits to Accurately Send the Probability Amplitudes, (α, β)
- Alice cannot Measure her Qubit to Observe the Probability Amplitudes since it would Collapse into the Observable Eigenvector
- Alice cannot Copy her Qubit into Another Qubit due to the "No Cloning" Theorem
- Assume that there Exists a Classical Communication Channel and a Quantum Communication Channel that Connect Alice and Bob

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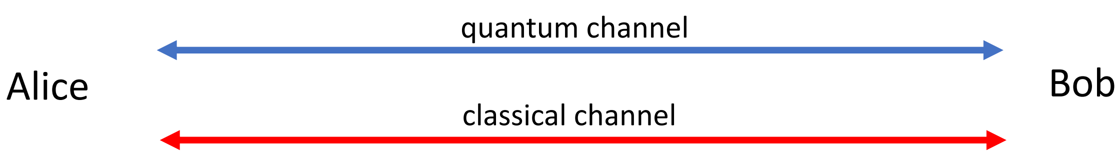
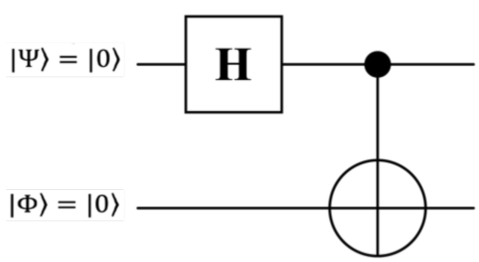
Quantum Teleportation Channels



- Alice Prepares an Entangled Pair $|\Psi\Phi\rangle$ by Initializing the Pair to a Ground State, $|\Psi\Phi\rangle = |00\rangle$, and Evolving them with a Bell State Generator:



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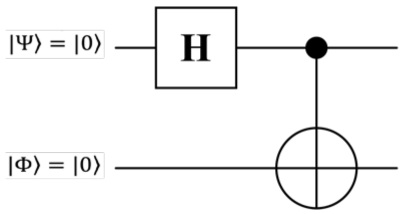
Alice's Entangled Pair



$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The overall Transfer Matrix is:

$$\mathbf{T} = \mathbf{C_X}(\mathbf{H} \otimes \mathbf{I}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\right) \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

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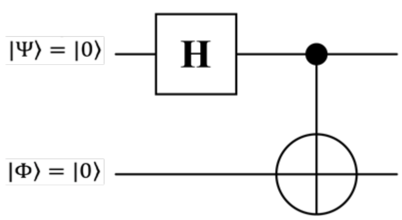
Alice's Entangled Pair



$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The Entangled Pair $|\Psi\Phi\rangle$ state becomes:

$$\mathbf{T}|\Psi\Phi\rangle = \mathbf{T}|00\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

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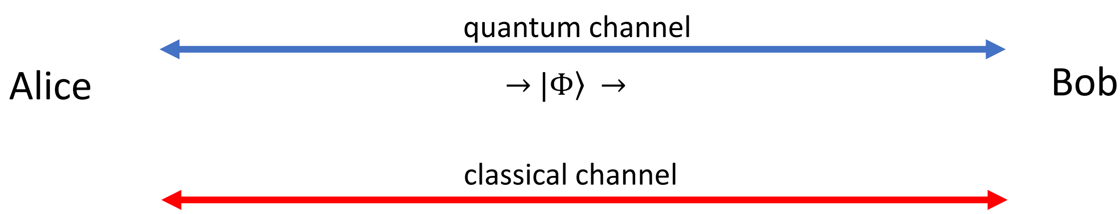
Alice's Combined State

- Alice's Combined 3-qubit State is:

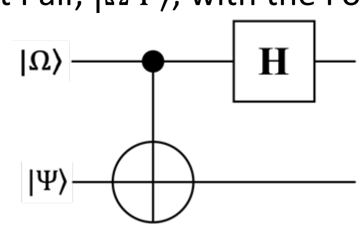
$$|\Omega\Psi\Phi\rangle = (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

- Alice Retains $|\Psi\rangle$ and Sends Bob $|\Phi\rangle$ Over the Quantum Channel
- Alice has $|\Omega\Psi\rangle$ and Bob has $|\Phi\rangle$



Alice's Next Step

- Alice Evolves her Qubit Pair, $|\Omega\Psi\rangle$, with the Following Circuit:



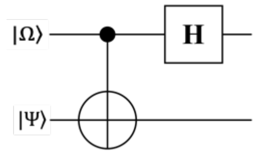
- The Transfer Matrix for this Circuit is:

$$\mathbf{T}_r = (\mathbf{H} \otimes \mathbf{I})\mathbf{C}_\mathbf{X} = \left(\frac{1}{\sqrt{2}}\right)\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Alice's Next Step (cont.)

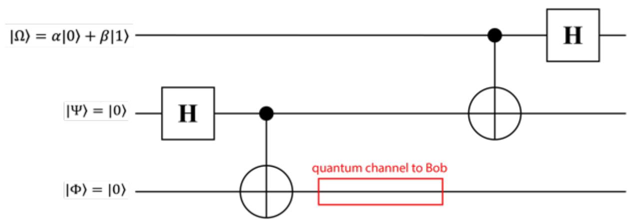
- Alice Evolves her Qubit Pair, $|\Omega\Psi\rangle$, with the Following Circuit:



- The Transfer Matrix for this Circuit is:

$$\mathbf{T}_r = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

- Although Alice Possesses $|\Omega\Psi\rangle$ and Bob Possesses $|\Phi\rangle$, when Alice Evolves her Pair, it Affects the State of Bob's Qubit, $|\Phi\rangle$, since $|\Psi\Phi\rangle$ are Entangled.
- We Must Consider all Three Qubits, $|\Omega\Psi\Phi\rangle$, the Joint State is given by this overall circuit:



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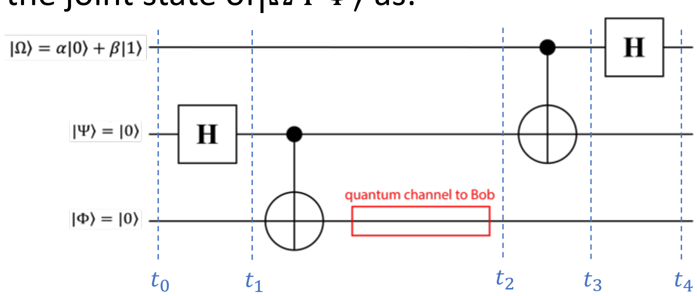
The 3-Qubit Joint State

- We compute the joint state of $|\Omega\Psi\Phi\rangle$ as:



- At time t_0:

$$|\Omega\Psi\Phi(t_0)\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle$$

- At time t_1:

$$|\Omega\Psi\Phi(t_1)\rangle = (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle$$

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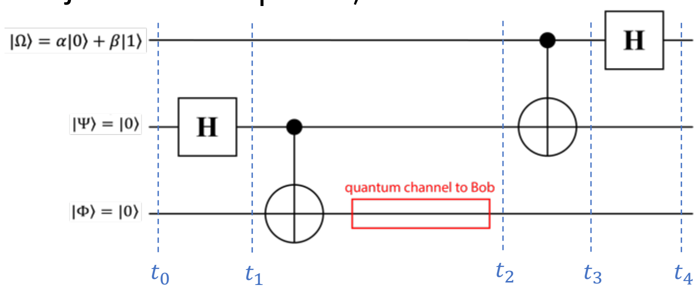
The 3-Qubit Joint State (cont.)

- We compute the joint state of $|\Omega\Psi\Phi\rangle$ as:



- At time t_2:

$$|\Omega\Psi\Phi(t_2)\rangle = (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

Entangled state

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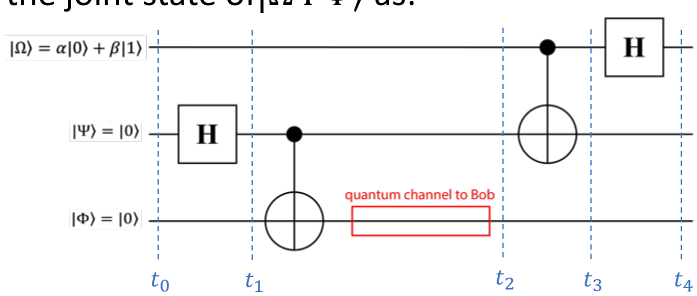
The 3-Qubit Joint State (cont.)

- We compute the joint state of $|\Omega\Psi\Phi\rangle$ as:



- At time t_3:

$$|\Omega\Psi\Phi(t_3)\rangle = (\mathbf{C_X} \otimes \mathbf{I}_2)(\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$\mathbf{C_X} \otimes \mathbf{I}_2 = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)$$

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The 3-Qubit Joint State (cont.)

- At time t_3:

$$|\Omega\Psi\Phi(t_3)\rangle = (\mathbf{C_X} \otimes \mathbf{I}_2)(\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$\mathbf{C_X} \otimes \mathbf{I}_2 = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

Using Explicit Notation:

$$\mathbf{C_X} \otimes \mathbf{I}_2 = \left(\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix}\right)$$

$$\otimes \left(\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$$

$$\mathbf{C_X} \otimes \mathbf{I}_2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}$$

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The 3-Qubit Joint State (cont.)

- At time t_3:

$$|\Omega\Psi\Phi(t_3)\rangle = (\mathbf{C_X} \otimes \mathbf{I}_2)(\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

Using Dirac's Notation:

$$\mathbf{C_X} \otimes \mathbf{I}_2 = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|)(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\begin{aligned}&\mathbf{C_X} \otimes \mathbf{I}_2\\ &= |000\rangle\langle 000| + |010\rangle\langle 010| + |100\rangle\langle 110| + |110\rangle\langle 100|\\ &+ |001\rangle\langle 001| + |011\rangle\langle 011| + |101\rangle\langle 111| + |111\rangle\langle 101|\end{aligned}$$

- Notice how the matrices, in terms of BraKet outer products, combine in the above Equation
- The Kets combine as rightmost term and the Bras combine as rightmost term

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The 3-Qubit Joint State (cont.)

- At time t_3:

$$|\Omega\Psi\Phi(t_3)\rangle = (\mathbf{C_X} \otimes \mathbf{I}_2)(\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$|\Omega\Psi\Phi(t_3)\rangle = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{C_X} \otimes \mathbf{I}_2)(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

$$\begin{aligned}&|\Omega\Psi\Phi(t_3)\rangle \\ &= \left(\frac{1}{\sqrt{2}}\right)\big(|000\rangle\langle 000| + |010\rangle\langle 010| + |100\rangle\langle 110| + |110\rangle\langle 100| \\ &+ |001\rangle\langle 001| + |011\rangle\langle 011| + |101\rangle\langle 111| + |111\rangle\langle 101|\big)(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)\end{aligned}$$

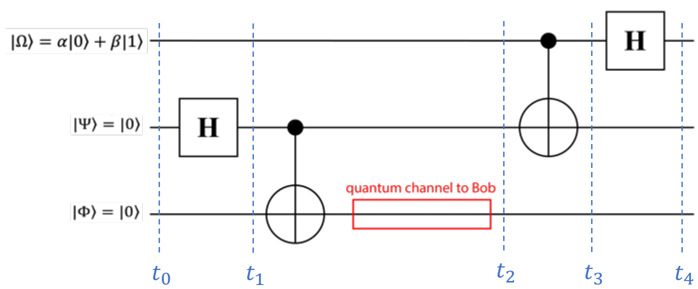
$$|\Omega\Psi\Phi(t_3)\rangle = \left(\frac{1}{\sqrt{2}}\right)\big(\alpha|000\rangle\langle 000|000\rangle + \beta|110\rangle\langle 100|100\rangle + \alpha|011\rangle\langle 011|011\rangle + \beta|101\rangle\langle 111|111\rangle\big)$$

$$|\Omega\Psi\Phi(t_3)\rangle = \left(\frac{1}{\sqrt{2}}\right)(\alpha|000\rangle + \beta|110\rangle + \alpha|011\rangle + \beta|101\rangle)$$

The 3-Qubit Joint State (cont.)

- At time t_3:

$$|\Omega\Psi\Phi(t_3)\rangle = \left(\frac{1}{\sqrt{2}}\right)(\alpha|000\rangle + \beta|110\rangle + \alpha|011\rangle + \beta|101\rangle)$$



- At time t_4:

$$|\Omega\Psi\Phi(t_4)\rangle = (\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)\left(\frac{1}{\sqrt{2}}\right)(\alpha|000\rangle + \beta|110\rangle + \alpha|011\rangle + \beta|101\rangle)$$

The 3-Qubit Joint State (cont.)

- At time t_4:

$$|\Omega\Psi\Phi(t_4)\rangle = (\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)\left(\frac{1}{\sqrt{2}}\right)(\alpha|000\rangle + \beta|110\rangle + \alpha|011\rangle + \beta|101\rangle)$$

$$\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2 = \mathbf{H} \otimes \mathbf{I}_4$$
$$= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|)$$

$$\mathbf{H} \otimes \mathbf{I}_4$$
$$= \frac{1}{\sqrt{2}}(|000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| - |100\rangle\langle 100| + |001\rangle\langle 001| + |001\rangle\langle 101| + |101\rangle\langle 001|$$
$$- |101\rangle\langle 101| + |010\rangle\langle 010| + |010\rangle\langle 110| + |110\rangle\langle 010| - |110\rangle\langle 110| + |011\rangle\langle 011| + |011\rangle\langle 111|$$
$$+ |111\rangle\langle 011| - |111\rangle\langle 111|)$$

$$|\Omega\Psi\Phi(t_4)\rangle$$
$$= \frac{1}{2}(|000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| - |100\rangle\langle 100| + |001\rangle\langle 001| + |001\rangle\langle 101| + |101\rangle\langle 001|$$
$$- |101\rangle\langle 101| + |010\rangle\langle 010| + |010\rangle\langle 110| + |110\rangle\langle 010| - |110\rangle\langle 110| + |011\rangle\langle 011| + |011\rangle\langle 111|$$
$$+ |111\rangle\langle 011| - |111\rangle\langle 111|)(\alpha|000\rangle + \beta|110\rangle + \alpha|011\rangle + \beta|101\rangle)$$

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The 3-Qubit Joint State (cont.)

- At time t_4:

$$|\Omega\Psi\Phi(t_4)\rangle$$
$$= \frac{1}{2}(|000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| - |100\rangle\langle 100| + |001\rangle\langle 001| + |001\rangle\langle 101| + |101\rangle\langle 001|$$
$$- |101\rangle\langle 101| + |010\rangle\langle 010| + |010\rangle\langle 110| + |110\rangle\langle 010| - |110\rangle\langle 110| + |011\rangle\langle 011| + |011\rangle\langle 111|$$
$$+ |111\rangle\langle 011| - |111\rangle\langle 111|)(\alpha|000\rangle + \beta|110\rangle + \alpha|011\rangle + \beta|101\rangle)$$

$$|\Omega\Psi\Phi(t_4)\rangle$$
$$= \frac{1}{2}(|000\rangle\langle 000|\alpha|000\rangle + |100\rangle\langle 000|\alpha|000\rangle + |001\rangle\langle 101|\beta|101\rangle - |101\rangle\langle 101|\beta|101\rangle$$
$$+ |010\rangle\langle 110|\beta|110\rangle - |110\rangle\langle 110|\beta|110\rangle + |011\rangle\langle 011|\alpha|011\rangle + |111\rangle\langle 011|\alpha|011\rangle)$$

$$|\Omega\Psi\Phi(t_4)\rangle = \frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \beta|001\rangle - \beta|101\rangle + \beta|010\rangle - \beta|110\rangle + \alpha|011\rangle + \alpha|111\rangle)$$

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The 3-Qubit Joint State (cont.)

- At time t_4:

$$|\Omega\Psi\Phi(t_4)\rangle = \frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \beta|001\rangle - \beta|101\rangle + \beta|010\rangle - \beta|110\rangle + \alpha|011\rangle + \alpha|111\rangle)$$

$$|\Omega\Psi\Phi(t_4)\rangle =$$
$$\frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle)$$
$$+\frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle)$$
$$+\frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle)$$
$$+\frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$$

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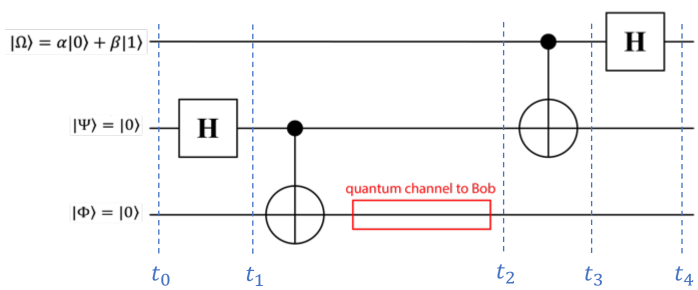
The 3-Qubit Joint State (cont.)

- At time t_4:

$$|\Omega\Psi\Phi(t_4)\rangle =$$
$$\frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle)$$
$$+\frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle)$$
$$+\frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle)$$
$$+\frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$$

Alice's qubits

Bob's qubit



- At time t_4, Alice caused her qubits to evolve into a (computational) basis state
- Since Alice retained qubit $|\Psi\rangle$, which was entangled with Bob's qubit, $|\Phi\rangle$, the rightmost $\mathbf{C_X}$ gate served to entangle Alice's qubit $|\Omega\rangle$ with Bob's qubit $|\Phi\rangle$
- This entangling operation "teleported" the probability amplitudes of Alice's $|\Omega\rangle$ to Bob's $|\Phi\rangle$

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Eliminating the Superposition in the Joint State

- At time t_4, the three qubits are in (perfect) Superposition since the Four possible states each have probability amplitudes of one-fourth.

$$|\Omega\Psi\Phi(t_4)\rangle = \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle(\alpha|1\rangle + \beta|0\rangle) + \frac{1}{2}|10\rangle(\alpha|0\rangle - \beta|1\rangle) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle)$$

- Alice can Measure her two qubits $|\Omega\Psi\rangle$ with respect to the computational basis by using the Pauli-$\mathbf{Z}$ observable
- This measurement will force Alice's two qubits to collapse into one of the four basis states, $|00\rangle$, $|01\rangle$, $|10\rangle$ or $|11\rangle$ with equal probability
- When Alice performs her measurement, this causes Bob's qubit to collapse into one of the following four states:

$$|\Phi_{00}\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\Phi_{01}\rangle = \alpha|1\rangle + \beta|0\rangle, \quad |\Phi_{10}\rangle = \alpha|0\rangle - \beta|1\rangle, \quad |\Phi_{11}\rangle = \alpha|1\rangle - \beta|0\rangle$$

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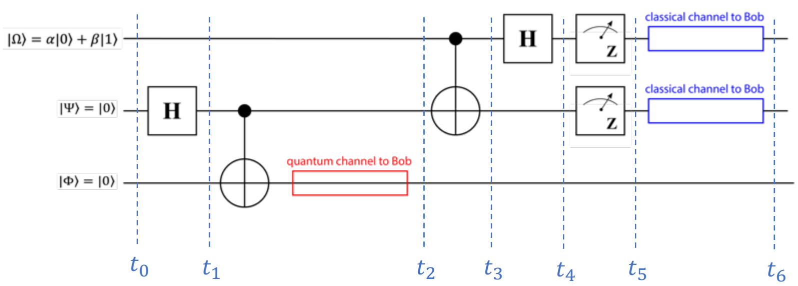
Alice Measures her Two Qubits and Tells Bob the Outcome



- At time t_5, Alice has measured her two qubits causing them to collapse into one of the four (4-dimensional) basis states: $|00\rangle$, $|01\rangle$, $|10\rangle$ or $|11\rangle$
- Alice's measurement "collapses" the 3-qubit joint superposition and causes Bob's qubit to likewise "collapse" into one of these four states (that is still in superposition)

$$|\Phi_{00}\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\Phi_{01}\rangle = \alpha|1\rangle + \beta|0\rangle, \quad |\Phi_{10}\rangle = \alpha|0\rangle - \beta|1\rangle, \quad |\Phi_{11}\rangle = \alpha|1\rangle - \beta|0\rangle$$

- At time t_6, Alice has has told Bob which of the four outcomes, 00, 01, 10 or 11, that resulted

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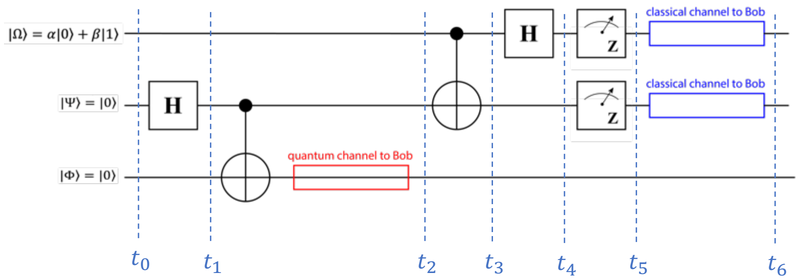
Alice Measures her Two Qubits and Tells Bob the Outcome

- At time t_6, Alice has told Bob, using the classical channel, which of the four outcomes, 00, 01, 10 or 11, resulted from her measurements
- When Bob receives Alice's measurement results, (00, 01, 10 or 11), he knows that his qubit $|\Phi\rangle$ has one of these forms:

$$|\Phi_{00}\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\Phi_{01}\rangle = \alpha|1\rangle + \beta|0\rangle, \quad |\Phi_{10}\rangle = \alpha|0\rangle - \beta|1\rangle, \quad |\Phi_{11}\rangle = \alpha|1\rangle - \beta|0\rangle$$

- The desired result is for Bob to evolve his qubit $|\Phi\rangle$ such that it assumes the form of $|\Omega(t_0)\rangle = \alpha|0\rangle + \beta|1\rangle$, the original state that Alice is "teleporting" to Bob



How can Bob Evolve his Teleported Qubit ?

- At time t_6, Bob possesses $|\Phi(t_6)\rangle$ that is one of these states:

$$|\Phi_{00}\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\Phi_{01}\rangle = \alpha|1\rangle + \beta|0\rangle, \quad |\Phi_{10}\rangle = \alpha|0\rangle - \beta|1\rangle, \quad |\Phi_{11}\rangle = \alpha|1\rangle - \beta|0\rangle$$

- At time t_6, Bob knows which state he has since Alice sent him a classical 2-bit value indicating which one he has:

Alice sent Bob:	Bob knows his qubit state:	Bob wants to have:	What operator(s) does Bob need?
00	$\lvert\Phi_{00}\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$	$\lvert\Phi(t_7)\rangle = \lvert\Omega(t_0)\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$	
01	$\lvert\Phi_{01}\rangle = \alpha\lvert 1\rangle + \beta\lvert 0\rangle$	$\lvert\Phi(t_7)\rangle = \lvert\Omega(t_0)\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$	
10	$\lvert\Phi_{10}\rangle = \alpha\lvert 0\rangle - \beta\lvert 1\rangle$	$\lvert\Phi(t_7)\rangle = \lvert\Omega(t_0)\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$	
11	$\lvert\Phi_{11}\rangle = \alpha\lvert 1\rangle - \beta\lvert 0\rangle$	$\lvert\Phi(t_7)\rangle = \lvert\Omega(t_0)\rangle = \alpha\lvert 0\rangle + \beta\lvert 1\rangle$	

How can Bob Evolve his Teleported Qubit ?

- At time t_6, Bob possesses $|\Phi(t_6)\rangle$ that is one of these states:

  $|\Phi_{00}\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\Phi_{01}\rangle = \alpha|1\rangle + \beta|0\rangle$, $|\Phi_{10}\rangle = \alpha|0\rangle - \beta|1\rangle$, $|\Phi_{11}\rangle = \alpha|1\rangle - \beta|0\rangle$
- At time t_6, Bob knows which state he has since Alice sent him a classical 2-bit value indicating which one he has:

Alice sent Bob:	Bob knows his qubit state:	Bob wants to have:	What operator(s) does Bob need?
00	$\|\Phi_{00}\rangle = \alpha\|0\rangle + \beta\|1\rangle$	$\|\Phi(t_7)\rangle = \|\Omega(t_0)\rangle = \alpha\|0\rangle + \beta\|1\rangle$	$\mathbf{I}_2$, (no operator)
01	$\|\Phi_{01}\rangle = \alpha\|1\rangle + \beta\|0\rangle$	$\|\Phi(t_7)\rangle = \|\Omega(t_0)\rangle = \alpha\|0\rangle + \beta\|1\rangle$	$\mathbf{X}$, (bit-flip)
10	$\|\Phi_{10}\rangle = \alpha\|0\rangle - \beta\|1\rangle$	$\|\Phi(t_7)\rangle = \|\Omega(t_0)\rangle = \alpha\|0\rangle + \beta\|1\rangle$	$\mathbf{Z}$, (phase-flip)
11	$\|\Phi_{11}\rangle = \alpha\|1\rangle - \beta\|0\rangle$	$\|\Phi(t_7)\rangle = \|\Omega(t_0)\rangle = \alpha\|0\rangle + \beta\|1\rangle$	$\mathbf{Z}$ & $\mathbf{X}$, (phase- & bit-flip)

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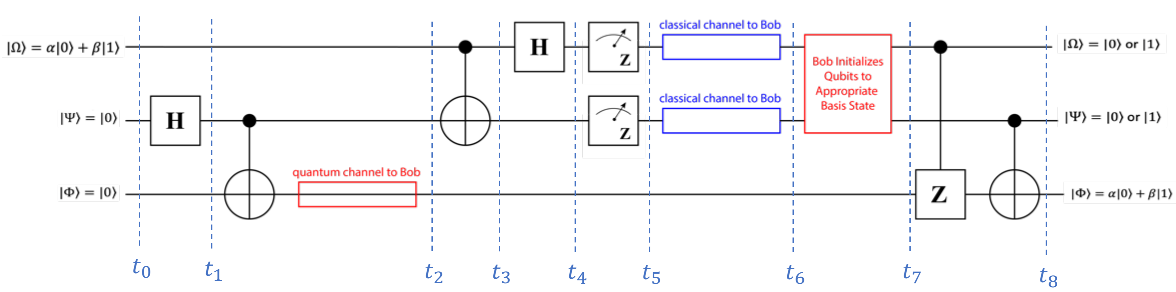
Bob Evolves his Qubit to the Teleported State

- At time t_6, Bob Initializes his own version of Qubits $|\Omega\rangle$ and $|\Psi\rangle$ into the Basis States indicated by Alice's Classical Communication to Him:
- At time t_7, Bob has Evolved his Qubit, $|\Phi(t_7)\rangle$, with the Controlled-$\mathbf{Z}$ and Controlled-$\mathbf{X}$ gates, $\mathbf{C_Z}$ and $\mathbf{C_X}$:

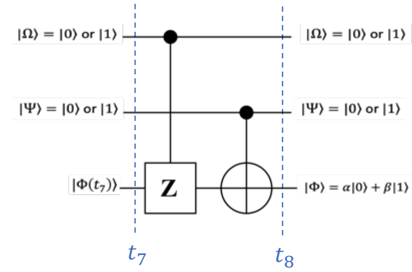


- At time t_8, Bob possesses the qubit $|\Phi(t_8)\rangle = |\Phi(t_0)\rangle = \alpha|0\rangle + \beta|1\rangle$, and the Only Information Alice sent Bob was an Entangled Qubit, $|\Phi(t_2)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and Two Bits of Classical Information, 00, 01, 10, or 11.

26

Bob's Circuit

- What is the transfer function of Bob's circuit?



$$\mathbf{C_Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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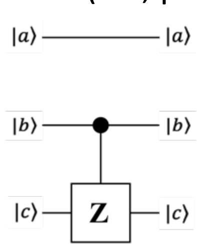
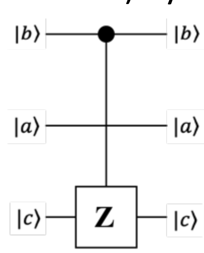
Bob's Circuit

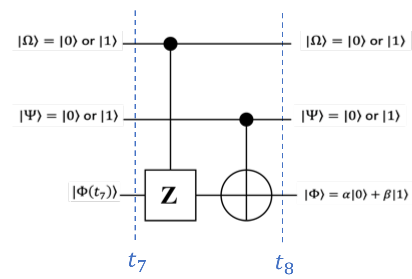
- What is the transfer function of Bob's circuit?
- Must Account for the "middle" qubit in the $\mathbf{C_Z}$ gate
- One way to determine the 3-qubit transfer function is to use a permutation matrix, $\mathbf{P}$
- Compare these two circuits where the leftmost is represented by transfer matrix, $\mathbf{T}_1$, and the rightmost (*i.e.*, part of Bob's circuit) by $\mathbf{T}_2$



$\mathbf{T}_1$



$\mathbf{T}_2$



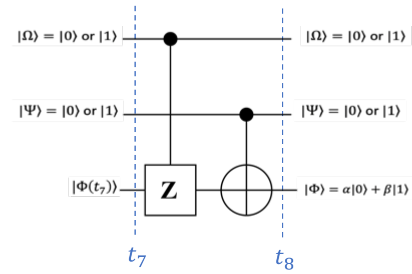
$$\mathbf{C_Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

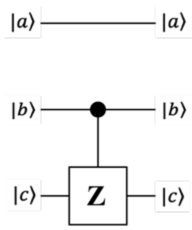
$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

28

Bob's Circuit – Controlled-$\mathbf{Z}$ Operator

- Transfer function of $\mathbf{C_Z}$ with no "middle" qubit is:





$$\mathbf{T_1} = \mathbf{I_2} \otimes \mathbf{C_Z}$$

$$\mathbf{C_Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|)\,(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|)$$

$$= |000\rangle\langle 000| + |001\rangle\langle 001| + |010\rangle\langle 010| - |011\rangle\langle 011|$$

$$+|100\rangle\langle 100| + |101\rangle\langle 101| + |110\rangle\langle 110| - |111\rangle\langle 111|$$

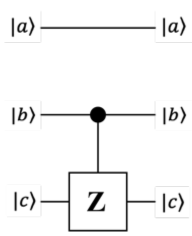
$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

29

Bob's Circuit – Controlled-$\mathbf{Z}$ Operator

- Transfer function of $\mathbf{C_Z}$ with no "middle" qubit is:



$$\mathbf{T_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

output state Kets

input state Bras

$$\begin{array}{cc} & \begin{array}{cccccccc} |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \end{array} \\ \begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

- Consider the input/output relationship where the evolved matrix "Ket" (column vectors) are possible output states for a given "Bra" (row vector) is the un-evolved input state, when the input state is also a basis vector.
- We can label the Ket and Bra vectors of the transfer matrix

- Each term in the Dirac form of the Transfer matrix is of the form:

$$|abc\rangle\langle abc|$$

- We Permute above matrix with:

$$|bac\rangle\langle bac|$$

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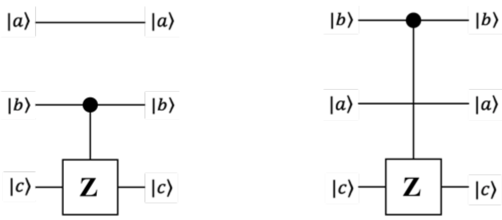
Permuting the matrix

- We Permute above matrix with $|bac\rangle\langle bac|$



- Interchanging the ab values with ba in the Ket labels: $|000\rangle\ |001\rangle\ |100\rangle\ \ |101\rangle\ |010\rangle\ |011\rangle\ |110\rangle\ \ |111\rangle$
- Indicates matrix Kets 010 switched with 100, and 011 is switched with 101
- Switching the Kets (column vectors) results in the lower right matrix

$$\begin{array}{c c}
 & \begin{array}{cccccccc} |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \end{array} \\
\begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}$$

"switched" output state Kets

$$\begin{array}{c c}
 & \begin{array}{cccccccc} |000\rangle & |001\rangle & |100\rangle & |101\rangle & |010\rangle & |011\rangle & |110\rangle & |111\rangle \end{array} \\
\begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}$$

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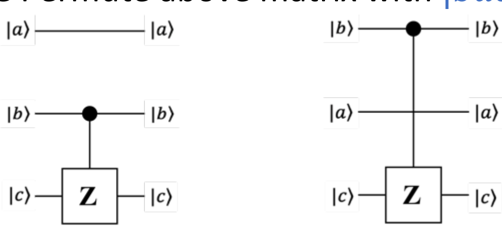
Permuting the matrix (cont.)

- We Permute above matrix with $|bac\rangle\langle bac|$



- We must likewise interchange the appropriate row vectors
- Finally, we relabel the Ket and Bra vectors to be in sequential order as shown on following slide

"switched" output state Kets

$$\begin{array}{c c}
 & \begin{array}{cccccccc} |000\rangle & |001\rangle & |100\rangle & |101\rangle & |010\rangle & |011\rangle & |110\rangle & |111\rangle \end{array} \\
\begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}$$

"switched" output state Kets

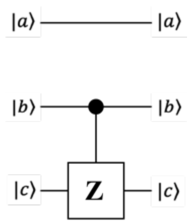
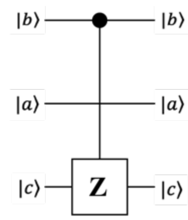
"switched" input state Bras

$$\begin{array}{c c}
 & \begin{array}{cccccccc} |000\rangle & |001\rangle & |100\rangle & |101\rangle & |010\rangle & |011\rangle & |110\rangle & |111\rangle \end{array} \\
\begin{array}{c} |000\rangle \\ |001\rangle \\ |100\rangle \\ |101\rangle \\ |010\rangle \\ |011\rangle \\ |110\rangle \\ |111\rangle \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}$$

32

Permuting the matrix (cont.)

- We Permute above matrix with $|bac\rangle\langle bac|$

"switched" output state Kets

"switched" input state Bras

$$\begin{array}{c} \\ |000\rangle \\ |001\rangle \\ |100\rangle \\ |101\rangle \\ |010\rangle \\ |011\rangle \\ |110\rangle \\ |111\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} |000\rangle & |001\rangle & |100\rangle & |101\rangle & |010\rangle & |011\rangle & |110\rangle & |111\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

"relabeled" output state Kets

"relabeled" input state Bras

$$\begin{array}{c} \\ |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} |000\rangle & |001\rangle & |010\rangle & |011\rangle & |100\rangle & |101\rangle & |110\rangle & |111\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

- We must likewise interchange the appropriate row vectors
- Finally, we relabel the Ket and Bra vectors to be in sequential order as shown on following slide
- This is the Transfer matrix for the $\mathbf{C_Z}$ gate with the "middle qubit"

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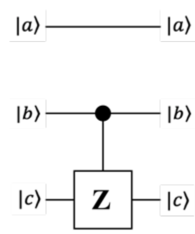
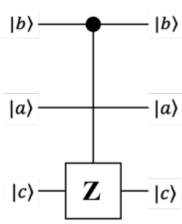
Bob's Circuit – Controlled-$\mathbf{Z}$ Operator



- Transfer function of $\mathbf{C_Z}$ with no "middle" qubit is:



$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- This is the same thing as multiplying $\mathbf{T}_1$ with a permutation matrix, $\mathbf{P}$, that interchanges the appropriate Kets and Bras and with $\mathbf{P}^\mathrm{T}$ that interchanges appropriate Bras with Kets

$$\mathbf{P} = |000\rangle\langle 000| + |001\rangle\langle 001| + |100\rangle\langle 010| + |101\rangle\langle 011| + |010\rangle\langle 100| + |011\rangle\langle 101| + |110\rangle\langle 110| + |111\rangle\langle 111|$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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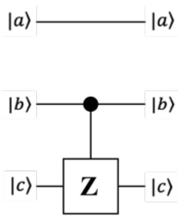
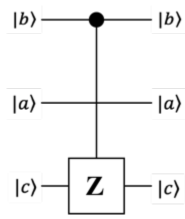
Verify Permutation: $\mathbf{T}_2 = \mathbf{P}\mathbf{T}_1\mathbf{P}^\mathrm{T}$

- Transfer function of $\mathbf{C_Z}$ with no "middle" qubit is:



$$\mathbf{T}_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix}$$



$$\mathbf{T}_2 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix}$$

$$\mathbf{T}_2 = \mathbf{P}\mathbf{T}_1\mathbf{P}^\mathrm{T} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

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Verify Permutation: $\mathbf{T}_2 = \mathbf{P}\mathbf{T}_1\mathbf{P}^\mathrm{T}$ (cont.)

$$\mathbf{T}_2 = \mathbf{P}\mathbf{T}_1\mathbf{P}^\mathrm{T} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{bmatrix} = \mathbf{T}_2$$

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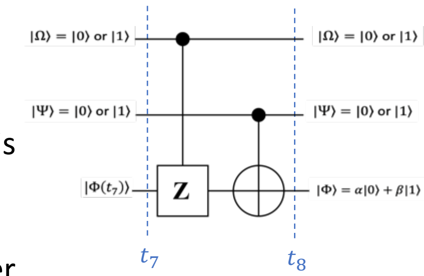
Bob's Circuit



- What is the transfer function of Bob's circuit?
- Now we know that the $\mathbf{C_Z}$ with the "middle" qubit is represented with:
$$\mathbf{T}_2 = \mathbf{P}\mathbf{T}_1\mathbf{P}^{\mathrm{T}} = \mathbf{P}(\mathbf{I}_2 \otimes \mathbf{C_Z})\mathbf{P}^{\mathrm{T}}$$
- The transfer function for Bob's circuit, shown in upper left, denoted by $\mathbf{T}_3$, is:
$$\mathbf{T}_3 = (\mathbf{I}_2 \otimes \mathbf{C_X})\mathbf{T}_2 = (\mathbf{I}_2 \otimes \mathbf{C_X})\mathbf{P}(\mathbf{I}_2 \otimes \mathbf{C_Z})\mathbf{P}^{\mathrm{T}}$$
- The explicit form of $(\mathbf{I}_2 \otimes \mathbf{C_X})$ is:

$$\mathbf{C_Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{I}_2 \otimes \mathbf{C_X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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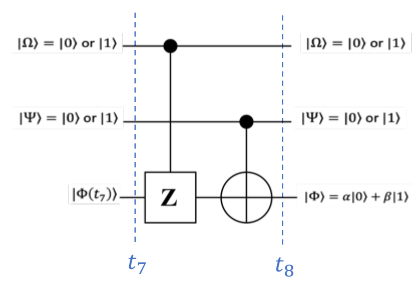
Bob's Circuit (cont.)



- What is the transfer function of Bob's circuit?
- The explicit form of Bob's circuit is:
$$\mathbf{T}_3 = (\mathbf{I}_2 \otimes \mathbf{C_X})\mathbf{P}(\mathbf{I}_2 \otimes \mathbf{C_Z})\mathbf{P}^{\mathrm{T}}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C_Z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{C_X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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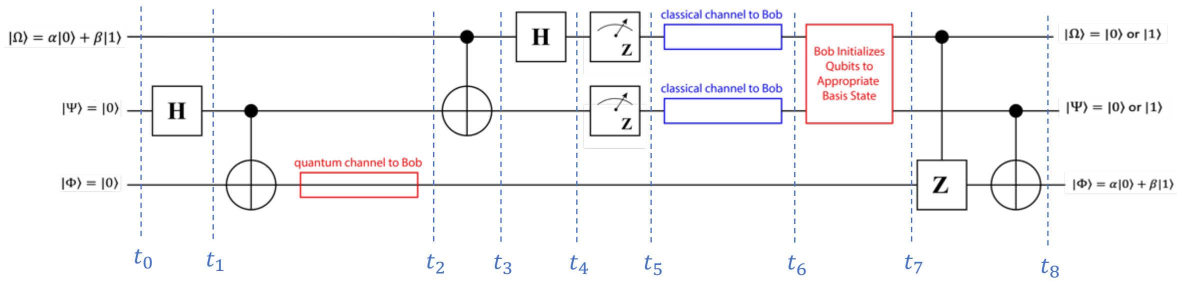
Transfer Matrix of Entire Teleportation Circuit



- Alice's circuit from time t_0 to t_4 is represented by transfer matrix $\mathbf{T}_0$:

$$\mathbf{T}_0 = (\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)(\mathbf{C}_\mathbf{X} \otimes \mathbf{I}_2)(\mathbf{I}_2 \otimes \mathbf{C}_\mathbf{X})(\mathbf{I}_2 \otimes \mathbf{H} \otimes \mathbf{I}_2)$$

$$= \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)$$
$$\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)$$

Transfer Matrix of Entire Teleportation Circuit (cont.)

- Alice's circuit from time t_0 to t_4 is represented by transfer matrix $\mathbf{T}_0$:

$$\mathbf{T}_0 = (\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)(\mathbf{C}_\mathbf{X} \otimes \mathbf{I}_2)(\mathbf{I}_2 \otimes \mathbf{C}_\mathbf{X})(\mathbf{I}_2 \otimes \mathbf{H} \otimes \mathbf{I}_2)$$

$$= \frac{1}{2}\left(\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right.$$
$$\left.\otimes \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\right)$$

Transfer Matrix of Entire Teleportation Circuit (cont.)

- Alice's circuit from time t_0 to t_4 is represented by transfer matrix $\mathbf{T}_0$:

$$\mathbf{T}_0 = (\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)(\mathbf{C_X} \otimes \mathbf{I}_2)(\mathbf{I}_2 \otimes \mathbf{C_X})(\mathbf{I}_2 \otimes \mathbf{H} \otimes \mathbf{I}_2)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

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Transfer Matrix of Entire Teleportation Circuit (cont.)

- Alice's circuit from time t_0 to t_4 is represented by transfer matrix $\mathbf{T}_0$:

$$\mathbf{T}_0 = (\mathbf{H} \otimes \mathbf{I}_2 \otimes \mathbf{I}_2)(\mathbf{C_X} \otimes \mathbf{I}_2)(\mathbf{I}_2 \otimes \mathbf{C_X})(\mathbf{I}_2 \otimes \mathbf{H} \otimes \mathbf{I}_2)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

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Transfer Matrix of Entire Teleportation Circuit (cont.)

- Alice's circuit from time t_0 to t_4 is represented by transfer matrix $\mathbf{T}_0$:

$$\mathbf{T}_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

- Bob's circuit from time t_7 to t_8 is represented by transfer matrix $\mathbf{T}_3$:

$$\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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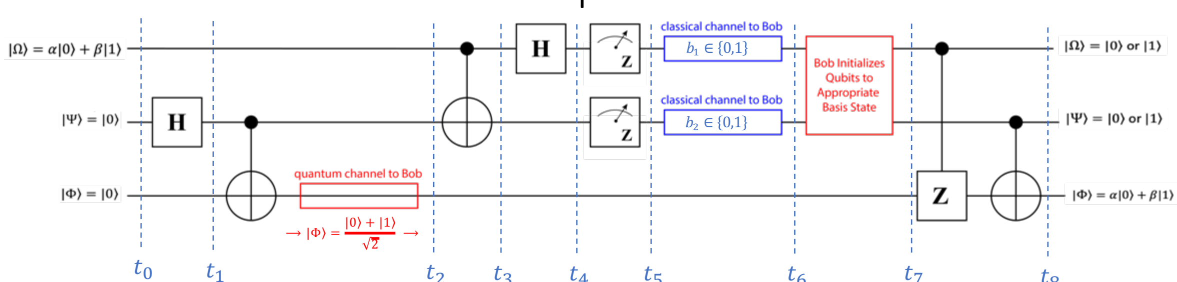
Transfer Matrix of Entire Teleportation Circuit



$$\mathbf{T}_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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Quantum Teleportation: Summary

- Quantum Teleportation Exploits Entanglement to (theoretically) Instantly Transfer Information
  - information is NOT transmitted over a channel either wirelessly or over a wireline
- It does NOT instantly transfer matter or energy
- It does instantly transfer an energy state
- Requires Transmission of Matter/energy over a channel
- Successfully Demonstrated Experimentally
- Applications in Cyber Security
  - EXAMPLE: Secure Encryption Key Distribution
- Does NOT violate Speed-of-Light Transmission Limits (special relativity) since Information is NOT transmitted, but a quantum state host and 2 Classical bits are transmitted

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