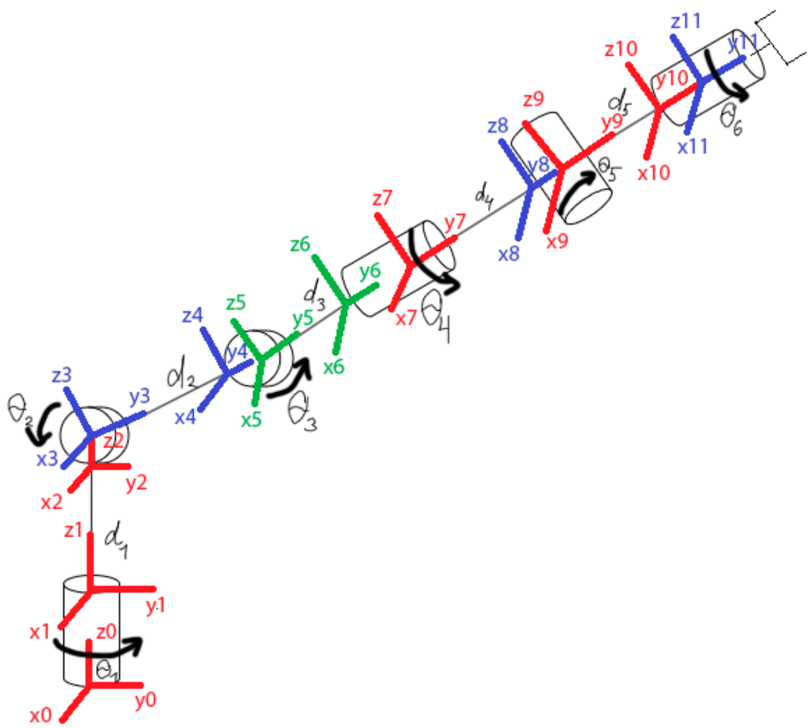
Chosen manipulator

For this task I decided to chose an anthropomotphic manipulator with yzy spherical wrist.



Forward kinematics

I found position of the end effector in this way:

$$P_{end} = P_{base} R_z(\theta_1) T_z(d_1) R_x(\theta_2) T_y(d_2) R_x(\theta_3) T_y(d_3) R_y(\theta_4) T_y(d_4) R_z(\theta_5) T_y(d_5) R_y(\theta_6)$$

where P_{base} is an identity 4x4 matrix, $R_{axis}(\theta)$ is a rotation matrix in homogeneous representation on given angle θ around given axis and $T_{axis}(d)$ is a translation matrix in homogeneous representation on given axis on given distance.

Inverse kinematics

First, we should find the position of the wrist:

$$P_w = P_{desired} T_y(d_5)$$

Where P_w is a position of the wrist in homogeneous form and $P_{desired}$ is desired position.

We can express this position using first three generalized coordinates:

$$\begin{cases} P_x = c_1(d_2 c_2 + d_3 c_{23}) \\ P_y = s_1(d_2 c_2 + d_3 c_{23}) \\ P_z = d_1 + d_2 s_2 + d_3 s_{23} \end{cases}$$

Then, we can find θ_1:

$$\theta_1 = Atan2(P_y, P_x)$$

To find θ_2, I summed squares of P_x, P_y and P_z:

$$\begin{aligned} P_x^2 + P_y^2 + P_z^2 = (d_2 c_2 + d_3 c_{23})^2 + (d_1 + d_2 s_2 + d_3 s_{23})^2 = \\ = d_1^2 + d_2^2 + d_3^2 + 2 d_2 d_3 c_3 + 2 d_1 d_2 s_2 + 2 d_1 d_3 s_3 \end{aligned}$$

At this point I gave up and decided to set $d_1 = 0$ as they did in Siciliano, hence:

$$c_3 = \frac{P_x^2 + P_y^2 + P_z^2 - d_2^2 - d_3^2}{2 d_2 d_3}$$

$$s_3 = \sqrt{1 - c_3^2}$$

I took s_3 with positive sign just because I want to find any solution.

$$\theta_3 = Atan2(s_3, c_3)$$

After that, summing squares of P_x and P_y, we get:

$$P_x^2 + P_y^2 = (d_2 c_2 + d_3 c_{23})^2$$

$$d_2 c_2 + d_3 c_2 c_3 - d_3 s_2 s_3 = \sqrt{P_x^2 + P_y^2}$$

Using equation of P_z:

$$P_z = d_2 s_2 + d_3 s_2 c_3 + d_3 s_3 c_2$$

$$s_2 = \frac{P_z - d_3 s_3 c_2}{d_2 + d_3 c_3}$$

Substituting it to the previous equation we get:

$$
\begin{aligned}
\sqrt{P_x^2+P_y^2} &= d_2c_2 + d_3c_2c_3 - \frac{d_3s_3P_z - d_3^2s_3^2c_2}{d_2 - d_3c_3} \\
(d_2+d_3c_3)\sqrt{P_x^2+P_y^2} &= c_2(d_2+d_3c_3)^2 - d_3s_3P_z + d_3^2s_3^2c_2 \\
c_2 &= \frac{(d_2+d_3c_3)\sqrt{P_x^2+P_y^2} + d_3s_3P_z}{d_2^2+d_3^2+2c_3d_2d_3} \\
s_2 &= \frac{P_z(d_2+d_3c_3) - d_3s_3\sqrt{P_x^2+P_y^2}}{d_2^2+d_3^2+2d_2d_3c_3}
\end{aligned}
$$

Now we should find θ_4, θ_5 and θ_6
Knowing other three coordinates we can find rotation matrix R_3^0.

$$R_6^0 = R_3^0R_6^3 \longrightarrow R_6^3 = (R_3^0)^{-1}R_6^0$$

We can represent R_6^3 as $R_y(\theta_4)R_z(\theta_5)R_y(\theta_6)$:

$$R_6^3 = \begin{pmatrix} -s_4s_6 + c_4c_5c_6 & -s_5c_4 & s_4c_6 + s_6c_4c_5 \\ s_5c_6 & c_5 & s_5s_6 \\ -s_4c_5c_6 - s_6c_4 & s_4s_5 & -s_4s_6c_5 + c_4c_6 \end{pmatrix}$$

Hence

$$
\begin{aligned}
\theta_4 &= Atan2(R_{3,2}, R_{1,2}) \\
\theta_5 &= Atan2(\sqrt{R_{2,1}^2 + R_{2,3}^2}, R_{2,2}) \\
\theta_6 &= Atan2(R_{2,3}, R_{2,1})
\end{aligned}
$$

Plots obtained

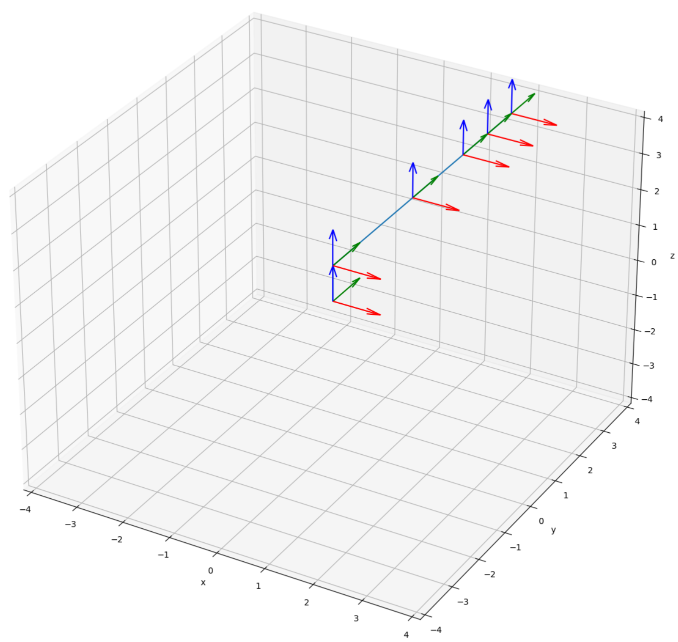


Figure 1: Obtained positions for zero angles

As you can see on figure 3, manipulator did not reach the desired position (you can see it on figure 3 as blue point), hence the algorithm for inverse kinematics did not work.

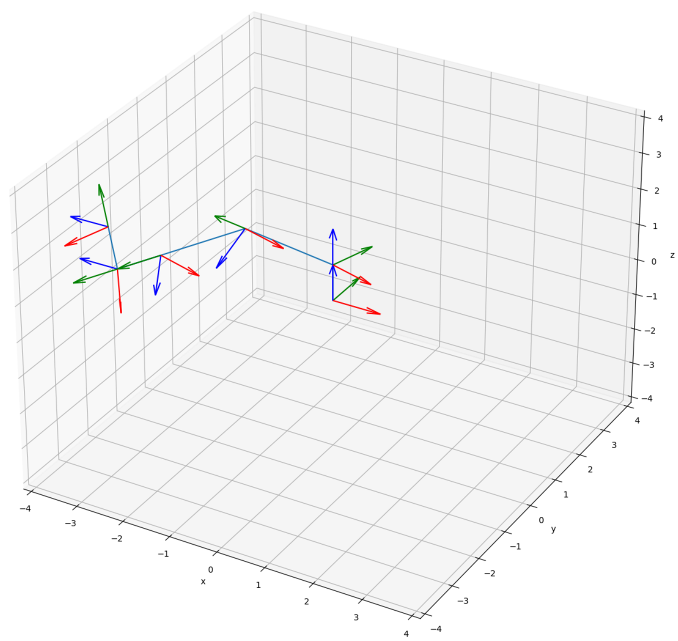


Figure 2: Obtained positions for random angles

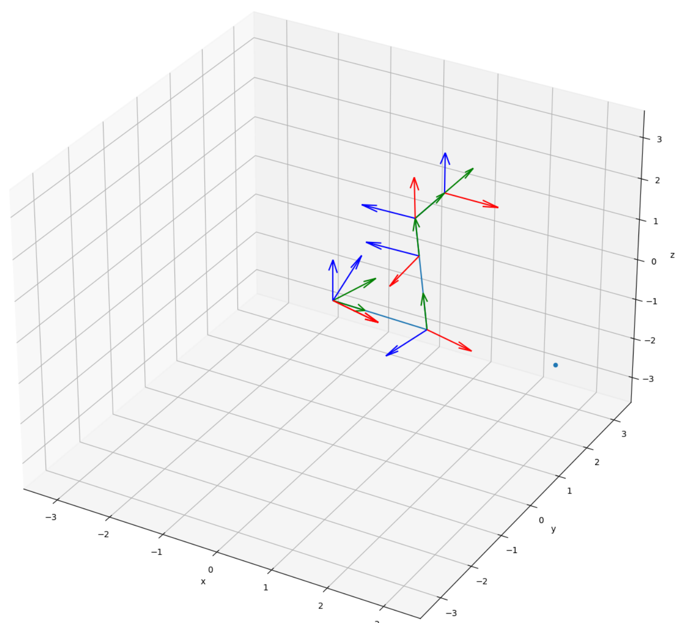


Figure 3: Obtained inverse kinematics for position (4, 0, 0)