Confined particle on a ring

Anikin Evgeny

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Let us consider the particle on a ring with delta potential in presence of magnetic flux:

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - \frac{eA}{c}\right)^2 - \frac{\kappa_0}{m}\delta(x - x_0),\ x + L = x \tag{1}$$

We are interested in the Berry phase acquired by a ground state after a slow rotation of x_0 around the ring.

Assume that we know the wavefunction of the ground state. The Berry phase is given by the expression

$$\phi = \int \mathcal{A} dx_0 = L\mathcal{A} \tag{2}$$

The Berry connection is

$$\mathcal{A} = i\langle\psi|\partial_{x_0}|\psi\rangle = \langle\hat{p}\rangle \tag{3}$$

As $\hat{p} = m\hat{v} + \frac{eA}{c}$,

$$\mathcal{A} = m\langle\psi|\hat{v}|\psi\rangle + \frac{eA}{c} \tag{4}$$

Furthermore, we can notice that

$$\hat{v} = -\frac{c}{e}\frac{\partial\hat{H}}{\partial A} \tag{5}$$

Therefore, by Ehrenfest theorem,

$$\langle\hat{v}\rangle = -\frac{c}{e}\frac{\partial E}{\partial A}, \tag{6}$$

where E is the energy of the ground state. Thus,

$$\mathcal{A} = -\frac{mc}{e}\frac{\partial E}{\partial A} + \frac{eA}{c} \tag{7}$$

Before any computations it is obvious that in the limit $\kappa L \to \infty$ the mean velocity is zero, or equivalently, the energy does not depend on A. Therefore, in this limit

$$\phi = \frac{eLA}{c} = \frac{e\Phi}{c} \tag{8}$$

Now let us perform the computation of the ground state energy. For convenience we set $x_0 = 0$. Therefore, in the region $0 < x < L$ the wavefunction takes the form

$$\psi = \alpha \exp\left(\kappa x + \frac{ieAx}{c}\right) + \beta \exp\left(-\kappa x + \frac{ieAx}{c}\right), \tag{9}$$

where $E = -\frac{\kappa^2}{2m}$. The boundary conditions read

$$\begin{gathered} \psi(0) = \psi(L) \\ \left(\frac{\partial}{\partial x} - \frac{ieA}{c}\right)\psi\Big|_{L-0}^{+0} = -2\kappa_0 \psi(0) \end{gathered} \tag{10}$$

After some calculations, we obtain the equation which determines the eigenvalues:

$$\det \begin{pmatrix} \exp\left(\kappa L + \frac{ieAL}{c}\right) - 1 & \exp\left(-\kappa L + \frac{ieAL}{c}\right) - 1 \\ \exp\left(\kappa L + \frac{ieAL}{c}\right) - 1 - \frac{2\kappa_0}{\kappa} & -\exp\left(-\kappa L + \frac{ieAL}{c}\right) + 1 - \frac{2\kappa_0}{\kappa} \end{pmatrix} = 0 \tag{11}$$

The latter can be simplified:

$$\cosh \kappa L - \frac{\kappa_0}{\kappa} \sinh \kappa L = \cos \frac{eAL}{c} \tag{12}$$

So,

$$\frac{d\kappa}{dA} = \frac{e\kappa}{c\kappa_0} \cdot \frac{\sin \frac{eAL}{c}}{\cosh \kappa L - \left(\frac{\kappa}{\kappa_0} + \frac{1}{\kappa L}\right) \sinh \kappa L} \tag{13}$$

The Berry phase is

$$\phi = \frac{e\Phi}{c} + \frac{\kappa^2 L}{\kappa_0} \frac{\sin \frac{e\Phi}{c}}{\cosh \kappa L - \left(\frac{\kappa}{\kappa_0} + \frac{1}{\kappa L}\right) \sinh \kappa L} \tag{14}$$

At large $\kappa_0 L$

$$\phi \approx \frac{e\Phi}{c} - \frac{\kappa_0 L \sin \frac{e\Phi}{c}}{\sinh \kappa_0 L} \tag{15}$$