Quantum nonlinear oscillator

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Problem Hamiltonian

$$\hat{H} = \omega_0 a^\dagger a + \frac{\beta}{2} a^\dagger a^\dagger a a + E^* e^{i\Omega t} a + E e^{-i\Omega t} a^\dagger \tag{1}$$

After substitution $\Psi = e^{i\Omega t a^\dagger a}\widetilde{\Psi}$ the Schroedinger equation becomes

$$i\partial_t \widetilde{\Psi} = \hat{H}_{\text{eff}} \widetilde{\Psi}, \tag{2}$$

with the effective Hamiltonian

$$\hat{H}_{\text{eff}} = (\omega_0 - \Omega) a^\dagger a + \frac{\beta}{2} a^\dagger a^\dagger a a + E^* a + E a^\dagger \tag{3}$$

The quantum Heisenberg equation

$$i\frac{\partial \hat{a}}{\partial t} = -|\Delta|\hat{a} + \beta \hat{a}^\dagger \hat{a}\hat{a} + E$$

becomes

$$i\frac{\partial a}{\partial t} = -|\Delta| a + \beta |a|^2 a + E$$

Classical energy:

$$H = -\Delta|a|^2 + \frac{\beta}{2}|a|^4 + E^* a + E a^*$$

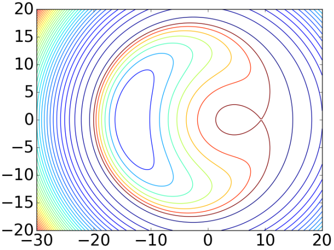


Figure 1: Phase portrait of the classical nonlinear oscillator in the complex a plane. $\Delta = -1$, $\beta = 1/12^2$, $E = 0.8\sqrt{\frac{4\Delta^3}{27\beta}} = 3.70$

Quantum description

Eigenvectors of the Hamiltonian have the form

$$|\psi_n\rangle = \sum c_k |k\rangle$$

With any vector, it is possible to construct the quantity $e^{-\frac{\bar{\xi}\xi}{2}}\langle\xi|\psi\rangle$, where $|\xi\rangle$ is a coherent state.

Reminder on coherent states

By definition, $\hat{a}|\xi\rangle = \xi|\xi\rangle$. That leads to a decomposition

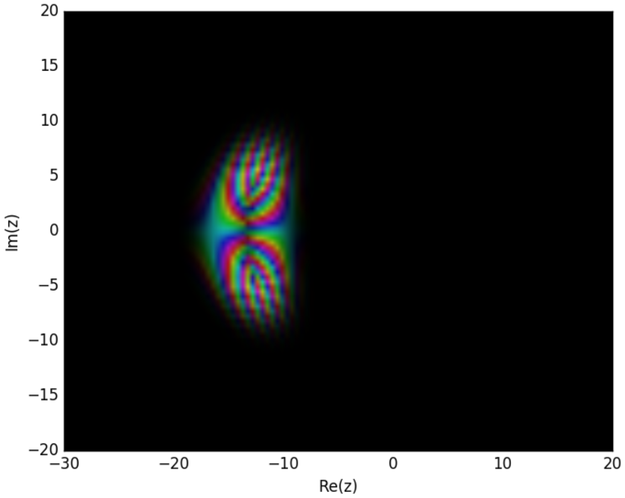
$$|\xi\rangle = e^{\xi \hat{a}^\dagger}|0\rangle = \sum \frac{\xi^n}{\sqrt{n!}}|n\rangle$$

An enormously important relation:

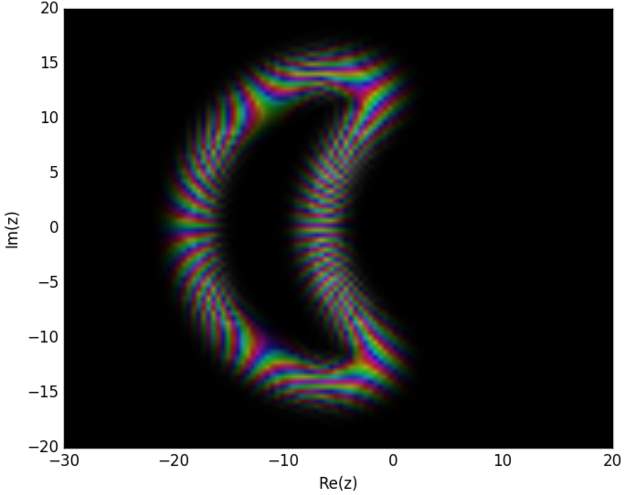
$$\int d\bar{\xi} d\xi \, e^{-\bar{\xi}\xi}|\xi\rangle\langle\xi| = \mathbb{1}$$

As a consequance, any state can be represented as an integral over coherent states.

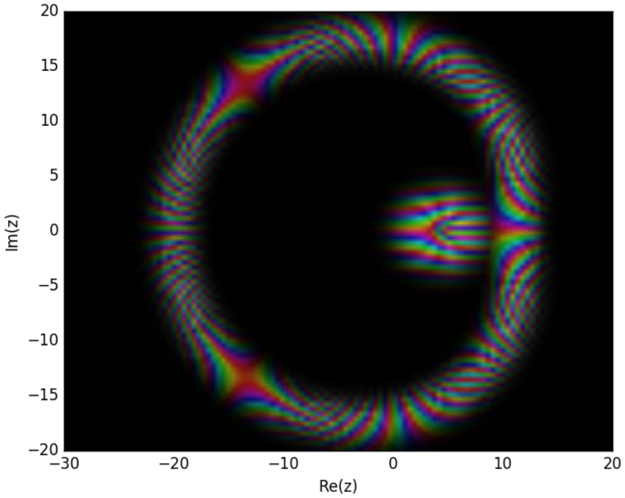
Eigenstates of the quantum Hamiltonian



Eigenstates of the quantum Hamiltonian



Eigenstates of the quantum Hamiltonian



Bistability

Bistability means that there exist two stationary points on the phase diagram. In the presence of dissipation, the oscillator evolves to one of them.

"Quantum" bistability

It's necessary to introduce some relaxation to understand bistability in quantum case. With relaxation, it is reasonable to expect two peaks in occupation numbers.

Oscillator with dissipation

A nonlinear oscillator interacting with a bath:

$$\hat{H}_{\text{full}} = \hat{H} + \hat{H}_{\text{bath}} + \hat{H}_{\text{int}},$$

$$\hat{H}_{\text{bath}} = \sum \omega_k b_k^\dagger b_k$$

$$\hat{H}_{\text{int}} = \sum \gamma_k a^\dagger b_k + \text{h.c.}$$

Oscillator with dissipation

In rotating frame,

$$\hat{H}_{\mathrm{eff}} = (\omega_0 - \Omega)a^\dagger a + \frac{\beta}{2}a^\dagger a^\dagger aa + E^* a + Ea^\dagger + \\ + \sum(\omega_k - \Omega)b_k^\dagger b_k + \sum \gamma_k a^\dagger b_k + \mathrm{h.c.}$$

Now the bath contains oscillators with negative energy!

Kinetic equation

According to the golden Fermi rule,

$$\frac{dP(m \leftarrow n)}{dt} \propto |\gamma_k \langle \psi_m | \hat{a} | \psi_n \rangle|^2,$$
$$\omega_k - \Omega = E_n - E_m$$

That allows to write a kinetic equation:

$$\frac{dP_m}{dt} = \sum_n \frac{dP(m \leftarrow n)}{dt} P_n - \frac{dP(n \leftarrow m)}{dt} P_m$$

Matrix elements $\langle \psi_m | \hat{a} | \psi_n \rangle$ can be computed numerically, and $\gamma(E)$ can be defined phenomenologically as $\propto \theta(E)$ or $\propto E$, $E > 0$.

Distributions of occupation numbers

$$\omega_0 = 1,\ \Omega = 1.025,\ \beta = 0.001$$

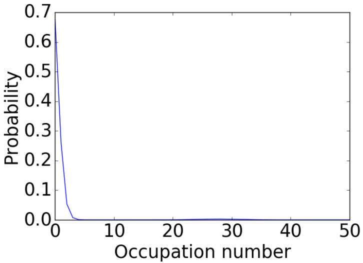


Figure 2: $E = 0.01516$

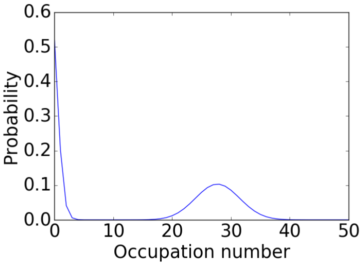


Figure 3: $E = 0.015165$

Distributions of occupation numbers

$$\omega_0 = 1,\ \Omega = 1.025,\ \beta = 0.001$$

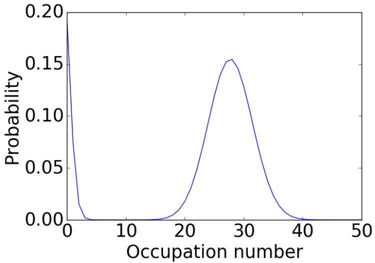


Figure 2: $E = 0.015170$

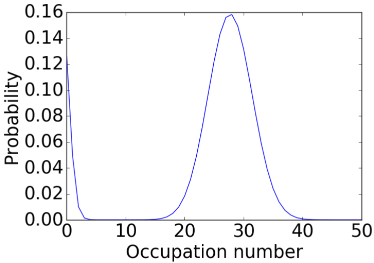


Figure 3: $E = 0.01575$