# Two spins in magnetic field

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## 1 Constant magnetic field

Two spins in magnetic field with exchange interaction are described by a Hamiltonian

$$\hat{H} = J\vec{S}_1 \cdot \vec{S}_2 + \vec{B} \cdot (\vec{S}_1 + \vec{S}_2) \tag{1}$$

First, let us find the eigenvalues and eigenvectors at constant magnetic field. Assume that $\vec{B} = (0, 0, B_z)$. Then the Hamiltonian becomes

$$\hat{H} = \frac{J}{2}(S^2 - S_1^2 - S_2^2) + BS_z, \tag{2}$$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$. It is a standard task to diagonalize the operator of total angular momentum. Here it's obvious that the total spin $s$ can be either 1 or 0.

At $s = 1$:

$$E_{1,m} = \frac{J}{4} + Bm,\ m \in \{1, 0, -1\} \tag{3}$$

$$\begin{gathered} |1,1\rangle = |\uparrow\rangle|\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \\ |1,-1\rangle = |\downarrow\rangle|\downarrow\rangle \end{gathered} \tag{4}$$

At $s = 0$:

$$E_{1,m} = -\frac{3J}{4} \tag{5}$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \tag{6}$$

## 2 Berry phase in rotating magnetic field

Now let us assume that the magnetic field slowly rotates around the $z$ axis: $\vec{B} = (B\sin\theta\cos\Omega t, B\sin\theta\sin\Omega t, B\cos\theta)$. After a $2\pi$ rotation the wavefunction generally acquires a phase. To compute this phase, it is possible to transform the wavefunction as follows:

$$|\psi\rangle = e^{-i\hat{s}_z\Omega t}|\psi'\rangle \tag{7}$$

After that, the dependence on time in Schrodinger equation cancels, but the additional term arises:

$$i\partial_t|\psi'\rangle = \hat{H}_{\text{eff}}|\psi'\rangle = \hat{H}_0|\psi'\rangle - \Omega\hat{s_z}|\psi'\rangle \tag{8}$$

Here $\hat{H}_0 = J\vec{S}_1 \cdot \vec{S}_2 + \vec{B}_0 \cdot (\vec{S}_1 + \vec{S}_2)$, $\vec{B}_0 = (B\sin\theta, 0, B\cos\theta)$.

It's obvious that the phase acquired by the wavefunction is

$$e^{i\beta} = e^{-2\pi i(E-E_0)/\Omega}, \tag{9}$$

where $E$ is the energy corresponding to $\hat{H}_{\text{eff}}$, and $E_0$ — to $\hat{H}_0$.

As $\Omega$ is small, we should treat $\Omega\hat{s_z}$ as a perturbation. In the first order $E - E_0 = \Omega\langle\hat{s_z}\rangle$, where $s_z$ is averaged over the unperturbed eigenfunctions. The latter were found in the previous section for $B \parallel Oz$, but it is easy to modify them for arbitrary $\vec{B}$.

The wavefunctions of the $\frac{1}{2}$ spins corresponding to the different projections on a given axis can be found by diagonalizing the operator $\vec{n} \cdot \vec{s}$:

$$\vec{n} \cdot \vec{s} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix} \tag{10}$$

The answer is below:

$$\begin{gathered} u_\uparrow = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \\ u_\downarrow = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} \end{gathered} \tag{11}$$

Now it is straightforward to compute the phase of any state after a rotation. The system should be initially in the ground state, so let us consider the possible scenarios for ground states. Without loss of generality, $B > 0$ and $\Omega > 0$.

If $J > B$, the ground state will be $|0,0\rangle$. As it is a scalar, $\langle 0,0|s_z|0,0\rangle = 0$. Therefore, **the phase is** $0$.

Otherwise, the ground state will be

$$|1,-1\rangle = u_\downarrow \otimes u_\downarrow \tag{12}$$

A simple calculation is below:

$$\langle 1,-1|s_z|1,-1\rangle = \langle 1,-1|(s_{1z} + s_{2z})|1,-1\rangle = 2\langle u_\downarrow|s_z|u_\downarrow\rangle = -\cos\theta \tag{13}$$

So,

$$\beta = 2\pi\langle s_z\rangle = -2\pi\cos\theta \tag{14}$$

As the phase is defined modulo $2\pi$,

$$\beta = 4\pi\sin^2\frac{\theta}{2} \tag{15}$$