Контрольная работа

Задание №1

$$f(z) = \log\left(\frac{z-1}{z+1}\right)$$

На бесконечности: $\log\left(\frac{z-1}{z+1}\right) \to \log 1 \to 0$ – однозначная т.
(при одном обороте)

$$f(z) = \log(z-1) - \log(z+1) = \left|\log\frac{(z-1)}{(z_0-1)}\right| + i\Delta\arg\varphi_1(z) - \left|\log\left(\frac{z+1}{z_0+1}\right)\right| - i\Delta\arg\varphi_2(z) =$$

$$= \left|\ln\frac{(z-1)}{(z_0-1)}\right| - \left|\ln\left(\frac{z+1}{z_0+1}\right)\right| + f(z_0) + i(\Delta\arg\varphi_1(z) - \Delta\arg\varphi_2(z))$$

Точки $z = \pm 1$ – точки ветвления

Задание №2

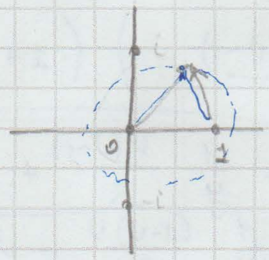
$$f(z) = \sqrt{\log(z+1)} = \sqrt{\omega(z)}$$

$$\varphi(z) = \underbrace{\sqrt{\frac{\omega(z)}{\omega(z_0)}}}_{m(z)}\sqrt{\omega(z_0)}\, e^{i\Delta\arg m(z)} = \sqrt{\left|\frac{\log(z+1)}{\log(z_0+1)}\right|}\sqrt{\log(z_0+1)}\, e$$

$z=0$, $\quad f(z) = \sqrt{\ln 1} = \sqrt{\ln e^{2\pi i n}} = \sqrt{2\pi i n} = \sqrt{2\pi i}\cdot n^{1/2}$

$n \in \mathbb{Z}$, например: $[n=0,\ f(z)=0]$, $[n=1,\ f(z) = \sqrt{2\pi i}]$ и т. д.

Задание №3



$$f(z) = \frac{1}{\sqrt{z}\,(z+1)^2} = \frac{1}{\sqrt{z}}\ \frac{1}{(z+1)^2}$$

Особые точки:

$z=0$: оборот на 2π вокруг данной точки

$\Delta\arg\varphi(z) = e^{-\frac{2\pi i}{2}} e^{-0\cdot 2} = e^{-\pi i} = -1$ – точка ветвления $(\Delta\arg = -\pi i)$

Точки ветвления: $z = 0, -1$

$z = 1$: оборот на 2π вокруг данной точки

$\Delta\arg\varphi(z) = e^{-\frac{0}{2}} e^{-(2\pi i)2} = e^{-4\pi i} = 1$ – точка однозначного характера $(\Delta\arg: -4\pi i)$

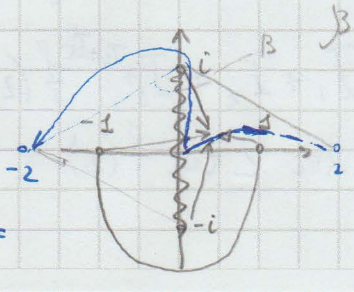
$z = \infty$. вокруг данной точки

$f(z) \to \frac{1}{z^{1/2}}\ \frac{1}{z^2} = \frac{1}{z^{5/2}}$: оборот на 2π

$\Delta\arg\varphi(z) = e^{-2\pi i\cdot\frac{5}{2}} = e^{-5\pi i} = -1$ – точка ветвления $(\Delta\arg: -5\pi i)$

Задание №4

$\beta = \operatorname{arctg}\frac{2}{1}$



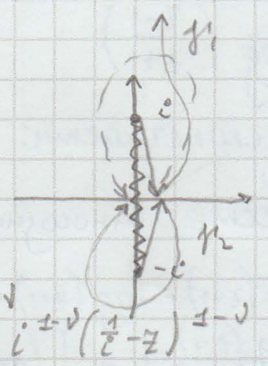
$$f(z) = \sqrt[4]{(z+i)(z-i)(z-1)(z+1)}, \quad f(+0) = 1$$

$$f(z) = \left|\frac{\omega(z)}{\omega(z_0)}\right| e^{i\Delta\arg\omega(z)}\ \omega(z_0)$$

$$f(2) = \left|\frac{\sqrt{(2+i)(2-i)(2-1)(2+1)}}{1}\right| e^{\frac{-\operatorname{arctg}2 + \operatorname{arctg}2 - \pi}{4}i}\cdot 1 =$$

$= \sqrt{15}\, e^{-i\frac{\pi}{4}} = \sqrt{15}\, e^{-i\frac{\pi}{4}}$

$$f(-2) = \left| \frac{\sqrt{(-2-i)(-2+i)(-2+1)(-2-1)}}{1} \right| 1 \cdot e^{\frac{\operatorname{arctg} 2 + 2\pi - \operatorname{arctg} 2 + \pi + 0}{4} i} = \sqrt{15}\, e^{\frac{3\pi i}{4}}$$



Задание №6

$$f(z) = (1+iz)^{\nu}(1-iz)^{1-\nu}, \quad f(+0) = 1$$

$$\gamma_1: f(-0) = \left| \frac{(1-i0)^{\nu}(1+i0)^{1-\nu}}{1} \right| 1\, e^{i(0\cdot(1-\nu)+2\pi\nu)} =$$

$$= e^{i2\pi\nu}$$

$$i^{\nu}\left(\frac{1}{i}+z\right)^{\nu} i^{1-\nu}\left(\frac{1}{i}-z\right)^{1-\nu} =$$

$$\gamma_2: f(-0) = \left| \frac{(1-i0)^{\nu}(1+i0)^{1-\nu}}{1} \right| 1\, e^{i(0\cdot\nu - 2\pi(1-\nu))} = i(-i+z)^{\nu}(-i-z)^{1-\nu} =$$

$$= i(-1)^{1-\nu}(z+i)^{1-\nu}(z-i)^{\nu}$$

$$= e^{-2\pi i + 2\pi i\nu} = e^{2\pi i\nu}$$

$$\operatorname*{res}_{z=\infty} f(z) = -C_{-1}$$

$z = iy,\ y \to \infty$

$$f(z) = \left| \frac{(1+iz)^{\nu}(1-iz)^{1-\nu}}{1} \right| 1\, e^{i(\pi\cdot\nu + 0(1-\nu))} = \left|(1-y)^{\nu}(1+y)^{1-\nu}\right| e^{i\pi\nu} =$$

$$= (y-1)^{\nu}(y+1)^{1-\nu} e^{i\pi\nu} = y^{\nu}\cdot y^{1-\nu}\left(1-\frac{1}{y}\right)^{\nu}\left(1+\frac{1}{y}\right)^{1-\nu} e^{i\pi\nu} = y\left(1 - \left(\frac{1}{y}\right)\cdot\nu + \right.$$

$$\left. + \frac{1}{y^2}\frac{(\nu)(\nu-1)}{2}\dots\right)\left(1 + (1-\nu)\frac{1}{y} + \frac{1}{y^2}\frac{(1-\nu)(1-\nu-1)}{2}\dots\right) =$$

$$= \left(y - \nu + \frac{1}{y}\frac{\nu(\nu-1)}{2} + \dots\right)\left(1 + (1-\nu)\frac{1}{y} - \frac{1}{y^2}\frac{(1-\nu)\nu}{2}\right) e^{i\pi\nu} =$$

$$= \dots - \frac{1}{y}\frac{(1-\nu)\nu}{2} - \nu(1-\nu)\frac{1}{y} + \frac{1}{y}\frac{\nu(\nu-1)}{2} + \dots = -\frac{1}{y}\left(\frac{(1-\nu)\nu}{2} + \nu(1-\nu) + \frac{\nu(1-\nu)}{2}\right) =$$

$$= -\frac{1}{y}\left(2\nu(1-\nu)\right) e^{i\pi\nu}$$

$$C^{y}_{-1} = -2\nu(1-\nu) = 2\nu(\nu-1) e^{i\pi\nu}$$

$$y = \frac{z}{i},\ z = iy \to C_{-1} = 2i\nu(\nu-1)e^{i\pi\nu}$$

$$\operatorname*{res}_{z=\infty} f(z) = -2i\nu(\nu-1)e^{i\pi\nu}$$

Задание №7

$$I = \int_0^{\infty} \frac{\log x}{x^3+1}\,dx = \cancel{\int_0^{\infty} \frac{\ln x}{x^3+1}\,dx}$$

$$\Phi = I_1 + I_2 + I_3 + I_4 + I_5, \quad \cancel{f(z) = \frac{\ln z}{z^3+1}}$$

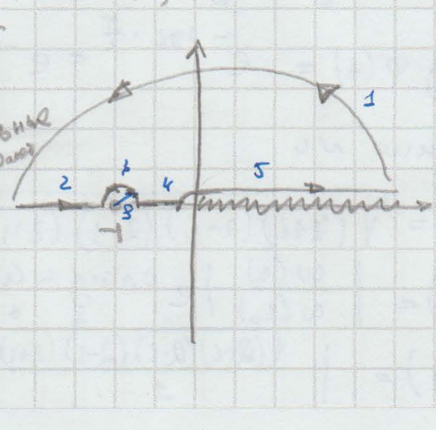
$$\Phi = 2\pi i \sum \operatorname{res} f(z) =$$

$z^3 + 1 = 0,\ z^3 = -1,\ z = e^{i\frac{2\pi + 2\pi n}{3}}$

$z_1 = e^{\frac{i\pi}{3}}$

$z_2 = e^{i\pi}$

z_3 Остальные не попадают в контур



$I_1 \xrightarrow{R\to\infty} 0$

$I_5 \xrightarrow{R\to\infty} I$

~~Сделаем параметризацию:~~

~~$z = re^{i\pi}$, $dz = -dz$~~

~~$\int_{\infty}^{0} \frac{\ln z}{z^3+1} dz = \int_{0}^{\infty} \ln z + i\pi$~~

Задание №7

$I = \int_0^\infty \frac{\ln x dx}{x^3+1}$ (Рассмотрим $I_{всп} = \int_0^\infty \frac{\ln^2 x dx}{x^3+1}$)

$\Phi = I_1 + I_2 + I_{\gamma_\rho} + I_{C_R}$, $x \to z$

$I_1 = \int_\rho^R \frac{\ln^2 z}{z^3+1} dz \xrightarrow[\rho\to 0]{R\to\infty} I_{всп}$

$I_2 = \int_R^\rho \frac{\ln^2 z + 4\pi i \ln z - 4\pi^2}{e^{6\pi i} z^3+1} \xrightarrow[\rho\to 0]{R\to\infty} \int_\infty^0 \frac{\ln^2 z + 4\pi i \ln z - 4\pi^2}{z^3+1} dz = -\int_0^\infty \frac{\ln^2 z dz}{z^3+1} -$

$- 4\pi i \int_0^\infty \frac{\ln z dz}{z^3+1} + 4\pi^2 \int_0^\infty \frac{dz}{z^3+1} = -I_{всп} - 4\pi i I + 4\pi^2 I_0$

$I_{\gamma_\rho} \to 0$ (параметризация вида $z = \rho e^{i\varphi}$, $\varphi \in [0, 2\pi]$, $\rho \to 0$)

$I_{C_R} \to 0$ (параметризация вида $z = Re^{i\varphi}$, $\varphi \in [0, 2\pi]$, $\left|\int_a^b f(z) dz\right| \leq \int_a^b |f(z)| |dz|$)

$\Phi = I_1 + I_2 = \cancel{I_{всп}} - \cancel{I_{всп}} - 4\pi i I + 4\pi^2 I_0 = -4\pi i I + 4\pi^2 I_0$

$\Phi = 2\pi i \sum \operatorname{res} f(z)$

$z^3 + 1 = 0$, $z^3 = -1 = e^{i(\pi + 2\pi n)}$, $z = e^{i\frac{\pi + 2\pi n}{3}}$

$z_1 = e^{i\pi/3}$, $z_2 = e^{i\pi}$, $z_3 = e^{i\frac{5\pi}{3}}$ (далее $z = e^{i\frac{7\pi}{3}} = e^{i\pi/3}$ - повторяется)

$\operatorname{res}_{z = e^{i\frac{\pi}{3}}} f(z) = \left.\frac{\ln z}{(z - e^{i\pi})(z - e^{i\frac{5\pi}{3}})}\right|_{z = e^{i\pi/3}} = \frac{i\pi/3}{\underbrace{(e^{i\pi/3} - e^{i\pi})}_{\frac{1}{2} + i\frac{\sqrt{3}}{2} + 1}\underbrace{(e^{i\pi/3} - e^{i\frac{5\pi}{3}})}_{\frac{1}{2} + i\frac{\sqrt{3}}{2} - (-\frac{1}{2} - i\frac{\sqrt{3}}{2})}}$

$\operatorname{res}_{z = e^{i\pi}} f(z) = \left.\frac{\ln z}{(z - e^{i\pi/3})(z - e^{i\frac{5\pi}{3}})}\right|_{z = e^{i\pi}} = \frac{i\pi}{(e^{i\pi} - e^{i\pi/3})(e^{i\pi} - e^{i\frac{5\pi}{3}})}$

$\operatorname{res}_{z = e^{i\frac{5\pi}{3}}} f(z) = \left.\frac{\ln z}{(z - e^{i\pi/3})(z - e^{i\pi})}\right|_{z = e^{i\frac{5\pi}{3}}} = \frac{i\frac{5\pi}{3}}{(e^{i\frac{5\pi}{3}} - e^{i\frac{\pi}{3}})(e^{i\frac{5\pi}{3}} - e^{i\pi})}$

$e^{\frac{i\pi}{3}} - e^{i\pi} = \frac{1}{2} + i\frac{\sqrt{3}}{2} - (-1) = \frac{3}{2} + i\frac{\sqrt{3}}{2}$

$e^{\frac{i\pi}{3}} - e^{\frac{i5\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2} - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

$e^{i\pi} - e^{i\frac{5\pi}{3}} = 1 - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

$$-2\pi^2 = -2\pi \sin\frac{\pi}{8} I_0 \quad , \quad \pi = \sin\frac{\pi}{8} I_0 \Rightarrow I_0 = \frac{\pi}{\sin\frac{\pi}{8}}$$

$$\sin\frac{\pi}{8} I = \pi \cos\frac{\pi}{8} I_0 = \pi \cos\frac{\pi}{8} \frac{\pi}{\sin \pi/8}$$

$$I = \pi^2 \frac{\cos \pi/8}{\sin^2 \pi/8}$$