Домашняя работа

Задание №1

1) $\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$

$$\|x\|_2 = \sqrt{\sum_i x_i^2}$$

$$\|x\| = \max |x_i|$$

$$\|x\|_2 = \sqrt{\sum_i x_i^2} \le \sqrt{\sum_i \max^2 |x_i|} = \sqrt{m}\,\max|x_i| = \sqrt{m}\,\|x\|_\infty$$

2) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$

$$\|A\|_2^2 = \max_x \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_x \frac{\sum_{j,i}(a_{ij}x_j)^2}{\|x\|_2^2} \ge \max_x \frac{\sum_{ij}(a_{ij}x_j)^2}{(\sqrt{n})^2\|x\|_\infty^2} \ge \max_x \frac{\max_i \sum (a_{ij}x_j)^2}{n\|x\|_\infty} =$$

$$= \max_x \frac{\|Ax\|_\infty^2}{n\,\|x\|_\infty^2} = \frac{1}{n}\|A\|_\infty^2$$

$$\|A\|_2 \ge \frac{1}{\sqrt{n}}\|A\|_\infty$$

Задание №2

a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = A$

$$\underset{m\times n}{A} = \underset{m\times r}{U}\,\underset{r\times r}{\Lambda}\,\underset{r\times n}{V^T}$$

$$AA^T = U\Lambda V^T V \Lambda^T U^T = U\Lambda\Lambda U^T$$

$$AA^T = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \quad \det[AA^T - \lambda E] = 0, \quad (9-\lambda)(4-\lambda) = 0$$

$$\lambda_1 = 9, \quad \lambda_2 = 4 \Rightarrow \sigma_1 = 3, \ \sigma_2 = 2$$

Собственные векторы:

$$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 9\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \vec{a_1} = \begin{bmatrix} a \\ 0 \end{bmatrix}, \quad a = \forall$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 4\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \vec{a_2} = \begin{bmatrix} 0 \\ a' \end{bmatrix}, \quad a' = \forall$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2) $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A_{m\times n} = U_{m\times 2} A_{2\times 2} V^T_{2\times n}$$

$$1) AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 2) A^TA = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$$1) \quad \lambda_{1,2,3} = (4-\lambda)\lambda^2 = 0 \qquad k_{1,2} = \lambda(4-\lambda) = 0$$

$$\lambda_{2,3} = 0, \ \lambda_1 = 4 \qquad k_1 = 4, \ k_2 = 0$$

$$\vec{a_1} = \begin{bmatrix} 0 \\ a' \\ a'' \end{bmatrix}, \quad \vec{a_1} = \vec{a_1'} + \vec{a_2'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{a_2} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$2) \quad \vec{z_1} = \begin{bmatrix} \hat{a} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{z_2} = \begin{bmatrix} 0 \\ \tilde{a} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_{m\times n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = AA^T$$

$$(2-\lambda)^2 - 4 = 0, \quad \cancel{4} - 4\lambda + \lambda^2 - \cancel{4} = 0, \quad \lambda_1 = 0, \ \lambda_2 = 4$$

$$\vec{a_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad \vec{a_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Задание №4

$$f(A) = A^{-1}X(X^TA^{-1}X)^{-1}$$

$$f(X\Omega X^T+\Delta) = (X\Omega X^T+\Delta)^{-1}X\left(X^T(X\Omega X^T+\Delta)^{-1}X\right)^{-1} = \left(\Delta^{-1}-\Delta^{-1}X(\Omega^{-1}+X^T\Delta^{-1}X)^{-1}X^T\Delta^{-1}\right)X\left(X^T(\Delta^{-1}-\Delta^{-1}X(\Omega^{-1}+X^T\Delta^{-1}X)^{-1}X^T\Delta^{-1})X\right)^{-1} =$$

$$=\left(\Delta^{-1}X-\Delta^{-1}X(\Omega^{-1}+X^T\Delta^{-1}X)^{-1}X^T\Delta^{-1}X\right)\left(\underbrace{X^T\Delta^{-1}X}_{S}-X^T\Delta^{-1}X(\Omega^{-1}+X^T\Delta^{-1}X)^{-1}X^T\Delta^{-1}X\right)$$

$$=\left(\Delta^{-1}X-\Delta^{-1}X(\Omega^{-1}+X^T\Delta^{-1}X)^{-1}X^T\Delta^{-1}X\right)\left(S-S(\Omega^{-1}+S)^{-1}S\right)^{-1} =$$

$$=\Delta^{-1}X\left(E-(\Omega^{-1}+\underbrace{X^T\Delta^{-1}X}_{S})^{-1}\underbrace{X^T\Delta^{-1}X}_{S}\right)\left(E-(\Omega^{-1}+S)^{-1}S\right)^{-1}S^{-1} =$$

$$=\Delta^{-1}X\left(\underbrace{E-(\Omega^{-1}+S)^{-1}S}_{M}\right)\underbrace{\left(E-(\Omega^{-1}+S)^{-1}S\right)^{-1}}_{M^{-1}}S^{-1} = \Delta^{-1}XS^{-1} =$$

$$=\Delta^{-1}X(X^T\Delta^{-1}X)^{-1} = f(\Delta)$$