Контрольная работа №1.

Задание №1

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{2021} = -\Big(\underbrace{\frac{1}{2}-\frac{\sqrt{3}}{2}i}_{\cos\varphi+i\sin\varphi}\Big)^{2021} = -\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)^{2021} =$$

$$= -\left(e^{-i\frac{\pi}{3}}\right)^{2021} = -e^{-i\frac{2021}{3}\pi} = -e^{-i\left(673+\frac{2}{3}\right)\pi} = -e^{-i\left(672+\frac{5}{3}\right)\pi} = -\underbrace{e^{-i672\pi}}_{1}\cdot e^{-i\frac{5}{3}\pi} =$$

$$= -e^{-i\frac{5}{3}\pi} = -\left(\cos\left(-\frac{5}{3}\pi\right)+i\sin\left(-\frac{5}{3}\pi\right)\right) = -\left(\cos\frac{5}{3}\pi - i\sin\frac{5}{3}\pi\right) =$$

$$= -\left(-\frac{1}{2}-i\left(-\frac{\sqrt{3}}{2}\right)\right) = -\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \frac{1}{2}-i\frac{\sqrt{3}}{2}$$

Задание №2

$$f(z) = \frac{\sinh^2(\pi z)}{(z^2+1)^4} = \frac{(-i\sin i\pi z)^2}{(z^2+1)^4} = \frac{-1\cdot\sin^2(i\pi z)}{(z^2+1)^4}$$

Точка $z_0=-i$ – полюс четвертого порядка

$$\operatorname{res}\left[f(z),-i\right] = \lim_{z\to z_0}\frac{1}{3!}\frac{d^3}{dz^3}\frac{\sinh^2(\pi z)}{(z-i)^4} = -\frac{i\pi^2}{8}$$

(Получено путем ужасного дифференцирования, заняло минут 5-7)

Задание №3

$$f(z) = \frac{z^3}{\sin^8(2z)}, \quad z=0$$

$$\frac{z^3}{\sin^8(2z)} \approx \frac{z^3}{\left(2z-\frac{8z^3}{6}+\frac{32z^5}{120}\right)^8} = \frac{z^3}{(2z)^8}\frac{1}{\left(1-\frac{4z^2}{6}+\frac{16z^4}{120}\right)^8} = \frac{z^3}{(2z)^8}\frac{1}{\left(1-\frac{2}{3}z^2+\frac{2}{15}z^4\right)^8} =$$

$$= \frac{1}{256z^5}\frac{1}{\Big(1-\underbrace{\left(\frac{2}{3}z^2-\frac{2}{15}z^4\right)}_{\varphi}\Big)^8} = \frac{1}{256z^5}\left(1-8\varphi+\frac{(-8)(-8-1)}{2}\varphi^2\right) =$$

$$= \frac{1}{256z^5}\left(1-8\varphi+36\varphi^2\right) = \frac{1}{256z^5}\left(1-8\left(\frac{2}{3}z^2-\frac{2}{15}z^4\right)+36\left(\frac{2}{3}z^2-\frac{2}{15}z^4\right)^2\right) =$$

$$= \frac{1}{256z^5}-\frac{1}{48z^3}+\underline{\frac{1}{240z}}+\underline{\frac{1}{16}\frac{1}{z}}-\frac{1}{40}z+\frac{1}{400}z^3 = \frac{1}{256z^5}-\frac{1}{48z^3}+$$

$$+\frac{0}{z^2}+\frac{1}{15z}-\frac{1}{40}z+\frac{1}{400}z^3$$

Лидирующие члены: $\frac{1}{256z^5}$ и $-\frac{1}{48z^3}$

Задание №4

$$f(z) = \frac{e^{\cancel{z}/z}\cdot e^{z}}{(z-1)^2}$$

Особые точки, $z_1=0$, $z_2=1$

$z_1=0$ – существенно особая точка, т. к.

$\lim\limits_{z\to 0-} f(z) = 0, \quad \lim\limits_{z\to 0+} f(z) = 1$

$z_2 = 1$ - полюс второго порядка

$\lim\limits_{z\to 1} f(z) = \infty$

Задание №5

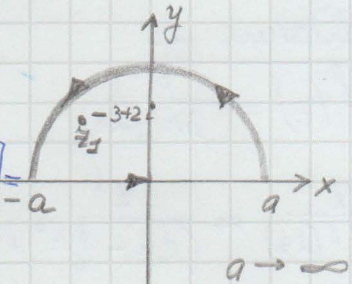
$$\int\limits_{|z|=4} (|z|+2)^3 \bar{z}\, dz = \left| \begin{array}{llll} z = re^{i\varphi}, & r=4, & z=4e^{i\varphi}, & dz = ire^{i\varphi}d\varphi = irz\,d\varphi \\ \bar{z} = re^{-i\varphi}, & r=4, & \bar{z}=4e^{-i\varphi}, & dz = 4iz\,d\varphi \end{array} \right| =$$

$$= \int\limits_0^{2\pi} (r+2)^3 \frac{\bar{z}\,4iz}{1} d\varphi = \int\limits_0^{2\pi} 6^3 \cdot 4i \cdot \underbrace{r^2}_{16} d\varphi = 27648\pi$$

Задание №6

$$J = \int\limits_{\mathbb{R}} \frac{\sin 3x}{(x^2+6x+13)^2} dx = \int\limits_{\mathbb{R}} \frac{\sin 3x}{(x-(\underbrace{-3+2i}_{z_1}))^2 (x - (\underbrace{-3-2i}_{z_2}))^2} dx$$

Сделаем переход $x \to z$ по контуру.



$$\int\limits_{-\infty}^{\infty} \frac{\sin 3x}{(x^2+6x+13)^2} dx = 2\pi \; Re\left[\sum_{Im z_k > 0} res\left(e^{i3z}\frac{1}{(z^2+6z+13)^2}\right)\right] =$$

$$= 2\pi\, Re\left[\lim_{z\to z_1} \frac{d}{dz} \frac{e^{i3z}}{(z-z_2)^2}\right] = 2\pi Re\left[\lim_{z\to z_1} \frac{(3iz - 3iz_2 - 2)e^{3iz}}{(z-z_2)^3}\right] =$$

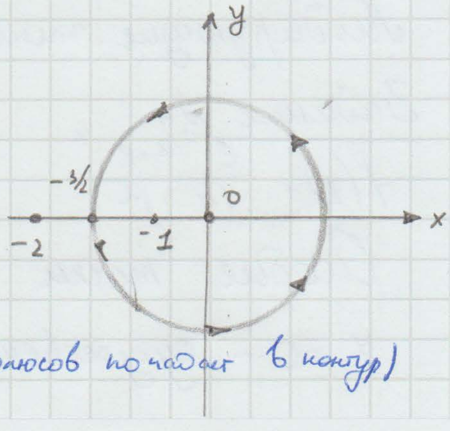
$$= 2\pi Re\left[\frac{(3i(2i-3) - 3i(-2i-3) - 2)e^{3iz_2}}{(2i-\cancel{3}+\cancel{3}+2i)^3}\right] = 2\pi Re\left[\frac{(-6 \cancel{-9i} -6 \cancel{+9i} -2)e^{3iz_1}}{-64i}\right] =$$

$$= 2\pi \cdot \frac{7}{32} Re\left[\frac{e^{3i(2i-3)}}{i}\right] = 2\pi \cdot \frac{7}{32} Re\left[\frac{e^{-6-9i}}{i}\right] = 2\pi \frac{7}{32} e^{-6} Re\left[\frac{e^{-9i}}{i}\right]$$

$$= 2\pi \cdot \frac{7}{32} e^{-6} (-\sin 9) = -\frac{7\pi e^{-6}}{16} \sin 9$$

Задание №8

$$\int\limits_{|z|=\frac{3}{2}} \frac{z^2 e^{1/z}}{(z+1)(z+2)} dz = J$$



$z_1 = 0$ - существенно особая точка.

$z_2 = -1$ - полюс первого порядка (только он у двух полюсов попадает в контур)

$\lim\limits_{z\to -1} = \infty$

$f(z) = \frac{z^2 e^{1/z}}{(z+1)(z+2)}$, $z=0$ $\qquad \frac{1}{2}\left(z^2 + z + \frac{1}{2} + \frac{1}{6z} + \dots\right)\left(1 - z + z^2 - z^3 \dots\right)\left(1 - \frac{z}{2} + \frac{z^2}{4} - \dots\right)$

$$\frac{z^2 e^{1/z}}{2(z+1)\left(\frac{z}{2}+1\right)} = \frac{1}{2} z^2 \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots\right)\left(1 - z + z^2 - z^3\right)\dots\left(1 - \frac{z}{2} + \frac{z^2}{4} - \dots\right)$$

$$\lim_{z\to+0} \frac{\overset{0\to}{z^2}\, e^{\overset{1/z\nearrow\infty}{}}}{(z+1)(z+2)} = \infty, \quad \lim_{z\to-0} \frac{z^2 e^{1/z}}{(z+1)(z+2)} = \left| k=-z \right| = \lim_{k\to+0} \frac{k^2 e^{-\frac{1}{k}}}{(1-k)(2-k)} = \lim_{k\to+0} \frac{k^2\cdot 1}{(1-k)(2-k)e^{1/k}} \overset{\to 0}{}$$

$$\operatorname{res}(f(z), -1) = \lim_{z\to-1} \frac{z^2 e^{1/z}}{z+2} = e^{-1} = \frac{1}{e}$$

$$\operatorname{res}(f(z), 0) = -C_{-1} \qquad \sum_k \operatorname{res} f(z_k) + \operatorname{res}_{z=\infty} f(z) = 0$$

(попавших в контур)

$$J = 2\pi i \sum_{z_k} \operatorname{res} f(z_k)$$

Задание № 7

$$J = \int_0^{\infty} \underbrace{\frac{3\cos 2x + 2\cos 3x - 5}{x^2}}_{\substack{f(x)\text{ — четная} \Rightarrow \\ \Rightarrow \text{ можем продлить} \\ \text{на всю ось}}} dx = \frac{1}{2} \underbrace{\int_{\mathbb{R}} \frac{3\cos 2x + 2\cos 3x - 5}{x^2} dx}_{J_0}$$

$$J_0 = \int_{\mathbb{R}} \left(\frac{3\cos 2x}{x^2} + \frac{2\cos 3x}{x^2} - \frac{5}{x^2} \right) dx \qquad \int_{\mathbb{R}} \frac{\sin u}{u} du = \pi$$

интеграл Дирихле

$$J_1 = \int_{\mathbb{R}} \frac{3\cos 2x}{x^2} dx = -\frac{3\cos 2x}{x} - \underbrace{\int_{\mathbb{R}} \frac{6\sin 2x}{x} dx}_{-6\pi}$$

$$J_3 = -\int_{\mathbb{R}} \frac{5}{x^2} dx = \frac{5}{x} \Big|_{-\infty}^{+\infty} = 0$$

интеграл Дирихле

$$J_2 = \int_{\mathbb{R}} \frac{2\cos 3x}{x^2} dx = -\underbrace{\cancel{\frac{2\cos 3x}{x}}}_{0} \Big|_{-\infty}^{+\infty} - \underbrace{\int_{\mathbb{R}} \frac{6\sin 3x}{x} dx}_{-6\pi}$$

$$J = \frac{1}{2} J_0 = \frac{1}{2}(-12\pi) = -6\pi$$