Self-check problems on Quantum information processing

Part 2: Measurements and evolution

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Problem 1

Consider a qubit, called "system", in some pure state $|\psi_0\rangle$, and another qubit, called "probe", in the initial state $|0\rangle$. Let the system and the probe interact according to a Hamiltonian

$$V = \frac{1}{2}\sigma_z \otimes \sigma_y \tag{1}$$

for time period t (the first tensor factor corresponds to the first qubit, the second factor – to the probe). Let then the probe be measured in $\{|+\rangle, |-\rangle\}$ basis (remember, that it corresponds to projective $\sigma_x$measurement).

1. Find evolution of the initial state for cases $|\psi_0\rangle = |0\rangle$ and $|\psi_0\rangle = |1\rangle$. Draw pictures of evolution of Bloch vectors for the system and the probe for the both cases.

2. Consider the general case of $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$. Find the reduced state of the system and probe. How do the Bloch vectors of the system and the probe evolve in this general case? What we can say about entanglement between the system and the probe? (Remember, that entanglement of a pure bipartite state is characterized by mixedness of its parties.)

3. Consider the case $|\psi_0\rangle = |0\rangle$ and find probabilities of obtaining outcomes $+1$ and -1 in the σ_x-measurement of the probe as function of time t. What about the case $|\psi_0\rangle = |1\rangle$?

4. Let's consider the general case $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ again. Let at time $t = \frac{\pi}{2}$ the probe be measured in the state $|+\rangle$ (with a corresponding outcome $+1$). What will be the resulting "collapsed" state of the system in this case? Consider the same story but with $t = \frac{\pi}{4}$. How the collapsed state changed?

5. Consider the case $|\psi_0\rangle = |+\rangle$ and $t = \frac{\pi}{2}$. Let the system be given to Alice, and the probe – to Bob. Remember what is the state of Alice's particle (you have calculated it already). Let Bob measure σ_x of the probe, but keep the result of his measurement in a secret from Alice. What is an "effective" state of Alice's particle (the system) without this information? Let then Bob phone Alice and tell that he obtained $+1$ outcome in his measurement (we assume that Bot is honest). What is the state of Alice's particle now?

6. Think a bit about the statement "Information is physical" :-)

7. Find POVM realized on the system by projective measurement of the probe for $t = \frac{\pi}{2}$, $t = \frac{\pi}{4}$.

Problem 2

Consider a qubit POVM

$$M = \left\{ \frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}\mathbf{1} \right\} \tag{2}$$

(here $\mathbf{1}$ is a 2×2 identity matrix).

1. Check that M is a valid POVM.
2. Design a projective measurement, which corresponds to the given POVM, in an extended space obtained by considering qubit states as states of some 4-level system.
3. Design a projective measurement, which corresponds to the given POVM, on a two qubit state $\rho \otimes |0\rangle\langle 0|$ (here ρ is a states measured with POVM).

Problem 3

Consider a realistic single-photon detector. Let the input state coming to detector live in the two-dimensional Hilbert space spanned by vectors $|0\rangle$ (no photon) and $|1\rangle$ (one photon). Our detector is characterized by two parameters: efficiency $\eta \in [0, 1]$ (probability that incoming photon will be observed by the detector), and dark-count probability $p_{\mathrm{dark}} \in [0, 1]$ (probability that in a given time-window the detector will "click" regardless presence of photon in the input channel). Write down POVM elements corresponding to outcomes "click" and "no click" of the detector. How the POVM will change if we extend the space of input states to higher photon numbers ($|2\rangle, |3\rangle, \ldots$)?

Problem 4

Design a construction of SIC-POVM elements for arbitrary d-dimensional Hilbert space ($d > 2$).

Problem 5

Consider a three-qubit system where the first qubit is prepared in an arbitrary state ρ and two other qubits in the fixed state $|00\rangle_{2,3}$. Let the initial state of qubits be affected by unitary operator

$$U = |000\rangle_{1,2,3}\langle 000| + |111\rangle_{1,2,3}\langle 100| + \ldots. \tag{3}$$

Find Kraus operators of a map $\Phi_{1\to 3}[\cdot]$ which outputs the resulting state of the third qubit depending on the initial state of the first qubit ρ. Find Kraus operators of a dual map $\Phi_{1\to(1,2)}[\cdot]$ which outputs the resulting state of the first and the second qubits depending on the initial state of the first qubit ρ. Check that the obtained operators satisfy normalization condition.

Problem 6

Let $\Phi[\rho] = \sum_i A_i \rho A_i^\dagger$ be a CPTP map. Let $u_{i,j}$ be elements of some unitary matrix U, and $B_i = \sum_i u_{ij} A_j$. Show that $\Phi[\rho] = \sum_i B_i \rho B_i^\dagger$.

Problem 7

Consider a CPTP map $\Phi[\cdot]$ and a corresponding Choi state

$$\rho_{\text{Choi}} = \sum_{i,j} |i\rangle_1\langle j| \otimes \Phi[|i\rangle_2\langle j|] \tag{4}$$

(here $\{|i\rangle\}$ is a orthonormal basis for the Hilbert space of inputs). Show that the action of $\Phi[\cdot]$ on some input state ρ is given by

$$\Phi[\rho] = \text{Tr}_1(\rho \otimes \mathbf{1}\rho_{\text{Choi}}^{\text{T}_1}) = \text{Tr}_1(\rho^{\text{T}_1} \otimes \mathbf{1}\rho_{\text{Choi}}), \tag{5}$$

where T_1 stands for a partial transpose in "the first space".

Problem 8

Consider a single-qubit Pauli transformation

$$\rho \to \sigma_\alpha \rho \sigma_\alpha, \quad \alpha \in \{x, y, z\}. \tag{6}$$

How this transformation affects the Bloch vector of ρ?

Show that

$$\frac{1}{4}\left(\mathbf{1}\rho\mathbf{1} + \sum_{\alpha\in\{x,y,z\}} \sigma_\alpha \rho \sigma_\alpha\right) = \frac{\mathbf{1}}{2} \tag{7}$$

for any ρ.

Hint: decompose ρ in sum of terms with Pauli matrices.

Problem 9

Write Choi matrices for dephasing, depolarizing, and damping channels.