Self-check problems on Quantum information processing

Part 1: Basics of Quantum Mechanics

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Problem 1

Let $\mathcal{H}$ be a d-dimensional Hilbert space. Denote it computational basis states as

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad |d-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{1}$$

Let

$$|\psi\rangle = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{d-1} \end{bmatrix}, \quad M = \begin{bmatrix} m_{0,0} & m_{0,1} & \ldots & m_{0,d-1} \\ m_{1,0} & m_{1,1} & \ldots & m_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{d-1,0} & m_{d-1,1} & \ldots & m_{d-1,d-1} \end{bmatrix}. \tag{2}$$

be some matrix.

Check the following facts.

1. $\langle i|j\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker symbol.
2. $\sum_{i=0}^{d-1} |i\rangle\langle i| = \mathbf{1}$, where $\mathbf{1}$ is the identity matrix.
3. $C_i = \langle i|\psi\rangle$.
4. $|\psi\rangle = \sum_{i=0}^{d-1} C_i|i\rangle$.
5. $M_{i,j} = \langle i|M|j\rangle$.
6. $M = \sum_{i,j} M_{i,j}|i\rangle\langle j|$.
7. $M|\psi\rangle = \sum_{k,l=0}^{d-1} M_{k,l}C_l|k\rangle$.

Problem 2

Consider standard Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Find their eigenvalues and eigenvectors. Check that each matrix can be represented in the form $\sum_{i=0}^{1} \lambda_i|\psi_i\rangle\langle\psi_i|$, where $\{\lambda_i\}$ and $\{|\psi_i\rangle\}$ are eigenvalues and eigenvectors correspondingly. Pay special attention to non-uniqueness of $\mathbf{1}$ spectral decomposition. Show the eigenvectors on the Bloch sphere.

Problem 3

Let $\{|\psi_i\rangle\}_{i=1}^d$ be a set of orthonormal states ($\langle\psi_i|\psi_j\rangle = \delta_{i,j}$). Check that

$$U = \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & \dots & |\psi_d\rangle \end{bmatrix} \tag{4}$$

is a unitary matrix. Check that a spectral decomposition $M = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i|$ can be written in the form

$$M = UDU^\dagger, \tag{5}$$

where D is diagonal matrix with $D_{i,i} = \lambda_i$

Problem 4

Let $f(\xi) : \mathbb{R} \to \mathbb{R}$ be some function. Here we would like to extend f on Hermitian matrices. Consider a Hermitian matrix M with a spectral decomposition $M = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$. Let

$$f(\xi) = \sum_{i=0}^{\infty} C_i \xi^i \tag{6}$$

be a Taylor decomposition of f. Check that

$$f(M) := \sum_{i=0}^{\infty} C_i M^i = \sum_k f(\lambda_k)|\psi_k\rangle\langle\psi_k|. \tag{7}$$

Problem 5

Check that $|\psi(t)\rangle = U(t)|\psi_0\rangle = \exp(-\mathrm{i}Ht)|\psi_0\rangle$ is solution of the Schroedinger equation

$$\mathrm{i}\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{8}$$

with initial condition $|\psi(0)\rangle = |\psi_0\rangle$.

Problem 6

Consider a Hamiltonian

$$H = \frac{\hbar\omega}{2}\sigma_x \tag{9}$$

and an initial state

$$|\psi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{10}$$

1. Find an evolution operator $U(t)$.
2. Find an evolution of initial state in the Schroedinger picture $|\psi(t)\rangle$. Show it on the Bloch sphere.
3. Find an evolution of mean value of σ_z.
4. Find an evolution of σ_z in the Heisenberg picture $\sigma_z^H(t)$. Express it in terms of other Pauli matrices.
5. Check that both pictures provide the same mean value of σ_z.

Problem 7

Check that making a Hamiltonian transformation

$$H \to H + \alpha \mathbf{I}, \tag{11}$$

where α is real and $\mathbf{I}$ is identity matrix of corresponding dimension, results in acquiring additional phase for the evolution operator. Show that this phase does not affect the result of evolution of a density matrix.

Problem 8

Let $\{|n\rangle_A\}_{n=0}^{d_A-1}$ and $\{|m\rangle_B\}_{m=0}^{d_B-1}$ be computational bases of finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ (d_A and d_B are corresponding dimensions). Write an expression for position of unit element inside the vector $|i\rangle_A \otimes |j\rangle_B$ in terms of i and j. Check applicability your result for two-qubit case considered in the lectures.

Problem 9

Consider a two qubit state with density matrix

$$\rho_{AB} = \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}, \tag{12}$$

where

$$X = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}, \quad Y = \begin{bmatrix} \rho_{02} & \rho_{03} \\ \rho_{12} & \rho_{13} \end{bmatrix}, \quad Z = \begin{bmatrix} \rho_{20} & \rho_{21} \\ \rho_{30} & \rho_{31} \end{bmatrix}, \quad W = \begin{bmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} \end{bmatrix}. \tag{13}$$

Show that

$$\rho_A = \mathrm{Tr}_B \rho_{AB} = \begin{bmatrix} \mathrm{Tr}X & \mathrm{Tr}Y \\ \mathrm{Tr}Z & \mathrm{Tr}W \end{bmatrix}, \quad \rho_B = \mathrm{Tr}_A \rho_{AB} = X + W. \tag{14}$$

Problem 9

Consider three-qubit W-state[1]

$$|\mathrm{W}\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle_{ABC} + |010\rangle_{ABC} + |001\rangle_{ABC}). \tag{15}$$

Write its Schmidt decomposition with respect to bipartite partitioning $A : BC$ (consider qubits B and C as single object). What is a Schmidt rank of the resulting decomposition? Show that A is not maximally entangled with BC. Design a three-qubit state $|\Psi\rangle_{ABC}$ where the first qubit A is maximally entangled with a pair BC.

[1] Note that this kind of states appears in Rydberg blockade.

Problem 10

Consider the following two-qubit states:

$$\rho_1 = \frac{\mathbf{1}}{2} \otimes \frac{\mathbf{1}}{2}, \quad \rho_2 = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \quad \rho_3 = |\Phi^+\rangle\langle\Phi^+|, \tag{16}$$

where $|\Phi^+\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$. Check that reduced state of both qubits coincide for all three states. Find mean values of observables

$$\sigma_z^A \equiv \sigma_z \otimes \mathbf{1}, \quad \sigma_z^B \equiv \mathbf{1} \otimes \sigma_z, \quad \sigma_z \otimes \sigma_z, \quad \sigma_x \otimes \sigma_x, \tag{17}$$

for each state ρ_i.

Problem 11

Find purifications of all three states (16). Do the state of ancillary purifying system ("environment") correlate with the purified state?