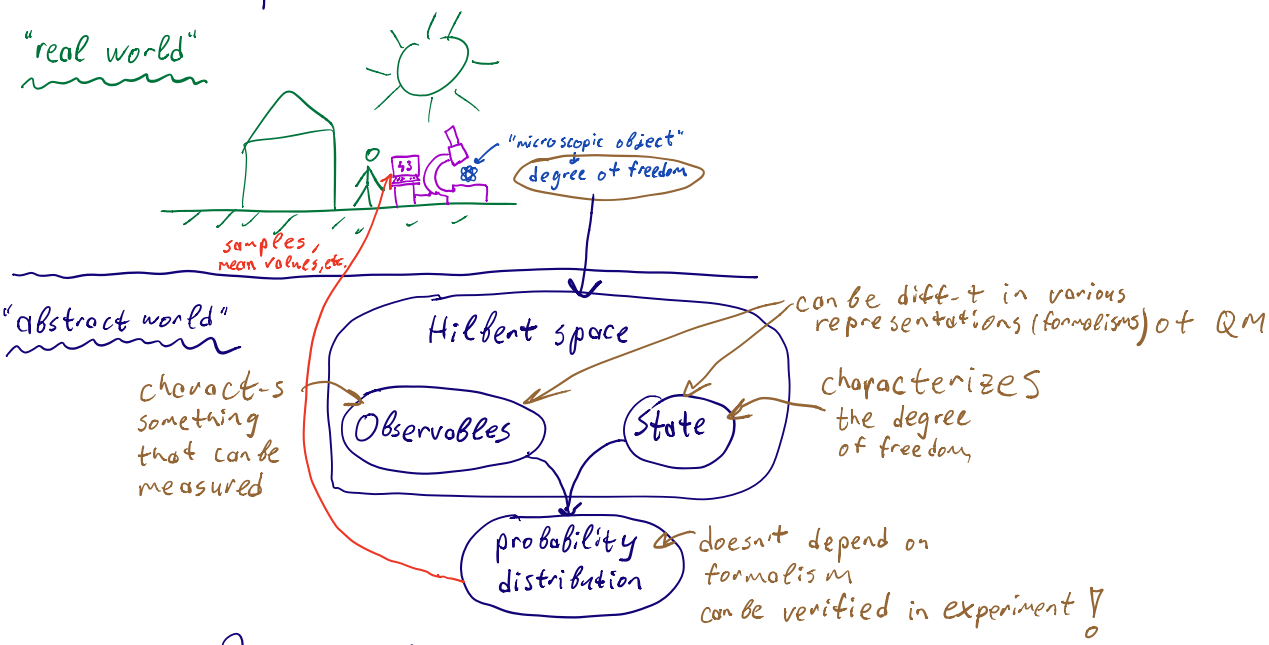
1. Basics of QM

1.1. General picture



1.2. States & observables

1.2.1. Hilbert space

In QM Hilbert space $\mathcal{H}$ is complex inner product space.

↳ $\mathcal{H}$ is vector (linear) space over $\mathbb{C}$

↳ abelian group with respect to $+$

↳ $\exists\, \vec{0}: \ \vec{a}+\vec{0}=\vec{a}$

↳ $\exists\, (-\vec{a}): \ \vec{a}+(-\vec{a})=\vec{0}$

↳ $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

↳ $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

↳ One can multiply vectors on complex scalars s.t.:

↳ $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$

↳ $1\cdot\vec{a}=\vec{a}$

↳ $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$

↳ $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$

↳ There is a scalar product $\langle \vec{a}, \vec{b} \rangle \in \mathbb{C}$

↳ $\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle^*$

↳ $\langle \vec{a}, \alpha\vec{b} + \beta\vec{c} \rangle = \alpha\langle \vec{a}, \vec{b} \rangle + \beta\langle \vec{a}, \vec{c} \rangle$

$\langle \alpha\vec{a} + \beta\vec{b}, \vec{c} \rangle = \alpha^*\langle \vec{a}, \vec{c} \rangle + \beta^*\langle \vec{b}, \vec{c} \rangle$ ← anti linear

↳ $\langle \vec{x}, \vec{x} \rangle \geq 0 \quad \langle \vec{x}, \vec{x} \rangle = 0 \Leftrightarrow \vec{x} = \vec{0}$

↳ $\mathcal{H}$ is complete with respect to $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

if $\{\forall \varepsilon > 0 : \exists N > 0 : \forall n, m > N : |\vec{a}_n - \vec{a}_m| < \varepsilon\}$ then $\lim\limits_{n \to \infty} \vec{a}_n = \vec{a}^* \in \mathcal{H}$.

In our course usually $\mathcal{H} = \mathbb{C}^d$, $d < \infty$

$\vec{a} \to |a\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix}$ – "ket" vector

$\mathcal{H}^*$ ($\mathcal{H}^+$) – linear forms on $\mathcal{H}$. $\mathcal{H} \simeq \mathcal{H}^*$.

$|a\rangle \Leftrightarrow \langle a| = |a\rangle^+ = (a^T)^* = [c_1^* \dots c_d^*]$ – "bra"-vector.

$|b\rangle = \begin{bmatrix} D_1 \\ \vdots \\ D_d \end{bmatrix} \qquad \langle a, b \rangle = \langle a| \cdot |b\rangle \equiv \langle a|b\rangle = \sum\limits_i c_i^* D_i$

1.1.2. Pure states.

Pure state = ray of normalized vectors in $\mathcal{H}$.

$|\psi\rangle \in \mathcal{H} : \langle \psi|\psi\rangle = 1 \to e^{i\varphi}|\psi\rangle$

Warning: $|\psi\rangle$ can not be measured!

density matrix

But $|\psi\rangle\langle\psi|$ can.

<u>Example</u>. $\mathcal{H} = \mathbb{C}^2$ – qubit

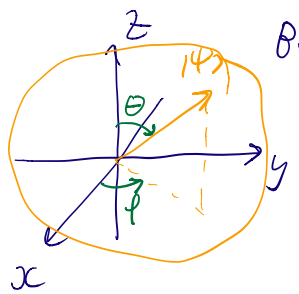
$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|\psi\rangle = e^{i\varphi_1}\cos\frac{\theta}{2}|0\rangle + e^{i\varphi_2}\sin\frac{\theta}{2}|1\rangle \to \cancel{e^{i\varphi_1}}\underbrace{\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)}_{\text{standard param. of pure qubit states}}$, $\varphi = \varphi_2 - \varphi_1$

$$|\psi\rangle\langle\psi| = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\varphi} \end{bmatrix} = \begin{bmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta e^{-i\varphi} \\ \frac{1}{2}\cos\theta e^{i\varphi} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

Note: $|\psi\rangle\langle\psi| \to |\psi\rangle\times\langle\psi|$

not $\times$, but $\rangle\langle$



Bloch sphere

each point on the Bloch sphere = pure state of qubit

1.1.3. Observables (von Neumann measurement)

Observable = Hermitian operator in $\mathcal{H}$

$\cancel{O} = \cancel{O}^{+} \iff \langle\varphi|O|\psi\rangle = \langle\psi|O|\varphi\rangle^{*}$

$O|\psi_i\rangle = \lambda_i|\psi_i\rangle$ one can set $\langle\psi_i|\psi_i\rangle = 1$

eig. vec. eig. val.

↳ $\lambda_i \in \mathbb{R}$

↳ $\langle\psi_i|\psi_j\rangle = \delta_{ij}$

↳ $\{|\psi_i\rangle\}_{i=1}^{d}$ forms a full basis in $\mathcal{H}$

↳ $O = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$ – spectral decomposition

$O = \sum_{i=1}^{M} \tilde{\lambda}_i \Pi_i$ projector $\Pi_i \Pi_j = \delta_{ij}\Pi_i$

$\tilde{\lambda}_i \neq \tilde{\lambda}_j$ $(|\cdot\rangle\langle\cdot| + |\cdot\rangle\langle\cdot| + \ldots)$

von Neumann measurement

$$|\psi\rangle \to \boxed{O} \to |\tilde{\psi}_i\rangle \equiv \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}$$

"collapsed state"

normalization coef. to have $\langle\tilde{\psi}_i|\tilde{\psi}_i\rangle = 1$

"Observed" value $\tilde{\lambda}_i$ with prob

Born rule → $p_i \equiv \langle\psi|\Pi_i|\psi\rangle = tr(|\psi\rangle\langle\psi|\Pi_i)$

if $\Pi_i = |\psi_i\rangle\langle\psi_i|$, then $p_i = \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle = |\langle\psi_i|\psi\rangle|^2$

Mean value $\langle O\rangle = \sum_i p_i \tilde{\lambda}_i = \sum_i \langle\psi|\Pi_i|\psi\rangle \cdot \tilde{\lambda}_i = \langle\psi|\sum_i \tilde{\lambda}_i \Pi_i|\psi\rangle = \langle\psi|O|\psi\rangle$

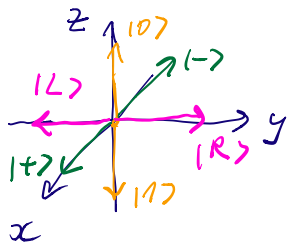
<u>Example</u>

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$; $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\sigma_x = |+\rangle\langle +| - |-\rangle\langle -|$, $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$

$\sigma_y = |R\rangle\langle R| - |L\rangle\langle L|$, $\underset{L}{|R\rangle} = \frac{1}{\sqrt{2}}\left(|0\rangle \underset{-}{+} i\,|1\rangle\right)$



$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle \longrightarrow \boxed{\sigma_x}$

$\to +1$, $|+\rangle$ with prob. $|\langle +|\psi\rangle|^2 = \frac{1}{2}(1+\sin\theta)$

$\to -1$, $|-\rangle$ with prob. $|\langle -|\psi\rangle|^2 = \frac{1}{2}(1-\sin\theta)$

1.1.4. Mixed states