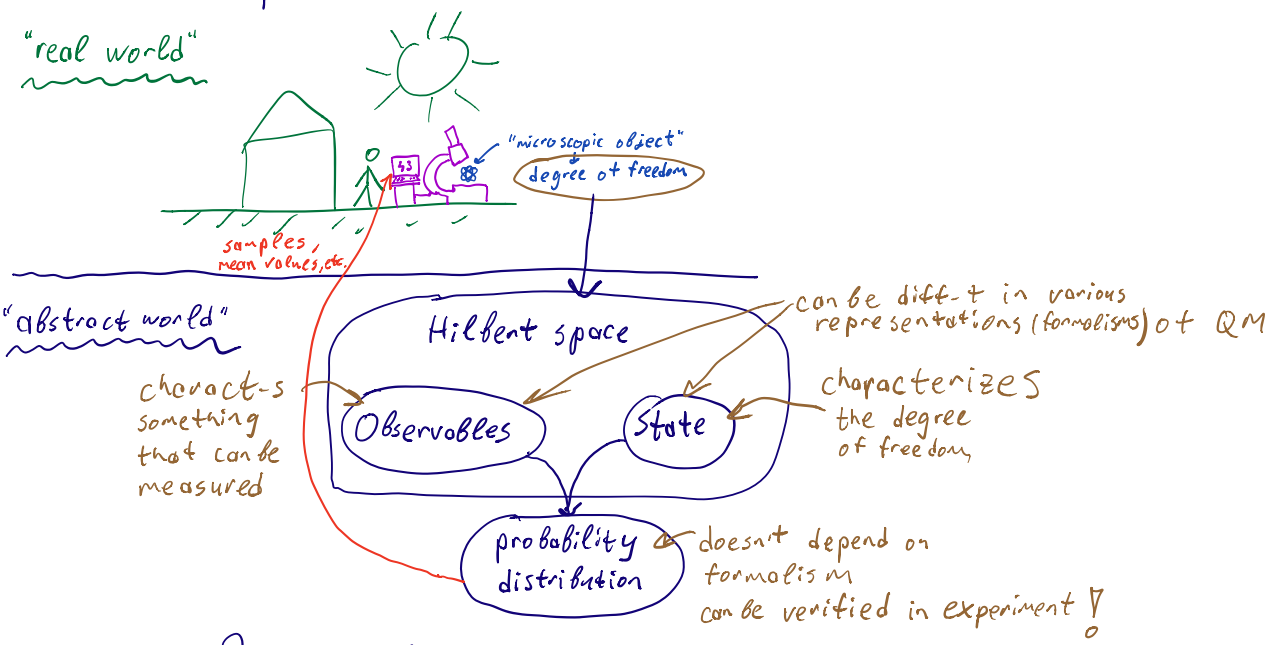
1. Basics of QM

1.1. General picture



1.2. States & observables

1.2.1. Hilbert space

In QM Hilbert space $\mathcal{H}$ is complex inner product space.

↳ $\mathcal{H}$ is vector (linear) space over $\mathbb{C}$

↳ abelian group with respect to $+$

↳ $\exists\, \vec{0}: \ \vec{a}+\vec{0}=\vec{a}$

↳ $\exists\, (-\vec{a}): \ \vec{a}+(-\vec{a})=\vec{0}$

↳ $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

↳ $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

↳ One can multiply vectors on complex scalars s.t.:

↳ $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$

↳ $1\cdot\vec{a}=\vec{a}$

↳ $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$

↳ $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$

↳ There is a scalar product $\langle \vec{a}, \vec{b} \rangle \in \mathbb{C}$

↳ $\langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{a} \rangle^*$

↳ $\langle \vec{a}, \alpha\vec{b} + \beta\vec{c} \rangle = \alpha\langle \vec{a}, \vec{b} \rangle + \beta\langle \vec{a}, \vec{c} \rangle$

$\langle \alpha\vec{a} + \beta\vec{b}, \vec{c} \rangle = \alpha^*\langle \vec{a}, \vec{c} \rangle + \beta^*\langle \vec{b}, \vec{c} \rangle$ ← anti linear

↳ $\langle \vec{x}, \vec{x} \rangle \geq 0 \quad \langle \vec{x}, \vec{x} \rangle = 0 \Leftrightarrow \vec{x} = \vec{0}$

↳ $\mathcal{H}$ is complete with respect to $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

if $\{\forall \varepsilon > 0 : \exists N > 0 : \forall n, m > N : |\vec{a}_n - \vec{a}_m| < \varepsilon\}$ then $\lim\limits_{n \to \infty} \vec{a}_n = \vec{a}^* \in \mathcal{H}$.

In our course usually $\mathcal{H} = \mathbb{C}^d$, $d < \infty$

$\vec{a} \to |a\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix}$ – "ket" vector

$\mathcal{H}^*$ ($\mathcal{H}^+$) – linear forms on $\mathcal{H}$. $\mathcal{H} \simeq \mathcal{H}^*$.

$|a\rangle \Leftrightarrow \langle a| = |a\rangle^+ = (a^T)^* = [c_1^* \dots c_d^*]$ – "bra"-vector.

$|b\rangle = \begin{bmatrix} D_1 \\ \vdots \\ D_d \end{bmatrix} \qquad \langle a, b \rangle = \langle a| \cdot |b\rangle \equiv \langle a|b\rangle = \sum\limits_i c_i^* D_i$

1.2.2. Pure states.

Pure state = ray of normalized vectors in $\mathcal{H}$.

$|\psi\rangle \in \mathcal{H} : \langle \psi|\psi\rangle = 1 \to e^{i\varphi}|\psi\rangle$

Warning: $|\psi\rangle$ can not be measured!

density matrix

But $|\psi\rangle\langle\psi|$ can.

<u>Example</u>. $\mathcal{H} = \mathbb{C}^2$ – qubit

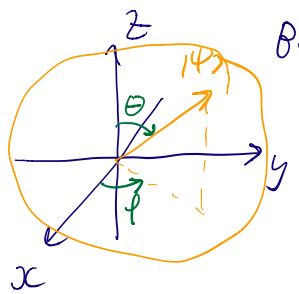
$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|\psi\rangle = e^{i\varphi_1}\cos\frac{\theta}{2}|0\rangle + e^{i\varphi_2}\sin\frac{\theta}{2}|1\rangle \to \cancel{e^{i\varphi_1}}\underbrace{\left(\cos\frac{\theta}{2}|0\rangle + e^{i\overset{=\varphi_2-\varphi_1}{\varphi}}\sin\frac{\theta}{2}|1\rangle\right)}_{\text{standard param. of pure qubit states}}$;

$$|\psi\rangle\langle\psi| = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\varphi} \end{bmatrix} = \begin{bmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta e^{-i\varphi} \\ \frac{1}{2}\cos\theta e^{i\varphi} & \sin^2\frac{\theta}{2} \end{bmatrix}$$

Note: $|\psi\rangle\langle\psi| \to |\psi\rangle\times\langle\psi|$

not $\times$, but $\rangle\langle$



Bloch sphere

each point on the Bloch sphere = pure state of qubit

1.2.3. Observables (von Neumann measurement)

Observable = Hermitian operator in $\mathcal{H}$

$\cancel{O} = \cancel{O}^{+} \iff \langle\varphi|O|\psi\rangle = \langle\psi|O|\varphi\rangle^{*}$

$O|\psi_i\rangle = \lambda_i|\psi_i\rangle$ one can set $\langle\psi_i|\psi_i\rangle = 1$

eig. vec. eig. val.

↳ $\lambda_i \in \mathbb{R}$

↳ $\langle\psi_i|\psi_j\rangle = \delta_{ij}$

↳ $\{|\psi_i\rangle\}_{i=1}^{d}$ forms a complete basis in $\mathcal{H}$

↳ $O = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$ – spectral decomposition

$O = \sum_{i=1}^{M} \tilde{\lambda}_i \Pi_i$ projector $\Pi_i \Pi_j = \delta_{ij}\Pi_i$

$\tilde{\lambda}_i \neq \tilde{\lambda}_j$ $(|\cdot\rangle\langle\cdot| + |\cdot\rangle\langle\cdot| + \ldots)$

von Neumann measurement

$$|\psi\rangle \to \boxed{O} \to |\tilde{\psi}_i\rangle \equiv \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}$$

"collapsed state"

normalization coef. to have $\langle\tilde{\psi}_i|\tilde{\psi}_i\rangle = 1$

"Observed value" $\tilde{\lambda}_i$ with prob

Born rule → $p_i \equiv \langle\psi|\Pi_i|\psi\rangle = tr(|\psi\rangle\langle\psi|\Pi_i)$

if $\Pi_i = |\psi_i\rangle\langle\psi_i|$, then $p_i = \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle = |\langle\psi_i|\psi\rangle|^2$

Mean value $\langle O\rangle = \sum_i p_i \tilde{\lambda}_i = \sum_i \langle\psi|\Pi_i|\psi\rangle \cdot \tilde{\lambda}_i = \langle\psi|\sum_i \tilde{\lambda}_i \Pi_i|\psi\rangle = \langle\psi|O|\psi\rangle$

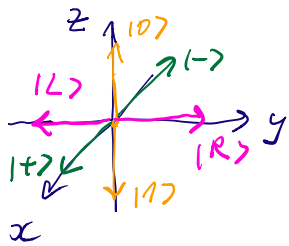
Example

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sigma_x = |+\rangle\langle +| - |-\rangle\langle -|, \quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

$$\sigma_y = |R\rangle\langle R| - |L\rangle\langle L|, \quad |\underset{L}{R}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \underset{-}{+} i\,|1\rangle\right)$$



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle \longrightarrow \boxed{\sigma_x}$$

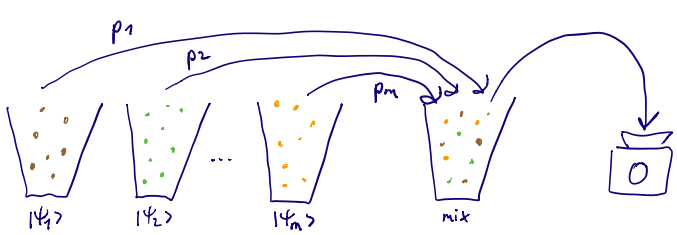
$\to +1$, $|+\rangle$ with prob. $|\langle +|\psi\rangle|^2 = \frac{1}{2}(1+\sin\theta)$

$\to -1$, $|-\rangle$ with prob. $|\langle -|\psi\rangle|^2 = \frac{1}{2}(1-\sin\theta)$

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle = (+1)\times\frac{1}{2}(1+\sin\theta) + (-1)\times\frac{1}{2}(1-\sin\theta) = \sin\theta$$

$(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$ – coordinates of Bloch vec. of $|\psi\rangle$.

1.2.4. Mixed states



$O = \ldots + \lambda \cdot \Pi + \ldots$

$$\mathbb{E}(\lambda) = \sum_i p_i \cdot \langle\psi_i|\Pi|\psi_i\rangle = \sum_i p_i \, \mathrm{tr}\left(\Pi_i |\psi_i\rangle\langle\psi_i|\right) =$$

$$= \mathrm{tr}\Big(\Pi \underbrace{\sum_i p_i |\psi_i\rangle\langle\psi_i|}_{\rho}\Big) = \mathrm{tr}(\rho\Pi)$$

density matrix of the mixed state

$\sum_i p_i = 1$ $\quad \langle\psi_i|\psi_j\rangle$ – arbitrary

Each ρ can be written in the form $\underbrace{\rho = \sum_{i=1}^{d=\dim \mathcal{H}} \tilde{p}_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|, \text{ s.t. } \sum_i \tilde{p}_i = 1,\ \tilde{p}_i \geq 0,\ \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle = \delta_{ij}}_{\text{spectral decomp. of } \rho}$

<u>Properties.</u>

↳ $\rho = \rho^\dagger$

↳ $\mathrm{tr}\,\rho = 1$ $\left(\Leftrightarrow \sum_i p_i = 1\right)$

↳ $\rho \geq 0$: $\forall |\psi\rangle$ $\langle\psi|\rho|\psi\rangle \geq 0$ $\left(\Leftrightarrow p_i \geq 0\right)$

positive semi-definite

"non-negative"

↳ $\mathrm{tr}(\rho^2) \leq 1$. $\mathrm{tr}(\rho^2) = 1 \Leftrightarrow \rho = 1 \cdot |\psi\rangle\langle\psi| + 0 \cdot |\ldots\rangle\langle\ldots| + 0 \ldots$ = "state is pure"

$\mathrm{tr}(\rho^2)$ – usefull measure of mixedness

$1 - \mathrm{tr}(\rho^2)$ – "linear entropy"

↳ mixed state ρ has inf. number of "decompositions":

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i' |\psi_i'\rangle\langle\psi_i'| = \ldots$$

<u>Example.</u>

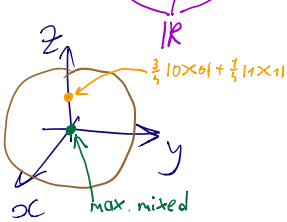
$$\rho_{mix} = \frac{\mathbb{1}}{d} = \begin{bmatrix} 1/d & & 0 \\ & 1/d & \\ & & \ddots \\ 0 & & & 1/d \end{bmatrix}$$ – maximally mixed state

Conside $d = 2$

$$\rho_{mix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -| = \ldots$$

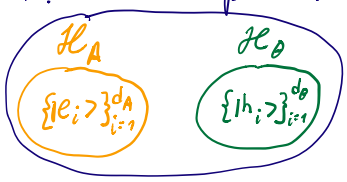
Eevery state of qubit can be represented in the form

$$\rho = \frac{\mathbb{1}}{2} + \frac{1}{2}\left(\underbrace{x}_{\mathbb{R}} b_x + \underbrace{y}_{\mathbb{R}} b_y + \underbrace{z}_{\mathbb{R}} b_z\right) \qquad x^2 + y^2 + z^2 \leq 1 \ (=1 \text{ for pure}, <1 \text{ for mixed})$$



1.3. Bipartite systems

1.3.1. Tensor product



$d_{A(B)} = \dim \mathcal{H}_{A(B)}$

$\mathcal{H}_A \otimes \mathcal{H}_B$

$\{|e_i\rangle \otimes |h_j\rangle\}_{i,j=1}^{d_A, d_B} \leftarrow$ o.n.b. $\left((\langle e_i| \otimes \langle h_j|)(|e_k\rangle \otimes |e_\ell\rangle) = \langle e_i|e_k\rangle \cdot \langle h_j|h_\ell\rangle = \delta_{ik} \cdot \delta_{k\ell} \right)$

$$|\Psi\rangle = \sum_{i,j=1}^{d_A, d_B} C_{ij} \underbrace{|e_i\rangle \otimes |h_j\rangle}_{\equiv} \qquad \sum_{ij} |C_{ij}|^2 = 1$$

$$|e_i\rangle_A |h_j\rangle_B \equiv |e_i\rangle |h_j\rangle \equiv |e_i h_j\rangle$$

Ways of thinking about $\{C_{ij}\}$:

↳ Tensor $\{1 \dots d_A\} \times \{1 \dots d_B\} \to \mathbb{C}$

↳ $d_A \times d_B$ vector

$$|e_i\rangle = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{matrix} \Big\} d_A \qquad |h_j\rangle = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{matrix} \Big\} d_B$$

Tensor
"
Kronecker product

$$|e_i\rangle \otimes |h_j\rangle = \begin{bmatrix} 0 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\ \vdots \\ 1 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\ \vdots \\ 0 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow (i-1)d_A + j$$

$$A = \sum_{ij} A_{ij} |e_i\rangle\langle e_j| \qquad B = \sum_{ij} B_{ij} |h_i\rangle\langle h_j|$$

some operators

$$A = \begin{bmatrix} A_{11} \dots A_{1 d_A} \\ \vdots \\ A_{d_A 1} \dots A_{d_A d_A} \end{bmatrix}$$

$$B = \begin{bmatrix} B_{11} \dots B_{1 d_B} \\ \vdots \\ B_{d_B 1} \dots B_{d_B d_B} \end{bmatrix}$$

$$(A\otimes B)(|\psi\rangle\otimes|\varphi\rangle) = A|\psi\rangle\otimes B|\varphi\rangle \qquad A\otimes B = \sum_{i,j=1}^{d_A}\sum_{k,l=1}^{d_B} A_{ij}\cdot B_{kl}\cdot \overbrace{|e_i\rangle\langle e_j|\otimes|h_k\rangle\langle h_l|} =$$

$$\sum_{i,j=1}^{d_A}\sum_{k,l=1}^{d_B} A_{ij}\cdot B_{kl}\,(|e_i\rangle\otimes|h_k\rangle)(\langle e_j|\otimes\langle h_l|) = \begin{bmatrix} A_{11}\begin{bmatrix} B_{11} & \dots & B_{1d_B}\\ \vdots & & \\ B_{d_B1} & \dots & B_{d_Bd_B}\end{bmatrix} & \dots & A_{1d_A}\begin{bmatrix} B_{11} & \dots & B_{1d_B}\\ \vdots & & \\ B_{d_B1} & \dots & B_{d_Bd_B}\end{bmatrix} \\ A_{d_A1}\begin{bmatrix} B_{11} & \dots & B_{1d_B}\\ \vdots & & \\ B_{d_B1} & \dots & B_{d_Bd_B}\end{bmatrix} & \dots & A_{d_Ad_A}\begin{bmatrix} B_{11} & \dots & B_{1d_B}\\ \vdots & & \\ B_{d_B1} & \dots & B_{d_Bd_B}\end{bmatrix}\end{bmatrix}$$

<u>Example</u>

$\mathbb{C}^2\otimes\mathbb{C}^2 \qquad |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} \qquad \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad \sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$|00\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1\\0\end{bmatrix}\\ 0\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ ← 0, $\quad |01\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ ← 1 $\quad |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ ← 2 $\quad |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ ← 3

dec	bin
0	00
1	01
2	10
3	11

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$\sigma_z\otimes\sigma_x = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{bmatrix}$$

1.3.2. Partial trace

$\mathcal{H}$, $\dim\mathcal{H} = d$, $\{|i\rangle\}_{i=0}^{d-1}$ – o.n.b.

$$A = \sum_{i,j=1}^{d} A_{ij}\,|i\rangle\langle j| = \begin{bmatrix}A_{11} & \dots & A_{1d}\\ \vdots & & \\ A_{1d} & \dots & A_{dd}\end{bmatrix} \qquad A_{ij} = \langle i|A|j\rangle$$

$$\mathrm{Tr}\,A = \sum_{i=0}^{d-1} A_{ii} = \sum_i \langle i|A|i\rangle$$

Properties

↳ $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

Warning: $\mathrm{Tr}(ABC) \neq \mathrm{Tr}(ACB)$!

↳ $\mathrm{Tr}\, A = \sum_i \langle i|A|i\rangle = \sum_i \langle e_i|A|e_i\rangle$, where $\{|e_i\rangle\}$ is other o.n.b.

$\mathcal{H}_A$, $\dim \mathcal{H}_A = d_A$, $\{|i\rangle_A\}_{i=0}^{d_A-1}$ – o.n.b.

$\mathcal{H}_B$, $\dim \mathcal{H}_B = d_B$, $\{|i\rangle_A\}_{i=0}^{d_B-1}$ – o.n.b.

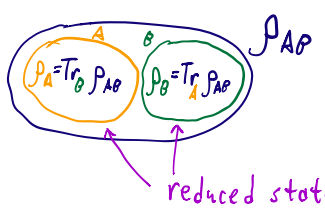
$\mathcal{H}_A \otimes \mathcal{H}_B$, $\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = d_A d_B$ $\{|i\rangle_A \otimes |j\rangle_B\}_{i,j=0}^{d_A-1, d_B-1}$ – o.n.b.

$$M = \sum_{ijkl} M_{ijkl}\, \underbrace{|i\rangle_A\langle j|}_{|i\rangle_A\langle j|_A} \otimes |k\rangle_B\langle l|$$

$\langle i|_X \cdot |j\rangle_X = \delta_{ij}$

$$\mathrm{Tr}\, M = \sum_{ij=0}^{d_A-1, d_B-1} \langle i|_A \langle j|_B\, M\, |i\rangle_A |j\rangle_B = \sum_{ij} M_{iijj} \in \mathbb{C}$$ ← Trace

$$\mathrm{Tr}_A M = \sum_{i=0}^{d_A-1} \langle i|_A\, M\, |i\rangle_A = \sum_i M_{iikl} |k\rangle_B\langle l| \in \mathcal{L}(\mathcal{H}_B)$$ ← Partial trace



Classical prob. distr-n

Intuition: $p_{AB}(a,b)$ → $p_A(a) = \sum_b p_{AB}(a,b)$, $p_B = \sum_a p_{AB}(a,b)$

Example

Pure !

$$|\psi\rangle_{AB} = \sqrt{\tfrac{3}{2}}|00\rangle_{AB} + \tfrac{1}{2}|11\rangle_{AB} \neq |\varphi\rangle_A \otimes |\chi\rangle_B$$ for any $|\varphi\rangle$ and $|\chi\rangle$

$$\rho_{AB} = |\psi\rangle_{AB}\langle\psi| = \left(\sqrt{\tfrac{3}{2}}|00\rangle_{AB} + \tfrac{1}{2}|11\rangle_{AB}\right)\left(\sqrt{\tfrac{3}{2}}\langle 00|_{AB} + \tfrac{1}{2}\langle 11|_{AB}\right) = \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{\sqrt{3}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\rho_A = \mathrm{Tr}_B |\psi\rangle_{AB}\langle\psi| = \frac{3}{4}|0\rangle_A\langle 0| + \frac{1}{4}|1\rangle_A\langle 1| = \begin{bmatrix} 3/4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\rho_B = \mathrm{Tr}_A |\psi\rangle_{AB}\langle\psi| = \frac{3}{4}|0\rangle_B\langle 0| + \frac{1}{4}|1\rangle_B\langle 1| = \begin{bmatrix} 3/4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

Mixed !

1.4. Evolution of closed systems

1.4.1. Schrödinger picture

$$i\cancel{\hbar}^{\,1}\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$ ← Sch. eq-n.

Hamiltonian

energy observable

Let H be time-independent

$$H = \sum_i E_i |\varphi_i\rangle\langle\varphi_i|$$

Consider $|\psi(t)\rangle = C(t)|\varphi_i\rangle \to i\dot{C}|\varphi_i\rangle = C E_i|\varphi_i\rangle \to \dot{C} = -iE_i C \to C(t) = e^{-iE_i t}\cdot C(t=0)$

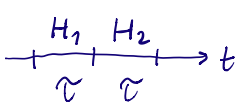
For $\underset{|\psi_0\rangle}{\underset{|||}{|\psi(t=0)\rangle}} = \sum C_i|\varphi_i\rangle = \sum\langle\varphi_i|\psi_0\rangle\cdot|\varphi_i\rangle \xrightarrow{\text{Linearity}} |\psi(t)\rangle = \sum_i e^{-iE_i t}\langle\varphi_i|\psi\rangle\cdot|\varphi_i\rangle =$

$$= \underbrace{\sum_i e^{-iE_i t}|\varphi_i\rangle\langle\varphi_i|}_{U(t)=e^{-iHt}}\cdot|\psi\rangle$$

← evolution operator

$$A = \sum_i \lambda_i |h_i\rangle\langle h_i|$$
$$f(A) = \sum_i f(\lambda_i)|h_i\rangle\langle h_i|$$
$$[f(A); A] = 0$$

Let $H = H(t)$. It seems that $U(t) \to e^{-i\int_0^t H(t')dt'}$, but it's not! (generally)



$$|\psi(2\tau)\rangle = \underbrace{e^{-iH_2\tau}e^{-iH_1\tau}}_{\neq\, e^{-iH_2\tau - iH_1\tau} \text{ if } [H_1;H_2]\neq 0}|\psi(t=0)\rangle$$

Baker-Campbell-Hausdorff formula:

$$e^{t(X+Y)} = e^{tX}\, e^{tY}\, e^{-\frac{t^2}{2}[X,Y]}\, e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\, e^{\frac{-t^4}{24}([[[X,Y],X],X]+3[[[X,Y],X],Y]+3[[[X,Y],Y],Y])} \cdots$$

$$e^X e^Y = e^Z \qquad Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,$$

For $H = H(t) \to U(t) = \underbrace{T_{\leftarrow}}_{\text{time ordering exponent}} e^{-i\int_0^t H(t')dt'} \equiv \mathbb{1} + \sum_{n=0}^{\infty}(-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, H(t_1)H(t_2)\ldots H(t_n) =$

$$= \mathbb{1} + \sum_{n=0}^{\infty}\frac{1}{n!}(-i)^n \int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n\, \underbrace{T}_{\text{time ordering operator}}[H(t_1)H(t_2)\ldots H(t_n)]$$

$$T[H(t_1)\ldots H(t_n)] = H(t_{i_1})\ldots H(t_{i_n}) \quad t_{i_1} > \ldots > t_{i_n}$$

Derivation of T-exp: http://scipp.ucsc.edu/~haber/ph215/TimeOrderedExp.pdf

$U(t)U(t)^{+} = \mathbb{1}$ ← unitary operator

$$i\frac{d}{dt}U(t)|\cancel{\psi_0}\rangle = H(t)U(t)|\cancel{\psi_0}\rangle \to \underbrace{i\frac{d}{dt}U(t) = H(t)U(t), \quad U(t=0) = \mathbb{1}}_{\text{equation on evolution operator}}$$

for any $|\psi_0\rangle$

What about density matrices?

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $|\psi_i\rangle = |\psi_i(t)\rangle$

$$\frac{d}{dt}\rho = \sum_i p_i |\dot{\psi}_i\rangle\langle\psi_i| + \sum_i p_i |\psi_i\rangle\langle\dot{\psi}_i| = -i\sum_i p_i H|\psi_i\rangle\langle\psi_i| + i\sum_i p_i |\psi_i\rangle\langle\psi_i| \overset{H}{\overset{\shortparallel}{H}}{}^{+} = -i[H,\rho]$$

$\frac{d}{dt}\rho = -i[H,\rho]$ ← von Neumann equation (Liouville)

Solution:
$$|\psi_i(t)\rangle = U(t)\,|\psi_i(t=0)\rangle$$
$$\langle\psi_i(t)| = \langle\psi_i(t=0)|U(t)^{+}$$
$$\rho(t) = U(t)\,\rho(t=0)\,U^{+}(t)$$

1.4.2. Heisenberg picture

$$Pr(\text{"some outcome"}) = Tr(\underbrace{U(t)\rho_0 U^{+}(t)}_{\rho(t)\text{ in Sch. picture}}\overset{\text{projector of "some outcome"}}{\Pi}) = tr(\rho_0 \underbrace{U^{+}(t)\Pi_i U(t)}_{\Pi_i^H(t)\text{ in Heis. picture}})$$

$$\Pi_i^H(t=0) = \Pi_i$$

Heis. pic-e Scr. pic-e

In general $\Pi_i = \Pi_i(t)$ in Scr. pic-e

Heis. Hamilt. $H(t)$

$$i\frac{d}{dt}\Pi_i^H(t) = i\frac{d}{dt}\left(U^\dagger(t)\,\Pi_i(t)\,U(t)\right) = -U^\dagger(t)\,H^\dagger(t)\,\Pi_i(t)\,U(t) +$$

$$i\frac{d}{dt}U(t) = H(t)U(t), \qquad -i\frac{d}{dt}U^\dagger(t) = U^\dagger(t)H(t)$$

$$\frac{\partial}{\partial t}\Pi_i^H(t) \equiv U^\dagger(t)\left(\frac{d}{dt}\Pi_i(t)\right)U(t) +$$

$$U^\dagger(t)\,\Pi_i(t)\,H(t)\,U(t) \ominus$$

$$\ominus\, U^\dagger(t)\left[\Pi_i(t), H(t)\right]U(t) + \frac{\partial}{\partial t}\Pi_i^H(t)$$

If H is time independent: $[H; U(t)] = 0$

$$i\frac{d}{dt}\Pi_i^H(t) = -U^\dagger(t)\,H\,\Pi_i(t)\,U(t) + U^\dagger(t)\,\Pi_i(t)\,H\,U(t) + \frac{\partial}{\partial t}\Pi_i^H(t) \ominus$$

$$\ominus\left[\Pi_i^H(t); H\right] + \frac{\partial}{\partial t}\Pi_i^H(t)$$

"full observable"

$$O = \sum_i \tilde{\lambda}_i \Pi_i \rightarrow i\frac{d}{dt}O^H(t) = \left[O^H(t); H\right] + \underbrace{\frac{\partial}{\partial t}O^H(t)}_{U^\dagger(t)\frac{d}{dt}O(t)\,U(t)}, \quad O^H(t=0) = O(t=0)$$

1.4.3. Dirac picture = Scr. + Heis.

$$H = \underset{\text{time ind-t}}{H_1} + \underset{\text{may be time dep-t}}{H_2} \qquad i\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$|\psi_{int}\rangle \equiv e^{iH_1 t}|\psi(t)\rangle$$

"feels H_2 only"

$$i\frac{d}{dt}|\psi_{int}(t)\rangle = \underbrace{i\cdot i}_{-1}e^{iH_1 t}\cancel{H_1}|\psi(t)\rangle + e^{iH_1 t}(\cancel{H_1} + H_2)|\psi(t)\rangle = \overbrace{e^{iH_1 t}H_2 \underbrace{e^{-iH_1 t}e^{iH_1 t}}_{\mathbb{1}}}^{V(t)}\overbrace{|\psi(t)\rangle}^{|\psi_{int}(t)\rangle}$$

$$i\frac{d}{dt}|\psi_{int}(t)\rangle = V(t)|\psi_{int}(t)\rangle$$

$V(t) = e^{iH_1(t)}H_2 e^{-iH_1(t)}$ ← H_2 in the "rotated frame" of H_1

1.5. Entanglement

1.5.1 Schmidt decomposition

$\mathcal{H}_A \quad \{|e_i\rangle\}_{i=1}^{d_A}$, $\mathcal{H}_B \quad \{|h_j\rangle\}_{j=1}^{d_B}$

o.n.b.

$$|\psi\rangle = \sum_{i,j=1}^{d_A\, d_B} C_{ij}\, |e_i\rangle|h_j\rangle$$

$$\rho_A = tr_B(|\psi\rangle\langle\psi|) = tr_B\Big(\sum_{ijkl} C_{ij} C^*_{kl}\, |e_i\rangle\langle e_k| \otimes |h_j\rangle\langle h_l|\Big) = \sum_{ijk} C_{ij} C^*_{kj}\, |e_i\rangle\langle e_k| = \underbrace{\sum_i p_i |\varphi_i\rangle\langle\varphi_i|}_{\text{spectral decomp-n}}$$

$\langle\varphi_i|\varphi_j\rangle = \delta_{ij}$

Let's use $\{|\varphi_i\rangle\}$ as a basis in $\mathcal{H}_A$

$$|\psi\rangle = \sum_{i,j=1}^{d} D_{ij}\, |\varphi_i\rangle|h_j\rangle = \sum_i |\varphi_i\rangle \underbrace{|\chi_i\rangle}_{= \sum_j D_{ij}|h_j\rangle}$$

$\langle\chi_i|\chi_j\rangle = ?$ ← not normalized

$$\rho_A = tr_B(|\psi\rangle\langle\psi|) = \sum_{ijk} |\varphi_i\rangle\langle\varphi_j| \cdot \underbrace{\langle h_k|\chi_i\rangle\langle\chi_j|h_k\rangle}_{\sum_k |h_k\rangle\langle h_k| = \mathbb{1}_B} = \sum_{ijk} |\varphi_i\rangle\langle\varphi_j| \cdot \langle\chi_j|\chi_i\rangle = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| \Rightarrow$$

$\Rightarrow \langle\chi_j|\chi_i\rangle = p_i \cdot \delta_{ij}$ for $i = 1 \ldots \mathrm{rank}(\rho_A)$; $\langle\chi_i|\chi_i\rangle = 0$, $|\chi_i\rangle = 0$ for $i > \mathrm{rank}(\rho_A)$

$$\langle\chi_j|\underbrace{\frac{1}{\sqrt{p_i}} \cdot \frac{1}{\sqrt{p_i}}|\chi_i\rangle}_{|\tilde{\chi}_i\rangle} = \delta_{ij}$$

$\{|\tilde{\chi}_i\rangle\}_{i=1}^{\mathrm{rank}(\rho_A)}$ ← orthonormal set

$$|\psi\rangle = \sum_{i=1}^{\mathrm{rank}(\rho_A)} \sqrt{p_i}\, |\varphi_i\rangle|\tilde{\chi}_i\rangle \quad \leftarrow \text{Schmidt decomp-n}$$

$e^{-i\theta_i} e^{i\theta_i}$

Sch. dec-n is not unique!

# of terms in Sch. dec-n = Sch. rank.

↳ Sch. rank $\leq \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$

↳ $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$ $\quad \rho_B = \mathrm{tr}_A(|\psi\rangle\langle\psi|) = \sum_i p_i |\tilde{\chi}_i\rangle\langle\tilde{\chi}_i|$

same spectrum!

↳ If Sch. rank > 1 then $|\psi\rangle$ is called entangled = non separable

$|\psi\rangle = \overset{\neq 0}{C_1} |\varphi_1\rangle|\tilde{\chi}_1\rangle + \overset{\neq 0}{C_2} |\varphi_2\rangle|\tilde{\chi}_2\rangle + \dots \neq |\psi_A\rangle|\psi_B\rangle$

↳ Reduced states of entangled states are mixed

$$\rho_A = \rho_B = \begin{bmatrix} \overset{\neq 0}{|C_1|^2} & & \\ & \overset{\neq 0}{|C_2|^2} & \\ & & \ddots \end{bmatrix}$$

in $\{|\varphi_i\rangle\}$ & $\{|\tilde{\chi}_i\rangle\}$ bases

mixedness(ρ_A) = mixedness(ρ_B) = entanglement$(|\psi\rangle)$;

e.g. $1 - \mathrm{tr}(\rho_\bullet^2)$

maximally entangled $|\psi\rangle \Leftrightarrow$ maximally mixed $\rho_A = \rho_B = \frac{1}{d}\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $\left(\text{e.g. } |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$

↳ $\rho_A = \rho_B = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1.5.2. Purification

Consider $\rho_S = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$ ← spectral decomp-n

"system"

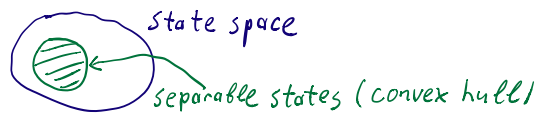
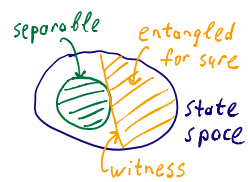
$\exists$ infinite number of pure states $|\psi\rangle_{SE}$, s.t. $\mathrm{tr}_E(|\psi\rangle_{SE}\langle\psi|) = \rho_S$, $\dim \mathcal{H}_E \geq \mathrm{rank}\, \rho_S$.

"Environment"

Idea: choose $\{|e_i\rangle_E\}_{i=1}^{\mathrm{rank}\,\rho_S}$, s.t. $\langle e_i | e_j\rangle = \delta_{ij}$.

$|\psi\rangle_{SE} = \sum_i \sqrt{p_i} |\varphi_i\rangle_S \otimes |e_i\rangle_E$ ← "purifies" ρ_S

1.5.3. Classification of states

	pure	mixed
separable	$\lvert\psi\rangle = \lvert\psi_A\rangle \otimes \lvert\psi_B\rangle$ $\underbrace{\lvert\psi\rangle\langle\psi\rvert}_{\rho} = \underbrace{\lvert\psi_A\rangle\langle\psi_A\rvert}_{\rho_A} \otimes \underbrace{\lvert\psi_B\rangle\langle\psi_B\rvert}_{\rho_B}$ S.r. $= 1$	convex combination $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$, $p_i \geq 0$, $\sum_i p_i = 1$ density matr.: $\rho^i_{A(B)} \geq 0$, $tr(\rho_{A(B)}) = 1$ $\rho = p\rho_{sep,1} + (1-p)\rho_{sep,2}$ – separable 
entangled	$\lvert\psi\rangle = C_1^{\neq 0}\lvert\psi_A^1\rangle \otimes \lvert\psi_B^1\rangle + C_2^{\neq 0}\lvert\psi_A^2\rangle \otimes \lvert\psi_B^2\rangle + \ldots$ $(\langle\psi^i_{A(B)}\rvert\psi^j_{A(B)}\rangle = \delta_{ij})$ $\rho_A = tr_B(\lvert\psi\rangle\langle\psi\rvert)$ $\rho_B = tr_A(\lvert\psi\rangle\langle\psi\rvert)$ are mixed S.r. > 1	$\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$, $p_i \geq 0 \ldots$ can not be represented as a mixed separable state The most difficult case! (NP-hard in some cases) Sufficient criteria for entanglement $\parallel$ entanglement witness  Nec. & suf. for 2×2 & 3×2 cases concurrence, negativity (partial transpose criterion)