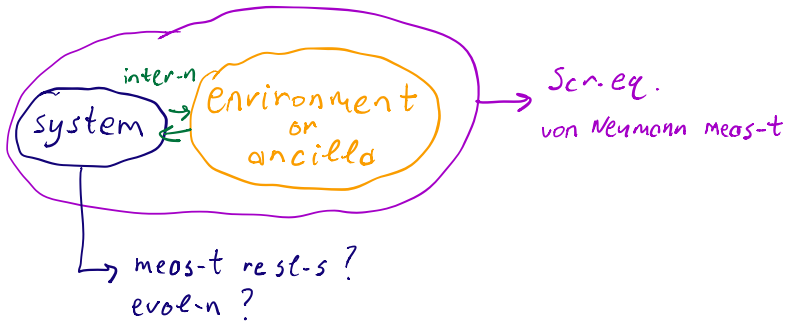
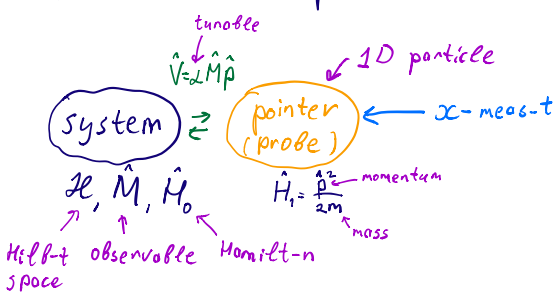
2. Measurements & evolution



2.1. Von Neumann pointer



$$\hat{M}=\sum_j \lambda_j |\varphi_j\rangle\langle\varphi_j| \quad \leftarrow \text{spectral decomp-n}$$

$$\hat{H}=\hat{H}_0+\frac{\hat{p}^2}{2m}+\alpha\hat{M}\hat{p}$$

$$\hat{H}_0\otimes\hat{1} \qquad \hat{1}\otimes\frac{\hat{p}^2}{2m} \qquad \hat{M}\otimes\hat{p}$$

$$m\to\infty \quad \frac{\hat{p}^2}{2m}\to 0$$

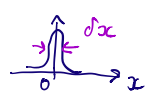
$$\hat{H}=\hat{H}_0+\alpha\hat{M}\hat{p}$$

Let $[\hat{M};\hat{H}_0]=0$ or $\alpha\hat{M}\hat{p}\gg\hat{H}_0$

$$\hat{H}_{int}=e^{+i\hat{H}_0 t}\alpha\hat{M}\hat{p}\,e^{-i\hat{H}_0 t}=\alpha\hat{M}\hat{p} \qquad \hat{H}\approx\alpha\hat{M}\hat{p}$$

$$\hat{U}=e^{-i\alpha\hat{M}\hat{p}t}=\sum_j\int dp\, e^{-i\alpha\lambda_j p t}|\varphi_j\rangle\langle\varphi_j|\otimes|p\rangle\langle p|=\sum_j|\varphi_j\rangle\langle\varphi_j|\otimes e^{-i\alpha\lambda_j\hat{p}t}$$

$$|\Psi_0\rangle=\left(\sum C_j|\varphi_j\rangle\right)\otimes|\psi_0\rangle \qquad \langle x|\psi_0\rangle=$$



$$|\varphi_j\rangle\langle\varphi_j|\otimes e^{-i\alpha\lambda_j\hat{x}t}|\Psi_0\rangle=C_j|\varphi_j\rangle\otimes\underbrace{e^{-i\alpha\lambda_j\hat{p}t}|\psi_0\rangle}_{|\psi_j\rangle}$$

Δx_j

$$\langle x|\underbrace{e^{-i\Delta x_j \hat{p}}|\psi_0\rangle}_{|\psi_j\rangle} = \langle x|\int dp\, e^{-i\Delta x_j p}|p\rangle\langle p|\psi_0\rangle = \langle x|\iint dx'dp\, e^{-i\Delta x_j p}|p\rangle\langle p|x'\rangle\langle x'|\psi_0\rangle =$$

$\frac{1}{\sqrt{2\pi}}e^{ipx}$ $\quad$ $\frac{1}{\sqrt{2\pi}}e^{-ipx'}$ $\quad$ $\psi_0(x')$ $\quad$ $\mathbb{1}$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx}$$

de Broglie wave

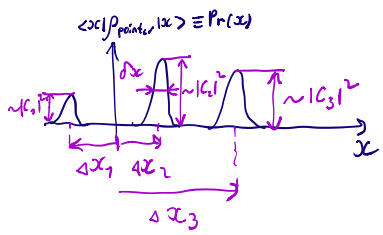
$$= \int\underbrace{\left(\int\frac{1}{2\pi}e^{ip(x-\Delta x_j-x')}dp\right)}_{=\delta(x-\Delta x_j-x')}\psi_0(x')\,dx' = \underbrace{\psi(x-\Delta x_j)}_{=\psi_j(x)} =$$

$$\hat{U}|\Psi_0\rangle = \sum_\alpha C_\alpha|\varphi_\alpha\rangle|\psi_j\rangle \equiv |\Psi\rangle \leftarrow \text{entangled state}$$

$$\rho_{pointer} = tr_{system}\left(\sum_{\alpha\beta} C_\alpha C_\beta^*\, |\varphi_\alpha\rangle\langle\varphi_\beta| \otimes |\psi_\alpha\rangle\langle\psi_\beta|\right) = \sum_\alpha |C_\alpha|^2\, |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$Pr(x) = \langle x|\rho_{pointer}|x\rangle = \sum_\alpha |C_\alpha|^2\, |\psi(x-\Delta x_\alpha)|^2$$

Case #1: $\delta x \ll |\Delta x_\alpha - \Delta x_\beta| \quad \forall\, \alpha\neq\beta$



resulting system's state $\qquad$ Projective meas-t!

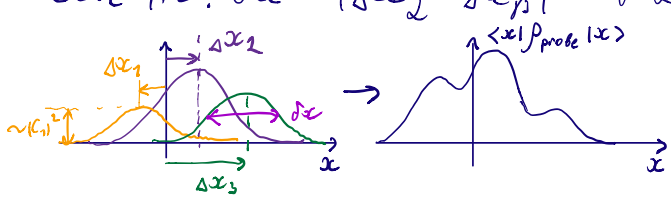
$$\underbrace{\text{Meas-d value } x \text{ "near" } \Delta x_i}_{Pr = |C_i|^2} \rightarrow \langle x|\Psi\rangle \sim \frac{1}{\underset{\text{normalization factor}}{n.f.}}\sum_\alpha C_\alpha\,\psi_\alpha(x)|\varphi_\alpha\rangle \approx |\varphi_i\rangle$$

$(\psi_j(x)\simeq 0 \text{ for } j\neq i)$

$$\sum_{ij} C_i C_j^*|\varphi_i\rangle\langle\varphi_j| = \begin{bmatrix} |C_1|^2 & C_1C_2^* \dots & C_1C_d^* \\ C_2C_1^* & |C_2|^2 \dots & C_2C_d^* \\ \vdots & \ddots & \\ C_dC_1^* & C_dC_2^* \dots & |C_d|^2 \end{bmatrix} \rightarrow \begin{bmatrix} |C_1|^2 & & 0 \\ & \ddots & \\ 0 & & |C_d|^2 \end{bmatrix}$$

"decoherence"

Case #2: $\delta x \sim |\Delta x_\alpha - \Delta x_\beta| \quad \forall\, \alpha\neq\beta$



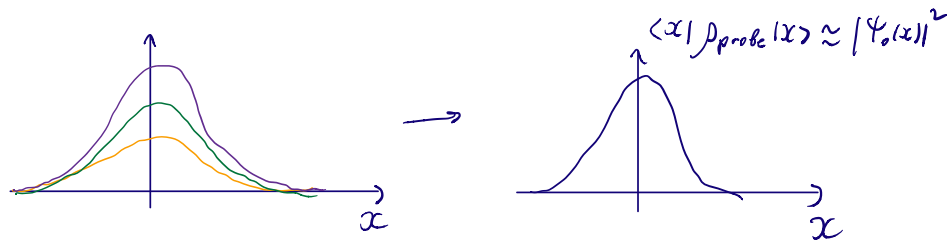
$$\langle x|\Psi\rangle = \frac{1}{n.f.}\sum_\alpha C_\alpha \cdot \underbrace{\Psi_\alpha(x)}_{\text{several terms survive}} |\varphi_\alpha\rangle$$ ← "soft meas-t"

$$\sum_{ij} C_i C_j^* |\varphi_i\rangle\langle\varphi_j| = \begin{bmatrix} |C_1|^2 & C_1C_2^* \dots & C_1C_d^* \\ C_2C_1^* & |C_2|^2 \dots & C_2C_d^* \\ \vdots & \ddots & \\ C_dC_1^* & C_dC_2^* \dots & |C_d|^2 \end{bmatrix} \to \begin{bmatrix} |C_1|^2 & D_{12} & \dots & D_{1d} \\ D_{12}^* & |C_2|^2 & \dots & D_{2d} \\ \vdots & & \ddots & \\ C_{1d}^* & \dots & & |C_d|^2 \end{bmatrix} \qquad |D_{\alpha\beta}| < |C_\alpha C_\beta^*|$$

$$\rho_{system} = tr_{probe}\left(\sum_{\alpha\beta} C_\alpha C_\beta^* |\varphi_\alpha\rangle\langle\varphi_\beta| \otimes |\Psi_\alpha\rangle\langle\Psi_\beta|\right) =$$

$$= \sum_{\alpha\beta} C_\alpha C_\beta^* |\varphi_\alpha\rangle\langle\varphi_\beta| \cdot \underbrace{\int_x \Psi_\alpha(x)\Psi_\beta^*(x)\,dx}_{\approx 1}$$

Case #3: $\delta x \gg |\Delta x_\alpha - \Delta x_\beta|, \alpha \neq \beta$



$$U|\Psi_0\rangle = \sum_\alpha C_\alpha |\varphi_\alpha\rangle|\Psi_\alpha\rangle \approx \underbrace{\sum_\alpha C_\alpha |\varphi_\alpha\rangle}_{\text{unchanged}} \otimes |\Psi_0\rangle$$ ← no measurement, no decoherence

$$\Delta x_i = \alpha t \lambda_i \leftrightarrow \delta x$$

tuning parameter to shift between cases

2.2. Quantum eraser

system (qubit) $\xleftrightarrow{U}$ probe (qubit)

$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\qquad$ $|\Psi_0\rangle = |0\rangle$

$U: |00\rangle \to |00\rangle$

$|10\rangle \to |11\rangle$

$$|\psi\rangle_s|\psi_0\rangle_p \xrightarrow{U} \alpha|00\rangle_{sp} + \beta|11\rangle_{sp} \xrightarrow{\sigma_z \text{ of probe}} \begin{cases} |0\rangle_s & \text{with prob. } |\alpha|^2 \\ |1\rangle_s & \text{with prob. } |\beta|^2 \end{cases}$$

σ_z-meas-t of the probe = σ_z meas-t of the system

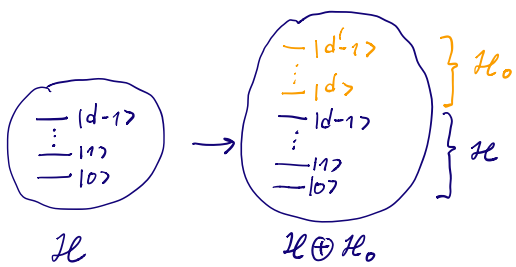
$$\alpha|00\rangle_{sp} + \beta|11\rangle_{sp} \xrightarrow{\sigma_x \text{ meas. of probe}} \left| \alpha|0\rangle_s \frac{|+\rangle_p + |-\rangle_p}{\sqrt{2}} + \beta|1\rangle_s \frac{|+\rangle_p - |-\rangle_p}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\alpha|0\rangle_s + \beta|1\rangle_s)|+\rangle_p + \frac{1}{\sqrt{2}}(\alpha|0\rangle_s - \beta|1\rangle_s)|-\rangle_p \right| \rightarrow$$

$$\rightarrow \begin{cases} \alpha|0\rangle_s + \beta|1\rangle_s & \text{with prob. } \frac{1}{2} \\ \alpha|0\rangle_s - \beta|1\rangle_s & \text{with prob. } \frac{1}{2} \end{cases} \xrightarrow{\text{apply } \sigma_z \text{ on system if the probe was meas-d in } |1\rangle} \underbrace{\alpha|0\rangle_s + \beta|1\rangle_s}_{\text{system's state restored, } \sigma_z \text{ information erased}}$$

indep-t from α & β!

2.3. POVM = positive operator-valued measure

2.3.1. Direct sum



$$\begin{bmatrix} c_0 \\ \vdots \\ c_{d-1} \end{bmatrix} \rightarrow \begin{bmatrix} c_0 \\ \vdots \\ c_{d-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2.3.2. Measurement in extended space

$|\psi\rangle \in \mathcal{H}$, $\dim \mathcal{H} = d$ $\quad |\psi\rangle = \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$

$\mathcal{H}_{ext} = \mathcal{H} \oplus \mathcal{H}_0$ $\quad \dim \mathcal{H}_0 = d' - d > 0$

$O = \sum_{i=1}^{d'} \lambda_i |\varphi_i\rangle\langle\varphi_i|$ (some obser-le), $\mathcal{H}_{ext} \ni |\varphi_i\rangle = |\tilde{\varphi}_i\rangle \oplus |\varphi_i^\perp\rangle$ $\quad \delta_{ij} = \langle\tilde{\varphi}_i|\tilde{\varphi}_j\rangle + \langle\varphi_i^\perp|\varphi_j^\perp\rangle$

$$Pr(\lambda_i) = |\langle\varphi_i|\underbrace{(|\psi\rangle \oplus |\overset{\overset{d}{\updownarrow}}{0}\rangle)}_{= \begin{bmatrix} c_1 \\ \vdots \\ c_d \\ 0 \\ \vdots \\ 0 \end{bmatrix} \updownarrow d'}| = |\langle\tilde{\varphi}_i|\psi\rangle|^2 = tr(|\psi\rangle\langle\psi|\underbrace{|\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i|}_{M_i}) = tr(|\psi\rangle\langle\psi| M_i)$$

$$M_i = |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i| = \underbrace{\Pi_{\mathcal{H}}}_{\text{projector on } \mathcal{H}} |\varphi_i\rangle\langle\varphi_i| \Pi_{\mathcal{H}}$$

$$\sum_i M_i = \Pi_{\mathcal{H}} \underbrace{\left(\sum_i |\varphi_i\rangle\langle\varphi_i|\right)}_{\mathbb{1} \text{ in } \mathcal{H}_{ext}} \Pi_{\mathcal{H}} = \mathbb{1}_{\mathcal{H}}$$

$$\{M_i\}_{i=1}^{d'} \text{ — POVM}$$

$d' \to \dim \mathcal{H}$

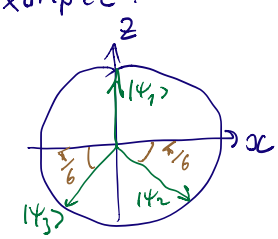
M_i live in $\mathcal{H}$

↳ $M_i \geq 0$

↳ $\sum_i M_i = \mathbb{1}$

$\mathrm{tr}(M_i M_j) = \langle\tilde{\varphi}_i|\tilde{\varphi}_j\rangle$ may not be eq. to δ_{ij} !

Example:



$$|\psi_1\rangle = |0\rangle$$

$$|\psi_2\rangle = \cos\frac{\pi}{3}|0\rangle + \sin\frac{\pi}{3}|1\rangle$$

$$|\psi_3\rangle = \cos\frac{\pi}{3}|0\rangle - \sin\frac{\pi}{3}|1\rangle$$

$$M_i = \frac{2}{3}|\psi_i\rangle\langle\psi_i|$$

$$\sum_i M_i = \frac{2}{3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}\frac{1}{4} & \frac{\sqrt{3}}{4}\\ \frac{\sqrt{3}}{4} & \frac{3}{4}\end{bmatrix} + \begin{bmatrix}\frac{1}{4} & -\frac{\sqrt{3}}{4}\\ -\frac{\sqrt{3}}{4} & \frac{3}{4}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$|\varphi_1\rangle = \begin{bmatrix}\sqrt{2/3}\\ 0\\ \sqrt{1/3}\end{bmatrix} \quad |\varphi_2\rangle = \begin{bmatrix}\sqrt{1/6}\\ \sqrt{1/2}\\ -\sqrt{1/3}\end{bmatrix} \quad |\varphi_3\rangle = \begin{bmatrix}\sqrt{1/6}\\ -\sqrt{1/2}\\ -\sqrt{1/3}\end{bmatrix}$$

← qutrit meas-t