Self-check problems on Quantum information processing

Part 1: Basics of Quantum Mechanics

Last update: October 1, 2020

Problem 1

Let $\mathcal{H}$ be a d-dimensional Hilbert space. Denote it computational basis states as

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad |d-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{1}$$

Let

$$|\psi\rangle = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{d-1} \end{bmatrix}, \quad M = \begin{bmatrix} m_{0,0} & m_{0,1} & \ldots & m_{0,d-1} \\ m_{1,0} & m_{1,1} & \ldots & m_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{d-1,0} & m_{d-1,1} & \ldots & m_{d-1,d-1} \end{bmatrix}. \tag{2}$$

be some matrix.

Check the following facts.

1. $\langle i|j\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker symbol.
2. $\sum_{i=0}^{d-1} |i\rangle\langle i| = \mathbf{1}$, where $\mathbf{1}$ is the identity matrix.
3. $C_i = \langle i|\psi\rangle$.
4. $|\psi\rangle = \sum_{i=0}^{d-1} C_i|i\rangle$.
5. $M_{i,j} = \langle i|M|j\rangle$.
6. $M = \sum_{i,j} M_{i,j}|i\rangle\langle j|$.
7. $M|\psi\rangle = \sum_{k,l=0}^{d-1} M_{k,l}C_l|k\rangle$.

Problem 2

Consider standard Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Find their eigenvalues and eigenvectors. Check that each matrix can be represented in the form $\sum_{i=0}^{1} \lambda_i|\psi_i\rangle\langle\psi_i|$, where $\{\lambda_i\}$ and $\{|\psi_i\rangle\}$ are eigenvalues and eigenvectors correspondingly. Pay special attention to non-uniqueness of $\mathbf{1}$ spectral decomposition. Show the eigenvectors on the Bloch sphere.

Problem 3

Let $\{|\psi_i\rangle\}_{i=1}^d$ be a set of orthonormal states ($\langle\psi_i|\psi_j\rangle = \delta_{i,j}$). Check that

$$U = \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & \dots & |\psi_d\rangle \end{bmatrix} \tag{4}$$

is a unitary matrix. Check that a spectral decomposition $M = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i|$ can be written in the form

$$M = UDU^\dagger, \tag{5}$$

where D is diagonal matrix with $D_{i,i} = \lambda_i$

Problem 4

Let $f(\xi) : \mathbb{R} \to \mathbb{R}$ be some function. Here we would like to extend f on Hermitian matrices. Consider a Hermitian matrix M with a spectral decomposition $M = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$. Let

$$f(\xi) = \sum_{i=0}^{\infty} C_i \xi^i \tag{6}$$

be a Taylor decomposition of f. Check that

$$f(M) := \sum_{i=0}^{\infty} C_i M^i = \sum_k f(\lambda_k) |\psi_k\rangle\langle\psi_k|. \tag{7}$$

Problem 5

Check that $|\psi(t)\rangle = U(t)|\psi_0\rangle = \exp(-\mathrm{i}Ht)|\psi_0\rangle$ is solution of the Schroedinger equation

$$\mathrm{i}\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{8}$$

with initial condition $|\psi(0)\rangle = |\psi_0\rangle$.

Problem 6

Consider a Hamiltonian

$$H = \frac{\hbar\omega}{2}\sigma_x \tag{9}$$

and an initial state

$$|\psi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{10}$$

1. Find an evolution operator $U(t)$.
2. Find an evolution of initial state in the Schroedinger picture $|\psi(t)\rangle$. Show it on the Bloch sphere.
3. Find an evolution of mean value of σ_z.
4. Find an evolution of σ_z in the Heisenberg picture $\sigma_z^H(t)$. Express it in terms of other Pauli matrices.
5. Check that both pictures provide the same mean value of σ_z.

Problem 7

Check that making a Hamiltonian transformation

$$H \to H + \alpha \mathbf{I}, \tag{11}$$

where α is real and $\mathbf{I}$ is identity matrix of corresponding dimension, results in acquiring additional phase for the evolution operator. Show that this phase does not affect the result of evolution of a density matrix.

Problem 8

Let $\{|n\rangle_A\}_{n=0}^{d_A-1}$ and $\{|m\rangle_B\}_{m=0}^{d_B-1}$ be computational bases of finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ (d_A and d_B are corresponding dimensions). Write an expression for position of unit element inside the vector $|i\rangle_A \otimes |j\rangle_B$ in terms of i and j. Check applicability your result for two-qubit case considered in the lectures.

Problem 9

Consider a two qubit state with density matrix

$$\rho_{AB} = \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}, \tag{12}$$

where

$$X = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}, \quad Y = \begin{bmatrix} \rho_{02} & \rho_{03} \\ \rho_{12} & \rho_{13} \end{bmatrix}, \quad Z = \begin{bmatrix} \rho_{20} & \rho_{21} \\ \rho_{30} & \rho_{31} \end{bmatrix}, \quad W = \begin{bmatrix} \rho_{22} & \rho_{23} \\ \rho_{32} & \rho_{33} \end{bmatrix}. \tag{13}$$

Show that

$$\rho_A = \mathrm{Tr}_B \rho_{AB} = \begin{bmatrix} \mathrm{Tr}X & \mathrm{Tr}Y \\ \mathrm{Tr}Z & \mathrm{Tr}W \end{bmatrix}, \quad \rho_B = \mathrm{Tr}_A \rho_{AB} = X + W. \tag{14}$$

Problem 10

Consider three-qubit W-state[1]

$$|\mathrm{W}\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle_{ABC} + |010\rangle_{ABC} + |001\rangle_{ABC}). \tag{15}$$

Write its Schmidt decomposition with respect to bipartite partitioning $A : BC$ (consider qubits B and C as single object). What is a Schmidt rank of the resulting decomposition? Show that A is not maximally entangled with BC. Design a three-qubit state $|\Psi\rangle_{ABC}$ where the first qubit A is maximally entangled with a pair BC.

[1] Note that this kind of states appears in Rydberg blockade.

Problem 11

Consider the following two-qubit states:

$$\rho_1 = \frac{\mathbf{1}}{2} \otimes \frac{\mathbf{1}}{2}, \quad \rho_2 = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \quad \rho_3 = |\Phi^+\rangle\langle\Phi^+|, \tag{16}$$

where

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{17}$$

Check that reduced state of both qubits coincide for all three states. Find mean values of observables

$$\sigma_z^A \equiv \sigma_z \otimes \mathbf{1}, \quad \sigma_z^B \equiv \mathbf{1} \otimes \sigma_z, \quad \sigma_z \otimes \sigma_z, \quad \sigma_x \otimes \sigma_x, \tag{18}$$

for each state ρ_i.

Problem 12

Find purifications of all three states (16). Do the state of ancillary purifying system ("environment") correlate with the purified state?

Problem 13

Show that mixture of mixed separable states is also mixed separable state.

Problem 14

Consider maximally entangled state $|\Phi^+\rangle$ from Eq. (17). Write $\rho = |\Phi^+\rangle\langle\Phi^+|$ in the form

$$\rho = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \quad p_i \geq 0, \quad \sum_i p_i = 1, \tag{19}$$

where $\rho_A^{(i)}$ and $\rho_B^{(i)}$ are *arbitrary* 2×2 matrices. Remember definition of separable mixed states. Draw conclusions :-)