Smooth 3D Velocity Field via Dirichlet Series under Impulsive Forcing

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Introduction

We consider a special case of the 3D Navier–Stokes equations for an incompressible fluid on a cubic domain $[0, L]^3$ with Dirichlet boundary conditions ($\mathbf{u} = 0$ on the boundaries). The goal is to construct an analytical velocity field that reacts to a short impulsive force and then decays smoothly.

Problem Statement

The simplified governing equation for the velocity field $\mathbf{u} = (u, v, w)$ is:

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} + \mathbf{f}(x, y, z, t), \quad \mathbf{u}|_{\partial[0,L]^3} = 0$$

where ν is the kinematic viscosity, and $\mathbf{f}$ is an external impulsive force.

Analytical Solution via Dirichlet Series

We expand the velocity field in a sine series satisfying the Dirichlet boundaries:

$$\phi_{nml}(x, y, z) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \sin\left(\frac{l\pi z}{L}\right), \quad n, m, l = 1, 2, \ldots$$

with Laplacian eigenvalues:

$$\lambda_{nml} = \frac{\pi^2}{L^2}(n^2 + m^2 + l^2)$$

The velocity components are expressed as:

$$
\begin{aligned}
u(x,y,z,t) &= \sum_{n,m,l} a_{nml}(t)\, \sin\frac{n\pi x}{L}\, \sin\frac{m\pi y}{L}\, \sin\frac{l\pi z}{L} \\
v(x,y,z,t) &= \sum_{n,m,l} a_{nml}(t)\, \sin\frac{m\pi x}{L}\, \sin\frac{l\pi y}{L}\, \sin\frac{n\pi z}{L} \\
w(x,y,z,t) &= \sum_{n,m,l} a_{nml}(t)\, \sin\frac{l\pi x}{L}\, \sin\frac{n\pi y}{L}\, \sin\frac{m\pi z}{L}
\end{aligned}
$$

Time Evolution of Coefficients

Each coefficient $a_{nml}(t)$ satisfies the ODE:

$$\frac{da_{nml}}{dt} = -\nu\lambda_{nml}a_{nml} + F_{nml}(t)$$

where $F_{nml}(t)$ is the projection of the external force on the corresponding mode:

$$F_{nml}(t) = \int_0^L \int_0^L \int_0^L f(x,y,z,t)\, \phi_{nml}(x,y,z)\, dx\, dy\, dz$$

For a short impulsive force, $F_{nml}(t) \neq 0$ only for $t \leq t_0$, and $F_{nml}(t) = 0$ afterwards. The solution of the ODE is:

$$\boxed{a_{nml}(t) = \frac{F_0}{nml\,\lambda_{nml}} \left(1 - e^{-\nu\lambda_{nml}\min(t,t_0)}\right) e^{-\nu\lambda_{nml}\max(0,t-t_0)}}$$

Physical Interpretation

- During $0 \leq t \leq t_0$, the impulsive force accelerates the field, producing a "velocity explosion".
- After $t > t_0$, all modes decay exponentially with factor $\exp(-\nu\lambda_{nml}(t - t_0))$.
- The velocity field remains smooth and satisfies Dirichlet boundary conditions at all times.

Numerical Illustration

The Python script ‘simulation.py‘ computes the velocity modulus:

$$|\mathbf{u}(x,y,z,t)| = \sqrt{u^2 + v^2 + w^2}$$

for a small grid and prints the values to the console at each time step, illustrating the initial acceleration and subsequent decay.

Conclusion

Analytical solution for a 3D velocity field under an impulsive external force using Dirichlet sine series. The solution explicitly shows the acceleration ("explosion") due to the force and the exponential decay afterwards. This provides a reproducible model suitable for educational or computational demonstrations.