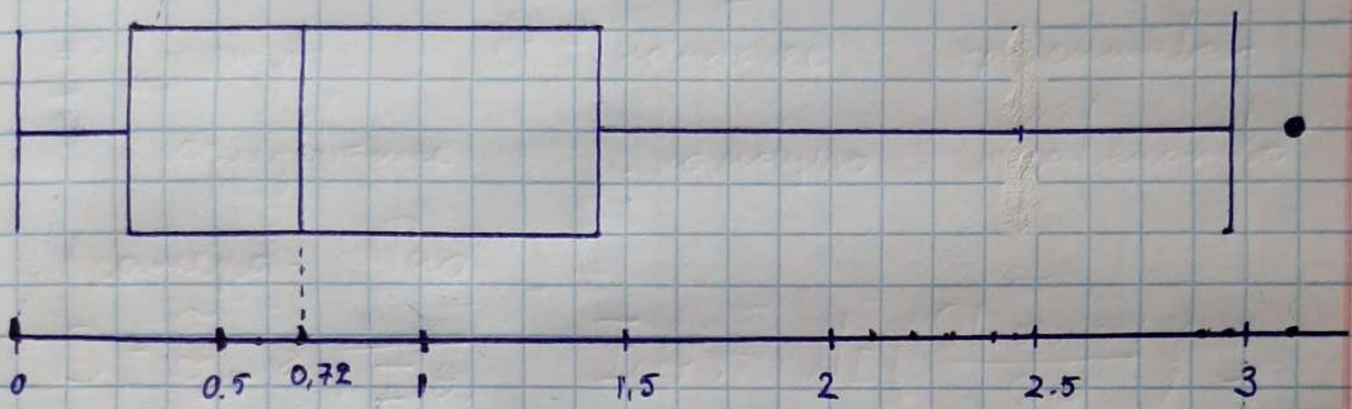
Ш-1.

Boxplot:



Совместная плотность распределения i и j элементов:

$$p(x_i, x_j) = p(x,y) = p(x)p(y) =$$

$$= \Big[25\, C_{24}^{i-1} e^{-x} (1-e^{-x})^{i-1} (e^{-x})^{25-i} \cdot$$

$$\cdot 25\, C_{24}^{j-1} e^{-x} (1-e^{-x})^{j-1} (e^{-x})^{25-j} \Big]$$

III-2

$\xi \sim R(0;\theta) \Rightarrow p(x) = \frac{1}{\theta}_{\theta} \; (0;\theta)$

$M[\xi] = \int_{-\infty}^{+\infty} \frac{1}{\theta} x\,dx = \int_0^{\theta} \frac{1}{\theta} x\,dx = \frac{x^2}{2\theta}\Big|_0^{\theta} = \frac{\theta}{2}$

$M[\xi^2] = \int_{-\infty}^{+\infty} \frac{1}{\theta} x^2\,dx = \int_0^{\theta} \frac{1}{\theta} x^2\,dx = \frac{x^3}{3\theta}\Big|_0^{\theta} = \frac{\theta^2}{3}$

$D[\xi] = M[\xi^2] - M^2[\xi] = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$

Первая оценка: $\tilde{\theta}_1 = \frac{2}{n}\sum_{i=1}^{n} x_i$.

а) Несмещённость:

$M[\tilde{\theta}_1] = M[\frac{2}{n}\sum_{i=1}^{n} x_i] = \frac{2}{n}\sum_{i=1}^{n} M[\xi] = \frac{2}{n}\, n M[\xi] =$

$= 2M[\xi] = 2\frac{\theta}{2} = \theta \Rightarrow \cancel{M[\xi]}\; M[\tilde{\theta}_1] = \theta \Rightarrow$

$\Rightarrow \tilde{\theta}_1$ – несмещённая

б) Состоятельность:

Проверка достаточного условия:

$D[\tilde{\theta}_1] = D[\frac{2}{n}\sum_{i=1}^{n} x_i] = \frac{4}{n^2}\sum_{i=1}^{n} D[\xi] =$

$= \frac{4}{n^2}\, n D[\xi] = \frac{4}{n} D[\xi] = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$

$\Rightarrow \tilde{\theta}_1$ – состоятельна.

Вторая оценка: $\tilde{\Theta}_2 = x_{min}$

$\Phi_{(x)} = 1 - (1 - F(x))^n = 1 - (1 - \int_0^x \frac{1}{\Theta} dt)^n$ - функция распределения min.

$\varphi(x) = \Phi'_{(x)} = -n(1 - \int_0^x \frac{1}{\Theta} dt)^{n-1} (-\frac{1}{\Theta}) =$

$= \frac{n}{\Theta}(1 - \frac{x}{\Theta})^{n-1}$ $(0; \Theta)$ - плотность распределения

а) Несмещённость:

$M[\tilde{\Theta}_2] = M[x_{min}] = \int_{-\infty}^{+\infty} n \frac{x}{\Theta}(1 - \frac{x}{\Theta})^{n-1} dx = \{x = \Theta t\}$

$= \int_{-\infty}^{+\infty} n t (1-t)^{n-1} \Theta dt = \int_0^1 n \Theta t (1-t)^{n-1} dt =$

$= B(2, n) \cdot \Theta n = \Theta n \frac{\Gamma(n)\Gamma(2)}{\Gamma(n+2)} = \frac{\Theta n \Gamma(n)}{(n+1)(n)\Gamma(n)} =$

$= \frac{\Theta}{n+1} \Rightarrow M[\tilde{\Theta}_2] = \frac{\Theta}{n+1} \Rightarrow \tilde{\Theta}_2$ - смещена

Коррекция: $\tilde{\Theta}_2' = (n+1)\tilde{\Theta}_2 = (n+1) x_{min}$

б) Состоятельность:

Проверка достаточного условия:

$D[\tilde{\Theta}_2] = M[\tilde{\Theta}_2^2] - M^2[\tilde{\Theta}_2]$

$M[\tilde{\Theta}_2^2] = \int_{-\infty}^{+\infty} n \frac{x^2}{\Theta}(1 - \frac{x}{\Theta})^{n-1} dx = \{t = \frac{x}{\Theta}\} =$

$= \int_0^1 n \Theta^2 t^2 (1-t)^{n-1} dt = B(3, n)\, n\Theta^2 =$

$$\Theta^2 n \cdot \frac{\Gamma(3)\Gamma(n)}{\Gamma(n+3)} = \frac{2\Theta^2 n\ \Gamma(n)}{(n+2)(n+1)n\Gamma(n)} = \frac{2\Theta^2}{(n+2)(n+1)}$$

$$D[\widetilde{\Theta}_2] = \frac{2\Theta^2}{(n+2)(n+1)} - \frac{\Theta^2}{(n+1)^2} =$$

$$= \frac{2(n+1)\Theta^2 - (n+2)\Theta^2}{(n+2)(n+1)^2} = \frac{2n\Theta^2 + 2\Theta^2 - n\Theta^2 - 2\Theta^2}{(n+2)(n+1)^2} =$$

$$= \frac{n\Theta^2}{(n+2)(n+1)^2}$$

Дисперсия несмещённой оценки:

$$D[\widetilde{\Theta}_2'] = D[(n+1)\widetilde{\Theta}_2] = (n+1)^2 D[\widetilde{\Theta}_2] =$$

$$= (n+1)^2 \frac{n\Theta^2}{(n+2)(n+1)^2} = \frac{n\Theta^2}{n+2} \xrightarrow[n\to\infty]{\not} 0$$

Достаточное условие не выполнено.

По определению:

$$\widetilde{\Theta}_2' \xrightarrow{P} \Theta$$

$$P(|\widetilde{\Theta}_2' - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|\widetilde{\Theta}_2' - \Theta| \geq \varepsilon) \geq P(\widetilde{\Theta}_2' \geq \Theta + \varepsilon) =$$

$$= P((n+1)x_{min} \geq \Theta + \varepsilon) =$$

$$P(x_1 \geq \frac{\Theta+\varepsilon}{n+1}, \dots, x_n \geq \frac{\Theta+\varepsilon}{n+1}) = \prod_{i=1}^{n} P(\xi_i \geq \frac{\Theta+\varepsilon}{n+1}) =$$

$$= (1 - P(\xi < \frac{\Theta+\varepsilon}{n+1}))^n = (1 - F(\xi < \frac{\Theta+\varepsilon}{n+1}))^n =$$

$$= (1 - \frac{1}{\Theta}\frac{\Theta+\varepsilon}{n+1})^n \xrightarrow[n\to\infty]{} e^{-\frac{\Theta+\varepsilon}{\Theta}} \Rightarrow$$

$\Rightarrow \tilde{\Theta}_2$ - не состоятельная оценка

<u>Третья оценка:</u> $\tilde{\Theta}_3 = x_{max}$

$$\Phi(x) = (F(x))^n = \left(\frac{x}{\Theta}\right)^n$$

$$\varphi(x) = \Phi'(x) = n\left(\frac{x}{\Theta}\right)^{n-1}\frac{1}{\Theta} = \frac{nx^{n-1}}{\Theta^n}$$

а) Несмещённость:

$$M[\tilde{\Theta}_3] = \int_0^{\Theta}\frac{nx^n}{\Theta^n}dx = \frac{n}{n+1}\frac{x^{n+1}}{\Theta}\Big|_0^{\Theta} = \frac{n}{n+1}\Theta \Rightarrow$$

$\Rightarrow \tilde{\Theta}_3$ - смещена

Коррекция: $\tilde{\Theta}_3' = \frac{n+1}{n}\tilde{\Theta}_3 = \frac{n+1}{n}x_{max}$

б) Состоятельность

Проверка достаточного условия:

$$M[\tilde{\Theta}_3^2] = \int_0^{\Theta}\frac{nx^{n+1}}{\Theta^n} = \frac{n}{n+2}\frac{x^{n+2}}{\Theta^n}\Big|_0^{\Theta} = \frac{n}{n+2}\Theta^2$$

$$D[\tilde{\Theta}_3] = M[\tilde{\Theta}_3^2] - M^2[\tilde{\Theta}_3] = \frac{n}{n+2}\Theta^2 - \frac{n^2}{(n+1)^2}\Theta^2 =$$

$$= n\left(\frac{(n+1)^2\Theta^2 - n(n+2)\Theta^2}{(n+2)(n+1)^2}\right) = \cancel{n^2\Theta^2}$$

$$= n\left(\frac{n^2\Theta^2 + 2n\Theta^2 + \Theta^2 - n^2\Theta^2 - 2n\Theta^2}{(n+2)(n+1)^2}\right) = \frac{n\Theta^2}{(n+2)(n+1)^2}$$

Дисперсия несмещённой оценки:

$$D[\tilde{\theta}_3'] = D\left[\frac{n+1}{n} x_{max}\right] =$$

$$= \frac{(n+1)^2}{n^2} D[\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \tilde{\theta}_3'$ – состоятельна.

Четвёртая оценка: $\tilde{\theta}_4 = x_{min} + x_{max}$

а) Несмещённость:

$$M[\tilde{\theta}_4] = M[\tilde{\theta}_2 + \tilde{\theta}_3] = M[\tilde{\theta}_2] + M[\tilde{\theta}_3] =$$

$$\frac{\theta}{n+1} + \frac{n}{n+1}\theta = \theta \Rightarrow \tilde{\theta}_4 \text{ – не смещена}$$

б) Состоятельность

Проверка достаточного условия:

$$D[\tilde{\theta}_4] = D[\tilde{\theta}_2] + D[\tilde{\theta}_3] + 2\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3)$$

Плотность распределения:

$$g(y,z) = n(n-1)\left(\frac{z}{\theta} - \frac{y}{\theta}\right)^{n-1} \frac{1}{\theta^2} \quad (z \geq y,\ z, y \in [0, \theta])$$

$$M[\tilde{\theta}_2 \tilde{\theta}_3] = \iint\limits_{-\infty}^{+\infty} yz\, g(y,z) = \int\limits_0^{\theta} dz \int\limits_0^{z} yz\, n(n-1)\left(\frac{z}{\theta} - \frac{y}{\theta}\right)^{n-1} \frac{dy}{\theta^2}$$

Внутренний интеграл:

$$\int\limits_0^{z} yz\, n(n-1)\left(\frac{z}{\theta} - \frac{y}{\theta}\right)^{n-1} \frac{1}{\theta^2} dy = \left\{t = \frac{y}{z}\right\} =$$

$$= \int_0^1 z^2 t (n-1) n (1-t)^{n-2} \frac{1}{\Theta^n} z^{n-2} \cdot z\,dt =$$

$$= z^{n+1} \frac{n(n-1)}{\Theta^n} B(2, n-1) = \frac{z^{n+1}}{\Theta^n} n(n-1) \frac{\Gamma(2)\Gamma(n-1)}{\Gamma(n+1)} =$$

$$= \frac{z^{n+1}}{\Theta^n} n(n-1) \frac{\Gamma(n-1)}{n\Gamma(n)} = \frac{z^{n+1}}{\Theta^n} n(n-1) \frac{\Gamma(n-1)}{n(n-1)\Gamma(n-1)} =$$

$$= \frac{z^{n+1}}{\Theta^n}$$

Внешний интеграл:

$$\int_0^\Theta \frac{z^{n+1}}{\Theta^n} dz = \frac{1}{n+2} \frac{z^{n+2}}{\Theta^n} \Big|_0^\Theta = \frac{\Theta^2}{n+2} \Rightarrow$$

$$\Rightarrow \operatorname{cov}(\widetilde{\Theta}_2, \widetilde{\Theta}_3) = M[\widetilde{\Theta}_2 \widetilde{\Theta}_3] - M[\widetilde{\Theta}_2] M[\widetilde{\Theta}_3] =$$

$$= \frac{\Theta^2}{n+2} - \frac{n\Theta^2}{(n+1)^2} = \frac{n^2\Theta^2 + 2n\Theta^2 + \Theta^2 - n^2\Theta^2 - 2n\Theta^2}{(n+2)(n+1)^2} =$$

$$= \frac{\Theta^2}{(n+2)(n+1)^2}$$

$$D[\widetilde{\Theta}_4] = \frac{2n\Theta^2}{(n+2)(n+1)^2} + \frac{2\Theta^2}{(n+2)(n+1)^2} = \frac{2\Theta^2}{(n+2)(n+1)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$

$\Rightarrow \widetilde{\Theta}_4$ – состоятельна.

Пятая оценка: $\widetilde{\Theta}_5 = x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i$

а) Несмещённость

$$M[\widetilde{\Theta}_5] = M[x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i] = M[\xi] + \frac{1}{n-1} \sum_{i=2}^{n} M[\xi] =$$

$$= M[\xi] + \frac{1}{n-1}(n-1) M[\xi] = 2M[\xi] = \Theta \Rightarrow$$

$\Rightarrow \tilde{\Theta}_5$ – не смещена

б) Состоятельность

Проверка достаточного условия:

$$D[\tilde{\Theta}_5] = D[x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i] =$$

$$= D[\xi] + \frac{1}{(n-1)^2}\sum_{i=2}^{n} D[\xi] = D[\xi] + \frac{1}{n-1} D[\xi] =$$

$$= \frac{n}{n-1} D[\xi] \xrightarrow[n\to\infty]{\not} 0$$

По определению:

$$x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} \xi + M[\xi] \xrightarrow[n\to\infty]{\not} 0 \Rightarrow$$

$\Rightarrow \tilde{\Theta}_5$ – не состоятельна

Определение наиболее эффективной оценки:

$$D[\tilde{\Theta}_1] = \frac{\Theta^2}{3n}$$

$$D[\tilde{\Theta}_3'] = \frac{\Theta^2}{n(n+2)}$$

$$D[\tilde{\Theta}_4] = \frac{2\Theta^2}{(n+2)(n+1)}$$

$$\frac{1}{n(n+2)} \leq \frac{1\cdot 2}{(n+2)(n+1)}$$

$$\frac{1}{n} \leq \frac{2}{n+1} \; ; \; n+1 \leq 2n \; ; \; n \leq 2n-1 \quad \forall n \Rightarrow$$

=> третья оценка самая эффективная

П-3

$$\xi \sim p(x) = \begin{cases} \frac{1}{\Theta}\exp(-\frac{x}{\Theta}), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

<u>Первая оценка:</u> $\tilde{\Theta}_1 = \bar{x}$

$$M[\xi] = \int_0^{+\infty} \frac{x}{\Theta}\exp(-\frac{x}{\Theta})dx = -x\exp(-\frac{x}{\Theta})\Big|_0^{+\infty} +$$

$$+ \int_0^{+\infty}\exp(-\frac{x}{\Theta})dx = -\Theta\exp(-\frac{x}{\Theta})\Big|_0^{+\infty} = \Theta$$

$$D[\xi] = M[\xi^2] - M^2[\xi]$$

$$M[\xi^2] = \int_0^{+\infty}\frac{x^2}{\Theta}\exp(-\frac{x}{\Theta})dx = -x^2\exp(-\frac{x}{\Theta})\Big|_0^{+\infty} +$$

$$2\cdot\int x\exp(-\frac{x}{\Theta})dx = -2\Theta x\exp(-\frac{x}{\Theta})\Big|_0^{+\infty} +$$

$$+ 2\Theta\int_0^{+\infty}\exp(-\frac{x}{\Theta})dx = -2\Theta^2\exp(-\frac{x}{\Theta})\Big|_0^{+\infty} =$$

$$2\Theta^2$$

$$D[\xi] = 2\Theta^2 - \Theta^2 = \Theta^2$$

Несмещённость:

$$M[\tilde{\Theta}_1] = M[\bar{x}] = M[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n}\sum_{i=1}^{n} M[\xi] =$$

$$= \frac{1}{n}\cdot n\,\Theta = \Theta \Rightarrow M[\tilde{\Theta}_1] = \Theta \Rightarrow \text{несмещённая}$$

$$D[\tilde{\Theta}_1] = D[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n^2}\sum_{i=1}^{n} D[\xi] = \frac{\Theta^2}{n}$$

<u>Вторая оценка:</u> $\tilde{\Theta}_2 = \frac{x_{min} + x_{max}}{2}$

$$M[\tilde{\Theta}_2] = \frac{1}{2}(M[x_{min}] + M[x_{max}])$$

Считаем $M[x_{min}]$:

Функция распределения: $F_{min}(x) = 1-(1-F(x))^n =$

$$= 1-(1-\int_0^x \frac{1}{\Theta}e^{-\frac{t}{\Theta}}dt)^n = 1-(\cancel{1}\cancel{-1} + e^{-\frac{x}{\Theta}})^n =$$

$$= 1 - e^{-\frac{xn}{\Theta}}$$

$$p(x) = (1-e^{-\frac{xn}{\Theta}})' = \frac{n}{\Theta}e^{-\frac{xn}{\Theta}} = \frac{3}{\Theta}e^{-\frac{3x}{\Theta}}$$

$$M[x_{min}] = \int_0^{+\infty} \frac{3x}{\Theta}e^{-\frac{3x}{\Theta}}dx = \{t = \frac{3x}{\Theta}\} =$$

$$= \int_0^{+\infty} \frac{\Theta}{3} t e^{-t} dt = \frac{\Theta}{3}\Gamma(2) = \frac{\Theta}{3}$$

Считаем $M[x_{max}]$:

Функция распределения: $F_{max}(x) = (1-e^{-\frac{x}{\Theta}})^n$

$$p(x) = n(1-e^{-\frac{x}{\theta}})^{n-1}\frac{1}{\theta}e^{-\frac{x}{\theta}} = \frac{3}{\theta}e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})^2$$

$$= \frac{3}{\theta}e^{-\frac{x}{\theta}}(1-2e^{-\frac{x}{\theta}}+e^{-\frac{2x}{\theta}}) =$$

$$= \frac{3}{\theta}(e^{-\frac{x}{\theta}} - 2e^{-\frac{2x}{\theta}} + e^{-\frac{3x}{\theta}})$$

$$M[x_{max}] = \int_0^{+\infty}\frac{3x}{\theta}e^{-\frac{x}{\theta}}dx - \cancel{\frac{8}{\theta}}3\int_0^{+\infty}\frac{2}{\theta}x e^{-\frac{2x}{\theta}}dx +$$

$$+ \int_0^{+\infty}\frac{3x}{\theta}e^{-\frac{3x}{\theta}}dx = 3\int_0^{+\infty}\theta t e^{-t}dt - 3\int_0^{+\infty}\frac{\theta}{2}te^{-t}dt$$

$$+ \int_0^{+\infty}\frac{\theta}{3}te^{-t}dt = 3\theta - \frac{3}{2}\theta + \frac{\theta}{3} = \frac{3}{2}\theta + \frac{\theta}{3}$$

$$M[\tilde{\theta}_2] = \frac{1}{2}\left(\frac{3}{2}\theta + \frac{\theta}{3} + \frac{\theta}{3}\right) = \frac{1}{2}\left(\frac{3}{2}\theta + \frac{2}{3}\theta\right) =$$

$= \frac{13}{12}\theta \Rightarrow$ оценка смещена.

Коррекция: $\tilde{\theta}_2' = \frac{12}{13}\tilde{\theta}_2 = \frac{6}{13}(x_{min} + x_{max})$

Дисперсия:

$$D[x_{min}] = M[x_{min}^2] - M^2[x_{min}];$$

$$M[\xi^2] = \int_0^{+\infty}\frac{3}{\theta}x^2 e^{-\frac{3x}{\theta}}dx = \left\{t = \frac{3x}{\theta}\right\} = \int_0^{+\infty}\frac{\theta^2}{9}t^2e^{-t}dt =$$

$$= \frac{\theta^2}{9}\Gamma(3) = \frac{2}{9}\theta^2$$

$$D[\xi] = \frac{2}{9}\theta^2 - \frac{\theta^2}{9} = \frac{\theta^2}{9}$$

$$D[x_{max}] = M[x_{max}^2] - M^2[x_{max}]$$

$$M[x_{max}^2] = \int_0^{+\infty} \frac{3x^2}{\theta} e^{-\frac{x}{\theta}} dx - 3\int_0^{+\infty} \frac{2x^2}{\theta} e^{-\frac{2x}{\theta}} dx +$$

$$+ \int_0^{+\infty} \frac{3x^2}{\theta} e^{-\frac{3x}{\theta}} dx = 3\theta^2 \Gamma(3) - \frac{3}{4}\theta^2 \Gamma(3) +$$

$$+ \frac{\theta^2}{9}\Gamma(3) = 2\left(3\theta^2 - \frac{3}{4}\theta^2 + \frac{1}{9}\theta^2\right) =$$

$$= \frac{9}{2}\theta^2 + \frac{2}{9}\theta^2$$

$$D[x_{max}] = \frac{9}{2}\theta^2 + \frac{2}{9}\theta^2 - \left(\frac{3}{2}\theta + \frac{\theta}{3}\right)^2 =$$

$$= \frac{9}{2}\theta^2 + \frac{2}{9}\theta^2 - \frac{9}{4}\theta^2 - \theta^2 - \frac{1}{9}\theta^2 =$$

$$= \frac{9}{4}\theta^2 - \frac{4}{4}\theta^2 + \frac{1}{9}\theta^2 = \frac{5}{4}\theta^2 + \frac{1}{9}\theta^2 = \frac{49}{36}\theta^2$$

$$D[\tilde{\theta}_2] = \frac{1}{4}D[x_{min}] + \frac{1}{4}D[x_{max}] + \frac{2}{4}\mathrm{cov}(x_{min}, x_{max})$$

$$\mathrm{cov}(x_{min}, x_{max}) = M[x_{min} \cdot x_{max}] - M[x_{min}]M[x_{max}]^2$$

$$M[x_{min} x_{max}] = \iint_{-\infty}^{+\infty} g(y,z)\, yz\, dy\, dz$$

$$g(y,z) = n(n-1)(F(z) - F(y))^{n-2} p(z)p(y), \quad z \geq y$$

$$g(y,z) = 6\left(1 - e^{-\frac{z}{\theta}} - 1 + e^{-\frac{y}{\theta}}\right)\frac{1}{\theta^2} e^{-\frac{z}{\theta}} \cdot e^{-\frac{y}{\theta}} =$$

$$= 6\left(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}}\right)\frac{1}{\theta^2} e^{-\frac{z}{\theta}} \cdot e^{-\frac{y}{\theta}}$$

$$M[x_{min}\, x_{max}] = \iint\limits_{-\infty}^{+\infty} yz\cdot 6\left(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}}\right)\frac{1}{\theta^2} e^{-\frac{z}{\theta}}\cdot e^{-\frac{y}{\theta}}\, dy\,dz =$$

$$= \int_0^{+\infty} dz \int_0^z 6yz\left(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}}\right)\frac{1}{\theta^2} e^{-\frac{z}{\theta}}\cdot e^{-\frac{y}{\theta}}\, dy$$

~~Внутренний интеграл:~~

$$\int_0^z \frac{6}{\theta^2} yz\left(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}}\right) e^{-\frac{z}{\theta}}\cdot e^{-\frac{y}{\theta}}\, dy =$$

$$= \int_0^z \frac{3}{\theta} z e^{-\frac{z}{\theta}}\, \frac{2y}{\theta} e^{-\frac{2y}{\theta}}\, dy - \int_0^z \frac{6}{\theta} z e^{-\frac{2z}{\theta}}\, \frac{y}{\theta} e^{-\frac{y}{\theta}}\, dy =$$

$$= \int_0^{\frac{z\theta}{2}} \frac{3}{\theta}\cdot\frac{\theta}{2} z e^{-\frac{z}{\theta}}\, te^{-t}\, dt - \int_0^{z\theta} \frac{6}{\theta}\cdot\theta z e^{-\frac{2z}{\theta}}\, te^{-t}\, dt$$

$$= \frac{3}{2} z e^{-\frac{z}{\theta}} \int_0^{\frac{z\theta}{2}} te^{-t}\, dt - 6ze^{-\frac{2z}{\theta}} \int_0^{z\theta} te^{-t}\, dt$$

Внутренний интеграл:

$$\frac{6}{\theta^2} z e^{-\frac{z}{\theta}} \int_0^z \left[ye^{-\frac{2y}{\theta}} - ye^{-\frac{y}{\theta}}\cdot e^{-\frac{z}{\theta}}\right] dy =$$

$$= \frac{6}{\theta^2} z e^{-\frac{z}{\theta}}\left(\frac{\theta^2}{4} - \theta^2 e^{-\frac{z}{\theta}} + \frac{3}{4}\theta^2 e^{-\frac{2z}{\theta}} + \frac{1}{2}\theta z e^{-\frac{2z}{\theta}}\right)\cancel{dz}$$

Внешний интеграл:

$$\int_0^{+\infty} \frac{6}{\theta^2} z e^{-\frac{z}{\theta}}\left(\frac{\theta^2}{4} - \theta^2 e^{-\frac{z}{\theta}} + \frac{3}{4}\theta^2 e^{-\frac{2z}{\theta}} + \frac{1}{2}\theta z e^{-\frac{2z}{\theta}}\right) dz =$$

$$= \frac{13}{18}\theta^2$$

$$M[x_{min}x_{max}] = \frac{13}{18}\theta^2$$

$$cov = \frac{13}{18}\theta^2 - \frac{\theta}{3}\cdot\frac{11\theta}{6} = \frac{2}{18}\theta^2$$

$$D[(x_{min}+x_{max})\cdot\frac{1}{2}] = \frac{1}{4}D[x_{min}] + \frac{1}{4}D[x_{max}] +$$

$$+ \frac{1}{2}cov(x_{min}, x_{max}) = \frac{1}{4}\frac{\theta^2}{9} + \frac{1}{4}\frac{49\theta^2}{36} + \frac{\theta^2}{18} =$$

$$= \frac{61}{144}\theta^2$$

$$D[\tilde{\theta}_2'] = \frac{144}{169}\cdot\frac{61}{144}\theta^2 = \frac{61}{169}\theta^2$$

<u>Третья оценка:</u> $\tilde{\theta}_3 = x_{(2)}$

$$p(x) = 6\cdot\frac{1}{\theta}e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})e^{-\frac{x}{\theta}} = \frac{6}{\theta}(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}})$$

$$M[\tilde{\theta}_3] = \int_0^{+\infty}\frac{6x}{\theta}(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}})dx = \frac{5}{6}\theta \Rightarrow \text{смещена}$$

Коррекция: $\tilde{\theta}_3' = \frac{6}{5}\tilde{\theta}_3$

Дисперсия:

$$D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3]$$

$$M[\tilde{\theta}_3^2] = \int_0^{+\infty}\frac{6x^2}{\theta}(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}})dx = \frac{19}{18}\theta^2$$

$$D[\tilde{\theta}_3] = \frac{19}{18}\theta^2 - \frac{25}{36}\theta^2 = \frac{13}{36}\theta^2$$

$D[\tilde{\theta}_3'] = \frac{36}{25} \frac{13}{36} \theta^2 = \frac{13}{25} \theta^2$

Исследование эффективности:

$D[\tilde{\theta}_1] = \frac{\theta^2}{3}$, $D[\tilde{\theta}_2'] = \frac{81}{169} \theta^2$, $D[\tilde{\theta}_3'] = \frac{13}{25} \theta^2 \Rightarrow$

$\Rightarrow$ при $n=3$ $\tilde{\theta}_1$ - наиболее эффективна

б) Исследование с помощью неравенства Крамера-Рао:

$D[\tilde{\theta}_i] \geq \frac{1}{n I_1(\theta)}$; $I_1(\theta) = M[(\frac{\partial \ln p(x,\theta)}{\partial \theta})^2]$

$\ln p(x,\theta) = \ln \frac{1}{\theta} e^{-\frac{x}{\theta}} = \ln \frac{1}{\theta} + \ln e^{-\frac{x}{\theta}} = \ln \frac{1}{\theta} - \frac{x}{\theta}$

$\frac{\partial \ln p(x,\theta)}{\partial \theta} = 1 / \frac{1}{\theta} \cdot (-\frac{1}{\theta^2}) + \frac{x}{\theta^2} = \frac{x}{\theta^2} - \frac{1}{\theta}$

$\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2 = \frac{x^2}{\theta^4} - \frac{2x}{\theta^3} + \frac{1}{\theta^2}$

$M[(\frac{\partial \ln p(x,\theta)}{\partial \theta})^2] = M[\frac{x^2}{\theta^4} - \frac{2x}{\theta^3} + \frac{1}{\theta^2}] =$

$= \int_0^{+\infty} \left(\frac{x^2}{\theta^4} - \frac{2x}{\theta^3} + \frac{1}{\theta^2} \right) \frac{1}{\theta} e^{-\frac{x}{\theta}} dx =$

$= \int_0^{+\infty} \frac{x^2}{\theta^5} e^{-\frac{x}{\theta}} dx - \int_0^{+\infty} \frac{2x}{\theta^4} e^{-\frac{x}{\theta}} dx + \int_0^{+\infty} \frac{1}{\theta^3} e^{-\frac{x}{\theta}} dx$

$= \frac{2}{\theta^2} - \frac{2}{\theta^2} - \frac{1}{\theta^2} e^{-\frac{x}{\theta}} \Big|_0^{+\infty} = \frac{1}{\theta^2}$

$$\frac{1}{n I(\theta)} = \frac{1}{n \frac{1}{\theta^2}} = \frac{\theta^2}{3} = D[\tilde{\theta}_1] \Rightarrow$$

$\Rightarrow \tilde{\theta}_1$ - эффективна по Крамеру - Рао.

Остальные оценки не эффективны по Крамеру Рао.