Randomized SVD for Hilbert-Schmidt operators low-rank approximation

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The problem:

Low rank approximation for the kernel function $G(x, y)$ of an integral Hilbert-Schmidt operator $\mathscr{F}$:

$$(\mathscr{F} f)(x) = \int_D G(x, y) f(y) \, \mathrm{d}y, \quad x \in D, \quad f \in L^2(D)$$

Applications:

1. Learning integral kernels such as Green functions associated with linear partial differential equations

2. Compressed storage of Integral operators

3. Fast Integral operators evaluation

Recently proposed solution

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A GENERALIZATION OF THE RANDOMIZED SINGULAR VALUE DECOMPOSITION

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The article proposes an algorithm for the generalization of randomized SVD for Hilbert-Schmidt operators with non-standard covariance kernels.

The algorithm

Algorithm 1 Randomized SVD for HS operators

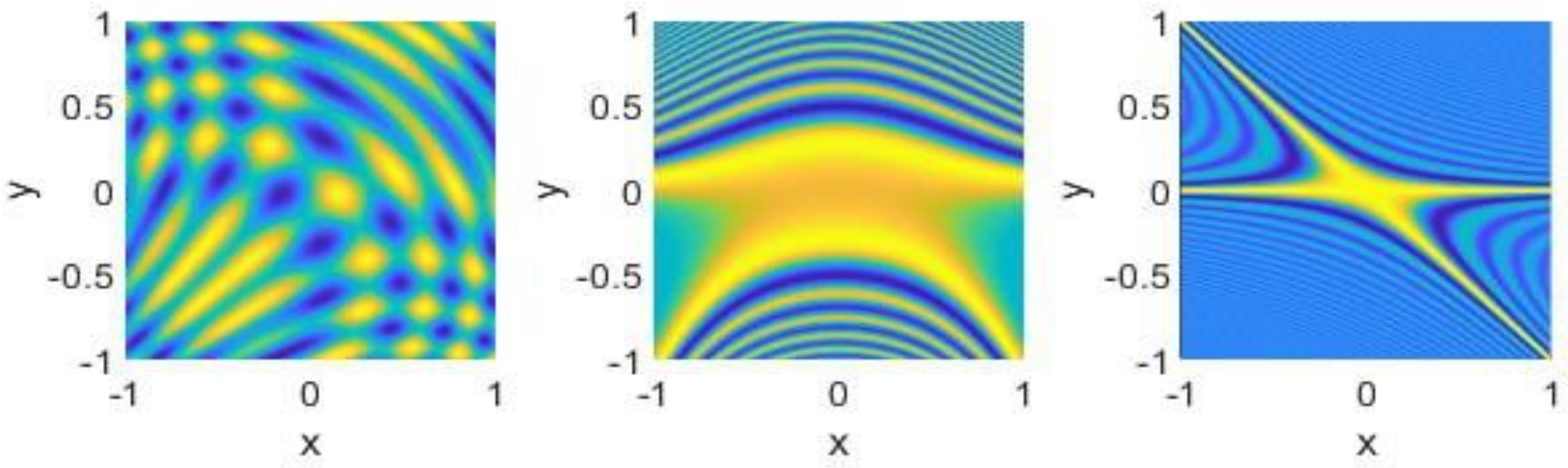
Input: HS integral operator $\mathscr{F}$ with kernel $G(x, y)$, number of samples $k > 0$
Output: Approximation G_k of G

Define a GP covariance kernel K
Sample the GP k times to generate a quasimatrix of random functions $\Omega = [f_1 \dots f_k]$
Evaluate the integral operator at Ω, $Y = [\mathscr{F}(f_1) \dots \mathscr{F}(f_k)]$
Orthonormalize the columns of Y, $Q = \mathrm{orth}(Y) = [q_1 \dots q_k]$
Compute an approximation to G by evaluating the adjoint of $\mathscr{F}$
Initialize $G_k(x, y)$ to 0
for $i = 1 : k$ **do**
    $G_k(x, y) \leftarrow G_k(x, y) + q_i(x) \int_D G(z, y) q_i(z)\, \mathrm{d}z$ // $G_k = Q(G^*Q)^*$
end for

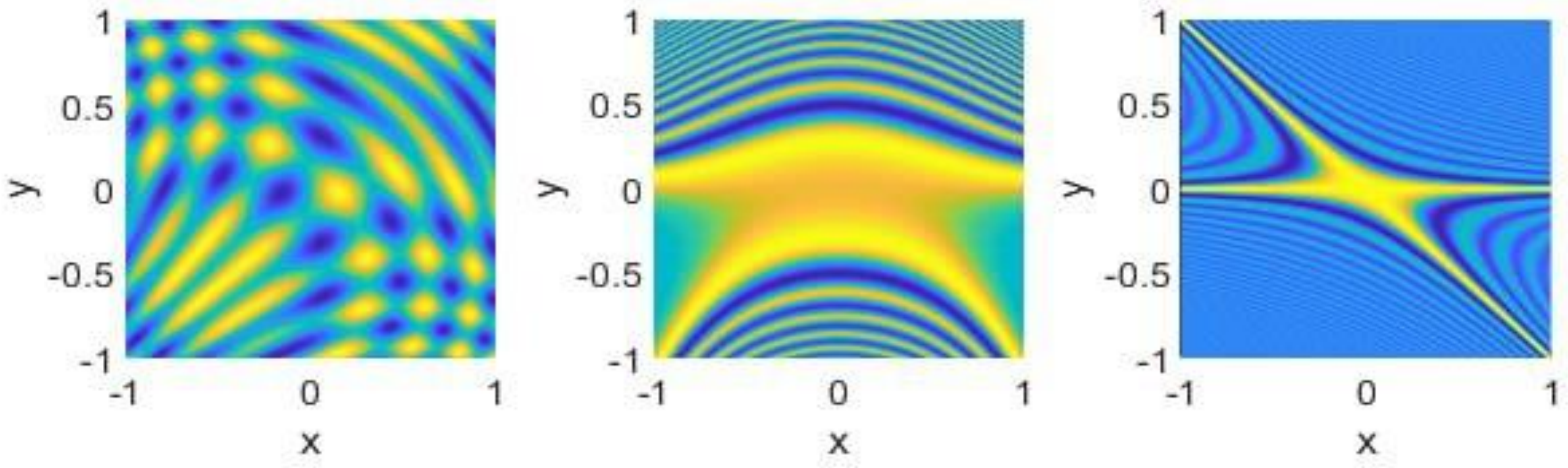
Experiments: original kernels vs obtained approximations

Kernel	Trigonometric	Airy	Bessel
Approx. rank	k = 5	k = 50	k = 100

Original



Learned

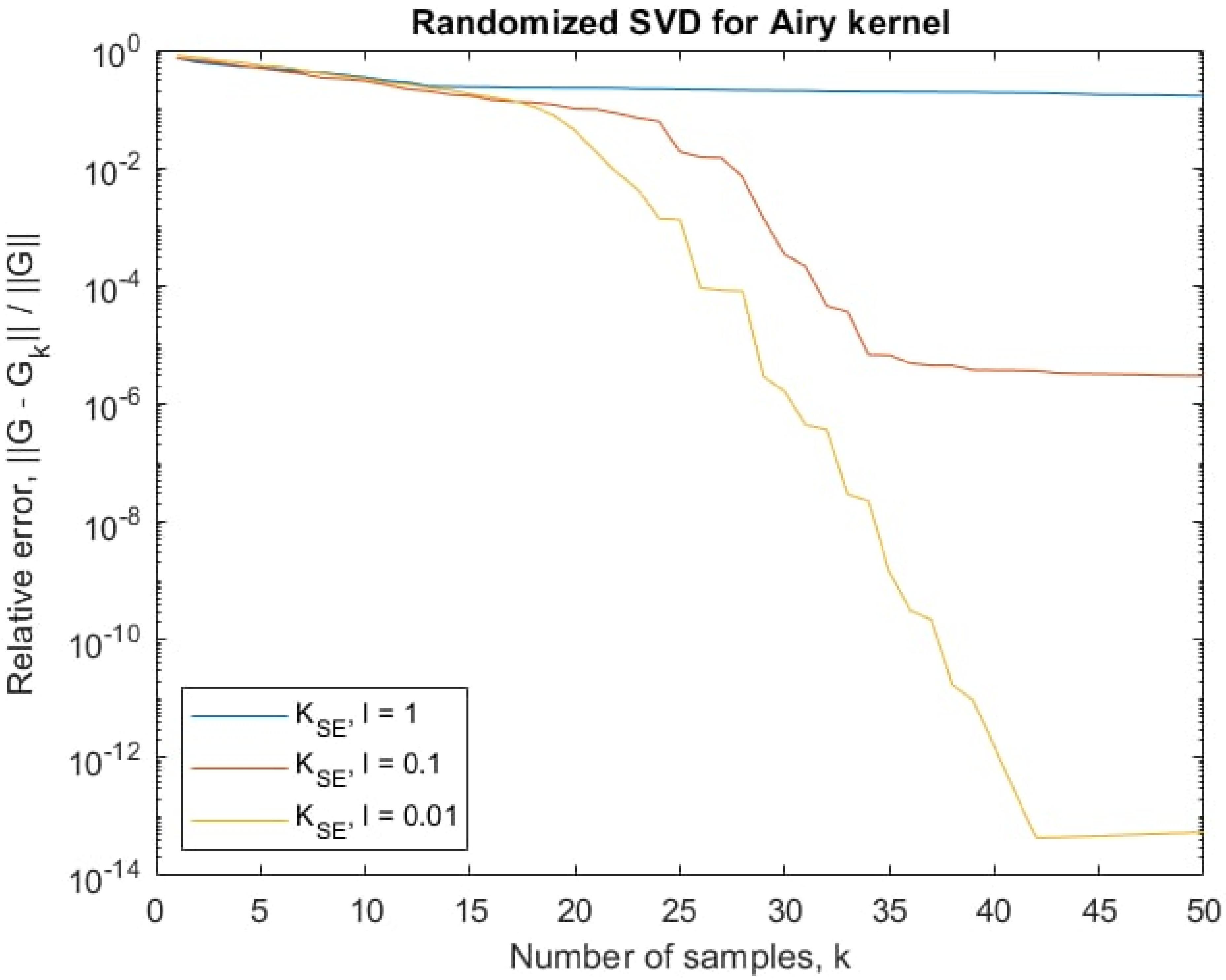


Experiments: accuracy of approximation for Airy kernel

Airy kernel:

$$G(x, y) = \mathrm{Ai}(-13(x^2 y + y^2))$$

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t, \quad x \in \mathbb{R}$$

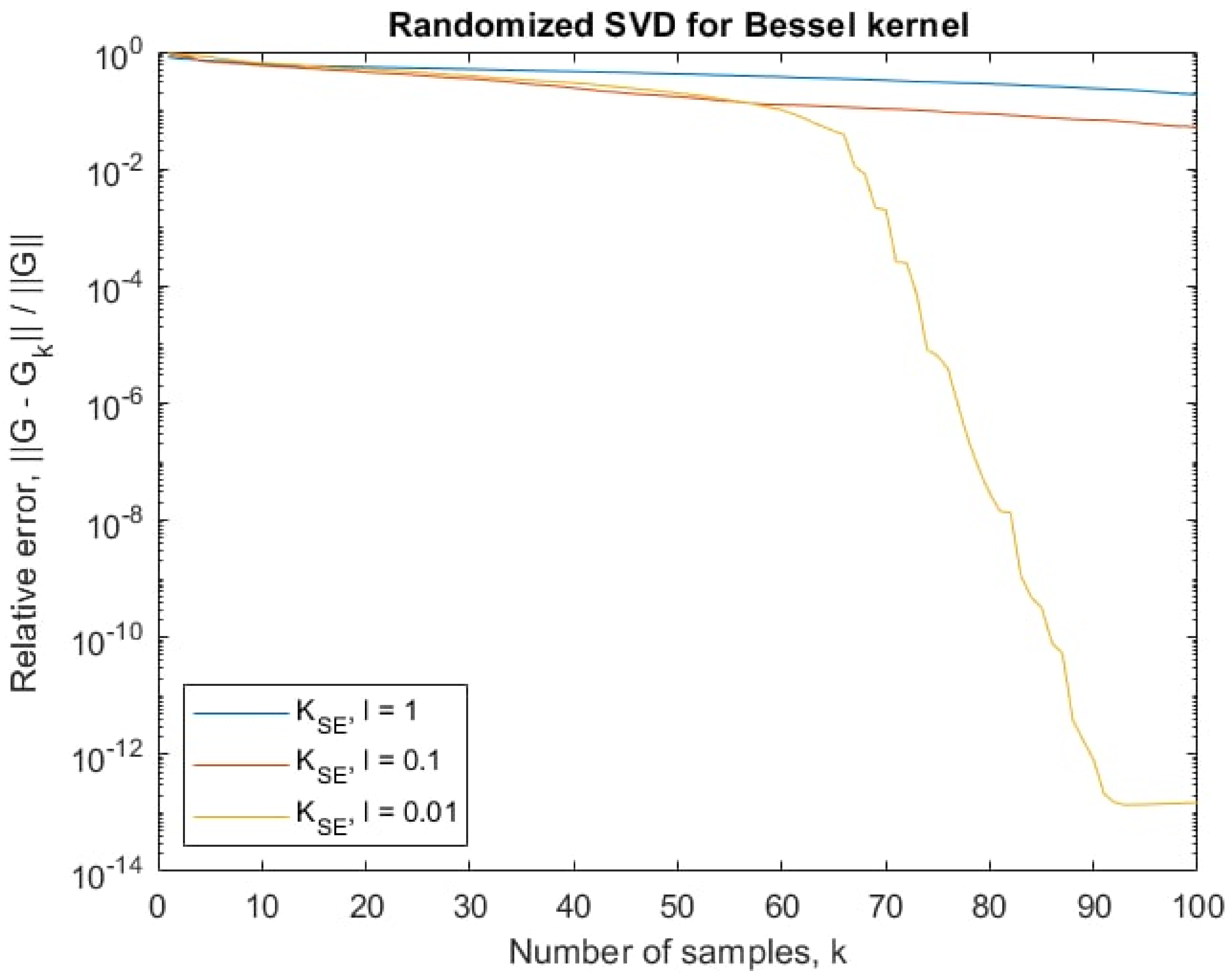


Experiments: accuracy of approximation for Bessel kernel

Bessel kernel:

$$G(x,y) = J_0(100(xy+y^2))$$

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t)\,\mathrm{d}t, \quad x \in \mathbb{R}$$



Approximation of matrices using non-standard covariance functions

Theorem 2 *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix,* $k \geq 1$ *an integer, and choose an oversampling parameter* $p \geq 4$. *If* $\mathbf{\Omega} \in \mathbb{R}^{n\times(k+p)}$ *is a Gaussian random matrix, where each column is sampled from a*

multivariate Gaussian distribution with covariance matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$, *and* $\mathbf{QR} = \mathbf{A\Omega}$ *is the economized QR decomposition of* $\mathbf{A\Omega}$, *then for all* $u, t \geq 1$,

$$\|\mathbf{A} - \mathbf{QQ}^*\mathbf{A}\|_{\mathrm{F}} \leq \left(1 + ut\sqrt{(k+p)\frac{3k}{p+1}\frac{\beta_k}{\gamma_k}}\right)\sqrt{\sum_{j=k+1}^{n}\sigma_j^2(\mathbf{A})}, \qquad (2)$$

with failure probability at most $t^{-p} + [ue^{-(u^2-1)/2}]^{k+p}$. *Here,* $\gamma_k = k/(\lambda_1 \operatorname{Tr}((\mathbf{V}_1^*\mathbf{K}\mathbf{V}_1)^{-1}))$ *denotes the covariance quality factor, and* $\beta_k = \operatorname{Tr}(\mathbf{\Sigma}_2^2\mathbf{V}_2^*\mathbf{K}\mathbf{V}_2)/(\lambda_1\|\mathbf{\Sigma}_2\|_{\mathrm{F}}^2)$, *where* λ_1 *is the largest eigenvalue of* $\mathbf{K}$.

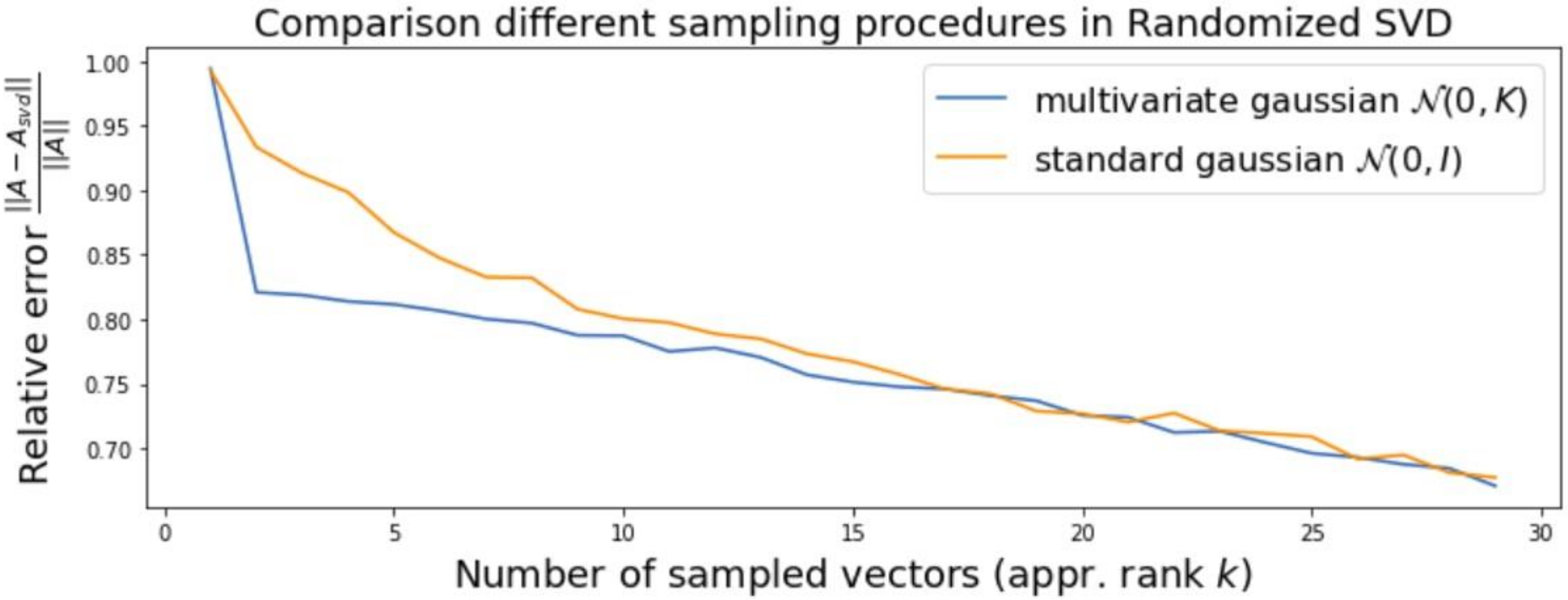
Highlight that, the more prior knowledge of the matrix A that can be incorporated into the covariance matrix, the better

Approximation of matrices using non-standard covariance functions

Operator: $\mathcal{L}u = d^2u/dx^2 - C \cdot \sin(Kx) \cdot u$, where $x \in [0, 1]$

Estimated matrix: the Green's function of $\mathcal{L}$

Kernel K is constructed by the Green's function of $\widehat{\mathcal{L}}u = -d^2u/dx^2$



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Team:

- **Emil Alkin** – performing experiments, presentation;
- **Arkadiy Vladimirov** – the idea, article analysis, algorithm realization;
- **Evgeny Gurov** – article analysis, algorithm realization, organization, github handling;
- **Aleksandr Tolmachev** – finite case algorithm realization.

Thx!

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