Derivatives

Definition and Notation

If $y=f(x)$ then the derivative is defined to be $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

If $y=f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x)=y'=\frac{df}{dx}=\frac{dy}{dx}=\frac{d}{dx}\left(f(x)\right)=Df(x)$$

If $y=f(x)$ all of the following are equivalent notations for derivative evaluated at $x=a$.

$$f'(a)=y'\big|_{x=a}=\frac{df}{dx}\bigg|_{x=a}=\frac{dy}{dx}\bigg|_{x=a}=Df(a)$$

Interpretation of the Derivative

If $y=f(x)$ then,

1. $m=f'(a)$ is the slope of the tangent line to $y=f(x)$ at $x=a$ and the equation of the tangent line at $x=a$ is given by $y=f(a)+f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x=a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x=a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(c\,f)'=c\,f'(x)$
2. $(f\pm g)'=f'(x)\pm g'(x)$
3. $(f\,g)'=f'\,g+f\,g'$ – **Product Rule**
4. $\left(\dfrac{f}{g}\right)'=\dfrac{f'\,g-f\,g'}{g^2}$ – **Quotient Rule**
5. $\dfrac{d}{dx}(c)=0$
6. $\dfrac{d}{dx}\left(x^n\right)=n\,x^{n-1}$ – **Power Rule**
7. $\dfrac{d}{dx}\left(f\left(g(x)\right)\right)=f'\left(g(x)\right)g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\dfrac{d}{dx}(x)=1$	$\dfrac{d}{dx}(\csc x)=-\csc x\cot x$	$\dfrac{d}{dx}\left(a^x\right)=a^x\ln(a)$
$\dfrac{d}{dx}(\sin x)=\cos x$	$\dfrac{d}{dx}(\cot x)=-\csc^2 x$	$\dfrac{d}{dx}\left(\mathbf{e}^x\right)=\mathbf{e}^x$
$\dfrac{d}{dx}(\cos x)=-\sin x$	$\dfrac{d}{dx}\left(\sin^{-1}x\right)=\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\left(\ln(x)\right)=\dfrac{1}{x},\ x>0$
$\dfrac{d}{dx}(\tan x)=\sec^2 x$	$\dfrac{d}{dx}\left(\cos^{-1}x\right)=-\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{d}{dx}\left(\ln\lvert x\rvert\right)=\dfrac{1}{x},\ x\neq 0$
$\dfrac{d}{dx}(\sec x)=\sec x\tan x$	$\dfrac{d}{dx}\left(\tan^{-1}x\right)=\dfrac{1}{1+x^2}$	$\dfrac{d}{dx}\left(\log_a(x)\right)=\dfrac{1}{x\ln a},\ x>0$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\dfrac{d}{dx}\left(\left[f(x)\right]^n\right)=n\left[f(x)\right]^{n-1}f'(x)$
2. $\dfrac{d}{dx}\left(\mathbf{e}^{f(x)}\right)=f'(x)\mathbf{e}^{f(x)}$
3. $\dfrac{d}{dx}\left(\ln\left[f(x)\right]\right)=\dfrac{f'(x)}{f(x)}$
4. $\dfrac{d}{dx}\left(\sin\left[f(x)\right]\right)=f'(x)\cos\left[f(x)\right]$
5. $\dfrac{d}{dx}\left(\cos\left[f(x)\right]\right)=-f'(x)\sin\left[f(x)\right]$
6. $\dfrac{d}{dx}\left(\tan\left[f(x)\right]\right)=f'(x)\sec^2\left[f(x)\right]$
7. $\dfrac{d}{dx}\left(\sec\left[f(x)\right]\right)=f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\dfrac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right)=\dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The Second Derivative is denoted as $f''(x)=f^{(2)}(x)=\dfrac{d^2f}{dx^2}$ and is defined as $f''(x)=\left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n$^{\text{th}}$ Derivative is denoted as $f^{(n)}(x)=\dfrac{d^nf}{dx^n}$ and is defined as $f^{(n)}(x)=\left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the (n-1)$^{\text{st}}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y}+x^3y^2=\sin(y)+11x$. Remember $y=y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$\mathbf{e}^{2x-9y}(2-9y')+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11$$
$$2\mathbf{e}^{2x-9y}-9y'\mathbf{e}^{2x-9y}+3x^2y^2+2x^3y\,y'=\cos(y)\,y'+11 \quad\Rightarrow\quad y'=\frac{11-2\mathbf{e}^{2x-9y}-3x^2y^2}{2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)}$$
$$\left(2x^3y-9\mathbf{e}^{2x-9y}-\cos(y)\right)y'=11-2\mathbf{e}^{2x-9y}-3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x=c$ is a critical point of $f(x)$ provided either **1.** $f'(c)=0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x)>0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x)<0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x)=0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x)>0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x)<0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x=c$ is a inflection point of $f(x)$ if the concavity changes at $x=c$.