Find the derivatives of the following formula,

(i) $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in R^d$

Ans: If,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{bmatrix}$$

Transpose of x,

$$x^T = [x_1 \ x_2 \ x_3 \ \dots \ x_d]$$

So, $x^T x = [x_1^2 + x_2^2 + x_3^2 + \dots + x_d^2]$

By applying the chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}(\log_e(1+z)) \cdot \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z} \frac{d}{dz}(1+z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + x_3^2 + \dots + x_d^2]$$

$$= \frac{1}{1+z}(0+1) \cdot (2x_1 + 2x_2 + 2x_3 + \dots + 2x_d)$$

$$= \frac{1}{1+z} \cdot 2(x_1 + x_2 + x_3 + \dots + x_d)$$

$$= \frac{2}{1+z} \cdot \sum_{i=1}^{d} x_i$$

$$\therefore \frac{df}{dx} = \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

(Ans)

(ii) $f(z) = e^{-\frac{z}{2}}$; where $z = g(y)$, $g(y) = y^T S^{-1} y$, $y = h(x)$, $h(x) = x - \mu$

Ans: Given,

$$z = g(y) = y^T S^{-1} y$$

$$y = h(x) = x - \mu$$

By using the chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Here,

$$\frac{df}{dz} = \frac{d}{dz}\left(e^{-\frac{z}{2}}\right) = e^{-\frac{z}{2}} \cdot -\frac{1}{2} = -\frac{e^{-\frac{z}{2}}}{2}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(h)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h \to 0} (y^T S^{-1} + S^{-1} y + h S^{-1}) = y^T S^{-1} + S^{-1} y$$

$$\frac{dy}{dx} = \frac{d}{dx}(x - \mu)$$

$$= 1 - 0$$

$$= 1$$

$$\therefore \frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= -\frac{1}{2} e^{-\frac{z}{2}} \cdot (y^T S^{-1} + S^{-1} y) \cdot 1$$

$$= -\frac{1}{2} e^{-\frac{z}{2}} S^{-1} (y^T + y)$$

(Ans)