# Celestial Mechanics

Corey Anderson

Monday 20th November, 2023

**Abstract**

This paper presents a detailed exploration of the fundamental principles underlying celestial mechanics, focusing on the derivation and implications of Kepler's Laws of Planetary Motion. Initially, the study delves into the historical and mathematical derivation of the inverse square law of gravitation, a cornerstone in understanding gravitational interactions. Subsequently, the paper examines the experiments that obtains the gravitational constant (G), a pivotal moment in the quantification of gravitational forces. The core of the paper is dedicated to a thorough analysis of Kepler's three laws of planetary motion. Each law is derived and discussed in detail, emphasizing their relevance and application in modern astronomy. The first law, concerning the elliptical orbits of planets, is explored through the lens of orbital mechanics. The second law, which deals with the areal velocity of a planet, is examined in the context of angular momentum conservation. Finally, the third law, relating to the harmonic relationship between the orbital period and the semi-major axis of an orbit, is investigated, highlighting its crucial role in the determination of celestial distances and the understanding of the solar system's dynamics.

## 1 Newton's Law of Universal Gravitation

### 1.1 The Inverse Square Rule

Newton used the orbit of the Moon around the Earth to deduce that gravitation was an inverse square rule. Newton knew that the gravitational acceleration on the surface of Earth was $g = 9.8\,\mathrm{m\,s^{-1}}$, and from measuring the Earth's shadow on the Moon, it can be shown that the Earth-Moon distance $R_{E,M}$ is approximately $60R_{Earth} \approx 3.84 \times 10^8\,\mathrm{m}$ . The Moon's orbital period was also measured to be 27.32 days. Therefore,

$$v_M = \frac{2\pi R_{E,M}}{T_M} = 1022\,\mathrm{m\,s^{-1}} \tag{1}$$

$$a_c = \frac{v_M^2}{R_{E,M}} = 0.002\,72\,\mathrm{m\,s^{-2}} \tag{2}$$

and thus the ratio between acceleration on the surface on Earth to the acceleration a distance $R_{E,M}$ away is

$$\frac{g}{a_c} \approx 60^2. \tag{3}$$

This tells us that the gravitational acceleration is $60^2$ times less on the Moon than the surface of the Earth. However, we also saw that $R_{E,M}/R_{Earth} = 60$. Therefore, the gravitational acceleration is likely to be inversely proportional to the square of the distance between two objects

$$a \propto \frac{1}{r^2}. \tag{4}$$

Newton verified that this equation holds from the observation of other planets around our solar system. By observing the solar system, it is evident that larger objects possess more gravitational force. so

$$a \propto \frac{M}{r^2}. \tag{5}$$

where $M$ is the object generating the gravitational field. Using Newton's second law on another object of mass $m$ that is within the gravitational field,

$$F \propto \frac{Mm}{r^2}, \tag{6}$$

$$\text{or} \quad F = \frac{GMm}{r^2} \tag{7}$$

where $G$ is a constant defined as the Universal gravitational constant.

## 1.2 The Universal Gravitational Constant

The experimental apparatus for determining the universal gravitational constant is shown in Figure 1. The two large objects of mass $M$ attracts the smaller masses $m$ which causes rod to rotate and induces a torsion force in the wire.

The torque on the torsion wire is $\tau_w = \kappa\theta$ where $\theta$ is the deflection angle of the rod and $\kappa$ is the torsion coefficient. The torque in the opposite direction is generated by the gravitational force $F$ of the masses. Thus at equilibrium:

$$\kappa\theta = \frac{L}{2}F + \frac{L}{2}F \tag{8}$$
$$= LF \tag{9}$$
$$= L\frac{GMm}{r^2}. \tag{10}$$

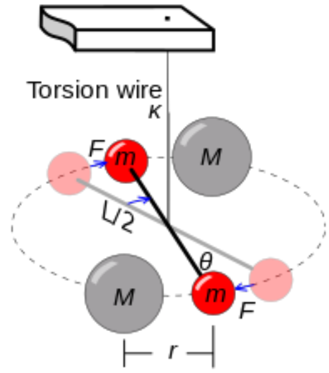


Figure 1: The torsion balance

To measure the torsion coefficient of the rod, we can measure the natural resonant oscillation period $T$ of the torsion balance:

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{11}$$

where $I$ is the moment of inertia of the two small balls assuming the mass of the torsion beam is negligible. Thus

$$T = 2\pi\sqrt{\frac{mL^2}{2\kappa}}. \tag{12}$$

Solving for $\kappa$ and substituting this into Equation 10 gives

$$G = \frac{2\pi^2 L r^2 \theta}{MT^2} \tag{13}$$

where the universal gravitational constant can be calculated to be $G = 6.67 \times 10^{-11}\,\mathrm{N\,kg^{-2}\,m^2}$.

## 2 Kepler's Laws of Planetary Motion

Newton's law of Universal Gravitation tells us that

$$m\ddot{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}}. \tag{14}$$

The vertical and horizontal components are

$$m\ddot{x} = -G\frac{Mm}{r^2}\cos\theta \tag{15}$$

$$m\ddot{y} = -G\frac{Mm}{r^2}\sin\theta. \tag{16}$$

Since an orbit is in a two dimensional plane, we can use polar coordinates and its corresponding time derivatives:

$$x = r\cos\theta \tag{17}$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{18}$$

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta \tag{19}$$

$$y = r\sin\theta \tag{20}$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{21}$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta \tag{22}$$

We shall now derive $\ddot{x}\cos\theta + \ddot{y}\sin\theta$ in two different forms. From Equation 15 and 16, we have

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = -\frac{GM}{r^2} \tag{23}$$

and from Equations 19 and 22, we have

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r\dot{\theta}^2 \tag{24}$$

. Thus,

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}. \tag{25}$$

Note that if $\ddot{r} = 0$ and $r$ is a constant, then this just becomes the equation for a circular orbit with $a_c = r\dot{\theta}^2$. Equation 25 is the foundation in which we will derive Kepler's three laws.

## 2.1 Law 2: Equal Areas in Equal Time

I know what you're thinking. Why are we starting with law 2 instead of law 1? This is because an important result in the derivation of law 2 will be needed for law 1. In the previous section, if we instead solved for $\ddot{x}\sin\theta + \ddot{y}\cos\theta$, we would get

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{26}$$

Multiplying both sides by $r$, we find

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \tag{27}$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{28}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0. \tag{29}$$

Thus, the angular momentum $L$ of the planet is conserved. This result is not surprising because there is no torque acting on the system:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = r\frac{GMm}{r^2}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = 0. \tag{30}$$

If we integrate Equation 29, we get $mr^2\dot{\theta} = L$, where $L$ is a constant defined to be the angular momentum. Integrating once more gives

$$\int L\,dt = m\int r^2\frac{d\theta}{dt}\,dt \tag{31}$$

$$= m\int_{\theta_i}^{\theta_f} r^2\,d\theta \tag{32}$$

$$= 2m\Delta A \tag{33}$$

Performing this integral gives us Kepler's 2nd Law:

$$\boxed{\Delta A = \frac{L\Delta t}{2m}} \tag{34}$$

Segments of orbits seek out equal areas in equal intervals of time.

## 2.2 Law 1: Elliptical Orbits

We need to solve the differential equation for $r$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}. \tag{35}$$

Making the substitution $r \to u^{-1}$, we have

$$\frac{dr}{dt} = -u^{-2}\frac{du}{dt} \tag{36}$$

$$= -u^{-2}\frac{du}{d\theta}\frac{Lu^2}{m} \tag{37}$$

$$= -\frac{L}{m}\frac{du}{d\theta}. \tag{38}$$

Differentiating gives us

$$\frac{d^2r}{dt^2} = -\frac{L}{m}\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} \tag{39}$$

$$= -\left(\frac{L}{m}\right)^2 u^2\frac{d^2u}{d\theta^2}. \tag{40}$$

Using this result, Equation 25 becomes

$$-\left(\frac{L}{m}\right)^2 u^2\frac{d^2u}{d\theta^2} - \left(\frac{L}{m}\right)^2 u^3 = -GMu^2 \tag{41}$$

which becomes

$$-\frac{d^2u}{d\theta^2} + \frac{GMm^2}{L^2} = u \tag{42}$$

Equation 42 has the general solution

$$u = A\cos(\theta + \delta) + \frac{GMm^2}{L^2}. \tag{43}$$

For simplicity, we can set $\delta$ to be zero. Now in terms of $r(\theta)$, we have

$$r(\theta) = \frac{1}{A\cos\theta + \frac{GMm^2}{L^2}} \tag{44}$$

$$= \frac{1}{\frac{GMm^2}{L^2}(e\cos\theta + 1)} \tag{45}$$

where $e$ is defined as $e = \frac{AL^2}{GMm^2}$. Equation 45 is an ellipse with $e$ being the eccentricity. It is clear that the aphelion and perihelion are given respectively by:

$$r_{max} = \frac{1}{\frac{GMm^2}{L^2}(1-e)}, \qquad r_{min} = \frac{1}{\frac{GMm^2}{L^2}(1+e)} \tag{46}$$

and the semi-major axis is given by

$$a = \frac{1}{2}(r_{min} + r_{max}) = \frac{L^2}{GMm^2}(1-e^2)^{-1}. \tag{47}$$

Since both $r_{min}$ and $r_{max}$ are distances from the Sun, the Sun is at one focus of the orbit. Which is indeed Kepler's 1st law: orbits are elliptical.

## 2.3 Law 3: Period of Motion

The centripetal force of an orbit is the gravitational force, so

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \tag{48}$$

$$\frac{GM}{r} = v^2 \tag{49}$$

$$= \frac{4\pi^2 r^2}{T^2}. \tag{50}$$

Rearranging gives

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM}. \tag{51}$$

Therefore, we have derived Kepler's third law: the period of an orbit squared is proportional to the radius cubed.