Projectile Motion Question

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Question: A projectile is launched off of a cliff that is 50m high. The initial velocity of the projectile is 300m/s and it lands 2000m away from the base of the cliff. Find the angle that the projectile was launched at.

From the kinematic equations:

$$x = v_0 t \cos\theta \tag{1}$$

and

$$y = y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2. \tag{2}$$

Substituting Equation 1 into Equation 2:

$$\begin{align} y &= y_0 + \left(x \frac{v_0 \sin\theta}{v_0 \cos\theta} \right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta} \right)^2 \tag{3} \\ &= y_0 + x \tan\theta - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\theta}. \tag{4} \end{align}$$

As a sanity check, when $x = 0$, Equation 4 gives $y = y_0 = 50\,\mathrm{m}$ which is what we expect. Now when $y = 0$, we know that $x = \Delta x$ where Δx is the range of the ball from the base of the cliff. So:

$$\begin{align} 0 &= y_0 + \Delta x \tan\theta - \frac{1}{2} g \frac{\Delta x^2}{v_0^2 \cos^2\theta} \tag{5} \\ 0 &= y_0 + \Delta x \tan\theta - \frac{K}{\cos^2\theta} \tag{6} \end{align}$$

where $K = \frac{g \Delta x^2}{2 v_0^2}$. This is the point in which I would resort to computational methods to solve for θ, but since you're probably studying for your exam I will continue with the algebraic solution via two different methods.

1 Quadratic Method

Equation 5 can be written as

$$\begin{align} 0 &= y_0 + \Delta x \tan\theta - K\left(\tan^2\theta + 1\right) \tag{7} \\ 0 &= -K \tan^2\theta + \Delta x \tan\theta + (y_0 - K). \tag{8} \end{align}$$

By the quadratic formula,

$$\tan\theta = \frac{-\Delta x \pm \sqrt{\Delta x^2 + 4K(y_0 - K)}}{-2K} \tag{9}$$

Substituting $\Delta x = 2000\,\mathrm{m}$, $y_0 = 50\,\mathrm{m}$, $v_0 = 300\,\mathrm{m\,s^{-1}}$, we get that $\theta = 83.73°$ and $4.84°$ which are two initial angles that give the same range of $2000\,\mathrm{m}$.

2 Phase Angle Method

Definitely go with the previous method, but here's an alternative method just for fun.

Multiplying Equation 6 by $\cos^2\theta$,

$$0 = y_0 \cos^2\theta + \Delta x \sin\theta\cos\theta - K \tag{10}$$

$$\begin{aligned} 0 &= \frac{1}{2}y_0(1+\cos 2\theta) + \frac{1}{2}\Delta x \sin 2\theta - K && (11)\\ &= \frac{1}{2}y_0\cos 2\theta + \frac{1}{2}\Delta x \sin 2\theta + \left(\frac{1}{2}y_0 - K\right) && (12)\\ &= y_0 \cos 2\theta + \Delta x \sin 2\theta + (y_0 - 2K) && (13) \end{aligned}$$

The sum of a cosine wave plus sine wave of the same frequency gives another sinusoidal wave. Therefore we can write $A\cos x + B\sin x = C\cos(x-\phi)$, where $C = \sqrt{A^2+B^2}$ and $\phi = \tan^{-1}\left(\frac{B}{A}\right)$. Using this, Equation 9 becomes:

$$\begin{aligned} 0 &= \sqrt{y_0^2 + \Delta x^2}\cos\left(2\theta - \tan^{-1}\left(\frac{\Delta x}{y_0}\right)\right) + (y_0 - 2K) && (14)\\ \frac{2K - y_0}{\sqrt{y_0^2+\Delta x^2}} &= \cos\left(2\theta - \tan^{-1}\left(\frac{\Delta x}{y_0}\right)\right) && (15)\\ \pm\cos^{-1}\left(\frac{2K-y_0}{\sqrt{y_0^2+\Delta x^2}}\right) &= 2\theta - \tan^{-1}\left(\frac{\Delta x}{y_0}\right) && (16)\\ \theta &= \pm\frac{1}{2}\cos^{-1}\left(\frac{\frac{g\Delta x^2}{v_0^2} - y_0}{\sqrt{y_0^2+\Delta x^2}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\Delta x}{y_0}\right) && (17) \end{aligned}$$

Substituting $\Delta x = 2000\,\mathrm{m}$, $y_0 = 50\,\mathrm{m}$, $v_0 = 300\,\mathrm{m\,s^{-1}}$, we get that $\theta = 83.73°$ and $4.84°$ which are the solutions we got using the Quadratic Method.