A Derivation of the Legendre Transform

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The Lagrangian is defined as the difference between the kinetic and potential energy

$$\mathcal{L} = T - V. \tag{1}$$

The Lagrangian is dependent on the position, velocity and my have explicit time-dependence

$$\mathcal{L} = \mathcal{L}(q, \dot{q}, t). \tag{2}$$

Using the chain rule to find the time-derivative of the Lagrangian:

$$\begin{align}
\frac{d\mathcal{L}}{dt} &= \frac{\partial \mathcal{L}}{\partial q}\frac{dq}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial \mathcal{L}}{\partial t} \tag{3}\\
&= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} + \frac{\partial \mathcal{L}}{\partial t} \quad \text{from the Euler-Lagrange Equation} \tag{4}\\
&= \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial \mathcal{L}}{\partial t} \quad \text{from the reverse product rule} \tag{5}\\
0 &= \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}\right) + \frac{\partial \mathcal{L}}{\partial t} \tag{6}\\
0 &= \frac{d}{dt}(\dot{q}p - \mathcal{L}) + \frac{\partial \mathcal{L}}{\partial t} \tag{7}\\
& \tag{8}
\end{align}$$

Now let the Hamiltonian be

$$\boxed{H = \dot{q}p - \mathcal{L}} \tag{9}$$

Equation 9 is known as the Legendre Transform and it allows us to transform between working with the Hamiltonian and the Lagrangian. To build some intuition behind the Legendre Transform, assume all forces are conservative such that the Lagrangian takes the general form

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - V(q) \tag{10}$$

and using the Legendre Transform, the Hamiltonian becomes

$$\begin{align}
H &= \dot{q}m\dot{q} - \frac{1}{2}m\dot{q}^2 + V(q) \tag{11}\\
&= \frac{1}{2}m\dot{q}^2 + V(q) \tag{12}
\end{align}$$

which is just the familiar form of kinetic energy plus potential energy. It is worth noting from Equation 7 that

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t} \tag{13}$$

so if the Lagrangian has no explicit time-dependence, the Hamiltonian is constant. In other words, energy is conserved if $\mathcal{L} = T - V$ does not depend on time. This is one of the conserved quantities as a result of Noether's Theorem.