Lagrangian Mechanics: Unraveling the Euler-Lagrange Equation and its Applications in Physics

Corey Anderson

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Abstract

This paper explores Lagrangian mechanics, focusing on the derivation and application of the Euler-Lagrange equation, a fundamental tool in classical and quantum physics. Connecting the equation with physics, the paper examines captivating examples, including the Brachistochrone problem, Noether's theorem, and the hanging chain problem.

The Euler-Lagrange equation embodies the principle of least action, describing a system's dynamics through a set of differential equations. The Brachistochrone problem showcases its application in finding the shortest time path for a sliding particle under gravity, while Noether's theorem reveals the connection between symmetries and conservation laws in physical systems. Additionally, the hanging chain problem demonstrates its versatility in tackling non-conservative systems.

Lagrangian mechanics serves as a powerful framework, offering profound insights into the laws governing the universe and providing a profound appreciation for its significance in physics.

Contents

1 The Euler-Lagrange Equation

1.1 Theorem

If a function $\mathcal{L}$ is defined by an integral of the form

$$S = \int \mathcal{L}(t, q, \dot{q})dt, \tag{1}$$

Then S has the stationary value if the differential equation

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0 \tag{2}$$

is satisfied. This differential equation is known as the Euler-Lagrange equation.

1.2 Proof

Consider two points $(t_1, q_1), (t_2, q_2)$, our goal is to connect the points while minimising some quantity. The optimal path function $\bar{q}(t)$ connecting the two points is the path that minimises the quantity. $\bar{q}(t)$ is given in terms of an arbitrary function $q(t)$ and an error function $\eta(t)$ such that

$$\bar{q}(t) = q(t) + \varepsilon\eta(t), \ \varepsilon \in \mathbb{R}. \tag{3}$$

Lets define the functional S to be

$$S = \int \mathcal{L}(t, q, \dot{q})dt, \tag{4}$$

this quantity is has a stationary point when

$$\begin{aligned}
0 &= \left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} &(5)\\
&= \int_{t_1}^{t_2} \frac{d}{d\varepsilon}\left[\mathcal{L}(t, \bar{q}, \dot{\bar{q}})\right]\Big|_{\varepsilon=0} dt &(6)\\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial t}\frac{\partial t}{\partial \varepsilon} + \frac{\partial \mathcal{L}}{\partial \bar{q}}\frac{\partial \bar{q}}{\partial \varepsilon} + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}}\frac{\partial \dot{\bar{q}}}{\partial \varepsilon}\right)\Bigg|_{\varepsilon=0} dt &(7)\\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \bar{q}}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}}\dot{\eta}(t)\right)\Bigg|_{\varepsilon=0} dt &(8)\\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\eta(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{\eta}(t)\right) dt &(9)\\
&= \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial q}\eta(t)dt + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\eta(t)\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\eta(t)dt &(10)\\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right)\eta(t)dt &(11)
\end{aligned}$$

Now since $\eta(t)$ is an arbitrary function, the condition must be

$$\boxed{\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0.} \tag{12}$$

1.3 The Lagrangian

An important point is that, from section 1.2, the partial derivatives are with respect to the variable on the vertical axis whereas the total derivative is with respect to the variable on the horizontal axis. This means that depending on the problem, we are free to swap variables. First lets take the vertical axis to be the position x and the horizontal axis to be the time. So we are effectively considering the time evolution of an object's position. That is, $q \to x$. This gives us:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0 \quad \Longrightarrow \quad \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \tag{13}$$

Lets call this form of the Euler-Lagrange equation the *temporal form.* Note that the temporal form has a similar form to Newton's second law.

Newton's second law

$$F = \frac{dp}{dt} \tag{14}$$

Euler-Lagrange Equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \tag{15}$$

This motivates the following definitions:

$$F = \frac{\partial \mathcal{L}}{\partial x} \tag{16}$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{17}$$

Which results in:

$$\begin{align} \mathcal{L} &= \int F dx + \phi(\dot{x}) \tag{18} \\ &= -V + \phi(\dot{x}) \tag{19} \end{align}$$

$$\begin{align} \mathcal{L} &= \int p d\dot{x} + \theta(x) \tag{20} \\ &= T + \theta(x) \tag{21} \end{align}$$

where T and V is the kinetic and potential energy respectively. Combining equations 19 and 21 gives us the following definition for what we call the *Lagrangian*:

$$\boxed{\mathcal{L} = T - V.} \tag{22}$$

To see the power behind this result, we will explore a fundamental principle of physics called the Principle of Least Action.

2 The Principle of Least Action

The Principle of Least Action states that a system will always behave in such way as to minimise the Action S, given by Equation 1. For example, consider a ball is launched from point A to point B in some gravitational field. If you consider *any* trajectory the ball did not take between A and B, and you calculate the kinetic energy and subtract the potential energy at every point along the path, then integrate it over the total duration of flight, you'll find that the result is always larger than that of the actual motion of the ball.

According to Theorem 1.1, to minimise the action, the Euler-Lagrange equation must be satisfied[1]. Therefore, the Euler-Lagrange equation is true for any physical system. I hope you're impressed because this result is significant. In Newtonian mechanics, we rely on forces to analyse problems. But this method can quickly get complicated because forces are vector quantities which point in different directions. However, using the Lagrangian we deal with scalar quantities (since they are energies) so they can be easily manipulated. In order to find the equations of motion, all we need is the potential and kinetic energy of the system, and using the Euler-Lagrange equation will provide the equations of motion of the system. This method is preferred over Newton's Laws for complex systems where the analysis of forces become tedious and complicated.

2.1 The Simple Spring

Consider a simple spring with mass m and spring constant k. We have

$$T = \frac{1}{2}m\dot{x}^2, \tag{23}$$
$$V = \frac{1}{2}kx^2. \tag{24}$$

So

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{25}$$

This spring system will behave in such way as to minimise the action S, for this to occur, the Euler-Lagrange equation must be satisfied:

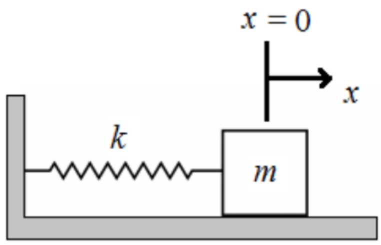

Figure 1: A spring connected to a rigid wall

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) && (26)\\ &= -kx - \frac{d}{dt}(m\dot{x}) && (27)\\ &= -kx - m\ddot{x} && (28)\end{aligned}$$

$$\implies \ddot{x} = -\frac{k}{m}x = -\omega^2 x, \quad \text{where} \;\; \omega = \sqrt{\frac{k}{m}}. \tag{29}$$

Evidently, this is just the familiar simple harmonic motion. Applying the Euler-Lagrange equation is equivalent to applying Newton's second law. But instead we deal with energies!

[1] Note that in section 1.1 we said the action is stationary when the Euler-Lagrange equation is satisfied. However, if the action is stationary it must be minimised since action as no upper bound. For example, kinetic energy can be as high as it wants.

2.2 The Beltrami Identity

Before we consider more complex physical systems, we require a very important identity called the *Beltrami Identity.* Lets consider a system with no time dependence, so we are only concerned with spatial variables. This means we have the familiar x and y axis. Using the change of variables, we can define the *spatial form* of the Euler-Lagrange equation:

$$0 = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right), \tag{30}$$

where $y' = \frac{dy}{dx}$. Now lets change this equation as follows:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) && (31) \\ &= \frac{\partial \mathcal{L}}{\partial y} y' - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) y'. && (32) \end{aligned}$$

From the multivariable chain rule,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \mathcal{L}}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \mathcal{L}}{\partial x}\frac{\partial x}{\partial x} && (33) \\ \implies \frac{\partial \mathcal{L}}{\partial y'} y' &= \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{L}}{\partial y'} y'', \quad \text{if } \mathcal{L} \text{ has no } x\text{-dependence } \left(\frac{\partial \mathcal{L}}{\partial x} = 0\right). && (34) \end{aligned}$$

Substituting Equation 34 into Equation 32 gives

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{L}}{\partial y'} y'' - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) y' && (35) \\ &= \frac{d}{dx}\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial y'} y'\right) && (36) \end{aligned}$$

$$\boxed{\mathcal{L} - \frac{\partial \mathcal{L}}{\partial y'} y' = \text{const}} \tag{37}$$

Equation 37 is called the Beltrami Identity and it is a special case of the spatial Euler-Lagrange equations when $\mathcal{L}$ has no x-dependence.

2.3 The Brachistochrone Curve

This is perhaps one of the most elegant curves in existence, it has three remarkable properties.

1. Suppose you have two points at different heights, name them A and B. Point A is at a greater height than point B. Your task is to draw a curve connecting A and B such that the ball gets from one point to the other in the least amount of time within a gravitational field, then that curve would be the brachistochrone.

2. No matter where you release the ball on the brachistochrone, it always takes the same amount of time to reach B.

3. The Brachistochrone can be constructed by placing a light source to the rim of the bicycle wheel and when the bike moves forward, the path the light traces out is the brachistochrone!

The next few pages are concerned with the derivation of these three properties.

2.3.1 The Property of Least Time

The ball has mass m, the coordinates of the start/end points are $A(x_A, y)$, $B(x_B, 0)$ and let s be the distance along the curve. The period the ball takes to roll from point A to point B is given by

$$T = \int_0^T dt \tag{38}$$
$$= \int_{x_A}^{x_B} \frac{ds}{v}. \tag{39}$$

By the conservation of energy

$$\frac{1}{2}mv^2 = mgy \longrightarrow v = \sqrt{2gy}, \tag{40}$$

and the arc length is

$$ds = \sqrt{dx^2 + dy^2} \tag{41}$$
$$= \sqrt{1 + \frac{dy}{dx}}dx \tag{42}$$

Putting these results into Equation 39 gives

$$T = \int_{x_A}^{x_B} \frac{1 + y'}{\sqrt{2gy}}dx \tag{43}$$
$$= \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \sqrt{\frac{1 + y'}{y}}dx, \tag{44}$$
$$\text{so } \mathcal{L}(y, y') = \sqrt{\frac{1 + y'}{y}} \tag{45}$$

Now from the theorem in section 1.1, we know that the value of y must satisfy the Euler-Lagrange equation. We can simplify this even further by noting that the Lagrangian has no x-dependence so the Euler-Lagrange equations reduce down to the Beltrami identity (Equation 37). Thus,

$$\text{const} = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial y'}y' \tag{46}$$
$$= \sqrt{\frac{1 + y'}{y}} - \frac{y'^2}{\sqrt{y(1 + y'^2)}}. \tag{47}$$

Solving this differential equation gives us the following solution:

$$\begin{cases} x = r(\phi - \sin\phi) \\ y = r(1 - \cos\phi) \end{cases} \quad \text{for } r \in \mathbb{R} \tag{48}$$

2.3.2 Property of Time Invariance

Suppose the ball is released from (x_0, y_0), then

$$dT = \frac{ds}{\dot{s}} \tag{49}$$

But

$$\frac{1}{2}m\dot{s}^2 = mg(y - y_0) \tag{50}$$

$$\rightarrow \dot{s} = \sqrt{2g(y - y_0)} \tag{51}$$

So

$$dT = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2g(y - y_0)}} \tag{52}$$

$$\tag{53}$$

Now using Equation 48 to change to angular coordinates:

$$T = \int_{\phi_0}^{\pi} \sqrt{\frac{2r^2(1 - \cos\phi)}{2rg(\cos\phi_0 - \cos\phi)}} d\phi \tag{54}$$

$$= \pi\sqrt{\frac{r}{g}} \tag{55}$$

which has no time dependence. Thus, the duration for the ball to travel to B is independent of the initial position of the ball.

2.3.3 Geometry Property

Consider a circle rolling along a smooth surface to the right with no sliding. Point B is initially at the origin, and after some time it is at an angle of ϕ to the vertical. This gives us:

$$\begin{cases} x = r\phi - r\sin\phi = r(\phi - \sin\phi) \\ y = r - r\cos\phi = r(1 - \cos\phi) \end{cases} \tag{56}$$

Which is identical to Equation 48. This shows that the path a point on the circumference traces out is just the brachistochrone curve.

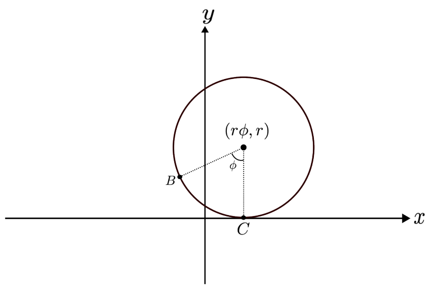

Figure 2: Circle rolling to the right with a point B on the circumference.

3 Lagrange Multipliers

Before moving on to the hanging chain problem, we need to introduce a new mathematical concept called Lagrange Multipliers. This is used to analyse systems where there exists a constraint. In other words, we want to extremise $f = f(x, y, z)$ subject to the constraint $g(x, y, z) = c,$ for $c \in \mathbb{R}$. The proof is as follows:

Suppose x_P is an extreme point. Let $\mathbf{r}(t)$ be any curve passing through x_P (i.e., $\mathbf{r}(t_0) = x_P$) on the constraint. Then $\boldsymbol{\nabla} g(x_P) \perp \mathbf{r}(t_0)$ since the gradient function is perpendicular to level curves. Now define $h(t) = f(\mathbf{r}(t)) \Rightarrow h'(t) = \boldsymbol{\nabla} f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ by the chain rule. Since h has an extremal point at t_0, then $h'(t_0) = \boldsymbol{\nabla} f(x_P) \cdot \mathbf{r}'(t_0) = 0$. Thus we have $\boldsymbol{\nabla} f(x_P) \perp \mathbf{r}'(t_0)$, and since $\mathbf{r}'(t)$ is the tangent function of any point passing through x_P, we have $\boldsymbol{\nabla} f(x_P) \perp g(x_P)$. Therefore, $\boldsymbol{\nabla} f(x_P) \parallel \boldsymbol{\nabla} g(x_P)$, or written in a more useful form:

$$\boxed{\boldsymbol{\nabla} f(x_P) = \lambda \boldsymbol{\nabla} g(x_P)} \tag{57}$$

This equation allows us to an analyse a problem where certain parameters are constrained.

4 The Hanging Chain

We can now apply Lagrange multipliers to the hanging chain problem. The system involves a chain hanging between two points within a gravitational field. We aim to use Lagrangian mechanics to derive the function describing the shape of the chain.

We begin by writing the action

$$S = \int_A^B mgy\, ds \tag{58}$$

$$= \int_{x_1}^{x_2} mgy\sqrt{1 + y'^2}\, dx\,. \tag{59}$$

Since the length of the chain is constant, we introduce a constraint:

$$g = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx = l. \tag{60}$$

Since we have only y dependence, the gradient operator in Equation 57 becomes a one dimensional derivative and the equation becomes:

$$0 = \delta(I + \lambda J) \tag{61}$$

$$= \delta\left\{ \int_{x_1}^{x_2} (mgy + \lambda)\sqrt{1 + y'^2}\, dx \right\} \tag{62}$$

This means that our constrained Lagrangian is given by

$$\mathcal{L} = (mgy + \lambda)\sqrt{1 + y'^2}. \tag{63}$$

The Lagrangian has no x dependence so we can use the Beltrami identity

$$\text{const} = \alpha = \frac{\partial \mathcal{L}}{\partial y'} y' - \mathcal{L} \tag{64}$$

$$= \frac{y'(mgy + \lambda)}{\sqrt{1 + y'^2}} y' - (mgy + \lambda)\sqrt{1 + y'^2}. \tag{65}$$

This gives

$$y' = \frac{dy}{dx} = \sqrt{\left(\frac{mgy + \lambda}{\alpha}\right)^2 - 1} \tag{66}$$

Integrating by substituting $\frac{mgy+\lambda}{\alpha} = \cosh\theta$ gives

$$y = \alpha \cosh\left(\frac{x + x_0}{\alpha}\right) + \beta, \tag{67}$$

for constants α, β and x_0. Note that I have absorbed factors into the constants for simplicity. Now to find the three constants, we require three constraints:

1.
$$l = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \tag{68}$$
$$= \alpha[\sinh(\alpha x_2 + x_0) - \sinh(\alpha x_1 + x_0)] \tag{69}$$

2.
$$x = x_1 \longrightarrow y = y_1 \tag{70}$$

3.
$$x = x_2 \longrightarrow y = y_2 \tag{71}$$

The purpose of the first constraint is to keep the length of the chain constant. Constraints 2 and 3 specify the stationary endpoints of the chain. This means that if we are given the values describing the system, we can use these three constraints to find the values of the three constants and form an exact function of the hight of the chain in terms of the horizontal position.