Homework 2

Evan Fox

2/13/2023

Morgan Prior and I worked together

Problem 1. *Let G be a graph and let λ_1 be the largest eigenvector of the ajacency matrix A. Show*

$$2\frac{m}{n} \leq \lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}$$

Proof. We will refer to the first inequality as (1) and the second as (2). (1) will follow clearly from teqniques discussed in class, namely, let $j = (1, \dots, 1)^T$. We will have $Aj = (d(1), \dots, d(n))^T$ and $\|j\| = \sqrt{n}$. Hence,

$$\frac{j^T}{\|j\|} A \frac{j}{\|j\|} = \frac{1}{n} \sum_{k=1}^{n} d(k) = d(G)$$

But since $\frac{j}{\|j\|}$ is a vector of norm 1, and we know that λ_1 is the maximum of the qudratic form $x^T Ax$ over all x such that $\|x\| = 1$, the first inequality follows.

$$\lambda_1 > d(G) = \frac{2m}{n}$$

Now (2) can be proven via an application of the Cauchy-Swartz inequality. Recall that for eigen values $\lambda_1, \dots, \lambda_n$,

1. $\sum_{k=1}^{n} \lambda_k = 0$
2. $\sum_{k=1}^{n} \lambda_k^2 = 2m$

So we have

$$\lambda_1 = -\sum_{k=2}^{n} \lambda_k$$

$$\lambda_1^2 = \left(-\sum_{k=2}^{n} \lambda_k\right)^2 \leq (n-1) \sum_{k=2}^{n} \lambda_k^2$$

Where the last inequality follows from C-S. Now replacing $\sum_{k=2}^{n} \lambda_k^2$ with $2m - \lambda_1^2$ we obtain

$$(n-1) \sum_{k=2}^{n} \lambda_k^2 = (n-1)(2m - \lambda_1^2) = 2mn - n\lambda_1^2 - 2m + \lambda_1^2$$

so that

$$\lambda_1^2 \le 2mn - n\lambda_1^2 - 2m + \lambda_1^2$$

Solving for λ_1 gives

$$\lambda_1 = \sqrt{\frac{2m(n-1)}{n}}$$

as desired.

$\square$

Problem 2. *Prove that a tree can have at most one perfect matching.*

Proof. Let T be a tree with a perfect matching, we prove that this matching is unique. I wil call an egde $e_v \in E(T)$ a *leaf edge* of the vertex v if it contains a vertice which is a leaf in T. We notice that in order for a matching M on T to be perfect, it must contain all leaf edges; otherwise, there would exist a leaf not covered by the matching. Let T_1 be the induced subgraph on all vertices $u \in V(T)$, under the condition that u is not contained in any leaf edge. In symbols $T_1 = T[u \in V(T) | u \notin e_v \forall v]$ where v runs over all leaves. We may define T_2 from T_1 in a similiar way and continue inductivly until we remove all vertices.

Now at each step it is clear that T_n remains a tree (and hence the matching will contain its leaf edges) , since we are only removeing leaves and all vertices ajacent to a leaf. Further, the perfect matching M on T is a perfect matching restricted to T_1(since for every element of the matching I remove, I remove both vertices it contains) , continuing inductivily, it follows that M restricted to T_n is a perfect matching.

Now assume that there exists another perfect matching M', then M' must contain all leaf edges in T as stated above. So that M and M' at least agree on the leaf edges of T. But then when we restrict M' to T_1, M' will also have to contain all leaf edges in T_1(otherwise it is not a perfect matching on T_1) which would be a contradiction). Continuing, we will see that M' must contain all leaf edges in T_n, but this will force M and M' to coincede, hence the perfect matching is unique (if it exits). $\square$

Problem 3. *Suppose that G is bipartite and that none of its eigen values are 0. Show that G has a perfect matching*

I am not sure how to use your hint to use the determinate but I think I found a nice proof

Proof. Let $G = X \dot{\cup} Y$. We prove that $|X| = |Y|$ then by halls condition, we will have a perfect matching. The trick will be to pick a labeling of the graph that makes the ajacency matrix have a nice form. Let the vertices in X be labeled from $\{1, \dots, k\}$ and the verticies in Y be labeled $\{n-k, \dots, n\}$. Then since X and Y are independant sets, when forming $A(G)$ we get the following block matrix

$$A(G) = \begin{pmatrix} 0_{k\times k} & a_{(n-k)\times k} \\ b_{k\times(n-k)} & 0_{(n-k)\times(n-k)} \end{pmatrix}$$

We know that 0 is not an egienvalue of this matrix, hence it is invertable, by inspection one can see the inverse of this matrix to be of the form

$$A(G)^{-1} = \begin{pmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{pmatrix}$$

but this will imply that $a_{(n-k)\times k}$ is left and right invertable, hence it is a square matrix (same is true for b). Hence $k = n - k$ and we get that there will exits a perfect matching.

To see that the converse is false, look at a 4-cycle. This is bipartite since it doesnt contain an odd cycle, further, it has a perfect matching. However, when you look at the adjacency matrix, you see that it is not invertable (row-echolen form has row of all zeros). $\square$

In case I need to be more cear in finding the inverse note

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \times \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

Then this gives

$$0x + ay = I$$

$$0z + aw = 0$$

$$bx + 0y = 0$$

$$bz + 0w = I$$

So $w = x = 0$ and y is the right inverse of a and z is the right inverse of b, multipling the other directions shows a and b to be left invertable.