Exam 1

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3/9/2023

Problem 1. *First I prove the following: Let $n \geq 4$ and G be on n vertices and let $4 < k < n$. Then if every induced subgraph on k vertices is disconnected, G is disconected.*

Proof. if $k = n - 1$, it is clear, since in any connected graph $n > 4$ there exists a vertex v such that $G \setminus v$ is conected. This can be seen first by removing any vertex of degree 1, if none exists then $\delta \geq 2$ and there is a cycle from which an appropiate vertice can be removed(not every edge is a brige). Then for any $4 < k < n$, if all induced subgraphs on k vertices are disconnected, it follows that any induced subgraph of order $k + 1$ is disconnected by the above. This will continue so that all induced subgraphs of order n-1 are disconnected and we see that G must be disconected.

Now to solve the question at hand, if n = 4, clearly P_3 is the only solution, by checking cases. Now let G be on n vertices and assume G and $\overline{G}$ are connected. Then if G does not contain P_3, the complement of every connected induced subgraph of order 4 must be disconnected (since P_3 is the only graph on 4 vertices with connected complement), these subgraphs are induced subgraphs in $\overline{G}$ and clearly every induced subgraph of order 4 in $\overline{G}$ is obtained this way, hence$\overline{G}$ is disconnected by the above remark; a contradiction.

So we may conclude that G contains a copy of P_3. $\square$

Problem 2. *Proof.* Suppose that G is connected and that there exits vertices a, b, c such that $\{b, c\} \in E(G)$ and $dist(a, b) = dist(a, c)$. We prove this holds iff there is an odd cycle. First if there is an odd clycle then clearly this will hold, fix a vertex v in the cycle, then the two furthest vertices in the cycle will have the same distence to v and they will be connected by an edge (since they are furthest). Conversly, let P be a minimal lenght path from a to b and Q be a minimal length path from a to c. let w be vertex with the largest distance from v such that both P and Q contain it. Then $wPbcQw$ is an odd cycle, since P and Q have the same lenght (this remains true if we start from w instead of v) the sum of their lengths is even, then the $\{b, c\}$ edge makes it an odd cycle. Hence G is not bipartite. $\square$

Problem 3. *Proof.* Let M and N be disjoint matchings and suppose that $|M| = |N| + k$. Then let $S \subset M$ satisfying $|S| = k - 1$. Then set

$$M_1 = N \cup S$$

$$N_1 = M \setminus S$$

We have that $|M_1| = |N \cup S| = |M| - k + (k - 1) = |M| - 1$, where I am using $S \cap N = \varnothing$ (since $S \subset M$) and $|N_1| = |M \setminus S| = |N| + k - (k - 1) = |N| + 1$, again since $S \subset M$. Clearly these are disjoint,

$$M_1 \cap N_1 = (N \cup S) \cap (M \setminus S) = (N \cap (M \setminus S)) \cup (S \cap (M \setminus S)) = \varnothing \cup \varnothing$$

and they have the same union, since

$$N \cup S \cup (M \setminus S) = N \cup M$$

□

Problem 4. *Proof.* First we assume $n = 2k$ then extend the result.

Let G be a graph on $2k$ vetices such that $\delta \geq k$. We show that $\alpha'(G) \geq k$ by finding a spanning path in G. If $k = 1$ then G is P_1 and the result is satisfied. Now for $k > 1$, first note that G is connected, since otherwise, the smaller component can have at most k vertices and hence has a minimal degree less than k.

Now consider a longest path $P = \{x_0, x_1, \ldots, x_l\}$ in G and order the vertices by their index. Both x_0 and x_l have at least k neighbors on the path, let E_1 denote the set of all edges whose maximal (by the ordering) vertex is adjacent to x_0 and let E_2 denote the set of all edges whose minimal vertex is adjacent to x_l. Then both $|E_1| \geq k \leq |E_2|$, since the number of edges in P is at most $2k - 1$, we have $E_1 \cap E_2 \neq \varnothing$. Hence we may fix $e = \{x_i, x_{i+1}\} \in E_1 \cap E_2$. Now the path $x_0 x_{i+1} P x_l x_i p x_1$ is a path in G that vists every vertex, since if there were a vertice w left unvisted, we would have a path Q from w to x_j then by travesring Q onto P, then going to the nearest end vertice, say x_l we construct a path $wQx_jPx_lx_iPx_0$ which contradics our assumption that P is the longest path. Now we have a path that vists all $2k$ vertices in G exactly once and doesnt repeat egdes. By constructing a matching M which takes every other edge from the path we get a matching of size k, since $\|P\| = 2k - 1$. As desired.

Now if $n > 2k$ the we can no longer prove that G is connected, if fact it may be the case that all connected components of G have less than $2k$ vertices. For each connected component C_i, we will consider the longest path in each C_i. If $|C_i| \leq 2k$ the above argument applies and we get a matching of size $\lfloor \frac{|C_i|}{2} \rfloor$ (which is k if $|C_i| = 2k$ which was the case above).

If there is C_i such that $|C_i| > 2k$, then the longest path is at least $2k$ vertices since if the longest path P has $|P| < 2k$, using the fact that we are in a connected component with minimal degree k along with the argument from the first paragraph will force P to be spannin, hence $|C_i| < 2k$; a contradiction.

Then by selecting alternating edges in the path (of length at least $2k-1$) we get a matching of sufficint size

So assume that each C_i has order less than $2k$ and let t be the number of components (clearly $t \geq 2$). We want to show that

$$k \leq \sum_{i=1}^{t} \lfloor \frac{|C_i|}{2} \rfloor$$

Now the minimal degree restricts how small $|C_i|$ can be; in oder to satisfy the minimal degree we must have $|C_i| > k$. So since $t \geq 2$ we have $t\frac{k}{2} \geq k$. And hence,

$$k \leq t\frac{k}{2} \leq \sum_{i=1}^{t} \frac{k}{2} \leq \sum_{i=1}^{t} \lfloor \frac{k+1}{2} \rfloor \leq \sum_{i=1}^{t} \lfloor \frac{|C_i|}{2} \rfloor$$

Hence there is a matching M of size k and so $\alpha'(G) \geq |M| = k$. As desired. $\square$