MTH 513 · LINEAR ALGEBRA

Problem Set 1

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DUE by 11:59pm on Sunday, 18 September 2022, via Brightspace.

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- Review *general submission guidelines* before submitting your assignment, in particular how to create a single pdf document from multiple handwritten pages, page numbering, problem statements, etc.
- Make this "cover page" the first page in your submitted pdf file.
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YOURLASTNAME–hw1–mth–513

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Name: ______________________________

1. Let $X, Y \in \mathbb{R}^{n\times n}$ be arbitrary and assume that a good-hearted oracle proved for you the fact that $(X \cdot Y)^T = Y^T \cdot X^T$. Use this fact to **prove** that if $A_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, k$, then for $k \geq 2$ the following equality holds

$$\Big(A_1 A_2 \cdots A_{k-1} A_k\Big)^T \;=\; A_k^T A_{k-1}^T \cdots A_2^T A_1^T\,. \tag{1}$$

Proof. We proceed with induction, the base case for $k \leq 2$ is given. Now for the induction step assume for some $k \geq 2$ we have $\Big(A_1 A_2 \cdots A_{k-1} A_k\Big)^T \;=\; A_k^T A_{k-1}^T \cdots A_2^T A_1^T$ Then we use the base case and our induction hypothesis to get

$$\begin{aligned}
\Big(A_1 A_2 \cdots A_k A_{k+1}\Big)^T \;&=\; \Big[\Big(A_1 A_2 \cdots A_k\Big)\Big(A_{k+1}\Big)\Big]^T \\
&= A_{k+1}^T \cdot \Big(A_1 A_2 \cdots A_k\Big)^T \\
&= A_{k+1}^T A_k^T \cdots A_2^T A_1^T
\end{aligned}$$

as desired.

□

Remark: This is essentially just asking you to set up *proof by induction* correctly. The "harder" part of this problem would be proving the base case, which you may assume is true for the purpose of this problem right now.

2.

Recall that the set of complex numbers is defined as

$$\mathbb{C} \;:=\; \Big\{a + b\mathtt{i} \;\Big|\; a, b \in \mathbb{R},\; \mathtt{i}^2 = -1\Big\}\,. \tag{2}$$

Moreover, the *addition* and the *multiplication*, $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, are defined as

$$\Big(a + b\mathtt{i}\Big) + \Big(c + d\mathtt{i}\Big) \;:=\; \Big(a + c\Big) + \Big(b + d\Big)\mathtt{i}\,, \tag{3}$$

$$\Big(a + b\mathtt{i}\Big) \cdot \Big(c + d\mathtt{i}\Big) \;:=\; \Big(ac - bd\Big) + \Big(bc + ad\Big)\mathtt{i}\,, \tag{4}$$

for all complex numbers $a + b\mathtt{i}$ and $c + d\mathtt{i}$. Clearly, the set of complex numbers whose imaginary part is zero represents the set of real numbers, that is, $\mathbb{R}$ is a proper subset of $\mathbb{C}$.

Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 real matrices and consider function $\phi : \mathbb{C} \to \mathbb{R}^{2\times 2}$ given by

$$\phi(a + b\mathtt{i}) \;:=\; \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad \text{for all } \; a + b\mathtt{i} \in \mathbb{C}\,. \tag{5}$$

(a) Show that ϕ is an injective (or one-to-one) function.

Proof. Let $a = x + y\mathtt{i}, b = w + z\mathtt{i} \in \mathbb{C}$ and assume $\phi(a) = \phi(b)$. then by definition,

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} w & -z \\ z & w \end{bmatrix} \tag{6}$$

Now since 2×2 matrices form a vector space, they form an Abelian group so we may subtract either matrix from both sides; this gives $x - w = 0$ and $y - z = 0$, which is to say that the real and imaginary parts of a and b are the same, but then $a = b$. Thus, ϕ is an injection. $\square$

(b) Describe the range/image of ϕ, that is, describe the set $\phi(\mathbb{C}) = \{\phi(z) \mid z \in \mathbb{C}\}$.

$\phi(\mathbb{C})$ contains all matrices of the form $\left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(c) Prove or disprove: $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

Proof. Let $z_1, z_2 \in \mathbb{C}$ then let $z_1 = a_1 + b_1\mathtt{i}$ and $z_2 = a_2 + b_2\mathtt{i}$. It follows that,

$$\phi(z_1 + z_2) = \begin{bmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \phi(z_1) + \phi(z_2) \tag{7}$$

as desired. $\square$

(d) Prove or disprove: $\phi(z_1 \cdot z_2) = \phi(z_1) \cdot \phi(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

Proof. Let $z_1, z_2 \in \mathbb{C}$ then let $z_1 = a_1 + b_1\mathtt{i}$ and $z_2 = a_2 + b_2\mathtt{i}$. Then $z_1 z_2 = a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)\mathtt{i}$. Using the definiton of matrix multiplication in reverse gives,

$$\phi(z_1 z_2) = \begin{bmatrix} a_1a_2 - b_1b_2 & -(a_1b_2 + a_2b_1) \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{bmatrix} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \phi(z_1) \cdot \phi(z_2) \tag{8}$$

completing the argument that ϕ is injective homomorphism. Given the above results we see that the vector space of 1×1 complex matrices is isomorphic to a subspace of the vector space of 2×2 real matrcies, this can be generalized to say that $\mathcal{M}_{n\times n}(\mathbb{C})$ is isomorphic to a subspace of $\mathcal{M}_{2n\times 2n}(\mathbb{R})$, thus we can always rewrite complex matrices as real matrices.

$\square$

3. The "same" way the set of real numbers was extended into the set of complex numbers, one can also continue this process and look for an extension of complex numbers. To that end, let us consider the set

$$\mathbb{H} := \left\{ a + b\mathtt{i} + c\mathtt{j} + d\mathtt{k} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}, \tag{9}$$

where the following multiplication conditions are imposed:

(i) $\mathtt{i}^2 = \mathtt{j}^2 = \mathtt{k}^2 = -1$,

(ii) $\mathtt{ij} = \mathtt{k},\ \mathtt{ji} = -\mathtt{k},\ \mathtt{jk} = \mathtt{i},\ \mathtt{kj} = -\mathtt{i},\ \mathtt{ki} = \mathtt{j},\ \mathtt{ik} = -\mathtt{j}$,

(iii) every $a \in \mathbb{R}$ commutes with $\mathtt{i}, \mathtt{j}, \mathtt{k}$.

Now the addition of two elements in $\mathbb{H}$ can be defined analogous to complex numbers where we simply add the corresponding parts. The multiplication is slightly more difficult, but it can be completely defined given the conditions (i)-(iii) from above along with the distributive law.

Observation: $\mathbb{H}$ is *NOT* a field since multiplication is not commutative, for example, condition (ii) gives us $\mathtt{i}\mathtt{j} = \mathtt{k} \neq -\mathtt{k} = \mathtt{j}\mathtt{i}$. However, except the commutativity of multiplication $\mathbb{H}$ does satisfy all other conditions to be a field.

Example: Multiply $(\mathtt{i}+\mathtt{j})(\mathtt{i}-\mathtt{j})$ assuming the distribute law and the conditions (i)-(iii).

$$(\mathtt{i}+\mathtt{j})(\mathtt{i}-\mathtt{j}) \;=\; \mathtt{i}^2 - \mathtt{i}\mathtt{j} + \mathtt{j}\mathtt{i} - \mathtt{j}^2 \;=\; -1 - \mathtt{k} - \mathtt{k} - (-1) \;=\; -2\mathtt{k}\,,$$

while $\mathtt{i}^2 - \mathtt{j}^2 = -1 - (-1) = 0$.

(a) Let $q = a + b\mathtt{i} + c\mathtt{j} + d\mathtt{k}$ and $w = e + f\mathtt{i} + g\mathtt{j} + h\mathtt{k}$ be two arbitrary elements of $\mathbb{H}$. Write the product $q\,w$ in the form $z_0 + z_1\mathtt{i} + z_2\mathtt{j} + z_3\mathtt{k}$, where $z_1, z_2, z_3, z_4 \in \mathbb{R}$.
We just multiply normally (i.e just use the distributive property) and treat $\mathtt{i}^2 = \mathtt{j}^2 = \mathtt{k}^2$ as -1.

$$qw = aw + b\mathtt{i}w + c\mathtt{j}w + d\mathtt{k}w =$$

then

$$\begin{aligned} aw &= ae + af\mathtt{i} + ag\mathtt{j} + ah\mathtt{k} \\ b\mathtt{i}w &= b\mathtt{i}e - bf + bg\mathtt{k} - bh\mathtt{j} \\ c\mathtt{j}w &= c\mathtt{j}e - cf\mathtt{k} - cg + ch\mathtt{i} \\ d\mathtt{k}w &= d\mathtt{k}e + df\mathtt{j} - dg\mathtt{i} - dh \end{aligned}$$

then collecting like terms shows

$$qw = (ae-bf-cg-dh)+(af+be+ch-dg)\mathtt{i}+(ag-bh+ce+df)\mathtt{j}+(ah+bg-cf+de)\mathtt{k}$$

(b) Find $A \in \mathbb{R}^{4\times 4}$ and $\mathbf{b} \in \mathbb{R}^4$ (both obviously related to q and/or w) such that

$$A\,\mathbf{b} \;=\; \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}.$$

Let $\mathbf{b} = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$ and $A = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -d \\ d & -c & b & a \end{bmatrix}$
Then computing the matrix vector product, gives the desired result.

Remark: Part (b) may have to wait until the second homework, but we can discuss it in class on Tuesday.

4. Consider the following system of equations over the finite field $\mathbb{Z}_3$, that is, all coefficients and operations are done over $\mathbb{Z}_3$,

$$\begin{array}{ccccccc} x & + & 2y & + & z & = & 1 \\ x & + & & & z & = & 1 \\ x & + & y & + & z & = & 1 \end{array} \qquad (10)$$

(a) What is the reduced row echelon form of the associated augmented matrix? Write down the sequence of operations you performed to obtain the reduced row echelon form.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[2\rho_1+\rho_3]{2\rho_1+\rho_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{\rho_2+\rho_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\rho_2+\rho_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Clearly describe the solution set and state how many different solutions are there.

The above reduced echelon form shows that $x_1 = 1 + 2x_3\,, x_2 = 0$ and that x_3 is free.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+2x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \mid x_3 \in \mathbb{Z}_3.$$

Since there are only 3 elements in $\mathbb{Z}_3$, we can just write out the enitre solution set but subing into x_3. Let S be the set of solutions, then

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \right\}$$