MTH 316 Homework 8

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Question 1.

Let $\phi : G \to H$ and $\psi : H \to K$ be homomorphisms.

(a) Prove that $\psi \circ \phi : G \to K$ is a homomorphism

Proof. It is clear that the composistion is a function from G to K so we must show that it is a homomorphism. Let $a, b \in G$ and using the fact that ϕ and ψ are both homomorphisms we get,

$$\psi \circ \phi(ab) = \psi(\phi(ab)) = \psi(\phi(a)\phi(b)) = \psi(\phi(a))\psi(\phi(b))$$

This completes the proof. □

(b) Prove that $\ker \phi \leq \ker(\psi \circ \phi)$.

Proof. Since we have established $\psi \circ \phi$ is a homomorphism, we know that its kernel forms a subgroup. By definition if $a \in \ker \phi$ then $\phi(a) = e_H$, recalling the fact that ψ is a homomorphism, we know that it must map e_H to e_K. So for $a \in \ker \phi$ we have

$$(\psi \circ \phi)(a) = \psi(\phi(a)) = \psi(e_H) = e_K$$

then it is clear $a \in \ker \psi \circ \phi$, but this is exactly what we needed to show. □

Question 2.

Ler G and H be groups with idenities e_G and e_H.

(a) Prove that $\phi : G \oplus H \to H$ defined by $\phi(g, h) = h$ is a homomorphism

Proof. Define ϕ as above. Then all we need to show is that it preserves group structure.

$$\phi(g_1 g_2, h_1 h_2) = h_1 h_2 = \phi(g_1, h_1)\phi(g_2, h_2)$$

as desired. □

(b) Prove that $(G \oplus H)/(G \oplus \{e_H\}) \cong H$

Proof. We will use the first isomorphism theorem. We have in the last part established a homomorphism ϕ from $G \oplus H$ to H, we define ϕ here as it is defined above. We want to find the kernel of this homomorphism; since ϕ maps the ordered pair (g, h) to h, we see that any element of the form (g, e_H) where g is arbitrary will map to the idenity in H. So $\ker \phi = \{(g, e_H) | g \in G\} = G \oplus \{e_H\}$. Then the result will follow directly from the first isomorphism theorem. □