MTH 548 Homework 6

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Problem 1. *Let G be m-regular with a cut vertex, show that G is class 2.*

I will make heavy use of the following

$$\chi'(G) \geq \lceil \frac{e(G)}{\alpha'(G)} \rceil.$$

Proof. let G be a graph on n vertices that is m-regular and has a cut vertex x. First we must have matching of size $\left\lceil \frac{n}{2} \right\rceil$, since if $\alpha'(G) < \lfloor \frac{n}{2} \rfloor$ we have

$$\chi'(G) \geq \left\lceil \frac{e(G)}{\alpha'(G)} \right\rceil > \left\lceil \frac{e(G)}{\lfloor \frac{n}{2} \rfloor} \right\rceil \geq \frac{\frac{n}{2}}{\lfloor \frac{n}{2} \rfloor} m \geq m$$

The case where $|G|$ is odd is handeled below, so we can assume that $|G|$ is even and hence that G has a perfect matching. Since G has a perfect matching tuttes condidtion is satisfied. Then consider $G \setminus x$, Tuttes condition together with our assumption that $|G|$ is even, implies that there is exactly one odd component of $|G \setminus x|$. Moreover, since removing x splits G into at least 2 components (cut vertex), there exists a component of $G \setminus x$ with even order, call it E. Now define $E^\star = E \cup x$, this is a graph on an odd number of vertices where every vertex execpt x has degree m. let c_0 denote the degree of x in $E^\star$, so we have $1 \leq c_0 \leq m - 1$. we will show that $E^\star$ cannot be m colorable.
To see this, let $|E^\star| = 2k + 1$ and notice that

$$\chi'(G) \geq \left\lceil \frac{e(G)}{\alpha'(G)} \right\rceil \geq \frac{\frac{1}{2}(2km + c_0)}{k} > m + \frac{c_0}{2k} > m$$

so we are done. Clearly, the existance of a subgraph with $\chi'(E^\star) > m$ implies the result for G. □

Problem 2. *Let G be a m-regular graph with $|G| = 2k + 1$, Then G is class 2.*

Proof. We know $\alpha'(G) \leq k$. Then we have that

$$\chi'(G) \geq \frac{e(G)}{\alpha'(G)} = \frac{m(2k+1)}{2\alpha'(G)} \geq \frac{m(2k+1)}{2k} > m$$

where $e(G) = \frac{m(2k+1)}{2}$ follows from the fact that G is regular and an application of the hand shaking lemma. □

Problem 3. *Let G be a graph with $|G| \geq 3$, such that for all $k < n$ we have that all non adjacent x, y satisfy $deg(x) + deg(y) \geq k$. Then there exists a path of length k*

Proof. Let $n \geq 3$ and let G be a graph on n vertices. Further take some $k < n$ such that for all $x, y \in V(G)$ non-adjaicent, we have $deg(x) + deg(y) \geq k$. We want to find a path of length k, so suppose no such path exists; then the longest path in G has at most $k - 1$ vertices. Let m be the length of the longest path P in G.

Case I: if $x_1 x_m \in E$, then adding this edge to the path creates a hamiltonian cycle. Since if any vertice is un visted, there exists a path to some vertex in P, then looping around the cycle contradicts our assumption about P being the longest path. Then this cycle visits n vertices and one can obtain a path of length k

Case II: if $x_1 x_m \notin E$, then by our assumption we have the condidtion that $d(x_1) + deg(x_m) \geq k$. Now order the vertices in the path P by their index, we prove that there exists a pair x_i, x_{i+1} such that $x_1 x_{i+1} \in E$ and $x_i x_m \in E$. First define

$$S = \{i | x_1 x_{i+1} \in E\}$$

and

$$T = \{i | x_i x_m \in E\}$$

then

$$|S \cup T| + |S \cap T| = |S| + |T| = deg(x_1) + deg(x_m) \geq k$$

and since there is no $x_1 x_m$ edge, we have $|S \cup T| < k - 2$. Hence $|S \cap T| \geq 1$ as desired. Then, this cycle must be hamiltonian as argued in the book and in class. Hence it is a cycle of length n, contradicting the assumption that the longest path has $k - 1$ vertices.

$\square$