MTH 513 · LINEAR ALGEBRA

Problem Set 2

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- Review *general submission guidelines* before submitting your assignment, in particular how to create a single pdf document from multiple handwritten pages, page numbering, problem statements, etc.
- Make this "cover page" the first page in your submitted pdf file.
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YOURLASTNAME–hw2–mth–513

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1. Consider the function $\texttt{trace} : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by

$$\texttt{trace}(A) \;:=\; \sum_{i=1}^{n} a_{ii}\,, \qquad \forall A \in \mathbb{R}^{n\times n}\,, \tag{1}$$

that is, $\texttt{trace}(A)$ is the sum of diagonal entries of A.

Prove that if $\delta \in \mathbb{R}$ and $A, B \in \mathbb{R}^{n\times n}$ are all arbitrary, then

$$\texttt{trace}(\delta A \,+\, B) \;=\; \delta\texttt{trace}(A) \,+\, \texttt{trace}(B)\,. \tag{2}$$

Proof. Let $\delta \in \mathbb{R}$ and $A, B \in \mathbb{R}^{n\times n}$. Then by the above definition, we have that

$$\texttt{trace}(\delta A + B) = \sum_{i=1}^{n} [\delta A + B]_{ii}$$

Then since $[\delta A + B]_{ii} = [\delta A]_{ii} + [B]_{ii}$ and $[\delta A]_{ii} = \delta [A]_{ii}$, it follows that

$$\sum_{i=1}^{n} [\delta A + B]_{ii} = \sum_{i=1}^{n} \delta [A]_{ii} + [B]_{ii}$$

now we may split this sum and factor out δ to get

$$\texttt{trace}(\delta A + B) = \delta \sum_{i=1}^{n} [A]_{ii} + \sum_{i=1}^{n} [B]_{ii}.$$

But the left hand side of the last equation is precisely $\delta\texttt{trace}(A) + \texttt{tarce}(B)$ which is what we wanted to prove. □

Remark: You should be using definitions of addition and scalar multiplication on pages 82-83 of our textbook.

2. **(Optional)** Consider the function $\texttt{vec} : \mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ such that for any $A \in \mathbb{C}^{m\times n}$

$$\texttt{vec}(A) \;=\; \begin{bmatrix} A_{*1} \\ A_{*2} \\ \vdots \\ A_{*n} \end{bmatrix}. \tag{3}$$

Prove that for arbitrary $A, B \in \mathbb{C}^{m\times n}$ and $\delta \in \mathbb{C}$, we have

$$\texttt{vec}(\delta A + B) \;=\; \delta\,\texttt{vec}(A) \,+\, \texttt{vec}(B)\,. \tag{4}$$

Proof. Considering the image as a block matrix we will have that $\texttt{vec}(\delta A + B)_{i*} = [\delta A + B]_{*i}$, where on the left side, i denotes the i^{th} block and on the right, i denotes the i^{th} column. Then a simple appliction of the definition of matrix addition and scalar multiplictation, we get $[\delta A + B]_{*i} = \delta [A]_{*i} + [B]_{*i} = \delta\texttt{vec}(A)_{i*} + \texttt{vec}(B)_{i*}$. So, $\texttt{vec}(\delta A + B) = \delta\texttt{vec}(A) + \texttt{vec}(B)$. □

3. **Definition:** The *tensor product* of matrices $A_{m\times n}$ and $B_{p\times q}$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq} . \tag{5}$$

(a) Let $A_{m\times n}$, $B_{p\times q}$, $C_{n\times k}$, and $D_{q\times r}$. Prove that $(A \otimes B)(C \otimes D) = AC \otimes BD$.

Proof. We will show that the corosponding blocks are equivalent. It follows from our definition of matrix multiplication that

$$[(A \otimes B)(C \otimes D)]_{ij} = [(A \otimes B)]_{i*}[C \otimes D]_{*j}$$

which is simply

$$\begin{bmatrix} a_{i1}B & \dots & a_{in}B \end{bmatrix} \begin{bmatrix} c_{1j}D \\ \vdots \\ c_{nj}D \end{bmatrix}$$

Multiplying this out using the definition of matrix multiplication gives the sum

$$a_{i1}c_{1j}BD + \cdots + a_{in}c_{nj}BD = [AC]_{ij} \cdot BD = [(AC) \otimes (BD)]_{ij}.$$

Thus, the corrosponding blocks in $(A \otimes B)(C \otimes D)$and $(AC) \otimes (BD)$ are equal, which is precisely what we wanted to show.

□

(b) Prove that if $A_{m\times m}$ and $B_{n\times n}$ are nonsingular matrices, then so is $A \otimes B$. What is the inverse of $A \otimes B$?

Proof. We prove that $A^{-1} \otimes B^{-1}$ is the inverse. Using the above result, we have

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = I \otimes I = I.$$

Hence the matrix $A \otimes B$ is invertible with invese $A^{-1} \otimes B^{-1}$. □

(c) Prove that for any two square matrices $A_{m\times m}$ and $B_{n\times n}$ the following equality holds

$$\text{trace}(A \otimes B) = \text{trace}(A) \cdot \text{trace}(B)\,.$$

Proof. We begin by noting that the diagonal elements of $A \otimes B$ are the diagonal elements of the diagonal blocks when we consider $A \otimes B$ as a block matrix. So it is clear that the $\texttt{trace}(A \otimes B)$ is equal to the sum of the trace of the diagonal blocks. By our definiton of tensor product we have

$$\texttt{trace}(A \otimes B) = \texttt{trace}(a_{11}B) + \cdots + \texttt{trace}(a_{mm}B)$$

Now using the linearity of trace proved above we may take out the scalars a_{ii}.

$$\texttt{trace}(A \otimes B) = \sum_{i=1}^{m} a_{ii} \cdot \texttt{trace}(B) = \left(\sum_{i=1}^{m} a_{ii}\right) \cdot \texttt{trace}(B) = \texttt{trace}(A) \cdot \texttt{trace}(B)$$

as desired. □

4. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$ be given and let $X \in \mathbb{C}^{n\times p}$ be unknown. Show that the *matrix equation*

$$AXB = C \tag{6}$$

is equivalent to the linear system of qm equations in np unknowns given by

$$(B^T \otimes A)\mathtt{vec}(X) = \mathtt{vec}(C), \tag{7}$$

that is, $\mathtt{vec}(AXB) = (B^T \otimes A)\mathtt{vec}(X)$.

Proof. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$ and let $X \in \mathbb{C}^{n\times p}$ be unkown. We prove that $AXB = C$ is equivalent to $(B^T \otimes A)\mathtt{vec}(X) = \mathtt{vec}(AXB)$. First we note that by viewing the matrix vector product as a linear combintions of the columns, it is clear that

$$XB_{*j} = \sum_{k=1}^{p} b_{kj}X_{*k} \tag{8}$$

Now we want to show

$$(B^T \otimes A)\mathtt{vec}(X) = \mathtt{vec}(AXB)$$

which can be written as

$$\begin{bmatrix} b_{11}A & \dots & b_{p1}A \\ \vdots & \ddots & \vdots \\ b_{1q}A & \dots & b_{pq}A \end{bmatrix} \cdot \begin{bmatrix} X_{*1} \\ \vdots \\ X_{*p} \end{bmatrix} = \begin{bmatrix} (AXB)_{*1} \\ \vdots \\ (AXB)_{*q} \end{bmatrix} \tag{9}$$

Then it is sufficient to prove that the i^{th} row of the product $(B^T \otimes A)\mathtt{vec}(X)$ (considering the product as a block matrix) is equal to the i^{th} column of AXB. In symbols we want to show,

$$\sum_{k=1}^{p} b_{ki}AX_{*k} = (AXB)_{*i}$$

Now it follows using equations 8 and 9, that

$$(AXB)_{*i} = A(XB)_{*i} = A(XB_{*i}) = A\left(\sum_{k=1}^{p} b_{ki}X_{*i}\right) = \sum_{k=1}^{p} b_{ki}AX_{*i}$$

Where is the last step we may commute A with b_{ki} since it is a scaler. But shown above is exactly what we needed to prove.

$\square$

5. **Definition:** Matrix $A \in \mathbb{C}^{m\times n}$ is said to be **right invertible** if there exists a matrix A^{-R} such that $A \cdot A^{-R} = I_m$. Similarly, $A \in \mathbb{C}^{m\times n}$ is said to be **left invertible** if there exists a matrix A^{-L} such that $A^{-L} \cdot A = I_n$. Matrices A^{-R} and A^{-L} are referred to as a *right inverse* and a *left inverse* of A, respectively.

(a) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. If possible, find a right inverse of A.

It is easy to find such an inverse, we want a, b, c, d, e, f such that

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Multiplying out, shows we want $2a + b = 1$, $2d + e = 0$, $3c = 0$, and $3f = 1$. So we may choose $c = 0$ and $f = \frac{1}{3}$, then choose $a = 0$. $b = 1$ and $d = 0$, and $e = 0$, then we we the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

multplying the given matrix on the right shows that this is indeed a right inverse.

(b) Give an example of *nonzero* matrices $C_0, C_1, C_2 \in \mathbb{R}^{3\times 2}$ such that any matrix R of the form

$$R = C_0 + \alpha C_1 + \beta C_2,$$

is a right inverse of A, for all $\alpha, \beta \in \mathbb{R}$.

We use our result in the last problem. Note that $2d + e = 0$ shows us that d can be solved for directly in therms of e, In general we use the relations on a, b, c, d, f to write down a general solution.

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} a & \frac{-e}{2} \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} \frac{1-b}{2} & \frac{-e}{2} \\ b & e \\ 0 & \frac{1}{3} \end{bmatrix}$$

Since $c = 0$ and $f = \frac{1}{3}$ are unqiuely determined. Now we can have the matrix in terms of b, e, these are the two degrees of fredom we need to produce the solution, now we just seperate the matrices under addition.

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} \frac{1-b}{2} & \frac{-e}{2} \\ b & e \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} + e \begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} \frac{-1}{2} & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

6. (a) **Prove/Disprove:** If A is an $m \times n$ matrix that is right invertible, then $\text{rank}(A) = m$.

Proof. Assume A is right invertable and suppose $\text{rank}(A) < m$. If A already has a row of all zeros, it is quickly seen that A cannot be right invertable since the product with any other matrix will contain a row of zeros. If not, then by one of our charactizations of rank, A can be reduced to a matrix with a row of all zeros, A_z. Now this reduced matrix is clearly singular with respect to right multiplication, since comuputing $A_z \cdot X$ for any conformable X, will get a row of all zeros in the product so it cannot be the identity. But since $PA = A_z$, and PA is the product of right invertable matrices we get that A_z is is right invertable, a contradiction. So the rank cannot be less than m, but it also cannot be greater than m, since we have $\text{rank}(A) \leq \min\{m, n\}$. So $\text{rank}(A) = m$. □

I use the fact the product of right ivertable matrices is right invertable without justification, but it is clear $(AB)(B^{-r}A^{-r}) = I$.

(b) Show that if A is an $n \times n$ matrix with a *unique right inverse* A^{-R}, then A is invertible and $A^{-R} = A^{-1}$.

Possible Hint (but not required): Consider the expression $A(A^{-R} + A^{-R}A - I)$.

Proof. The hint makes it trivial, note by matrix algebra,

$$A(A^{-R} + A^{-R}A - I) + AA^{-R} + AA^{R}A - I = I + IA - I = A$$

so that $(A^{-R}+A^{-R}A-I)$ is a right inverse of A, by uniqueness, $A^{-R} = A^{-R}+A^{-R}A-I$ and solving with matrix algebra shows

$$A^{-R} = A^{-R} + A^{-R}A - I$$

$$A^{-R} - A^{-R} = A^{-R}A - I$$

$$0 = A^{-R}A - I$$

$$I = A^{-R}A$$

then, A^{-R} is a left inverse of A and by definition A is invertable with unique inverse A^{-R}.

I aslo tried to get an argument like this to work, assume $BA = I$, then

$$AB = AIB = ABAB$$

Then I want to say that there is some cancelation law, and that I is the only matrix satisfying $X^2 = X$. But this is only true if there are no zero divisors, which I cant assume without restricting to invertable matrices, but if I do that then there is nothing to prove, so this doesn't seem to quite work. □

7. Let C be an $n \times n$ upper triangular matrix. Show that if $CC^T = C^TC$, then C is a diagonal matrix.

Proof. We will proceed with induction, for $n = 1$ there is nothing to prove, so we start with $n = 2$. Let $A_{2\times 2}$ be upper triangular, then

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

Then computing $AA^T = A^TA$ we get

$$\begin{bmatrix} a^2+b^2 & bc \\ bc & c^2 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

Now subtracting shows that $b^2 = 0$, hence $b = 0$ and A was diagonal. This completes the base case, now suppose that for some $n \geq 2$ that if A is triangular and $AA^T = A^TA$ then A is diagonal. Now consider an $A_{n+1\times n+1}$. We may consider A as a block matrix, in the following form,

$$A = \begin{bmatrix} B_{n\times n} & \mathbf{x} \\ \vec{(0)}^T & b \end{bmatrix}$$

Taking the transpose gives

$$A^T = \begin{bmatrix} B^T & \vec{(0)} \\ \mathbf{x}^T & b \end{bmatrix}$$

After multiplying $AA^T = A^T A$ we get

$$\begin{bmatrix} BB^T + \mathbf{x}\mathbf{x}^T & b\mathbf{x} \\ b\mathbf{x}^T & b^2 \end{bmatrix} = \begin{bmatrix} B^T B & B^T \mathbf{x} \\ \mathbf{x}^T B & \mathbf{x}^T \mathbf{x} + b^2 \end{bmatrix}$$

Now subtracting both sides and looking into the bottom right entry, we see $b^2 - \mathbf{x}^T\mathbf{x} - b^2 = 0$ which implies $\mathbf{x}^T\mathbf{x} = \sum_{i=1}^n x_i^2 = 0$ and thus $\mathbf{x} = \vec{0}$. Now we may use this restiction on $\mathbf{x}$ to show $B^T B = BB^T$, which now gives all the conditions we need to use our induction hypothesis on B, thus B is diagonal, then since we also know that $\mathbf{x}$ was the zero vector, we can see that A is diagonal. Hence by the principal of mathematical induction, we have proven the desired result. $\square$

8. Let $\mathbb{R}[x]$ denote the set of all *polynomials* in variable x with real coefficients. Further, let $\mathbb{R}(x)$ be the set of all *rational functions* over $\mathbb{R}$, that is,

$$\mathbb{R}(x) := \left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{R}[x]\,,\ q(x) \not\equiv 0 \right\}. \tag{10}$$

Clearly $\mathbb{R}[x] \subsetneq \mathbb{R}(x)$. Finally, recall from the first class that $\mathbb{R}(x)$ is a field.

Let $A(x)$ be a 3×3 matrix with entires from $\mathbb{R}(x)$ given by

$$A(x) = \begin{bmatrix} -1 & x & 2+x \\ -x & -1+x^2 & -3+3x+x^2 \\ x^2 & -1-x-x^3 & -2x-x^2-x^3 \end{bmatrix}.$$

Determine if $A(x)$ is nonsingular/invertible over $\mathbb{R}(x)$, and if so, then find $A^{-1}(x)$.

I just use standard elimination and carefully did my arthimitic. Below are the row operations used.

(a) $R_2 \leftarrow -xR_1 + R_2$
(b) $R_3 \leftarrow x^2 R_1 + R_3$
(c) $R_3 \leftarrow (-1-x)R_2 + R_3$
(d) $R_3 \leftarrow \frac{1}{3} R_3$
(e) $R_2 \leftarrow (3-x)R_3 + R_2$
(f) $R_1 \leftarrow (-2-x)R_3 + R_1$
(g) $R_1 \leftarrow xR_2 + R_1$
(h) $R_2 \leftarrow -R_2$
(i) $R_1 \leftarrow -R_1$

$$A^{-1}(x) = \begin{bmatrix} -1+\frac{2}{3}x+\frac{5}{3}x^2-x^3+\frac{2}{3}x^4 & \frac{-2}{3}-x+\frac{1}{3}x^2-\frac{1}{3}x^3 & \frac{2}{3}-\frac{2}{3}x+\frac{1}{3}x^2 \\ \frac{-5}{3}x^2+\frac{2}{3}x^3 & \frac{2}{3}x-\frac{1}{3}x^2 & -1+\frac{1}{3}x \\ \frac{1}{3}x+\frac{2}{3}x^2 & \frac{-1}{3}-\frac{1}{3}x & \frac{1}{3} \end{bmatrix} \tag{11}$$

I realize after typing this I could have factored out a 1/3 and saved myself many frac commands.

Remark: When typing your answer, you do NOT need to give me all intermediate steps. Only include the sequence of row operations that helps you obtain the answer.

9. The problem I had in mind for here will now be a part of your third homework (last updated on September 21, 2022).