MTH 435: Analysis HW 1

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September 26, 2022

Question 1.

Show that there doesn't exist a rational number s such that $s^2 = 6$.

Proof. First we prove that $6|a^2 \implies 6|a$. Suppose that 6 does not divide a and let $a = p_1 p_2 \dots p_n$ be the prime factorization of a, then we know that 3 and 2 cannot both be common factors, But then since 2 and 3 are prime they cannot be the square of a prime number, hence 2 and 3 cannot both appear in $a^2 = p_1^2 p_2^2 \dots p_n^2$, hence 6 does not divide a^2. It now follows by contrapositive that if $6|a^2$ then $6|a$.

Now assume there exists $a, b \in \mathbb{Z}$ such that

$$6 = \left(\frac{a}{b}\right)^2$$

If a and b have any common factors we may cancel them out, so we assume $(a, b) = 1$. Then $6b^2 = a^2$ implies $6|a$ so we may fix $m \in \mathbb{Z}$ such that $6m = a$. Then $6b^2 = (6m)^2$ implies $b^2 = 6m^2$ and $6|b$; thus a and b share a common factor, a contradiction.

□

Question 2.

If $a, b \in \mathbb{R}$ show that $|a + b| = |a| + |b|$ iff $ab \geq 0$.

Proof. For the forward direction we use contrapositive, assume $ab < 0$ then without loss of generality assume $a > 0$ and $b < 0$, we show that

$$|a + b| \neq |a| + |b|$$

we have by our assumptions on a and b that $|a| + |b| = a - b$. Now there are three cases (by tricotomy) for what $|a + b|$ can map to, if $|a + b| = 0$ we are done since $|a| + |b| > 0$. If $a + b$ is positive $|a + b| = a + b$, but then $a + b = a - b$ would imply $b = 0$, contradicting our assumption on b. If $a + b$ is negative $|a + b| = -a - b$, but now $-a - b = a - b$ would force $a = 0$, contradicting our assumption on a. Hence, in every case equality does not hold. This proves the first direction.

Now assume $ab \geq 0$, if either is equal to zero we are done. If they are both positive it follows since $a + b$ will be positive giving

$$|a + b| = a + b = |a| + |b|.$$

Then if they are both negative, their sum will be negative so,

$$|a + b| = -a - b = |a| + |b|$$

completing the opposite direction. □

Question 3.

Find all $x \in \mathbb{R}$ that satisfy the inequality

$$4 < |x + 2| + |x - 1| < 5.$$

Proof. The terms $x + 2$, $x - 1$ have different signs only if $x \in (-2, 1)$ but it is clear no such x is a solution, so assume $x \notin (-2, 1)$, then the terms have the same sign and by the above we may add them to get

$$4 < |2x + 1| < 5$$

which we may split into $4 < |2x + 1|$ and $|2x + 1| < 5$. For the first case we have

$$4 < |2x + 1| \implies 4 < 2x + 1 \vee 2x + 1 < -4$$

which gives $3/2 < x$ and $x < -5/2$; written in interval notation as $(-\infty, -5/2) \cup (3/2, \infty)$. Now for

$$|2x + 1| < 5 \implies -5 < 2x + 1 < 5 \implies -6/2 < x < 4 = (-3, 2)$$

We need both conditions to be satisfied so we must take the intersection over our two solutions,

$$(-\infty, -5/2) \cup (3/2, \infty) \cap (-3, 2) = (-3, -5/2) \cup (3/2, 2)$$

□

Question 4.

(a) *Proof.* Assume without loss of generality that $b \leq a$, then $|a-b| = a-b$. Thus
$$\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = \frac{1}{2}(2a) = a$$
We also have
$$\frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = \frac{1}{2}(2b) = b$$
and we are done. $\square$

(b) Prove $\min\{a,b,c\} = \min\{\min\{a,b\},c\}$

Proof. Note $\min\{a,b\}$ is either a or b, then if c where such that $c < a$ and $c < b$, it is clear. Now suppose $a < b$ and $a < c$. then
$$\min\{a,b,c\} = a = \min\{a,c$$
$$= \min\{\min\{a,b\},c\}.$$
Then $b < a$ and $b < c$ is the same as above.

$\square$

Question 5.

Proof. $\inf S_4 = \frac{1}{2}$, $\frac{1}{2}$ is an element of S_4 that occurs for $n = 2$, for $n \neq 2$, if n is odd we will be adding to 1 which gives $\frac{1}{2} < 1 + 1/n$. If $n > 2$ is even we have $1/n < 1/2$ which implies $1 - 1/2 < 1 - 1/n$, so $1/2$ is the minimal element of S_4. It follows that if a set contains a minimal element it is the infinium since we have $1/2 < x$ for all $x \in S_4$ it is a lower bound and since it is an element of S_4, given any lower bound l, we must have $l \leq 1/2$.

$\sup S_4 = 2$, I proceed with a very similar argument as above, 2 appears in S_4 when $n = 1$, for any $n > 1$ either we are subtracting from 1, or adding a number smaller than 1 to 1, in either case we get something less than 2. Thus 2 is the maximal element of the set and hence it must be the supremum. $\square$

Question 6.

Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A+B = \{a+b : a \in A, b \in B\}$. Prove that $\sup(A+B) = \sup(A) + \sup(B)$

Proof. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. Then for all a, b we have $a \leq \alpha$ and $b \leq \beta$, thus

$$a + b \leq \alpha + \beta, \forall a \in A, b \in B.$$

So $\alpha + \beta$ is an upper bound for the set $A + B$. But then for $\epsilon > 0$ there exists $b_0 \in B$ such that $\beta - \frac{1}{2}\epsilon < b_0$ and $a_0 \in A$ such that $a - \frac{1}{2}\epsilon < a_0$, so $\alpha + \beta - \epsilon < a_0 + b_0$, Thus $\alpha + \beta = \sup(A+B)$. A very similar argument works for infinium. Let $a = \inf(A)$ and $b = \inf(B)$. Then $a + b$ is a lower bound for $A + B$ for the same reasons stated above. But then given $\epsilon > 0$, I can find elements a_0, b_0 in A and B respectively such that $a+b+\epsilon > a_0+b_0$, so $a + b = \inf(A+B)$. $\square$

Question 7.

I first reprove a result from class. If every element of a set B is an upper bound for a set A, then $\inf(B)$ is an upper bound for A.

Proof. Assume every element of B is an upper bound for A, then if $\inf(B) < a$ for some $a \in A$, there exists $\epsilon = a - \inf(B) > 0$ such that $\inf(B) + \epsilon = a$, but by the epsilon formulation of infinium, we have there exists an element of b with $b < a$, a contradiction. $\square$

Proof. We prove that every element of $F = \inf\{f(x)|x \in X\}$ is an upper bound for the set $G = \{g(y)|y \in Y\}$, then it will follow $\sup(G) \leq \inf(F)$. Let $y_0 \in Y$ be arbitrary, then for all $x \in X$ we have

$$g(y_0) \leq h(x, y_0) \leq f(x).$$

The first inequality holds since $g(y)$ is the infinium over all choices of x and the second holds since for each $x \in X$ we have defined $f(x)$ to be the supremum over $y \in Y$. Then for each $g(y) \in G$, we have $g(y) \leq f(x)$ for all $x \in X$, thus each element of F is an upper bound for G and by the above, the result follows. $\square$

Question 8.

If $u > 0$ and $x < y$ show there exists a rational number $x < ru < y$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. Then fix $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$, from this we immediately get $0 < 1 + xn < yn$. Now choose $m \in \mathbb{N}$ to be the integer such that $xn < m \leq xn + 1$. Then we have

$$xn < m \leq xn + 1 < yn \implies xn < m < yn \implies x < \frac{m}{n} < y.$$

Now note $\frac{x}{u}, \frac{y}{u} \in \mathbb{R}$ so there exists $s, t \in \mathbb{Z}$ such that

$$\frac{x}{u} < \frac{s}{t} < \frac{y}{u} \implies x < \frac{su}{t} < y$$

and we are done. $\square$