Understanding Analysis Chapter 1.2

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Question 1.

Give a definition for greatest lower bound and prove a lemma analagous to 1.3.8

Definition 1. *Let $A \subseteq \mathbb{R}$ and let $l \in \mathbb{R}$. We say that $l = \inf(A)$ if and only if*

1. *l is a lower bound for A (i.e. $l \leq a$ for all $a \in A$).*
2. *For an arbitrary lower bound L, we have that $L \leq l$.*

Lemma 1.1. *Assume that $l \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $l = \inf(A)$ if and only if, for all choices $\varepsilon > 0$, we have that $l + \varepsilon > a$ for some $a \in A$.*

Proof. Assume $l = \inf(A)$. Then note for all $\varepsilon \geq 0$ that $l < l + \varepsilon$. Then since l is the greatest lower bound for A by definiton, we have that $l + \varepsilon$ is not a lower bound. But then there must exist $a \in A$ such that $l + \varepsilon > a$.

To prove the other direction assume we have $l \in \mathbb{R}$ such that l is a lower bound for A with the property that for all $\varepsilon > 0$ we have $l + \varepsilon > a$ for some $a \in A$. For the sake of contradiction assume that we have $L \in \mathbb{R}$ such that $L > l$ and that $L = \inf(A)$. Then we note that by choosing $\varepsilon = -l + L$ we get that $L > a$ for some $a \in A$. Hence L is not a lower bound for A contradicting our assumtion. Hence $l = \inf(A)$. □

Question 2.

For each part either given an example or state that the request is impossible.

(a) A set B with $\inf(B) \geq \sup(B)$.

Consider the set $B = \{0\}$. It is clear that $B \subset \mathbb{R}$ and that $\inf(B) = \sup(B) = 0$.

(b) A set that contains its infinium but not it's supprenum.

Consider the set $[0, 1)$.

(c) A set $B \subseteq \mathbb{Q}$ that contains its supprenum but not its infinium.

Consider the set $B = \{x \in \mathbb{Q} | 0 < x \leq 1\}$.

Question 3.

(a) Let A be non-empty and bounded below. Then Define $B = \{b \in \mathbb{R} |$ b is a lower bound for A$\}$. Prove that $\sup(B) = \inf(A)$.

Proof. We fix $s \in \mathbb{R}$ such that $s = \inf(A)$. Then we know the $s \in B$ since s is a lower bound for A. We also have that for an arbitrary element $l \in B$, $s \geq l$; then s is an upper bound for B and must be the least upper bound since $s \in B$ and any arbitrary upper bound S must then satisfy $s \leq S$. □

(b) Use the result from (a) to argue why there is no need to assert that greatest lower bounds exist in the axiom of completness.

The result from part (a) shows shows us that we can define the greatest lower bound as the supprenum of the set of lower bounds. Hence the assertion that all sets bounded above have a least upper bound implys that all sets bounded below have a greatest lower bound.

Question 4.

Let $A_1, A_2, A_2, ...$ be a collection of non empty sets that are bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$ and extend it to finite unions

Claim: $\sup(A_1 \cup A_2) = \sup(\{\sup(A_1), \sup(A_2)\})$

Proof. Note that either $\sup(A_1) \leq \sup(A_2)$ or $\sup(A_2) \leq \sup(A_1)$ must hold. If we are in the first cases then it is clear that $\sup(A_2)$ is the supprenum of $A_1 \cup A_2$ and if we are in the latter case then $\sup(A_1)$ will be the supprenum. Hence the supprenum of the union will be the largest supprenum of the sets being unioned. □

To extend this to finite unions we claim

$$\sup(\bigcup_{i=0}^{n} A_i) = \sup(S)$$

where $S = \{\sup(A_i) | i \leq n\}$.

Question 5.

Let $A \in \mathbb{R}$ be bounded above and let $c \in \mathbb{R}$. Then define $cA = \{ca | a \in A\}$.

(a) For $c \geq 0$, prove $\sup(cA) = c\sup(A)$.

Proof. Let $c \geq 0$ and fix $s \in \mathbb{R}$ such that $s = \sup(A)$. Then $cs \geq ca$ for all $a \in A$. Then if b is an arbitrary upper bound of A then $s \leq b$, hence $cs \leq cb$ where cb is an arbitrary upper bound of cA. So $cs = c\sup(A) = \sup(cA)$. □

(b) For $c < 0$, prove $\sup(cA) = c\inf(A)$.

Proof. Assume $c < 0$ and fix $s, i \in \mathbb{R}$ such that $s = \sup(cA)$ and $i = \inf(A)$. Then since $i \leq a$ for all $a \in A$ we have $ci \geq ca$ for all $a \in A$. Hence i is a upper bound for cA. To prove $s = ci$ first suppose that $s < ci$; then for all $a \in A$ we have $ci > s \geq ca$ which is equivalent to $i < s \leq a$, contradicting $i = \inf(A)$. Now suppose that $s > ci$; since we already know that $ci \geq ca$ for all $a \in A$ we see this contradicts $s = \sup(cA)$. Hence we must have $s = ci$.

□

Below is my attempt to shorten the argument and do a direct proof. In general I feel like using contradiction is easier for me.

Proof. Assume $c < 0$ and let $L \in \mathbb{R}$ be an arbitrary lower bound for A, then we note that since $L \leq a$ for all $a \in A$, we have that $cL \geq ca$ for all $a \in A$; therefore cL is an arbitrary upper bound for cA. Now we fix $i \in \mathbb{R}$ such that $i = \inf(A)$, then for our arbitrary lower bound L we have $i \geq L$ and $ci \leq cL$. Hence, $ci = sup(cA)$. □

Question 6.

Prove that if $a \in A$ and a is an upper bound for A, then $a = \sup(A)$.

Proof. Fix $a \in \mathbb{R}$ such that $a \in A$ and a is an upper bound for a. Then to prove that $a = \sup(A)$, note that if b is an arbitrary upper bound for A then $x \leq b$ for all $x \in A$. Then since $a \in A$ it is clear that $a \leq b$. Hence, a satisfys the definition of a least upper bound. □

Question 7.

if $\sup(A) < \sup(B)$ then there is some element $b \in B$ that is an upperbound for A.

Proof. We use the contrapostive. Assume that there is no $b \in B$ that bounds A above. Then $\sup(A) > b$ for all $b \in B$. Hence $\sup(A)$ is an upper bound for B and so $\sup(B) \leq \sup(A)$. □

Question 8.

Definition 2 (The Cut Property)**.** *If $A, B \subseteq \mathbb{R}$ such that $A \cap B = \varnothing$ and $A \cup B = \mathbb{R}$ and for all $a \in A$ and $b \in B$ we have $a < b$ then then there exists $c \in \mathbb{R}$ such that $c \geq a$ for all $a \in A$ and $c \leq b$ for all $b \in B$.*

Prove that the cut property is equivalent to the axiom of completness (i.e. prove that they imply each other).

Proof. We begin by assuming the cut property so suppose that we have $A \subseteq \mathbb{R}$ then define $B = \{x \in \mathbb{R} | (\forall a \in A)(x > a)\}$. Then we may fix $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$. Then it is clear that c is an upper bound for A. Then to prove that c is the least upper bound we first condsider the case where $c \in A$ then by previous result it follows that $c = \sup(A)$. If $c \notin A$ then B must be the set of all upper bounds; then since we have $c \leq b$ for all $b \in B$ we see that $c = \sup(A)$ as desired.

To prove the other direction we assume the axiom of completness, that is, assume every bounded subset of $\mathbb{R}$ has a least upper bound. Then let $A \subseteq \mathbb{R}$ that is bounded above. Then we may fix $\alpha \in \mathbb{R}$ such that $a \leq \alpha$ for all

$a \in A$. Now we define $B = \{x \in \mathbb{R} | (\forall a \in A)(x > a)\}$. then since every element of B is an upper bound for A we must have $a \leq \alpha \leq b$ for all $a \in A$ and all $b \in B$. □

Question 9.

Prove or disprove the following

(a) Let A and B be nonempty subsets of $\mathbb{R}$ such that $A \subseteq B$. Then $\sup(A) \leq \sup(B)$.

Proof. Note that since $A \subseteq B$ we must have $\sup(B) \geq a$ for all $a \in A$; so $\sup(B)$ is an upper bound for A. Then $\sup(A) \leq \sup(B)$ by definition. □

(b) if $\sup(A) < \inf(B)$ then there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and all $b \in B$.

Proof. Since $\sup(A) < \sup(B)$ there exists $\varepsilon \in \mathbb{R}$ such that $\sup(A) + \varepsilon = \sup(B)$. Then consider $\sup(A) + \frac{\epsilon}{2}$; it is clear that $\sup(A) < \sup(A) + \frac{\epsilon}{2} < \sup(B)$ □

(c) if there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and all $b \in B$ then $\sup(A) < \inf(B)$.

This is false

Proof. Consider the counter example $A = (0, 2)$ and $B = (2, 4)$. □

Question 10.

Let A and B be nonempty subsets of $\mathbb{R}$. Define $A + B = \{a + b | a \in A \text{and} b \in B\}$. Prove that $\sup(A + B) = \sup(A) + \sup(B)$.

Proof. Let $s_a = \sup(A)$ and $s_b = \sup(B)$. Then $s_a + s_b \geq a + b$ for all $a \in A$ and $b \in B$; so $s_a + s_b$ is an upper bound for $A + B$. Then let u be an arbitrary upper bound for $A + B$ so that $u \geq a + b$ for all a and b. We then fix $a \in A$ such that for all $\varepsilon > 0$, $s_a - \varepsilon < a$. Then

$$u - a \geq b$$
$$u - a \geq s_b$$
$$u - \varepsilon > s_b + s_a$$

which implys

$$u \geq s_b + s_a$$

hence it follows that $s_a + s_b = \sup(A + B)$. $\square$