MTH 513

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I worked with Kyle and Carmine and also spoke briefly with Hannah and Karrisa.

Question 1.

Let $C \in \mathbb{R}^{n\times n}$ be a matrix such that $x^T Cx > 0$ for all non-zero $x \in \mathbb{R}^n$.

(a) Prove that C is non-singular.

Proof. Suppose that C is singular, then the reduced echelon form E_C contains a row of all zeros, since otherwise C would be nonsingular. Let $E_{C_{i*}}$ denote the row of all zeros and let P be such that $PC = E_C$. Then let $w = (0, \dots, w_i, \dots, 0)^T$, were $w_i \neq 0$. then

$$w^T E_c w = (w^T E_c)w = (0E_{C,1*} + \cdots + w_i E_{C,i*} + \cdots + 0)w = \vec{0}^T w = 0.$$

hence taking $x = P^T w$ gives

$$x^T Cx = w^T PCP^T w = w^T E_C P^T w = 0$$

and since $w \neq 0$ and P is invertible, $P^T w \neq 0$. Hence we have found an x contradicting our assumption. $\square$

(b) Prove that for each principal submatrix C_k we have $x^T C_k x > 0$ for all nonzero x.

Proof. Assume that for some $k \in 1, \dots, n$ we have that $x^T C_k x > 0$ for all nonzero x does not hold. Then there exists a $y \in \mathbb{R}^k$ such that $y = (y_1, \dots, y_k)^T$ and

$$y^T C_k y \leq 0$$

Now define $z \in \mathbb{R}^n$ by $z = (y_1, \dots, y_k, 0, \dots, 0)^T$. Then we will have

$$z^T Cz = \sum_{i=1}^{n} (z^T)_i (Cz)_i$$

$$= \sum_{i=1}^{n} \left[(z^T)_i \left(\sum_{j=1}^{n} C_{ij} z_j \right) \right]$$

$$= \sum_{i=1}^{n} \left[(z^T)_i \left(\sum_{j=1}^{k} C_{ij} y_j \right) \right] = S$$

were the last step follows since everything after the j^{th} index of z is zero and z and y agree of the first k indices. Then appling the same logic again,

$$S = \sum_{i=1}^{k} \left[(y^T)_i \left(\sum_{j=1}^{k} C_{ij} y_j \right) \right] = y^T C_k y \leq 0$$

a contradiction; so the result follows, using part (a), we see that every principal submatrix is invertable and thus C has an LU factorization.

$\square$

Question 2.

For a symmetric matrix A, A is positive definite if $A = LU$ where the diagonal of U is positive. The following are equivalent.

1. A is *positive definite*
2. A can be factored as $A = R^T R$ where R is upper triangular with positive entries on its diagonal.
3. $x^T A x > 0$ for all $x \neq 0$.

Proof. (1) $\iff$ (2) was done in class. We first prove (2) $\implies$ (3). So assume that $A = R^T R$ where R is upper triangular and has positive diagonal entries. Then consider a nonzero x. We have

$$x^T A x = x^T (R^T R) x = (Rx)^T (Rx) \geq 0$$

since this is just a dot product, it is zero if and only if $Rx = 0$ but the assumptions on R show that it has a row echelon form with a pivot in every column and thus is invertible, it follows that the only x for which $Rx = 0$ is the zero vector. Hence $(Rx)^T (Rx) \neq 0$ for our choice of x and then $x^T A x > 0$. Now, assume that $x^T A x > 0$ for all nonzero x. Then by problem (1) we know that $A = LU$ so we only need to show that the diagonal entries on U are all posistive. Since A is symmetric we may write $A = LDL^T$, so $x^T A x = x^T LDLx = (L^T x)^T D (L^T x)$. Then note for any diagonal matrix D and vector b, we have

$$b^T D b = b_1^2 D_{11} + \dots b_n^2 D_{nn}$$

hence if a element of the diagonal D_{ii} is not positive, we can choose $b = (0, \dots, 1, \dots 0)$ where 1 is in the i^{th} position. This will give us a product that is not positive.

Then going back to $(L^T x)^T D (L^T x)$, since L^T is upper triangluar with 1's on the diagonal it is invertable, from this it follows that $L^T x = 0$ iff $x = 0$ and that $L^T x = b$ is always consistent. Thus if any element of on the diagonal of D, say D_{ii} is not greater than 0 I choose an x such that $L^T x = b$ where b is a vector of all zeros execpt for the i^{th} posisiton. Then it will follow that $x^T A x \leq 0$; a contradiction. Hence the diagonal of D is positive and we have shown that (3) $\implies$ (1). This proves that all the statements are equalivalent.

$\square$

Question 3.

Show that the given matrix is not positive definite,

$$A == \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix} \tag{1}$$

Proof. The clear way of doing this is to compute a LU factorization and see if A has an appropriate LU factorization. Preforming the row operations $-2R_1 + R_2 \to R_2$ and $-R_2 + R_3 \to R_3$ shows that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -6 \end{bmatrix} \tag{2}$$

From this and the uniqueness of LU factorization we see that A cannot be given an LU factorization in which the diagonal entries of U are positive. $\square$

Question 4.

Let A be positive definite. Show that A^{-1} and A^2 are also positive definite.

Proof. First we prove that if A is symmetric so is its inverse. Assume that A is symmetric, then note

$$AA^{-1} = I = I^T = (A^{-1}A)^T = A^T {A^{-1}}^T = A{A^{-1}}^T$$

then multiplying on the left by A^{-1} shows A^{-1} is symmetric.

Now we will show $x^T A^{-1} x > 0$ for all nonzero x. Observe

$$x^T A^{-1} x = (x^T A^{-1}) A (A^{-1} x) = ({A^{-1}}^T x)^T A (A^{-1} x)$$

then since A^{-1} is symmetric this shows,

$$x^T A^{-1} x = (A^{-1} x)^T A (A^{-1} x)$$

Since x is nonzero and A^{-1} is nonsingular $A^{-1} X > 0$ so that the whole expression

$$x^T A^{-1} x > 0$$

. Thus A^{-1} is positive definite.

Now for A^2 note that

$$x^T A^2 x = x^T A^T A x = (Ax)^T (Ax)$$

and again this is a dot product of a nonzero vector with itself (since $x \neq 0$ and A is nonsingular), so $x^T A^2 x > 0$. for all $x \neq 0$.

□

Question 5.

Let A be a symmetric matrix given by

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{3}$$

Find an upper triangular matrix R with positive diagonal entries such that $A = R^T R$ or jusitfy why such R doesn't exist.

Proof. We compute an LDV factorization, since A is symmetric, this will alow us to produce such an R. Applying the row operations $\frac{1}{2}R_1 + R_2 \to R_2$ and $\frac{2}{3}R_2 + R_3 \to R_3$ and factoring out the diagonal gives us the following LDV factorization,

$$A == \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Since the diagonals are positive, we have that A is positive definite and that there exists an R such that $A = R^T R$, namely we pick

$$R = \begin{bmatrix} \sqrt(2) & 0 & 0 \\ 0 & \sqrt(3) & 0 \\ 0 & 0 & \frac{2\sqrt(3)}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

As described in class. □