MTH 451 Worksheet 4

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Question 1.

We must find a value of c such that f is a PDF.

First we compute $\int_S f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$.

$$\iint_0^\infty ce^{-(\frac{x}{2}+\frac{y}{4})}\,\mathrm{d}x\,\mathrm{d}y = c\int_0^\infty\int_0^\infty e^{-\frac{x}{2}}e^{-\frac{y}{4}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= c\int_0^\infty -2e^{-\frac{x}{2}}e^{-\frac{y}{4}}\Big|_0^\infty\,\mathrm{d}y = c\int_0^\infty 2e^{-\frac{y}{4}}\,\mathrm{d}y = c[-8e^{-\frac{y}{4}}\Big|_0^\infty]$$

so $8c = 1$ and thus $c = 1/8$. Then it is clear that f is always non-negative and less than one, so f is a PDF.

Question 2.

(a) Consider the region $R = [x_1, x_2] \times [y_1, y_2]$(for x_i, y_i in the support). Then

$$P((x,y) \in R) = \iint_R f(x,y)dxdy = \iint_R 2dxdy$$

intergrating this gives

$$P((x,y) \in R) = 2(x_2 - x_1)(y_2 - y_1)$$

but note that $(x_2 - x_1)(y_2 - y_1) = Area(R)$ and the result follows.

(b) for $P(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$ we have the region $0 \leq x \leq 1/2$ and $0 \leq y \leq 1/2$ so $Area(R) = 1/4$ Then multipling by 2 gives $P(x \leq \frac{1}{2}, y \leq \frac{1}{2}) = 1/2$.

(c) $P(X + Y > 2/3)$ To find the area of this region we graph the line $y = -x + 2/3$, then we see the desired region is the difference between the triangle with vertices $\{(1,0),(0,1),(0,0)\}$ and the triangle with vertices $\{(\frac{2}{3},0),(0,\frac{2}{3}),(0,0)\}$. Using the formula for the area of a triangle $(1/2bh)$ we get $Area(R) = \frac{2}{9}$ and so $P(x + Y > 2/3) = \frac{4}{9}$.

(d) $P(X > 2Y)$ we graph the function $\frac{1}{2}x = y$ and note we want the reqion enclosed by this line and the x-axis. The base of this trianlge is $b = 1$ and we can find the hight by calculating the intersection between $\frac{1}{2}x = y$ and $y = -x + 1$. We get the hight as $h = \frac{2}{3}$ so $P(x > 2Y) = 2 * \frac{1}{2}(1 * \frac{2}{3}) = \frac{1}{3}$.

Question 3.

We want the region $1 < x < \infty$ and $1 < y < 2x$. So we want to compute the intergral

$$P(Y \geq 2X) = \int_1^\infty \int_1^{2x} x^{-2}y^{-2}\,\mathrm{d}y\,\mathrm{d}x = \int_1^\infty -x^{-2}y^{-1}\Big|_1^{2x}\,\mathrm{d}x = I$$

$$I = \int_1^\infty x^{-2} - \frac{1}{2}x^{-3}\,\mathrm{d}x = \frac{1}{4x^2} - \frac{1}{x}\Big|_1^\infty = 0 - [\frac{1}{4} - 1] = \frac{3}{4}$$

Question 4.

Since $x + y < 1$ we have

$$f_X(x) = \int_0^{1-x} 24y - 24yx - 24y^2 \,\mathrm{d}y = 12y^2 - 12y^2x - 8y^3\Big|_0^{1-x}$$
$$= 12(1-x)^2 - 12x(1-x)^2 - 8(1-x)^3$$

Then

$$f_Y(y) = \int_0^{1-y} 24y - 24yx - 24y^2 \,\mathrm{d}x = 24yx - 12yx^2 - 24y^2x\Big|_0^{1-y}$$
$$= 24y(1-y) - 12y(1-y)^2 - 24y^2(1-y)$$