MTH 435: Analysis and Topology

Evan Fox

November 3, 2022

Contents

Chapter 1

Introduction

1.1 Algebraic and Order properties

The Real numbers, denoted $\mathbb{R}$ form a field, that is $\mathbb{R}$ is an Abelian group under addition and multiplication that has distinct identities and satisfies the distributive property. All important algebraic properties can be derived from the fact that $\mathbb{R}$ is a field.

Definition 1.1.1
Let $P \neq \varnothing$ be a subset of $\mathbb{R}$ not containing 0. We say P is the set of positive real numbers and it satisfies

  1. $a, b \in P \implies a + b \in P$

  2. $a, b \in P \implies ab \in P$

  3. $a \in P \implies a \in P \vee -a \in P \vee a = 0$

  We are now in a position to define the ordering we wish to place on $\mathbb{R}$

Definition 1.1.2
For $a, b \in \mathbb{R}$ such that $b - a \in P$ then we say $a < b$. If $b - a \in P \cup \{0\}$ then $a \leq b$

  It follows immediately from tricotomy that exactly one of $a < b, a = b, a > b$ must hold. It is also clear that this is a total ordering on $\mathbb{R}$, but we must check that this turns $\mathbb{R}$ into an ordered field.

Lemma 1.1.3
The ordering defined above is a strict total order.

Proof. The order is irreflexive since $a - a = 0 \notin P$, then if $a - b \in P$ and $b - a \in P$ we can add to get $0 \in P$ a contradiction. Now we must prove the transitive property, suppose $a, b, b \in \mathbb{R}$ satisfy $a < b$ and $b < c$, then $b - a, c - b \in P$ thus so is their sum, $c - a \in P$ which implies $a < c$ as desired. $\qquad\square$

Lemma 1.1.4
The ordering defined above turns $\mathbb{R}$ into an ordered field, let $a, b, c \in \mathbb{R}$.

  1. if $a < b$ then $a + c < a + b$

  2. if $c > 0$ and $a < b$ then $ac < bc$

3. if $c < 0$ and $a < b$ then $ac > bc$

Proof. For the first item, we have $b - a > 0$ then by adding and subtracting c we get

$$0 < b - a - c + c$$

$$a + c < b + c$$

Next assume $c > 0$, then we have $0 < b - a$, since both of theses are positive, so is their product,

$$0 < c(b - a) \implies 0 < cb - ca \implies ca < cb.$$

Now let $c < 0$, then $-c > 0$ and the argument is the same as above. □

Thus we have turned $\mathbb{R}$ into an ordered field.

Theorem 1.1.5
The natural numbers are all positive, we will prove this in the following steps.

1. If $a \in \mathbb{R}$ with $a \neq 0$ then $a^2 > 0$

2. $1 > 0$

3. $\mathbb{N} \subset P$

Proof. If $a > 0$ we are done, suppose $a < 0$, then $-a \in P$ and since $a^2 = (-a)(-a) \in P$ we have $a^2 \in P$. Now note $1^2 = 1$ so $1 \in P$, then since we defined the natural number n as $1 + \cdots + 1, n$ times, we see that all natural numbers are positive. □

Theorem 1.1.6
If $a \in \mathbb{R}$ satisfies $0 \leq a < \epsilon$ for all $\epsilon > 0$, then $a = 0$

Proof. Suppose $a > 0$, then let $\epsilon_0 = a/2$, then

$$0 < \epsilon_0 < a < \epsilon$$

a contradiction. □

Theorem 1.1.7
If $ab > 0$ then a, b are both positive or both negative.

Proof. Suppose $ab > 0$ and that at least one is negative, without loss of generality say $a < 0$, then if $b > 0$, we have $-ab > 0$ so $-(ab) \in P$ but we assumed $ab \in P$; a contradiction, thus b is negative. □

1.2   Absolute Value

Next we define a function of great importance on $\mathbb{R}$.

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0 \end{cases} \tag{1.1}$$

Now we prove some basic properties of the absolute value function,

Theorem 1.2.1
Basic properties of the absolute value.

1. $|ab| = |a||b|$

2. $|a|^2 = a^2$

3. If $c > 0$ then $|a| \leq c \Leftrightarrow -c \leq a \leq c$

4. $-|a| \leq a \leq |a|$

Proof. To prove (1) first note that if a or b is zero we are done. Then we just consider the four possible cases on the signs of a and b. For example if $a > 0$ and $b < 0$, we have $|ab| = -ab$ and $|a| = a, |b| = -b$ so $|a||b| = -ab$. The rest are left as an exercise. Proving (2) is similar, if $a > 0$ we are done and if $a < 0$, then $|a|^2 = (-a)^2 = a^2$. Now suppose $c > 0$ and $|a| \leq c$, if $a \leq 0$, then the result is clear. If $a < 0$ we have $|a| = -a$, so $-a < c$, rearranging gives $-c < a$. So we are finished. Now suppose $-c \leq a \leq c$, then $-a \leq c$ and $a \leq c$. But the absolute value maps to a or $-a$ so in either case we are done. Now for (4) we let $c = |a|$ and apply (3) to get the result. $\qquad\square$

Theorem 1.2.2 (Triangle Inequality)
For all $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b| \qquad\qquad (1.2)$$

Proof. By the above, we have that

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

implies

$$|a + b| \leq |a| + |b|$$

as desired. $\qquad\square$

The Triangle inequality is very important and equality hold only when a, b have the same sign.

Theorem 1.2.3
The following two inequalities hold for all $a, b \in \mathbb{R}$

1. $|a - b| \leq |a| + |b|$

2. $||a| - |b|| \leq |a - b|$

Proof. Proof of (1) follows from substituting $-b$ into the triangle inequality. To prove (2) start my applying the triangle inequality to $a = a - b + b$ to get $|a| \leq |a - b| + |b|$, and $b = b - a + a$ to get $|b| \leq |b - a| + |a|$, then subtracting gives

$$|a| - |b| \leq |a - b|$$

and

$$|b| - |a| \leq |a - b|.$$

We may multiply by -1 to get

$$-|b| + |a| \geq -|a - b|$$

and it follows,

$$-|a - b| \leq |a| - |b| \leq |a - b|.$$

Now we let $|a - b| = c$ and use the third result from theorem 1.2.1, to get

$$|a| - |b| \leq |a - b|$$

$\qquad\square$

1.3 Archimedean Property and Completeness

The completeness axiom is the last thing that we need in order to call are a complete ordered field.

Definition 1.3.1
A subset $A \subset \mathbb{R}$ is bounded is above if there exists $u \in \mathbb{R}$ such that for all $a \in A$ we have $a \leq u$, we say A is bounded below if the other inequality holds. We call u an upper bound or lower bound for the set A.

We say a set is bounded if it is bounded above and below.

Definition 1.3.2
We say that an upper bound $\alpha \in \mathbb{R}$ is a least upper bound if

1. *α is an upper bound.*

2. *For an arbitrary upper bound u, we have $\alpha \leq u$.*

Definition 1.3.3
Every subset $A \subset \mathbb{R}$ that is bounded above has a least upper bound.

We will see that this property is very important.

Theorem 1.3.4
Let $A \subset \mathbb{R}$, an upper bound $\alpha \in \mathbb{R}$ satisfies $\alpha = \sup(A)$ if and only if for all $\epsilon > 0$ there exists $a \in A$ such that $\alpha - \epsilon < a$

Proof. Let $\alpha = \sup(A)$ then for all $\epsilon > 0$, $\alpha - \epsilon < \alpha$ so it cannot be an upper bound. Conversely, let α be an upper bound with the desired property and let u be an upper bound, then if $u < \alpha$ we have $u = \alpha - \epsilon$ for some $\epsilon > 0$, but then by assumption there is an $a \in A$ such that $u < a$; a contradiction. $\square$

Lemma 1.3.5
$\mathbb{N}$ is not bounded above in $\mathbb{R}$

Proof. Assume that $\mathbb{N}$ is bounded, then there exists a least upper bound, let $\alpha = \sup(A)$, then there exists $n \in \mathbb{N}$ such that $\alpha - 1 < n$, but then $\alpha < n + 1$; a contradiction. $\square$

Theorem 1.3.6
For all $\epsilon \in \mathbb{R}_{>0}$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. By the unboundedness of n, we may choose $n \in \mathbb{N}$ such that $\frac{1}{e} < n$, then $\frac{1}{n} < \epsilon$. $\square$

Now we may look at the algebraic properties of sup

Theorem 1.3.7
let S, A, B be sets and $r \in \mathbb{R}$

1. $\sup(r + S) = r + \sup(S)$

2. $\sup(A + B) = \sup(A) + \sup(B)$

3. *if $a \leq b$ for all $a \in A, b \in B$ then $\sup(A) \leq \inf(B)$.*

Proof. exercise. See HW1 $\square$

Definition 1.3.8
A function $f : D \to \mathbb{R}$ is bounded in $\mathbb{R}$ is there exists $M \in \mathbb{R}$ such that

$$-M \leq f(x) \leq M \; \forall x \in D$$

Lemma 1.3.9
if f, g are bounded functions such that $f(x) \leq g(x)$ then $\sup(f(x)) \leq \sup(g(x))$, but $\sup(f(x)) \not\leq \inf(g(x))$ in general.

Proof. The first part is easy. To see that the second statement is false consider $f : [0,1] \to \mathbb{R}$ as $f(x) = x^2$ and $g : [0,1] \to \mathbb{R}$ as $f(x) = x$. Then $g(x) \geq f(x)$ but $\sup(f(x)) = 1 > 0 = \inf(g(x))$. $\square$

Now we turn to a very important fact about $\mathbb{R}$, That you can write it as the disjoint union of rationals and irrationals, and that the closure of $\mathbb{Q} = \mathbb{R}$.

Theorem 1.3.10
For $x, y \in \mathbb{R}$ with $x < y$ there exists $m, n \in \mathbb{Z}$ such that

$$x < \frac{m}{n} < y$$

That is $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with x, y. By the archimedean property fix $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. This gives

$$0 < 1 + nx < y$$

Now fix $m \in \mathbb{Z}$ such that $xn < m \leq xn + 1$. Then from $eq(1)$ it follows

$$nx < x \leq nx + 1 < ny \implies nx < m < ny \implies x < \frac{m}{n} < y.$$

$\square$

1.4 Intervals

Definition 1.4.1
An interval on $\mathbb{R}$ is a set of one of the following forms. Let $a < b$ be real numbers

1. $(a, b) = \{x | a < x < b\}$

2. $(a, b] = \{x | a < x \leq b\}$

3. $[b, a) = \{x | a \leq x < b\}$

4. $[a, b] = \{x | a \leq x \leq b\}$

We also alow rays, which start from a real number and go to positive or negative infinity. We will see in many theorems will will need to use facts about intervals, namely that they are connected topologically, this means that there is no separation of an interval into two disjoint open sets.

Now given an arbitrary subset of $\mathbb{R}$ we would like to be able to tell if it is an interval or not. The following theorem will help us in this regarded.

Theorem 1.4.2
Let $S \subset \mathbb{R}$ with at least two points $x, y \in S$ such that $x < y$ implies $(x, y) \subset S$. Then S is an interval.

Proof. We consider several cases

1. Assume that S is bounded, then fix $a = \sup(S)$ and $b = \inf(S)$. We prove that $S = (a, b)$ or $S = [a, b]$. So assume that S does not contain its maximal and minimal elements. We have clearly, $S \subset (a, b)$ since $a, b \notin S$. Now let $x \in (a, b)$, then $a < x < b$, but then we may fix $s, t \in S$ such that $a < s < x < t < b$, then by hypothesis $x \in (s, t) \subset S$.

2. Now Assume that S is bounded only above, then fix $a = \sup(S)$ and assume $a \notin S$, then $S \subset (-\infty, a)$. Take $x \in (-\infty, a)$ since this set is not bounded below I can fix $s, t \in \mathbb{R}$ such that $s < x < t$ then $x \in (s, t) \subset S$, by our assumption.

The other two cases where S is bounded above or is unbounded are similar. $\qquad \square$

1.5   Nested Interval Property

Now we will look at a very important and useful property of closed sets in $\mathbb{R}$.

Definition 1.5.1
A set of closed intervals of the form $I_n = [a_n, b_n]$ are nested if $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$.

Example 1.5.1
Consider the sets defined by $I_n = [\frac{-1}{n}, \frac{1}{n}]$. Then by computing the first few values we see $I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots$

Note that that the decision to force our sets to be closed is very important, and the property we are after doesn't hold in general for open intervals.

Theorem 1.5.2
If I_n is a sequence of nested intervals then there exists $\xi \in \bigcap_{i \in \mathbb{N}} I_n$.

Proof. Note that b_1 is an Upper bound for the set $A = \{a_k \,|\, k \in \mathbb{N}\}$ so that we may fix $\xi = \sup(A)$. Now we want to prove $\xi < b_n$ for all $n \in \mathbb{N}$. But since the intervals are nested if $b_n < a_k$ we will get a contradiction, so every b_n is a upper bound for A. $\qquad \square$

Now we have a special case, that is if the diameter of the interval goes to 0 then ξ is unique.

Corollary 1.5.3
If $\inf\{b_n - a_n\} = 0$ then ξ is unique.

Now we will demonstrate an application of the Nested Interval Property by proving the real numbers are uncountable. We do this by first showing that the interval $I = [0, 1]$ is uncountable, then since $\mathbb{R}$ contains I we must have that I is uncountable. It is also possible to exhibit a bijection between I and $\mathbb{R}$.

Theorem 1.5.4
I is uncountable.

Proof. Suppose not, then we may list the elements of I, say

$$I = \{x_1, x_2, \ldots, x_n, \ldots\}$$

Not we construct intervals I_n such that $x_n \notin I_n$ and $I_{n+1} \subset I_n$. Then applying the nested interval property gives $\xi \in \bigcap_{i \in \mathbb{N}} I_n$, but $\xi \neq x_n$ for any n so our list is incomplete. $\qquad\square$

Corollary 1.5.5
$\mathbb{R}$ *is uncountable.*

The assumption that we have made on $\mathbb{R}$, that every nonempty set that is bounded above has a least upper bound was all we needed to prove the nested interval property, and from that we see that we can show the real numbers are uncountable, this shows that the assumption that $\mathbb{R}$ is a complete metric space is what forces us to be uncountable. One may wonder, can one assume the nested interval property and prove that every nonempty set that is bounded above has a least upper bound? The answer is no, the nested interval property is weaker than compleness axiom.

Chapter 2

Topology

We will discuss topology in metric spaces such as $\mathbb{R}$ and in $\mathbb{R}^n$. Topology gives us the language we need to properly discuss the theorems of analysis. You will find later that it makes alot of things much more clear.

2.1 Euclidean Space

First we must define and discuss the elements of $\mathbb{R}^n$. We write the elements as ordered n-tuples. Addition and scalar multiplication are defined component wise and this turns the Euclidean space $\mathbb{R}^n$ into a vector space. Infact we can go further and turn $\mathbb{R}^n$ into an inner-product space by defining the following inner-product on $\mathbb{R}^n$. We will use boldface to denote vectors. Let $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. We define the *dot* product of $\mathbf{v}$ and bfw as

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i \leq n} v_i w_i. \tag{2.1}$$

One can now check that $\mathbb{R}^n$ equipped with this product becomes an inner product space. We are now in a place to define the *norm* of a vector

Definition 2.1.1
Let $\mathbf{v} \in \mathbb{R}^n$, we define the norm *of $\mathbf{v}$ as follows*

$$\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}.$$

Lets look at an example in $\mathbb{R}^2$

Example 2.1.1
Let $\mathbf{v} = (3, 4)$ then the norm of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} = \left(\sum_{i \leq 2} v_i^2 \right)^{\frac{1}{2}} = \sqrt{3^2 + 4^2}.$$

Which is clearly just the Pythagorean theorem.

The norm just defined is often reffered to as the Euclidean norm. Now we have turned $\mathbb{R}^n$ into a normed inner product space. for vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we define $\|\mathbf{v} - \mathbf{w}\|$ to be the distance from $\mathbf{v}$ to $\mathbf{w}$.

Theorem 2.1.2
Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and denote by $\|\mathbf{v}\|$ the norm of $\mathbf{v}$, then the following hold.

1. $\|\mathbf{v}\| \geq 0$ with equality iff $\mathbf{v} = \vec{0}$.

2. $\|\mathbf{v} - \mathbf{w}\| = \|\mathbf{w} - \mathbf{v}\|$.

3. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.

4. $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$.

5. $\|\mathbf{v} + \mathbf{w}\| \leq \|bfv\| + \|\mathbf{w}\|$.

Proof. The first three are just applications of the definition of the norm. To prove the fourth statement (which is called the Cauchy-Swartz inequality) we first agree that for $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$

$$\left(\sum_{i \leq n} a_i b_i\right)^2 \leq \left(\sum_{i \leq n} a_i^2\right)\left(\sum_{i \leq n} b_i^2\right) \tag{2.2}$$

is equal to what we want to prove. Now

$$\sum_k (a_k x + b_k)^2 \geq 0$$

since it is a sum of squares. Now multiplying out and factoring gives

$$\left(\sum_k a_k^2\right) x^2 + 2\left(\sum_k a_k b_k\right) x + \left(\sum_k b_k^2\right) \geq 0.$$

Introduce the substition $A = \sum_k a_k^2, B = \sum_k a_k b_k, C = \sum_k c_k^2$. This transforms the above into

$$Ax^2 + Bx + C \geq 0.$$

If $A = 0$ we are done so we may assume that $A > 0$. Then take $X = \frac{-B}{A}$, we get

$$\frac{-B^2}{A} + C \geq 0$$

and with some rearranging we see

$$AC \geq B^2$$

which is what we wanted to prove.

Now the proof of the triangle inequality follows easily, since we have

$$\|\mathbf{v} + \mathbf{w}\|^2 = \sum_k (\mathbf{v} + \mathbf{w})^2 = \sum_k \mathbf{v}_k^2 + 2\mathbf{v} \cdot \mathbf{w} + \sum_k \mathbf{w}_k^2$$

which is less than or equal to

$$\|\mathbf{v}\|^2 + \|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2$$

and factoring finishes the proofs. $\qquad\square$

2.2   Topology

Definition 2.2.1
Given a subset $S \subset \mathbb{R}$, we say that S is a maximal interval if

$$S \subset J \subsetneq \mathbb{R}$$

implies $S = J$.

Theorem 2.2.2
Let $S \subset \mathbb{R}$ be open. Then each $x \in S$ can belong to only one maximal interval.

Proof. For each $x \in S$ we define the following functions

$$a(x) = \inf\{x \mid (x, a) \subset S\}$$

and

$$b(x) = \sup\{x \mid (b, x) \subset S\}$$

Then $x \in (a(x), b(x))$ and if another interval with $x \in I$ contains $(a(x), b(x))$ it is clear the must be equal. $\qquad \square$

any element belongs to one and only one maximal interval
any open subset can be written as a union of disjoint maximal intervals

Theorem 2.2.3
Let S be an open subset of $\mathbb{R}$, then S can be written as a union of a countable collection of disjoint maximal intervals.

Proof. let $x \in S$ be in the maximal interval I_x, clearly the union covers S. Now if I_x and I_y both contain x, then their union $I_x \cup I_y$ is a maximal interval containg I_x and I_y, thus $I_x = I_x \sup I_y = I_y$. So they are disjoint.

To show that this collection is countable, consider the countable collection of rationals

$$\mathbb{Q} = \{x_1, x_2, \dots\}$$

while each maximal interval of S will contain infinitely many rationals, it only has one whose index is minimal. Construct the function ϕ from the collection of all maximal intervals to $\mathbb{N}$ which takes the interval I_n to n when n is the minimal index in I; if $\phi(I_m) = \phi(I_n) = n$ then they must share the rational whose index is n so ϕ is an injection into the natural numbers. $\qquad \square$

Remark 2.2.4
This representation is unique, thus an open interval I only has one representation as a union of maximal intervals, namely iteself; it follows that I cannot be written as a union of open intervals and as such is connected.

Theorem 2.2.5 (Bolzano-weierstrass)
let $S \subset \mathbb{R}$ be infinite and bounded, then it contains a limit point.

Proof. Since S is bounded it lies in some closed interval $[-a, a]$, either $[-a, 0]$ or $[0, a]$ must contain infinitely many points of S. Without loss of generality we may assume that it is $[-a, 0]$. We proceed with the same trick and again partition $[-a, 0]$ into $[-a, \frac{-a}{2}]$ and $[\frac{-a}{2}, 0]$ and assume that the first interval contains infinitely many points of S. repeating this gives us a sequence of nested intervals which then produces a unique element in the intersection ξ since the diameter of the intervals goes to zero. Then any nbhd of ξ will intersect S in infinitely many points. Thus ξ is a limit point. Then we have that $\qquad \square$

This theorem is extremely useful, for example we can use this to show that every convergent sequence contains a convergent subsequence. We may use the Bolzano-weierstrass theorem to prove the cantor intersection theorem.

Theorem 2.2.6 (Cantor Intersection)
Let $Q_1 \supset Q_2 \supset \ldots$ be decreasing chain of non-empty sets where Q_1 is bounded and Q_k is closed. Then the intersection is closed and non-empty

Proof. We will use the Bolzano-weierstrass thm, since if the Q_i's are finite, then the result is clear since the are decreasing and non-empty. Now, since we may assume that they are infinte and bounded, Let $A = \{x_1, x_2, \ldots, \}$ where $x_k \in Q_k$. Since A is contained in Q_1 it is bounded, and since it is infinte, we may apply the Bolzano-weierstrass theorm to produce a limit point of A, ξ. We want to show that

$$\xi \in \cap_{i=1}^{\infty} Q_i$$

Since ξ is a limit point, every nbhd intersects A in infinitely many points. But then due to the decrseasing nature of the Q_i's, if a nbhd around ξ contains infinitely many points of A, it must contain infinitely many points of some Q_k, since each Q_k contains all but a fininte number of points in A. Thus ξ is a limit point of each Q_k and since they are closed, it must belong to each Q_k. $\square$

Theorem 2.2.7 (Lindelof Covering Theorem)
If S is a closed and bounded subset of $\mathbb{R}^n$, then every open cover F contains a countable subcover.

Proof. This will follow since we know that $\mathbb{R}$ has a countable basis. Given a basis $\mathfrak{B}$ given by balls with rational radii, forms a basis for the standard topology on $\mathbb{R}$. It is then a trival result that a basis for $\mathbb{R} \times \mathbb{R}$ is given by $\mathfrak{B} \times \mathfrak{B}$. $\square$

The biggest use the the above theorem is to help us prove and even stronger result about closed and bounded sets in $\mathbb{R}^n$.

Theorem 2.2.8 (Hine-Borel Theorem)
If $A \subset \mathbb{R}$ is closed and bounded, then every open cover F of A contains a finite subcover.

Proof. Let F be an open cover for A, then by lindelof, we know that there is a countable subcover, $\{I_1, I_2, \ldots\}$. Now define

$$S_m = \cup_{i=1}^{m} I_i$$

we want to show that at some point $A \subset S_m$ for some m. Observe that since S_m is open $\mathbb{R}^n \setminus S_m$ is closed and as such, $A \cap (\mathbb{R}^n \setminus S_m)$ is closed. Now Define $Q_1 = A$ and for $m > 1$

$$Q_m = A \cap (\mathbb{R}^n \setminus S_m)$$

When $Q_m = \varnothing$ then we are done. Since the S_m are increasing, Q_m are decreasing, since A is bounded so we know that Q_m is bounded, and we also know that Q_m is closed. Now if we assume that Q_m is never emtpy we can apply the cantor intersectin theorem to get a point

$$x \in \cap_{i=1}^{\infty} Q_m$$

That is, $x \in A$, and $x \in R^n \setminus S_m$, Then $x \notin S_m$ for all $m \in \mathbb{N}$ but this contradics the fact that S_m is an open cover. Hence at some point Q_m must be empty. $\square$

Definition 2.2.9
Let X be a topological space, $S \subset X$ is compact if every open cover contains a finite subcover.

The Hine-Borel theorem tells us that every closed and bounded subset of $\mathbb{R}^n$ is compact, it happens to be true that in $\mathbb{R}^n$, the converse also holds so that the only compact sets are closed and bounded. Note that these notions do not generalize to abstract spaces since boundedness is not a topological property (see topology notes).

Theorem 2.2.10
If $A \subset \mathbb{R}^n$ is compact, then it is closed and bounded, equivalently, every infinte subset of A contains a limit point in A.

Proof. Suppose that A is compact, then for $x \in A$, $B(x, r_i)$, with $r_i = i \in \mathbb{N}$ is an open cover hence it contains a finite subcover. Taking the largest radii shows A is bounded. Now we must show that A is closed. Suppose there is a limit point of A, y which is not contained in A. Now for each $x \in A$ define $r_x = \frac{1}{2}\|x - y\|$, each r_k is positive since $y \notin A$, now the collection of open balls, $B(x, r_x)$ (where x ranges over A) form an open cover and so there exists a finite subcover, $B(x_1, r_1), \ldots, B(x_n, r_n)$. Now choose $r = min\{r_1, \ldots, r_n\}$ and consider $B(y, r)$; if $x_0 \in B(y, r)$, then for any $x \in A$

$$\|x - y\| \le \|x - x_0\| + \|x_0 - y\|$$

so

$$\|x - y\| - \|x_0 - y\| \le \|x - x_0\|$$

and since $\|x - y\| = 2r_x \ge r$ and $\|x_0 - y\| < r$, then $\|x - x_0\| \ge 2r_x - \|x_0 - y\| > r_x$ so that x_0 is not in the ball $B(x, r_x)$ and thus $B(y, r)$ contains no points in any $B(x, r_x)$. Hence we have found a nbhd of y disjoint from A; contradicting the assumption that y is a limit point of A.

Now if A is closed and bounded, then we may apply Bolzano-weierstrass to show that every infinite subset of A has a limit point in A. Conversely assume that every infinite subset has a limit point in A. We prove that A is closed and bounded. If A is finite it is clear, so assume that A is infinite. Let $x \in A'$. By considering balls of radius $\frac{1}{n}$ around x, we can produce an infinte sequence which will converge to x, so that x is the only limit point of this set, hence $x \in A$. Now we must show boundedness, if A is unbounded then for every $m > 0$ there exists x_m such that $\|x_m\| > m$. Now consider $T\{x_1, x_2, \ldots\}$, this is an infinte subset of A and so by assumption there exists an accumulation point $y \in A$. But for $m > 1 + \|y\|$ we will have that

$$\|x_m - y\| \ge \|x_m\| - \|y\| > m - \|y\| > 1 \tag{2.3}$$

contradicting the fact that y is a limit point of T. Hence A is bounded. $\qquad \square$

2.3 Metric Spaces

we defined subspaces in terms of restricting a metric and then proved the definition of the subspace topology, i.e. if $S \subset Y$ is a subspace, $A \subset S$ is open if and only if there exists $U \subset Y$ such that $A = S \cap U$.

Then we discuss compacteness in a more general setting. The 'big' theorems of the last section do not hold in more general metric spaces, since the proofs used facts about $\mathbb{R}^k$ more than just being a metric space. I will insert some of the relavent practice problems when I get a chance to think about this futher.

Theorem 2.3.1
Let (M, d) be a metric space. If $A \subset M$ is compact, then

1. *A is closed and bounded.*

2. Every infinite subset of A has a limit point in A.

It turns out that in general the Hine-Borel theorem (converse to 1) is false in general, but the second condidtion listed is acutally equivalent to compacteness.

Proof. The proof that if A is compact then it is closed and bounded follows from above. Now suppose that A is compact and assume that there exists an infinite subset of A, T, which doesn't have a limit point in A. Then every neighborhood around $x \in A$ has either a trivial intersection with T or $U \cap T = \{x\}$. Then by taking a neighborhood for each x, we get an open cover of A. By compactness, this open cover contains a finite subcover S. But this implies that T is finite; a contradiction. Hence every infinre subset of A has a limie point in A. □

This basically concludes the discussion of metric spaces in this class, we spent a day doing review problems, which I may add to these notes at some stage.

Chapter 3

Sequences

We have spent about the first 7-8 weeks of the course discussing the basic propertys of metric spaces and well as learning some topology. Now we get to what is more generally considered analysis, which begins with a rigorous study of sequences.

A sequence can be thought of as a function $x : \mathbb{N} \to \mathbb{R}$, we often write x_n for $x(n)$

Here are more additions ξ.