Understanding Analysis Chapter 1

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The first part of chapter one just went over some prelims/reviews. Here are my solutions to selected exercises.

Question 1.

(a) Prove that the $\sqrt{3} \notin \mathbb{Q}$

First, I prove that $3|x^2 \implies 3|x$ as I will need this fact in my proof of the irrationality of $\sqrt{3}$.

Proof. We use the contrapositive, so assume that $3 \nmid x$. Then there exists $k \in \mathbb{Z}$ such that $x = 3k+1$ or $x = 3k+2$. Then note that if $x = 3k+1$, then

$$x = 3k + 1$$
$$x^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

and in the other case we have

$$x = 3k + 2$$
$$x^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$$

So, in both cases we see that $3 \nmid x^2$ as desired. □

Now I prove that $\sqrt{3} \notin \mathbb{Q}$

Proof. For the sake of contradiction, assume that $\sqrt{3} \in \mathbb{Q}$. Then we may fix $m, n \in \mathbb{Q}$ such that $\sqrt{3} = \frac{m}{n}$. We then have that,

$$3 = \left(\frac{m}{n}\right)^2$$

by the fundamental theorem of arithmetic, we may write m^2 and n^2 in terms of their prime factors and cancel any factor(s) that they have in common, i.e., we reduce $\frac{m}{n}$ such that they have no common factors. Since m and n have no common factors we note that they cannot both be divisible by 3. Fist observe that

$$3n^2 = m^2 \tag{1}$$

and hence we have $3|m^2$ which implies $3|m$; so we fix $k \in \mathbb{Z}$ such that $m = 3k$ and then we substitute this expression for m back into (1).

$$3n^2 = (3k)^2 = 9k^2 \tag{2}$$

$$n^2 = 3k^2 \tag{3}$$

and we have that 3 divides n^2 and by extension 3 divides n. Thus we have a contradiction as desired. $\square$

(b) Does a similar argument work to prove that $\sqrt{6} \notin \mathbb{Q}$? Where does the proof breakdown for $\sqrt{4}$?

Yes, weather or not this method works for $\sqrt{x}$ is related to the prime factorization of x, since the prime factors of 6 both have an exponent of 1, $6|m^2 \implies 6|m$ will hold. This is because if 2 and 3 are prime factors of m^2 then they will have to be prime factors of m, otherwise how would they have been prime factors of m^2? The point is squaring a number doesn't add new prime factors; it just multiples the exponent of each prime factor by 2. When we try to apply this argument to $\sqrt{4}$ the problem is that $4|m^2 \implies 4|m$ doesn't hold since the exponent of the prime factor of 4 is 2. To give an example note that $4|36 = 6^2 = 2^2 3^2$ but $4 \nmid 6 = 3(2)$.

Question 2.

Prove that there is no rational number satisfying $2^r = 3$.

Proof. We use contradiction, so assume that there exists $r \in \mathbb{Q}$ such that $2^r = 3$. Then since r is rational by assumption we may fix $m, n \in \mathbb{Q}$ such that $r = \frac{m}{n}$. Substituting in for r gives

$$2^{\frac{m}{n}} = 3$$

then we raise both sides to the n^{th} power and get

$$2^m = 3^n$$

which contradicts the uniqueness of the fundamental theorem of arithmetic. □

Question 3.

The triangle inequality is given by $|a+b| \leq |a| + |b|$.

(a) Verify the triangle inequality in the special case that a and b have the same sign.

Proof. Let $a, b \in \mathbb{R}$ and assume that a and b have the same sign. Then if a and b are both negitive we see that $|a+b| = a+b$ and if a and b are both positive then $|a+b| = a+b$. Then note that regardless of the signs of a and b, $|a| + |b| = a + b$. It is then clear that the inequality holds. □

(b) Give a general proof the the triangle inequality.

Proof. First we prove that $(a+b)^2 \leq (|a|+|b|)^2$ by observing $ab \leq |ab|$. Then we multiply both sides by 2 and obtain $2ab \leq 2|ab|$, we then add $a^2 + b^2$ to both sides and get $a^2 + 2ab + b^2 \leq |a|^2 + 2|ab| + |b|^2$. Factoring this gives $(a+b)^2 \leq (|a|+|b|)^2$. We can now use this to prove the triangle inequality by taking the square root of both sides which yeilds,

$$|a+b| \leq ||a| + |b||$$

but since $|a| + |b| \geq 0$ we have

$$|a+b| \leq |a| + |b|$$

as desired.

□

(c) Use the triangle inequality to prove that $|a-b| \leq |a-c|+|c-d|+|d-b|$.

Proof. Let $x = (a - c)$ and let $y = (c - d) + (d - b)$, the the triangle inequality tells us that

$$|x + y| \leq |x| + |y|$$
$$|a - b| \leq |a - c| + |(c - d) + (d - b)|$$

Now we can apply the triangle inequality again to the second term on the right hand side of the last equation. So we let $z = (c - d)$ and let $w = (d-b)$, by the triangle inequality we the have $|c-b| \leq |c-d|+|d-b|$. Substituting back in gives,

$$|a - b| \leq |a - c| + |(c - d) + (d - b)| = |a - c| + |c - d| + |d - b|$$
$$|a - b| \leq |a - c| + |c - d| + |d - b|$$

as desired.

$\square$

Question 4.

Given a function f and a subset of its domain A, let $f[A]$ denote the range of f over A, i.e., $f[A] = \{f(x)|x \in A\}$

(a) let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Let $A = [0, 2]$ and let $B = [1, 4]$. Does $f[A \cap B] = f[A] \cap f[B]$? What about $f[A \cup B] = f[A] \cup f[B]$?

Proof. Both parts of this question are true. First I prove $f[A \cap B] = f[A] \cap f[B]$. First note that $A \cap B = [1, 2]$ then we may compute the range of f on this domain and we get that $\operatorname{ran} f = [1, 4]$.

To prove that $\operatorname{ran} f = [1, 4]$ first we prove that $\operatorname{ran} f \subseteq [1, 4]$ so let $y \in \operatorname{ran} f$. Then we fix $x \in \operatorname{dom} f$ such that $y = f(x) = x^2$. Then since f is increasing on the interval $[1, 2]$ it is clear that $y \in [1, 4]$. Now to prove that $[1, 4] \subseteq \operatorname{ran} f$ let $y \in [1, 4]$ be arbitrary. Then we must find a $x \in \operatorname{dom} f$ such that $f(x) = y$. Choosing $x = \sqrt{y}$ gives the desired result. Since $f(\sqrt{y}) = (\sqrt{y})^2 = y$. Since $\sqrt{y}$ is well defined on $[0, 4]$ we conclude that $\operatorname{ran} f = [1, 4]$. Hence $f[A \cap B] = [1, 4]$. $\square$

(b) Give an example where $f[A \cap B] = f[A] \cap f[B]$ doesn't hold.

Example: let $A = [-1, -2]$ and $B = [1, 2]$.

(c) Let $A, B \subseteq \mathbb{R}$ and prove that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, $g[A \cap B] \subseteq g[A] \cap g[B]$.

Proof. Let $y \in g[A \cap B]$ be arbitrary. Then there exists an $x \in A \cap B$ such that $y = g(x)$. Since $x \in A \cap B$ we know that $x \in A$ and $x \in B$. We also have that $y = g(x)$. Hence $y \in g[A]$ and $y \in g[B]$. Then we have that $y \in g[A] \cap g[B]$. $\square$

*side note: if I add the condidtion that g be injective, I think that you could prove $g[A \cap B] = g[A] \cap g[B]$. A counter example existed for $f(x) = x^2$ only because it is not injective (I think).

(d) Form a conjecture about $g[A \cup B]$ and $g[A] \cup g[B]$ and prove it.

Conjecture: $g[A \cup B] = g[A] \cup g[B]$.

Proof. Let $y \in g[A \cup B]$ then we may fix $x \in A \cup B$ such that $y = g(x)$. Since we have that $x \in A$ or $x \in B$, we know that either $y \in g[A]$ or $y \in [B]$. It is then clear that y is in the union. To prove the other direction assume that $y \in g[A] \cup g[B]$, then either $y \in g[A]$ or $y \in g[B]$. If $y \in g[A]$ then we may fix $x \in A$ such that $y = g(x)$. Then it follows that $x \in A \cup B$; So then $y \in g[A \cup B]$. If it is not the case that $y \in g[A]$ then $y \in g[B]$ must be true and the argument is the same. $\square$

Question 5.

Let $A, B \subseteq \mathbb{R}$ and let $g : D \to \mathbb{R}$ be arbitrary. Then we define $g^{-1}[B] = \{x \in D | g(x) \in B\}$. Prove that $g^{-1}[A \cap B] = g^{-1}[A] \cap g^{-1}[B]$.

Proof. Let $x \in g^{-1}[A \cap B]$, then we may fix $g(x) \in A \cap B$ such that $x \mapsto g(x)$. Then since $g(x) \in A \cap B$ we have that $g(x) \in A$ and $g(x) \in B$. It then follows by definition that $x \in g^{-1}[A]$ and $x \in g^{-1}[B]$; then again by definition we have $x \in g^{-1}[A] \cap g^{-1}[B]$.

To prove the other direction, let $x \in g^{-1}[A] \cap g^{-1}[B]$. Since $x \mapsto g(x)$ we have that $g(x) \in A$ and $g(x) \in B$. Thus $g(x) \in A \cap B$. Then by defintion we have that $x \in g^{-1}[A \cap B]$ as desired.

$\square$

Question 6.

let $y_1 = 6$ then for all $n \in \mathbb{N}$ define $y_{n+1} = \frac{2y_n - 6}{3}$.

(a) Prove that $y_n > 6$ for all $n \in \mathbb{N}$

Proof. We proceed with the principal of mathematical induction.

Base Case

We show that $y_2 > -6$, since we already have $y_1 = 6$ we compute y_2 using the definition and we get

$$y_2 = \frac{2(y_1) - 6}{3} = \frac{2(6) - 6}{3} = 2 > -6$$

as desired.

Induction Hypothesis

Assume that for some $k \geq 2$ that we have $y_k > -6$ we want to show that $y_{k+1} > -6$.

Induction Step

we have that $y_k > -6$ and can use simple algebra to get to the y_{k+1} case.

$$y_k > -6$$
$$2y_k > -12$$
$$2y_k - 6 > -18$$
$$\frac{2y_k - 6}{3} > -6$$
$$y_{k+1} > -6$$

and so by the principal of mathematical induction the proposition holds.

□

(b) Show that the sequence $\{y_1, y_2, y_3, ...\}$ is decreasing.

Proof. To prove that the sequence is decreasing we show that for any $n \in \mathbb{N}$ we have $y_n > y_{n+1}$. By the previous result we have that $y_n > -6$ for all $n \in \mathbb{N}$. Then it follows,

$$\begin{aligned} 3y_n - 2y_n &> -6 \\ 3y_n &> 2y_n - 6 \\ y_n &> \frac{2y_n - 6}{3} \end{aligned}$$

but then by defintion of the $n+1$th term we get

$$y_n > y_{n+1}$$

as desired.

□

Question 7.

We have DeMorgan's laws given by

$$A^c \cap B^c = (A \cup B)^c$$

and

$$A^c \cup B^c = (A \cap B)^c$$

(a) use induction to prove DeMorgan's laws for an arbitrary number of unions/intersections with $n > 1$.

$$\bigcap_{i=1}^{n} A_i^c = \left(\bigcup_{i=1}^{n} A_i \right)^c$$

Proof. **Base Case**

For a basecase of $n = 2$ we simply have DeMorgan's laws, to prove that $A_1^c \cap A_2^c = (A_1 \cup A_2)^c$ first let $x \in A_1^c \cap A_2^c$ then we have that $x \in A_1^c$ and that $x \in A_2^c$. By definiton this gives $x \notin A_1$ and $x \notin A_2$. It then follows that $x \notin A_1 \cup A_2$. Which then by definiton implies $x \in (A_1 \cup A_2)^c$. The proof of the other direction is similar.

Induction Hypothesis

Assume that for some $k \geq 2$ that we have $\bigcap_{i=1}^{k} A_i^c = \left(\bigcup_{i=1}^{k} A_i\right)^c$.

Induction Step

Since we have $\bigcap_{i=1}^{k} A_i^c = \left(\bigcup_{i=1}^{k} A_i\right)^c$ we intersect both sides with A_{k+1}^c and get

$$\left(\bigcap_{i=1}^{k} A_i^c\right) \cap A_{k+1}^c = \left(\bigcup_{i=1}^{k} A_i\right)^c \cap A_{k+1}^c$$

$$\bigcap_{i=1}^{k+1} A_i^c = \left(\bigcup_{i=1}^{k} A_i\right)^c \cap A_{k+1}^c$$

to deal with the right hand side we simply apply DeMorgan's law as proven in the basecase. So we have that $\left(\bigcup_{i=1}^{k} A_i\right)^c \cap A_{k+1}^c = \left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)^c$. Now Substituting in yeilds

$$\bigcap_{i=1}^{k+1} A_i^c = \left(\bigcup_{i=1}^{k+1} A_i\right)^c$$

as desired.

□

(b) Give an example showing that induction cannot be used to imply the validity of the infinite case.

Proof. Consider $B_i = (0, \frac{1}{i})$ we use induction to show that for all $n \in \mathbb{N}$ with $n > 1$ the intersection of B_i is non-empty.

Base Case

For $n = 2$ we have that $\bigcap_{i=1}^{2} B_i = (0, \frac{1}{2})$.

Induction Hypothesis

Assume that for some $k \geq 2$ that we have $\bigcap_{i=1}^{k} B_i \neq \varnothing$.

Induction Step

We know that $\bigcap_{i=1}^{k} B_i \neq \varnothing$, now observe that for all $n \in \mathbb{N}$ the fraction $\frac{1}{n} > 0$. Also note that $\frac{1}{n+1} < \frac{1}{n}$. Hence it follows that $B_{k+1} \neq \varnothing$ and that $B_{k+1} \subseteq B_k \subseteq \bigcap_{i=1}^{k} B_i$. then we have that

$$\bigcap_{i=1}^{k} B_i \cap B_{k+1} \neq \varnothing$$
$$\bigcap_{i=1}^{k+1} B_i \neq \varnothing$$

as desired.

Now we show that $\bigcap_{i=1}^{\infty} B_i = \varnothing$ by contradiction. So assume that there exists $x \in \bigcap_{i=1}^{\infty} B_i$. Then we would have $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. But then by the Archimedian property we may fix $N \in \mathbb{N}$ such that for all $n > N$ we have $\frac{1}{n} < x$. Hence $x \notin B_n$. But since we assumed that $x \in \bigcap_{i=1}^{\infty} B_i$ we have a contradiction. Thus we see that $\bigcap_{i=1}^{\infty} B_i = \varnothing$ must be true. Hence induction on $\mathbb{N}$ does not imply the infinite case.

□

(c) prove the infinte case of DeMorgan's laws.

Proof. We want to prove that

$$\bigcap_{i=1}^{\infty} A_i^c = \left(\bigcup_{i=1}^{\infty} A_i \right)^c$$

We proceed by showing that they are subsets of each other in the standard way. So assume that $x \in \bigcap_{i=1}^{\infty} A_i^c$ Which implys that $x \notin A_i$ for all $i \in \mathbb{N}$. Hence $x \notin \bigcup_{i=1}^{\infty} A_i$. Then by defintion x must be in the complement. Now we assume that $x \in (\bigcup_{i=1}^{\infty} A_i)^c$ then we have that $x \notin \bigcup_{i=1}^{\infty} A_i$, Thus $x \notin A_i$ for all $i \in \mathbb{N}$ which implys that $x \in A_i^c$ for all $i \in \mathbb{N}$. Then we get the desired result that $x \in \bigcap_{i=1}^{\infty} A_i^c$.

□