Homework 6

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Problem 1.

1. Let $[c, d]$ be an interval We show that $f_n(x) \xrightarrow{\text{unif.}} f(x)$ where $f(x) = x$ for all $x \in \mathbb{R}$. Take $\epsilon > 0$ then let $N > \frac{d}{\epsilon}$. It follows
$$|f_n(x) - f(x)| = |x(1 - \frac{1}{n}) - x| = \left|\frac{x}{n}\right| \leq \left|\frac{d}{n}\right| < \epsilon.$$
since $x \leq d$. Hence the sequence $f_n(x)$ is converging uniformly to f for all $x \in [c, d]$. Notice that we have to take a bounded interval since the convergence would not be uniform on $\mathbb{R}$, since the value of x would be unbounded.

   Now we prove the same result for $g_n(x)$, where we define
$$g(x) = \begin{cases} 0 & x \in \mathbb{I} \cup \{0\} \\ b & x \in \mathbb{Q} \setminus \{0\} \text{ and } x = \frac{a}{b} \end{cases} \tag{1}$$
first if $x \in \mathbb{I} \cup \{0\}$ then the result is clear (maybe I will fill this in later.) so suppose that we have $x \in \mathbb{Q} \setminus \{0\}$. Let $\epsilon > 0$ and take $N > \frac{1}{\epsilon}$, we have, for all $n \geq N$ that
$$\left|b + \frac{1}{n} - b\right| = \left|\frac{1}{n}\right| < \epsilon.$$
Therefore we again have uniform convergence.

2. Now we wish to show that h_n does not converge unifomly on $[c, d]$. when $x \in \mathbb{Q} \setminus \{0\}$ we have that $h_n(x) = x(1 + \frac{1}{n})(b + \frac{1}{n}) = xb + \frac{x}{n} + \frac{xb}{n} + \frac{x}{n^2}$. Since $h_n(x_0)$ converges to $x_0 b$, h_n converges pointwise to $h(x) = xb$. Then this must be our candidate for h (since uniformconvergence implies pointwise). Then taking the distance between $h_n(x)$ and $h(x)$ for $x \in \mathbb{Q} \setminus \{0\}$, we have,
$$|h_n(x) - h(x)| = |\ fracxn + \frac{xb}{n} + \frac{x}{n^2}|.$$
Since x is chosen in a bounded interval, the first and last terms will vanish as $n \to \infty$, however, the middle term, can be made arbitrarily large for any fixed $n \in \mathbb{N}$, since we can find rational numbers in any interval with large denominator (even in reduced form). Hence there can be no such n fixed. We see that h cannot converge uniformly.

Problem 2.

Proof. (a) Suppose that $f_n \xrightarrow{\text{unif.}} f$ and $g_n \xrightarrow{\text{unif.}} g$. Then for $\epsilon > 0$, there exists N_1, N_2 such that for all $x \in S$ we have $n \geq N_1$ implies
$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$
and $n \geq N_2$ implies for all $x \in S$ that,
$$|g_n(x) - g(x) < \frac{\epsilon}{2}$$
holds. Then for $n \geq \max\{N_1, N_2\}$ we will have for all $x \in S$
$$|(f_n(x) + g_n(x)) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) + g(x)| < \epsilon.$$
Hence the convergence is uniform.

(b) Let $|f_n(x)| \leq M_n$ and let $|g_n(x)| \leq T_n$ for all $n = 1, 2, \ldots$. Since f and g converge uniformly, they are cauchy. I will write f_n for $f_n(x)$ and similliarly for g. Observe,

$$|f_n g_n - f_m g_m| = |f_n g_n - f_n g_m + f_n g_m - f_m g_m| \leq |f_n(g_n - g_m) + g_m(f_n - f_m| \leq M|g_n - g_m| + T|f_n - f_m|$$

where M and T are uniform bounds on $\{f_n\}$ and $\{g_n\}$ (Im using prb 1 in the book here). And clearly, this shows the sequence $\{f_n g_n\}$ to be cauchy since f_n and g_n are.

$\square$

Problem 3.

Proof. (a) Clearly for $x \in (0, 1)$ we have $f_n(x) \to 0$ as $n \to \infty$. This convergence will not be uniform since after fixing $\epsilon > 0$, given any choice of $N \in \mathbb{N}$, we can find a small $x \in (0, 1)$ so that the expression $\frac{1}{nx+1} > \epsilon$ for all $n > N$.

(b) Let $\epsilon > 0$, let $N > \frac{1}{\epsilon}$, then for all $n \geq N$ we will have

$$\left|\frac{x}{nx+1}\right| < \left|\frac{x}{(\frac{1}{\epsilon})x+1}\right| < \left|\frac{1}{\frac{1}{\epsilon}+1}\right| = \left|\frac{\epsilon}{\epsilon+1}\right| < \epsilon$$

since $x \in (0, 1)$., hence the convergence is uniform.

$\square$

Problem 4.

Proof. If the convergence is uniform then the continuity of f is clear from results proved in class. Further, property (ii) is implies by uniform convergance taking $\delta = \epsilon$ and $m \in \mathbb{N}$ such that $n > m$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$.

To see sufficiency we make heavy use of the hint given in the book. Suppose that (i) and (ii) hold. By continuity, there exits ξ_1 and ξ_2 such that $|x - x_0| < \xi_1 \implies |f_k(x) - f_k(x_0)| < \epsilon_1 < \frac{\delta}{2}$ for all k, and $|x - x_0| < \xi_2 \implies |f(x) - f(x_0)| < \epsilon_2 < \frac{\delta}{2}$. Then by pointwise convergence fix k be such that $|f_k(x_0) - f(x_0)| < \delta - \epsilon_1 - \epsilon_2$, then for all $x \in S$ such that $|x - x_0| < min\{\xi_1, \xi_2\}$, we have $|f_k(x) - f(x)| < \epsilon_1 + (\delta - \epsilon_1 - \epsilon_2) + \epsilon_2 = \delta$, since the distance from $f_k(x)$ to $f_k(x_0)$ is at most ϵ_1 and similliarly the distance from $f(x)$ to $f(x_0)$ is at most ϵ_2.

Now, we have established that for each $x_0 \in S$, there is a nbhd, such that all x in this nbhd satisfy the same inequality ($|f_k(x_0) - f(x_0)| < \delta$, where k was fixed and depended only on x_0) that x_0 does. Letting x_0 vary, we obtain an open cover of S and by compactnes a finite number of these nbhds must conver S. Now associating the nbhd around x_0 to k, we get a finitly many values of k such that for any $x \in S$, $|f_k(x) - f(x)| < \delta$, for some choice of k. Since there are finitly many choices of k, fix the largest one. Then the inequaility $f_k(x) - f(x)| < \delta$ holds for all $x \in S$, that is, f_k converges uniformly.

$\square$

Problem 5.

Proof. (a) Clearly, we just need to show that (ii) is satisfied in the above problem. Let $\epsilon > 0$, then choice $\delta = \epsilon$ and $m = 1$. Then for $n > m$ we have that $|f_k(x) - f(x)| < \delta = \epsilon \implies |f_{k+1}(x) - f(x)| < |f_k(x) - f(x)| < \epsilon$. Since $f_{k+1}(x) \leq f_k(x)$ for all $x \in S$, and $f_n \to f$ pointwise.

(b) Note that the limit function in 9.5 is continuous, since it is constant. Furthmore, we have $f_n(x) \geq f_{n+1}(x)$ for all $x \in S$ and n, since

$$\frac{1}{nx+1} \geq \frac{1}{nx+x+1} = \frac{1}{(n+1)x+1}$$

for all $x \in (0, 1)$. The only condidtion not satisfied is the compacteness of S, since $(0, 1)$ is not compact.

$\square$

Problem 6.

Proof. I will use facts from 525. Abels test is proven for series in chapter 8, and is listed as thm 8.29. The proof relys on having a a metric space, as well as addition and multiplication so that the sums are all defined and also an ordering. Hence, I claim the the proof will hold in the metric space of functions on $\mathbb{R}$ in the uniform topology (where addidtion and multiplication are defined in the usual pointwise way). We know that convergence in the uniform topology on $\mathbb{R}^\infty$ (a point of which can be thought of as a function), is equaivalent to uniform convergence as functions (proven in 525). Hence applying Abels theorem from chapter 8 in this space, proves the theorem.

□

Problem 7.

Proof. (a) Defining $f(x) = \lim_{n\to\infty} f_n(x)$ we can see that $f(x) = 0$ for all x. Then by applying the quotient rule, we see that $\{f'_n(x)\} = \{\frac{1-nx^2}{(1+nx^2)^2}\}$. Then defining $g(x)$ to be the limit of the previous sequence we have $g(x) = 0$ for $x \neq 0$ and $g(0) = 1$.

(b) Since $f(x)$ is identically zero, the derivitive exists and is $f'(x) = 0$ for all x. Thus, from the above we can see that $f'(0) = 0 \neq 1 = g(0)$. However, equality will hold for all other choices of x (that is $x \in \mathbb{R} \setminus \{0\}$.

(c) $f_n \to f$ uniformly on $\mathbb{R}$, and every subinterval. since for all $x \in [a, b]$, let $\epsilon > 0$, then the function is bounded, letting $c \in [a, b]$ be a point obtaining the max, and letting $n > \frac{1}{c\epsilon}$,

$$|f_n(x) - f(x)| = |f_n(x)| = |\frac{x}{1+nx^2}| < |\frac{c}{1+\frac{1}{c\epsilon}c^2} < \epsilon$$

We dont go to infinity because of the 1 in the denominator, making the whole denomator greater than 1 even for x near 0.

(d) The situation for $f'_n \to g$ is not so as nice as the previous. We have that the convergence will be uniform on all intervals not containing 0. Consider an interval $[a, b]$ containing 0, Then the convergence is uniform if there exists an ϵ such that $|\frac{1-nx^2}{(1+nx^2)^2} - g(x)| < \epsilon$. but letting $\epsilon > 0$, then for each n, we may pick $x \neq 0$ (hence $g(x) = 0$) close enouph to 0, such that the fraction

$$\frac{1-nx^2}{(1+nx^2)^2} \to 1$$

hence the convergence is not uniform. On the other hand if $0 \notin [a, b]$, then there is no obstruction to the convergence being uniform, since the quadraic n^2 in the denominator will make the inequality above go to 0, no matter the choice of $x \in [a, b]$.

□

Problem 8.

Proof. by thm 9.8 in the book we know that

$$\int_0^1 f_n\, dx \to \int_0^1 f\, dx$$

now for each $n \in \mathbb{N}$, we may write

$$\int_0^1 f_n\, dx = \int_0^{1-\frac{1}{n}} f_n\, dx + \int_{1-\frac{1}{n}}^1 f_n\, dx$$

it follows that the limit of the left is equal to the limit of the right. Hence we will just need to show that the second term on the RHS is going to converge to 0, which seems reasonable. Since the length of the interval $[1-\frac{1}{n}, 1]$ is getting smaller as $n \to \infty$, the value $|f_n(x)|$ would need to keep growing in order to prevent the integral from converging to 0. But then $\lim_{n\to\infty} f_n(x) = \infty$ which contradicts that the f_ns converge to a function. For example it is easily seen that $\lim_{n\to\infty} \int_0^{\frac{1}{n}} n\, dx = 1$, but the functions $f_n(x) = n$ do not converge to any f. □

Problem 9.

Proof. I didnt complete this one but I will write the first part. To begin, note that

$$\sum_{n=1}^{\infty} \frac{1}{2n+1} - \frac{1}{2n+2} = \sum_{i=n}^{\infty} \frac{1}{(2n+1)(2n+2)} < \sum_{n=1}^{\infty} \frac{1}{n^2} = 2$$

So it converges when $x = 1$, by the comparision test, then for any other choice of x, it is clear that the partial sums form an increasing sequence bounded by the sum for $x = 1$, hence must converge. Saying that the convergence is not uniform is equivalent to saying that there exits $\epsilon > 0$ such that for all $n \in N$, we have $\sum_{k=n+1}^{\infty} \frac{x^{2k+1}}{2k+1} \frac{x^{k+1}}{2k+2} > \epsilon$ for some $x \in [0,1]$. Im not quite sure how to do this, morgan suggested to use taylor series, but I dont how that helps. $\square$

Problem 10.

Proof. Since $a_n n^{-s} \leq a_n$ for all $s \in [0, \infty)$, the uniform convergence follows from applying the weierstrauss M-test. The hypoethisis of the M-test is satisfied by assumption that $\sum_{i=1}^{\infty} a_i$ converges. Then we may apply results from the book to exchange the limit and summation in the secound part of the problem, then use the fact that $\lim_{s \to 0^+} n^{-s} = 1$ since it is a continuous function. $\square$