

MTH 316 Notes

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Chapter 1

Cosets and Lagrange's theorem

1.1 Cosets

Definition 1.1.1 (Cosets)

Let H be a subgroup of G then for each $a \in G$ we define the left coset of H as

$$aH = \{ah \in G | h \in H\}$$

We will find this definition usefull in our study of groups. First we prove basic propertys of cosets.

Theorem 1.1.2 (Properties of Cosets)

1. $aH = H$ iff $a \in H$.
2. $aH = bH$ iff $a \in bH$
3. $o(aH) = o(H)$.
4. $aH = bH$ or $aH \cap bH = \emptyset$.

Proof. (1) Assume $aH = H$, then since $e \in H$, $ae = a \in aH$ thus $a \in H$. Conversely assume that $a \in H$, then for $ah \in aH$ we have by closure $ah \in H$. Similarly if $h \in H$ then $ha^{-1} \in H$ and so $h \in aH$. For (2) The proof is similar. For (3) we want to find a bijection from aH to H . Define $\phi : H \rightarrow aH$ as $\phi(h) \mapsto ah$. Then one may easily show this is a bijection. (4) Assume $aH \neq bH$, then for the sake of contradiction assume that $aH \cap bH \neq \emptyset$. We fix x in the intersection and note that $x = ah_1$ and $x = bh_2$. Then $b = ah_1h_2^{-1} \in aH$, but then by (2) $aH = bH$; a contradiction. \square

Note that the last two parts of the theorem show that the cosets of a subgroup H partition the group G , later we will find out when it makes sense to define an operation on these equivalence classes.

1.2 Lagrange's Theorem

This is often called the A, B, Cs in finite group theory and one of the biggest results in MTH 316.

Theorem 1.2.1 (Lagrange)

Let $H \leq G$ then $o(H) | o(G)$.

Proof. Let $H \leq G$. Using the result from the last section we already know that the cosets of H partition G . Now consider the set of left cosets of H in G .

$$\{g_1H, g_2H, \dots, g_rH\}$$

Since $a \in aH$ we know that each element of G is in at least one coset, then since the cosets are disjoint,

$$ro(H) = o(G)$$

□