MTH 525: Topology

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Question 1.

Show that if A is closed in Y and Y is closed in X then A is closed in X.

Proof. Assume that A is closed in Y, then there exists a closed set of X, C such that $A = Y \cap C$. Then since this is the intersection of two closed sets in X, A is closed. $\square$

Question 2.

Let A, B, and A_α denote subsets of a space X. Prove the following:

1. If $A \subset B$, then $\bar{A} \subset \bar{B}$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$

Proof. (1) Let $A \subset B$. Let x be a limit point of A, then every nbhd of x intersects A in a point other than x, since $A \subset B$, every nbhd of x also must contain a point of B other than x, hence x is a limit point of B, it now follows

$$\overline{A} = (A \cup A') \subset (B \cup B') = \overline{B}$$

as desired.

(2) Let $x \in \overline{A \cup B}$ and suppose $x \notin \overline{B}$, then (since x is not a limit point) there exists a neigborhood of x, U which does not intersect B, now if there also exists a neighborhood V which doesn't intersect A in a point other than x, taking the intersection $U \cap V$ furnishes an open set containg x which doesn't intersect $A \cup B$ (in a point other than x), which is a contradiction. Hence every neighborhood or x must intersect A so that x belongs to the closure of A. Conversely, let $x \in \overline{A} \cup \overline{B}$, then suppose $x \in \overline{A} = A \cup A^prime$, if $x \in A$, then we are done so assume that x is a limit point of A. Then every

neighborhood will intersect A in a point other than x, so every neighborhood intersects $A \cup B$ in a point other than x so that x belongs to the closure of $A \cup B$. If $x \in \overline{B}$, the argument is simialliar.

(3)Let $x \in \bigcup \overline{A_\alpha}$, then $x \in \overline{A_\alpha}$ for some α, hence every neighborhood intersects A_α and thus intersects $\bigcup A_\alpha$, hence $x \in overline \bigcup A_\alpha$. To see that the converse is false, let $A_n = (0, \frac{n}{n+1})$ for $n \in \mathbb{N}$. Then $\bigcup_{n\in\mathbb{N}} A_n = (0,1)$ so $1 \in \overline{\bigcup_{n\in\mathbb{N}} A_n}$. But for any A_k, the $\epsilon-$ball of radius $\frac{1}{2}|\frac{k}{k+1} - 1|$, is a neighborhood around 1 which doesn't intersect A_k, hence $1 \notin \overline{A_k}$, since k was arbitrary, $1 \notin \bigcup \overline{A_k}$.

□

Question 3.

Let A, B, and A_α be as in the previous question. Determine if the following are true.

1. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
2. $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$.
3. $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

Proof. (1) Let $x \in \overline{A \cap B}$, if $x \in A \cap B$ the result is clear so suppose that x is a limit point; then every neighborhood of x intersects $A \cap B$. Hence every neighborhood will intersect A and B, thus x is a limit point of A and B so $x \in \overline{A} \cap \overline{B}$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. The converse is false since $1 \in \overline{(0,1)}$ and $1 \in \overline{(1,2)}$ But $1 \notin \overline{(0,1) \cap (1,2)}$.

(2) Let $x \in \overline{\bigcap A_\alpha}$, then an arbitrary neighborhood U of x intersects $\bigcap A_\alpha$, and so U intersects each A_α, thus $x \in \overline{A_\alpha}$ and so it belongs to $\bigcap \overline{A_\alpha}$. The converse is false since it was false in the finite case. For example let $A_1 = (0,1)$ and $A_n = (1,n)$. Then $1 \in \bigcap \overline{A_k}$, but not in $\overline{\bigcap A_k}$.

(3) We prove $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$. Let $x \in \overline{A} \setminus \overline{B}$. Then $x \notin B$. If $x \in A$, then $x \in A \setminus B \subset \overline{A \setminus B}$, so assume that x is a limit point of A. It follows that every neighborhood of x intersects A in a point not in B, since otherwise it would contradict our assumption on x. Then every neighborhood intersects $A \setminus B$ so $x \in \overline{A \setminus B}$. The converse is false. □

Question 4.

Let X and X' denote a single set in the two topologies $\mathfrak{T}$ and $\mathfrak{T}'$, respectively. Let $i : X' \to X$ be the identity function.

1. Show that i is continuous iff $\mathfrak{T}'$ is finer than $\mathfrak{T}$
2. Show that i is a homeomorphism iff $\mathfrak{T}' = \mathfrak{T}$

Proof. (1) Assume that i is continuous, then let U be open in X (i.e, $U \in \mathfrak{T}$). It follows from continuity that $i^{-1}(U) = U \subset X'$ is open, thus $\mathfrak{T} \subset \mathfrak{T}'$. Conversely, assume that $\mathfrak{T}'$ is finer than $\mathfrak{T}$. Then let U be open in X, since $\mathfrak{T}'$ is finer than $\mathfrak{T}$, we know that the preimage of U, which is itself, is open in X', hence i is continuous.

(2) If i is a homeomorphism, we know that it and its inverse are continuous, so we simply apply (1) in both directions to get $\mathfrak{T}' \subset \mathfrak{T}$ and $\mathfrak{T} \subset \mathfrak{T}'$ and hence $\mathfrak{T}' = \mathfrak{T}$. Now conversely, assume $\mathfrak{T}' = \mathfrak{T}$. Again by applying (1) in both directions we will get that $i : X' \to X$ is continuous and $i^{-1} : X \to X'$ is continuous, since it is also bijective, it is a homeomorphism. $\square$

Question 5.

Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

1. Show that the set $\{x | f(x) \leq g(x)\}$ is closed in X
2. Show that $h(x) = min\{f(x), g(x)\}$ is continuous.

Proof. (1) We prove $X - S$ is open. If it is empty we are done, so suppose there exists $x_0 \in X - S$, i.e. assume $f(x_0) > g(x_0)$. Since Y is in the order topology, it is Hausdorff, thus there exists disjoint nbhd's V_1, V_2 with $f(x_0) \in V_1$ and $g(x_0) \in V_2$. Since f and g are continuous functions, there exists $U_1, U_2 \subset X$ around x_0 such that

$$f(U_1) \subset V_1 \text{ and } g(U_2) \subset V_2$$

Now take $U = U_1 \cap U_2$. Then for $x \in U$, we have $f(x) \in V_1$ and $g(x) \in V_2$, since $f(x_0) > g(x_0)$ and $V_1 \cap V_2 = \varnothing$, it follows $f(x) > g(x)$ hence there is a nbhd around x_0 contained in $X - S$, so x_0 is an interior point. Since it was chosen arbitrarily, it follows that $X - S$ is open.

(2) Define $A = \{x | f(x) \leq g(x)\}$ and $B = \{x | g(x) \leq f(x)\}$. By the above argument both of these sets are closed and it is clear $A \cup B = X$. Further, $x \in A \cap B$ implies $f(x) = g(x)$. Now define $h(x) = f(x)$ when $x \in A$ and $h(x) = g(x)$ for $x \in B$. Then we see $h(x) = \texttt{min}\{f(x), g(x)\}$ and by the pasting lemma h is continous. $\square$

Question 6.

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation ...

1. Show that F is continuous in each variable separately
2. Compute $g(x) = F(x \times x)$
3. show that F is not continuous

Proof. (1) Without loss of generality fix $y \in \mathbb{R}$, if $y = 0$ the function just becomes the zero function which is continuous. If $y \neq 0$, then the function will never have a denominator of 0 since $x^2 + y^2 > 0$ for all x given $y \neq 0$. Then F just becomes a quotient of two continuous functions (polynomials are continuous) with a nonzero denominator on its domain and therefore is continuous. We can ignore that it was defined piecewise since the function will be zero iff $x = 0$. The situation for a fixed x is the same since there is clearly some symmetry with the variables, the proof would just be a relabeling of the above.

(2) if $y = x$ then $\frac{xy}{x^2+y^2} = \frac{x^2}{2x^2} = \frac{1}{2}$ so

$$g(x) = \left\{ \begin{array}{ll} \frac{1}{2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{array} \right\} \tag{1}$$

Note that g is not continuous at 0.

(3) Define $h : \mathbb{R} \to \mathbb{R}^2$ such that $h(x) = (x, x)$. Then since maps into products are continuous iff the coordinate functions are continuous, we see that h is continuous. Now assume that F is continuous, then $F \circ h : \mathbb{R} \to \mathbb{R}$ is continuous (composition of continuous functions), but $F \circ h = F(x \times x) = g(x)$ is discontinuous at 0; a contradiction. Hence, F is not continuous.

$\square$

Question 7.

Let $x_1, x_2, \dots$ be a sequence of the points of the product space ΠX_α. Show that this sequence converges to the point x if and only in the sequence $\pi_\alpha(x_1), \dots$ converges to $\pi_\alpha(x)$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Proof. Suppose $(x_n) \to x$, let $V_\alpha \subset X_\alpha$ be a nbhd around $\pi_\alpha(x)$. Then the preimage $\pi_\alpha^{-1}(V_\alpha) \subset \Pi X_\alpha$ is an open set containing x, and thus contains all but finitely many points of the sequence (x_n). But then V_α must contain all but finitely many points of the sequence $(\pi_\alpha(x_n))$. Hence $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$. Since I only used the fact that projections are continous this direction is true in the product or box topology. The converse is only true in the product topology. Assume that $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ for all α. Then let $U = U_{\alpha 1} \times \dots U_{\alpha m} \times X \times \dots.$ be a neighborhood around x. Then for each α_i there exists a k_i such that for all $k > k_i$, $\pi_{\alpha i}(x_k) \in U_{\alpha i}$. Now take $K = \mathtt{max}\{k_1, \dots k_m\}$, then for $k > K$, $x_k \in U$. Hence $(x_n) \to x$. To see that this is false in the box topology, let $X = \mathbb{R}^\omega$ and consider the neighborhood $A = (-1,1) \times (\frac{-1}{2}, \frac{1}{2}) \times \dots$ around 0, and define the sequence $x_n = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots)$. Then each projection converges to 0 in $\mathbb{R}$, but for each index k, $x_k \notin A$ since the $(k+1)^{th}$ index of x_k is not in $(\frac{-1}{k+1}, \frac{-1}{k+1})$. Hence the sequence cannot converge to zero.

□

Question 8.

Let $\mathbb{R}^\infty$ be the subset of R^ω consisting of all sequences that are eventually zero. What is the closure in the product and box topologies

Proof. First consider the product topology. Let $x \in \mathbb{R}^\omega$ and let $U = U_{\alpha 1} \times \dots \times U_{\alpha n} \times \mathbb{R} \times \dots$ be an open set of x. Then it is clear that U contains an element of $\mathbb{R}^\infty$, we can pick any element from each $U_{\alpha i}$ for $i = 1, \dots n$ and then just pick zeros for the rest. Hence every x is a limit point and thus $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$ so its closure is the whole space.

Now we consider the box topology. Let x be a limit point of $\mathbb{R}^\infty$ and assume that $x \notin \mathbb{R}^\infty$. We write $x = (x_\alpha)_{\alpha \in J}$. Since this sequence is never eventually zero, for each term not equal to zero we can select an ϵ nbhd $U_\alpha = (x_\alpha - \epsilon_\alpha, x_\alpha + \epsilon_\alpha)$ that does not contain zero. Then it is clear that this nbhd cannot contain an element of $\mathbb{R}^\infty$, hence x is not a limit point; a contradiction. Thus, $\mathbb{R}^\infty$ must contain all its limit points so, $\mathbb{R}^\infty$ is a closed subset in the box topology. $\square$

Question 9.

Proof. To show that h is a bijection, Let $(x_1, x_2, \dots) \in \mathbb{R}^\omega$, then let $x = (\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots)$. Since $a_i > 0$ each term is well defined. Then it is clear that $h(x) = (x_1, \dots)$, hence h is a surjection. Now suppose that $h(x_1, x_2, \dots) = h(x_1', x_2', \dots)$. Then

$$(a_1x_1 + b_1, a_2x_2 + b_2, \dots) = (a_1x_1' + b_1, a_2x_2' + b_2, \dots)$$

so $a_ix_i + b_i = a_ix_i' + b_i$ with $a_i \neq 0$, so $x_i = x_i'$. Thus h is a bijection. Now we must show that h is continuous with a continuous inverse.

Let $U = \Pi U_\alpha = U_{\alpha 1} \times \cdots \times U_{\alpha n} \times \mathbb{R} \times \dots$. Be an open set in the product topology and let $f_i(x) = a_ix + b_i$. Then

$$h^{-1}(U) = f_1^{-1}(U_{\alpha 1}) \times \cdots \times f_n^{-1}(U_{\alpha n}) \times f_{n+1}^{-1}(\mathbb{R}) \times \dots$$

Since $a_i \neq 0$ each f_i is a bijection, and thus for $k > n$ we have $f_k^{-1}(\mathbb{R}) = \mathbb{R}$ and by continuity of each polynomial f_i, each preimage is open. Hence $h^{-1}(U)$ is open in $\mathbb{R}^\omega$ under the product topology. To prove that the inverse is also continuous, we can write

$$h^{-1}(x_1, x_2, \dots) = \left(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}, \dots\right)$$

again using the fact that $a_i \neq 0$. Theses are all bijective polynomials and so nothing stops us from just reapplying the above argument.

In the box topology, let $U = \Pi U_\alpha$ be an open set. Then if

$$h(x_1, x_2, \dots) = (f_1(x), f_2(x), \dots)$$

then $h^{-1}(U) = f_1^{-1}(U_\alpha) \times f_2^{-1}(U_\beta) \times \dots$ as above. Since each f_i is continuous, we have that each preimage is open, thus the product is open in the box topology and h is continuous. To prove that the inverse of h is continuous, by computing a formula for the inverse as in the previous paragraph, we will again see that each component function is continuous and make a similar argument. $\square$