Topology: 525 Chapter

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Question 1.

Show that $\mathbb{R}_K$ and $\mathbb{R}_l$ are not comparable.

Proof. Let $x = 0$, and condisder $X = (-1, 1) - K \in \mathbb{R}_K$. For any $[a, b) \subset \mathbb{R}_l$, with $0 \in [a, b)$, then $0 < b$ and we may use the archimedian property of $\mathbb{R}$ to fix $N \in \mathbb{N}$ such that $n \geq N$ implies $\frac{1}{n} < b$, so $\frac{1}{n} \in [a, b)$ since $\frac{1}{n} \notin X$, there is no basis element of $\mathbb{R}_l$ containg zero that is a subset of X. Now consider the basis element of $|R_l$, $[0, b)$. We must show there is no basis element of $\mathbb{R}_K$ containg 0 that is a subset of $[0, b)$. In order for 0 to be in a basis element of $\mathbb{R}_K$. we must have $a < 0$, but then $a \notin [0, b)$. $\square$

Question 2.

If A is a basis for the topology on X then the topology generated by A equals the intersection of all topolgies containg A

Proof. Let A be a basis for the topolgy on X, $\mathfrak{T}$. Let $\{\mathfrak{D}_\alpha\}$ be the collection of all topologies containg A. We want to show that $\mathfrak{T} = \bigcap \mathfrak{D}_\alpha$. If $U \in \mathfrak{T}$ then U is equal to a union of elements of A, since A is contained in each $\mathfrak{D}_\alpha$ so is U, since each $\mathfrak{D}_\alpha$ is a topology. Now if $U \in \bigcap \mathfrak{D}_\alpha$, then it is in every topology containing A, since $\mathfrak{T}$ is one such topology we have $U \in \mathfrak{T}$. In the case that A is a subbasis generating the topology $\mathfrak{T}_A$. If $U \in \mathfrak{T}_A$, then U is a union of finite intersections of elements of A, then since $A \in \mathfrak{D}_\alpha$ it follows from the definiton of a topology that U will be in each $\mathfrak{D}_\alpha$ hence it is in the intersection. The second part of the argument is a repeat of the above.

$\square$

Question 3.

Show that $\mathfrak{B} = \{(a, b) | a, b \in \mathbb{Q}\}$ generates the standard topology on $\mathbb{R}$. Show that the this statement if false with the lower limit topology.

Proof. We have by earlier result that $\mathfrak{B}$ is a basis if for every open set U and for all $x \in U$ there exists an element of $\mathfrak{B}$ containg x that is a subset of U. So let $U \subset R$ be open, for $x \in U$, x must appear in some basis element, say $x \in (a,b)$ for $a, b \in \mathbb{R}$. Now by the density of the rationals in $\mathbb{R}$, there exists $s, t \in \mathbb{Q}$ such that

$$a < s < x$$

and

$$x < t < b$$

Thus $x \in (s,t) \subset (a,b) \subset U$; it follows that $\mathfrak{B}$ generates the topology on $\mathbb{R}$.

Now consider $\mathbb{R}_l$. We have that $[e, 3)$ is an open subset of $\mathbb{R}_l$, but there is no element of $\{[a,b) | a, b \in \mathbb{Q}\}$ containg e which is a subset of $[e, 3)$ since we must choose a as a rational if $e < a$ then we have a contradiction and if $e > a$ then $[a,b)$ is not a subset of $[e, 3)$. □

Question 4.

is the finite complement topology true if we replace finite with infinite?

Ans: No, under $\mathbb{Z}$ we have a collection of open sets

$$\mathfrak{U} = \{..., [-4, -3], [-2, -1], [1, 2], [3, 4], ...\}$$

whose union equals $(-\infty, -1] \cup [1, \infty)$. Then it is clear the complement is not infinite.

Question 5.

Let $\{\mathfrak{T}_\alpha\}$ be a collection of topologies on X prove that the union and intersection of this set are topologies on X.

Proof. It is clear that $X, \varnothing \in \bigcap\{\mathfrak{T}_\alpha\}$. Now suppose that $\{U_\beta\}_{\beta \in J} \subset \bigcap\{\mathfrak{T}_\alpha\}$. Since each U_β is in the intersection, $\{U_\beta\} \subset \mathfrak{T}_\alpha$ for all α. Then since $\mathfrak{T}_\alpha$ is a topology, the union is in each $\mathfrak{T}_\alpha$ and thus is contained in the intersection. Now let $U_1, ..., U_n$ be a finite collection of elements in $\bigcap\{\mathfrak{T}_\alpha\}$. Then each U_i is in all $\mathfrak{T}_\alpha$ and again by the definition of topology the finite intersection over U_i is in each $\mathfrak{T}_\alpha$ and thus contained in the intersection.

Now in general the union of two topologies need not be a topology. How ever, the union of topologies on X will form a subbasis for a topology on X. and this topology is the smallest one containg each of the topologies in the union. To see that the union is not always a topology just consider a case where the topologys are not comparable. The fact the the union forms a subasis is clear since the elements of any one topology for a subbais ($X \in \mathfrak{T}$ makes T a subbasis). It is also clear that the topology generated by the subbasis contains all topologys in the union. Now suppose there exists a topology $\mathfrak{T}$ such that $\mathfrak{T}_\alpha \subset \mathfrak{T}$ for all α. Then if U is open is the topology genererated by the subbasis $\cup\{\mathfrak{T}_\alpha\}$, U is a union of finite intersections of elements of $\cup\{\mathfrak{T}_\alpha\}$, hence by definiton of topology U is open in $\mathfrak{T}$.

$\square$