MTH 525: Topology

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Question 1.

Show that a first countable T_1 space is G_δ.

Proof. Let X be first countable and T_1 and let $x \in X$. Then let $\mathcal{A}$ be a countable basis at x. Then let $B = \cap_{A \in \mathcal{A}} A$, clearly B is a countable intersection of open sets, we show that $B = \{x\}$. Clearly $x \in B$, now suppose that $y \neq x$ and $y \in B$, then note that since X is a T_1 space, $X \setminus \{y\}$, is an open set around x. Now there must exist a basis element containing x contained in $X \setminus \{y\}$, hence there is an open set $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$, so that $y \notin B$ a contradiction. Hence $B = \{x\}$. $\square$

For an example consider $\mathbb{R}^\omega$ in the box toplogy, this space is not first countable, since given a countable collection of open sets about a point x, $\{U_n\}_{n \in \mathbb{N}}$. We start by selection an open set $V_1 \subset \pi_1(U_1)$ such that $x_1 \in V_1$, then we select $V_2 \subset \pi_2(U_2)$ with $x_2 \in V_2$ and so on, then the open set $V = \bigtimes_{i=1}^{\infty} V_i$ is an open set but does not contain any element U_i since $\pi_i(V) \subset \pi_i(U_i)$. Hence $\mathbb{R}^\omega$ is not first countable. On the other hand given x, the sets $A_n = \bigtimes_{i=1}^{\infty}(x_i - \frac{1}{n}, x_i + \frac{1}{n})$. It is clear that this gives a countable collection and that x is the only element of the intersection.

Question 2.

Show that $\mathbb{R}_\ell$ and I_o^2 are not metrizable

Proof. Note that $\mathbb{R}_\ell$ is not second countable, since given any countable collection of open sets of the form $[a_i, b_i)$ we can find a reall number $\xi \notin \{x | x = a_i\}$ since $\mathbb{R}$ is uncountable. Then the set $[\xi, b)$ is open by the definition of lower limit topology, but there is no basis element containing ξ contained in this open set, a contradiction. However $\mathbb{R}_\ell$ does have a countable dense subset, $\mathbb{Q}$, not that for an arbitrary $x \in \mathbb{R}$, every nbhd of x, $[a, b)$ will contain rational points, hence every real number is a limit point of $\mathbb{Q}$, hence $\mathbb{Q}$ is dense.

We know that if a space is metrizable then second countable is equivalent to having a countably dense subset. Since this is not the case for $\mathbb{R}_\ell$, it must be the case that $\mathbb{R}_\ell$ is not metrizable.

We will employ a similiar argument for the ordered squar I_o^2, first note that it cannot have a countable basis since $\{x\} \times (1/3, 2/3)$ is open for each $x \in [0, 1]$ and is an uncountable disjoint collection, and so for each point $x \times \frac{1}{2} \in \{x\} \times (\frac{1}{3}, \frac{2}{3})$, there must exist a basis element $B_x \subset \{x\} \times (\frac{1}{3}, \frac{2}{3})$. Thus the basis must be uncountble. But again $\mathbb{Q}^2$ restricted to the ordered square will give a countable dense subset. Thus the space cannot be metrizable becasue second countable and the existence of a countable dense subset are not equivalent.

□

Question 3.

Which of the four countablility axioms does S_ω and $\overline{S_\omega}$ satisfy?

Proof. S_Ω is first countable, given $a \in S_\Omega$, Then S_a is countable and since S_Ω is totally ordered, there is an immeadiate sucessor of a, say b_1 then define the imeadiate sucessor recursivly as $b_1 < b_2 < b_3 < \ldots$ wich also gives a countable collection. Then considering all open sets (x, y) with $x \in S_a$ and $y = b_i$ is a countable collection of open sets about a, and given an arbitrary open set around a, it is clear it must contain a set of this form.

Now we show that this set is not second countable by showing that it has no countable dense subset, note that any countable set in S_Ω is bounded, but S_Ω itself is uncountable and has no maximal element, hence there cannot be a countable dense subset, since given any countble set B, there exists $c, d \in S_\Omega$ such that for all $b \in B$, $b < c < d$ hence d will not be a limit point.

Now we show that this space is not lindelof, consider the convering S_a for all $a \in S_\Omega$, and suppose a countble subcollection covers S_Ω, then we get that the countable union of countable sets convers S_Ω, which cannot be the case since S_Ω is uncountable and the countable union of countable sets is countable.

The only thing that changes when considering $\overline{S_\Omega}$, is that the space is no longer first countable since we have included the point Ω, Given any countable collection of open sets around Ω, the set of lower bounds of these intervals forms a countable set and as such is bounded, then we may take an element larger than the upperbound and form an open set containg Ω that doesn't contain any element of our countable collect, hence $\overline{S_\Omega}$ fails to be first countable at the point Ω. $\square$

Question 4.

Let $P : X \to Y$ be closed continuous and surjective.

(a) Show that X Hausdorff implies the same for Y.

Proof. Let p be a closed continuous surjective map s.t. $p^{-1}(y)$ is compact for all $y \in Y$. Let $a_1, a_2 \in Y$, the since their pre images a disjoint compact sets, they can be seperated into disjoint open sets U and V. Then let $A = Y \setminus P(X \setminus U) \subset P(U)$. Note that A is open since U is open, $X \setminus U$ is closed and then its image is closed becasue p is a closed map, hence the complement in Y is open. We have $a_1 \in A$ and $A \cap P(V) = \varnothing$, since $p^{-1}(A) \subset U$ is disjoint from V. Then letting $A_2 = Y \setminus p(X \setminus V)$ gives a similiar open set about a_2, then we have that Y is Hausdorff.

$\square$

(b) Same but for regularity

Proof. Assume that X is regular and let $a \in Y$. We show that every nbhd of a, U has a open V such that $\overline{V} \subset U$. Note that $p^{-1}(U)$ is open and contains the compact set $P^{-1}(a)$. For each $x \in p^{-1}(a)$, by regularity there exists a nbhd V_x such that $x \in V_x$ and $\overline{V_x} \subset p^{-1}(U)$. Then These V_x's form an open cover of $p^{-1}(a)$ and hence there exists a finite subcover, $V = \bigcup_{i=1}^n V_{x_i}$. Since the finite union of closed sets are closed we also have $\bigcup_{i=1}^n \overline{V}$, which is a closed set containd in the pre image of U, then its image is a closed set contained in U. and the set $Y \setminus P(X \setminus V) \subset p(V)$ is an open set containg a whose closure is in U.

$\square$

(c) local compactness

Proof. Let X be locally compact and let $a \in Y$, then $p^{-1}(a)$ is compact and for all $x \in p^{-1}(a)$ there exists a compact C_x and an open U_x such that $U_x \subset C_x$. Then the U_x's form an open cover and hence a finite number of them must cover $p^{-1}(a)$. Then let $U = \bigcup_{i=1}^{n} U_i$ and $C = \bigcup_{i=1}^{n} C_i$, then $p(C)$ is compact since it is the image of a compact set and taking $A = Y \setminus p(X \setminus U)$ gives an open nbhd of a.

Thus Y is locally compact. □

(d) countable basis.

Proof. As in the given hint, let $\mathfrak{B}$ be a basis, and given a finite subset of $\mathfrak{B}$, J, let U_J be the union of all $p^{-1}(W)$ for W open in Y such that $p^{-1}(W) \subset \cup J$. Then we show that $p(U_J)$ is a basis for Y. Clearly the collection of $p(U_J)$ is countable since, since there are countably many finite subsets J of $\mathfrak{B}$. Now Let $V \subset Y$ be open. Then consider a open covering of $p^{-1}(V)$ by basis elements in $\mathfrak{B}$, for each $p^{-1}(a) \subset p^{-1}(V)$ we know that there exists a finite subcollection covering the compact set $p^{-1}(a)$, then unioning the finite subcover is a finite union of elements in $\mathfrak{B}$, call it B. Let U be the corrosponding open set consisting of the union of all $p^{-1}(W)$ where W is open and $p^{-1}(W)$ is contained in B. Then $p(U) \subset V$ since it is the union of open sets whose pre images lie in B and $B \subset p^{-1}(V)$. Hence by reapeating this process for each compact $p^{-1}(y)$ we can write V as the union of such open sets of the form $p(U)$, so that V is open in the topology generated by elements of the desired form. □

Question 5.

Topological groups.

Proof. First we prove normality, so assume that X is normal, Then by the given hint, we know that p is closed continuous and surjective. Let A_1, A_2 be disjoint closed sets in the quotient space X/G. Then since p is a continuous function, $p^{-1}(A_1), p^{-1}(A_2)$ are both closed and disjoint. By normality of X, they can be seperated by disjoint open sets U_1 and U_2 respectivily. Then using a similiar trick as above we define

$$V_1 = Y \setminus p(X \setminus U_1)$$

and

$$V_2 = Y \setminus p(X \setminus U_2)$$

Note that V_1 is open since U_1 is open, its complement is closed, then since p is a closed map the image of $X \setminus U_1$ is closed and hence its complement V_1 is open. We also have $A_1 \subset V_1$ Since $p^{-1}(A_1) \subset U_1$, equivalent statments hold for V_2, and since $p^{-1}(V_1) \subset U_1$ and $p^{-1}(V_2) \subset U_2$, V_1 and V_2 are disjoint. Hence X/G is normal.

Now to do the other ones we let $\overline{x} \in X/G$, then $p^{-1}(\overline{x}) = \alpha(G, x)$. Then since G is compact and α continuous, the image $\alpha(G, x)$ for fixed x is compact. That is the pre image of a fiber is compact. Also by the hint given we have that p is closed continuous and surjective. Thus it follows p is a perfect map and the above results in the previous question provide the proof. $\square$