

## MTH 513

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October 11, 2022

I worked with Kyle and Carmine and also spoke briefly with Hannah and Karrisa.

### Question 1.

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Let  $C \in \mathbb{R}^{n \times n}$  be a matrix such that  $x^T C x > 0$  for all non-zero  $x \in \mathbb{R}^n$ .

(a) Prove that  $C$  is non-singular.

*Proof.* Suppose that  $C$  is singular, then the reduced echelon form  $E_C$  contains a row of all zeros, since otherwise  $C$  would be nonsingular. Let  $E_{C_{i*}}$  denote the row of all zeros and let  $P$  be such that  $PC = E_C$ . Then let  $w = (0, \dots, w_i, \dots, 0)^T$ , where  $w_i \neq 0$ . then

$$w^T E_C w = (w^T E_C) w = (0E_{C,1*} + \dots + w_i E_{C,i*} + \dots + 0) w = \vec{0}^T w = 0.$$

hence taking  $x = P^T w$  gives

$$x^T C x = w^T P C P^T w = w^T E_C P^T w = 0$$

and since  $w \neq 0$  and  $P$  is invertible,  $P^T w \neq 0$ . Hence we have found an  $x$  contradicting our assumption.  $\square$

(b) Prove that for each principal submatrix  $C_k$  we have  $x^T C_k x > 0$  for all nonzero  $x$ .

*Proof.* Assume that for some  $k \in 1, \dots, n$  we have that  $x^T C_k x > 0$  for all nonzero  $x$  does not hold. Then there exists a  $y \in \mathbb{R}^k$  such that  $y = (y_1, \dots, y_k)^T$  and

$$y^T C_k y \leq 0$$

Now define  $z \in \mathbb{R}^n$  by  $z = (y_1, \dots, y_k, 0, \dots, 0)^T$ . Then we will have

$$z^T C z = \sum_{i=1}^n (z^T)_i (C z)_i$$

$$\begin{aligned}
&= \sum_{i=1}^n \left[ (z^T)_i \left( \sum_{j=1}^n C_{ij} z_j \right) \right] \\
&= \sum_{i=1}^n \left[ (z^T)_i \left( \sum_{j=1}^k C_{ij} y_j \right) \right] = S
\end{aligned}$$

were the last step follows since everything after the  $j^{th}$  index of  $z$  is zero and  $z$  and  $y$  agree of the first  $k$  indices. Then applying the same logic again,

$$S = \sum_{i=1}^k \left[ (y^T)_i \left( \sum_{j=1}^k C_{ij} y_j \right) \right] = y^T C_k y \leq 0$$

a contradiction; so the result follows, using part (a), we see that every principal submatrix is invertible and thus  $C$  has an  $LU$  factorization.

□

**Question 2.**

For a symmetric matrix  $A$ ,  $A$  is positive definite if  $A = LU$  where the diagonal of  $U$  is positive. The following are equivalent.

1.  $A$  is *positive definite*
2.  $A$  can be factored as  $A = R^T R$  where  $R$  is upper triangular with positive entries on its diagonal.
3.  $x^T A x > 0$  for all  $x \neq 0$ .

*Proof.* (1)  $\iff$  (2) was done in class. We first prove (2)  $\implies$  (3). So assume that  $A = R^T R$  where  $R$  is upper triangular and has positive diagonal entries. Then consider a nonzero  $x$ . We have

$$x^T A x = x^T (R^T R) x = (Rx)^T (Rx) \geq 0$$

since this is just a dot product, it is zero if and only if  $Rx = 0$  but the assumptions on  $R$  show that it has a row echelon form with a pivot in every column and thus is invertible, it follows that the only  $x$  for which  $Rx = 0$  is the zero vector. Hence  $(Rx)^T (Rx) \neq 0$  for our choice of  $x$  and then  $x^T A x > 0$ . Now, assume that  $x^T A x > 0$  for all nonzero  $x$ . Then by problem (1) we know that  $A = LU$  so we only need to show that the diagonal entries on  $U$  are all positive. Since  $A$  is symmetric we may write  $A = LDL^T$ , so  $x^T A x = x^T LDLx = (L^T x)^T D (L^T x)$ . Then note for any diagonal matrix  $D$  and vector  $b$ , we have

$$b^T D b = b_1^2 D_{11} + \dots + b_n^2 D_{nn}$$

hence if a element of the diagonal  $D_{ii}$  is not positive, we can choose  $b = (0, \dots, 1, \dots, 0)$  where 1 is in the  $i^{th}$  position. This will give us a product that is not positive.

Then going back to  $(L^T x)^T D (L^T x)$ , since  $L^T$  is upper triangular with 1's on the diagonal it is invertible, from this it follows that  $L^T x = 0$  iff  $x = 0$  and that  $L^T x = b$  is always consistent. Thus if any element of on the diagonal of  $D$ , say  $D_{ii}$  is not greater than 0 I choose an  $x$  such that  $L^T x = b$  where  $b$  is a vector of all zeros except for the  $i^{th}$  position. Then it will follow that  $x^T A x \leq 0$ ; a contradiction. Hence the diagonal of  $D$  is positive and we have shown that (3)  $\implies$  (1). This proves that all the statements are equivalent

□

**Question 3.**

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Show that the given matrix is not positive definite,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 1 \end{bmatrix} \quad (1)$$

*Proof.* The clear way of doing this is to compute a  $LU$  factorization and see if  $A$  has an appropriate  $LU$  factorization. Performing the row operations  $-2R_1 + R_2 \rightarrow R_2$  and  $-R_2 + R_3 \rightarrow R_3$  shows that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -6 \end{bmatrix} \quad (2)$$

From this and the uniqueness of  $LU$  factorization we see that  $A$  cannot be given an  $LU$  factorization in which the diagonal entries of  $U$  are positive.  $\square$

**Question 4.**

Let  $A$  be positive definite. Show that  $A^{-1}$  and  $A^2$  are also positive definite.

*Proof.* First we prove that if  $A$  is symmetric so is its inverse. Assume that  $A$  is symmetric, then note

$$AA^{-1} = I = I^T = (A^{-1}A)^T = A^T A^{-1T} = AA^{-1T}$$

then multiplying on the left by  $A^{-1}$  shows  $A^{-1}$  is symmetric.

Now we will show  $x^T A^{-1}x > 0$  for all nonzero  $x$ . Observe

$$x^T A^{-1}x = (x^T A^{-1})A(A^{-1}x) = (A^{-1T}x)^T A(A^{-1}x)$$

then since  $A^{-1}$  is symmetric this shows,

$$x^T A^{-1}x = (A^{-1}x)^T A(A^{-1}x)$$

Since  $x$  is nonzero and  $A^{-1}$  is nonsingular  $A^{-1}X > 0$  so that the whole expression

$$x^T A^{-1}x > 0$$

. Thus  $A^{-1}$  is positive definite.

Now for  $A^2$  note that

$$x^T A^2x = x^T A^T Ax = (Ax)^T (Ax)$$

and again this is a dot product of a nonzero vector with itself (since  $x \neq 0$  and  $A$  is nonsingular), so  $x^T A^2x > 0$ . for all  $x \neq 0$ .

□

**Question 5.**

Let  $A$  be a symmetric matrix given by

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad (3)$$

Find an upper triangular matrix  $R$  with positive diagonal entries such that  $A = R^T R$  or justify why such  $R$  doesn't exist.

*Proof.* We compute an  $LDV$  factorization, since  $A$  is symmetric, this will allow us to produce such an  $R$ . Applying the row operations  $\frac{1}{2}R_1 + R_2 \rightarrow R_2$  and  $\frac{2}{3}R_2 + R_3 \rightarrow R_3$  and factoring out the diagonal gives us the following  $LDV$  factorization,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

Since the diagonals are positive, we have that  $A$  is positive definite and that there exists an  $R$  such that  $A = R^T R$ , namely we pick

$$R = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

As described in class. □