MTh 535 Homework 4

Evan Fox

11/19

Problem 1.

Proof. Since f is a bounded measureable function on a set of finite measure, E, and $A \subseteq E$, $\int_A f$ exists and thus,

$$\sup\{\int_A \phi \,|\, \phi : A \to \mathbb{R} \text{ simple }, \phi \leq f\} = \int_A f$$

Similarly, $\int_E f \cdot \chi_A$ exits and we have,

$$\sup\{\int_E \psi \,|\, \psi : E \to \mathbb{R} \text{ simple }, \psi \leq f \cdot \chi_A\} = \int_E f \cdot \chi_A$$

Now for each such simple function $\phi : A \to \mathbb{R}$ with $\phi \leq f$ we may define $\tilde{\phi} : E \to \mathbb{R}$ to be ϕ for all $x \in A$ and 0 for all $x \in E \setminus A$. We have defined a map $\phi \to \tilde{\phi}$ and we have

$$\sup\{\int_E \psi \,|\, \psi : E \to \mathbb{R} \text{ simple }, \psi \leq f \cdot \chi_A\} = \sup\{\int_E \tilde{\phi} \,|\, \phi : A \to \mathbb{R} \text{ simple }, \phi \leq f\}$$

By construction, $\tilde{\phi}$ is a simple function defined on all of E and clearly $\tilde{\phi} \leq f \cdot \chi_A$, so the fact that the LHS of the above is greater or equal to the RHS is clear. To see the other direction, for any such $\psi : E \to \mathbb{R}$ simple satisfing $\psi \leq f \cdot \chi_A$, we can first restrict the domain of ϕ to A, to obtain a ϕ such that $\phi = \psi$ on A and then extending again we will have $\phi \leq \tilde{\phi} \leq f \cdot \chi_A$, since $\tilde{\phi} = f \cdot \chi_A$ on the complement of A in E. Thus we also have that the RHS is less or eqaul the LHS and this implies equality. All that remains is to argue

$$\int_A \phi = \int_E \tilde{\phi}$$

Since both ϕ and $\tilde{\phi}$ are a simple functions and since $\tilde{\phi}$ is an extension of ϕ to all of E which is identically zero on $E \setminus A$. Thier canonical representations differ by a single term, of the from $0 \cdot \chi_{E_0}$ where $E_0 = \{x \in E | \tilde{\phi}(x) = o\}$. Since this term has a leading coefficient 0, it follows from the definiton of the Lesbegue integral for simple functions, that the integrals above are equal. Then the result follows just by following the equalities. $\square$

Problem 2.

Proof. Let $A = \{x \in E \,|\, f(x) \neq 0\}$. We show $m(A) = 0$. By additivity over domains of integration we have

$$0 = \int_E f = \int_{E \setminus A} f + \int_A f = \int_A f$$

since $f = 0$ for all $x \in E \setminus A$, the integral over this set is zero. Now

$$0 = \int_A f = \sup\{\int_A h \,|\, h \text{ simple }, 0 \leq h \leq f\}$$

Thus for all such h, we have

$$0 = \int_A h = \sum_{i=1}^{n} a_i \cdot m(E_i)$$

for some canonical representation of $h = \sum_{i=1}^n a_i \chi_{E_i}$. If $m(A) > 0$, then there would exist a simple function $h \leq f$ defined on A such that $h = \sum_{i=1}^n a_i \cdot \chi_{E_i}$ where $a_i \neq 0$ for all i and $m(E_i) > 0$ for some i, thus $\int_A h > 0$ which implies $\int_A f > 0$ a contradiction. $\square$

Problem 3.

Proof. First we have a small lemma. Let $\{a_n\}_n \to a$ in $\mathbb{R}$ and let $\{b_n\}_n$ be any real valued sequence. Then

$$\liminf\{a_n + b_n\} = a + \liminf\{b_n\}$$

Let $\epsilon > 0$, then fix $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a - \epsilon \leq a_n \leq a + \epsilon$. Then for $n \geq N$, we have

$$a - \epsilon + \liminf\{b_n\} = \liminf\{a - \epsilon + b_n\} \leq \liminf\{a_n + b_n\} \leq \liminf\{a + \epsilon + b_n\} = a + \epsilon + \liminf\{b_n\}$$

Letting ϵ go to zero proves the result. Note that this prove works just as well for $\limsup$.

Let $E \subseteq \mathbb{R}$ be arbitrary. By Fatou's lemma we have

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

Using the above lemma we get

$$\limsup_{n\to\infty}\left(\int_E f_n\right) = \limsup_{n\to\infty}\left(\int_{\mathbb{R}} f_n - \int_{\mathbb{R}\setminus E} f_n\right) = \int_{\mathbb{R}} f + \limsup_{n\to\infty}\left(-\int_{\mathbb{R}\setminus E} f_n\right)$$

$$= \int_{\mathbb{R}} f - \liminf_{n\to\infty}\left(\int_{\mathbb{R}\setminus E} f_n\right) \leq \int_E f$$

where the last inequality follows from Fatou's lemma applied to the integral of f over $\mathbb{R} \setminus E$ and linearity of integration. Then since we have

$$\limsup_{n\to\infty} \int_E f_n \leq \liminf_{n\to\infty} \int_E f_n$$

the two are equal (the other inequality is always true) and this implies convegence of the sequene $\int_E f_n$. Thus

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

$\square$

Problem 4.

Proof. By Fatou's lemma we have that

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

And since for all $n \in \mathbb{N}$ we have $f_n \leq f$, by monotinicity, it follows $\int_E f_n \leq \int_E f$, thus

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f$$

and just as above we get that

$$\limsup_{n\to\infty} \int_E f_n \leq \liminf_{n\to\infty} \int_E f_n$$

which implies the result, as in the previous problem. $\square$

Problem 5.

Proof. First we prove it for non-negative functions on sets of arbitrary measure. Let f be a non-negative measureable function of E. Let $C \subseteq E$ be measureable. Given h, boundend, measureable, function on C with finite support which satisfies $0 \leq h \leq f$, then automatically h is a bounded, measureable, finite support function on E satifying $0 \leq h \leq f \cdot \chi_C$. Conversly, given a bounded, measureable function $h : A \to \mathbb{R}^+$ with finite support $A \subset E$ satisfying $0 \leq h \leq f \cdot \chi_C$ by restricting the domain of h to $A \cap C$ and setting $h = 0$ for all $C \setminus (A \cap C)$. We obtain a bdd., measu., finite spp function on C such that $0 \leq h \leq f$. Then it clearly follows

$$\sup\{\int_C h \,|\, h \text{ bdd., measu., finite spp. }, 0 \leq h \leq f\} = \sup\{\int_E h \,|\, h \text{ bdd., measu., sinite spp.}, 0 \leq h \leq f \cdot \chi_C\}$$

Thus by definition, $\int_C f = \int_E f \cdot \chi_C$.

Now, for the general case let f be integrable, then

$$\int_A f = \int_A f^+ - \int_A f^- = \int_E f^+ \cdot \chi_A - \int_E f^- \cdot \chi_A = \int_E f \cdot \chi_A \tag{1}$$

where the first and last equalities are by the defintion of integration and the middle inequality follows since f^+ and f^- are both non-negative measureable functions, we may apply the case proven above. □

Problem 6.

Proof. Fix a sequence $y_n \to 0^+$ and define $f_n(x) = f(x, y_n)$. Then $f_n(x)$ is a measureable function for each n. Since for each fixed value of x we have $\lim_{y \to 0^+} f(x, y) = f(x)$, we have $f_n(x) \to f(x)$ pointwise. Further, $|f_n(x)| \leq g(x)$ for all n, by assumption. Then by the Lesbeque dominated convergence theorem we have that

$$\lim_{y \to 0^+} \int_0^1 f(x, y) = \lim_{n \to \infty} \int_0^1 f_n(x) = \int_0^1 f(x)$$

□

Problem 7.

Proof. First note that by the definition of the derivative, we have

$$\frac{d}{dy}\left(\int_{[0,1]} f(x, y)dx\right) = \lim_{h \to 0}\left(\frac{\int_{[0,1]} f(x, y)dx - \int_{[0,1]} f(x, y + h)dx}{h}\right) \tag{2}$$

then by linearity we have

$$\frac{d}{dy}\left(\int_{[0,1]} f(x, y)dx\right) = \lim_{h \to 0}\left(\int_{[0,1]} \frac{f(x, y) - f(x, y + h)}{h} dx\right) \tag{3}$$

Now for each $y \in Q$, we have that $f(x, y)$ is a measureable function of x, thus so is $\frac{f(x,y) - f(x,y+h)}{h}$. Then

$$\lim_{h \to 0} \frac{f(x, y) - f(x, y + h)}{h} = \frac{\partial f}{\partial y}(x, y)$$

for each $(x, y) \in Q$ so we have pointwise convergence. All that remains is to exchange the limit and the integral sign. By assumption we have

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq g(x, y) \ \forall (x, y) \in Q \tag{4}$$

Fix $y \in [0, 1]$. This inequality can easily be made strict (i.e. redefine $G(x) = g(x, y) + 1$). Then if we fix a sequence $\{h_n\} \to 0$ there exists an $N \in \mathbb{N}$ such that

$$|F_n(x)| = \left|\frac{f(x, y) - f(x, y + h_n)}{h_n}\right| \leq G(x)$$

for all $n \geq N$. Then by starting the sequencce $\{F_n\}_{n=N}^{\infty}$ at N we get a sequence of measureable functions in x which are uniformly bounded by $G(x)$ which are converging point wise to the partial derivative of f w.r.t y. By Lesbeque Dominated convergence we have,

$$\lim_{h \to 0} \left(\int_{[0,1]} \frac{f(x,y) - f(x,y+h)}{h} dx \right) = \lim_{n \to \infty} \left(\int_{[0,1]} F_n(x) dx \right) = \int_{[0,1]} \frac{\partial f}{\partial y}(x,y) dx \tag{5}$$

Since $y \in [0,1]$ was arbitrary, the above holds for all $y \in [0,1]$. □