

# MTH 316 Notes

Evan Fox

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# Chapter 1

## Cosets and Lagrange's theorem

### 1.1 Cosets

**Definition 1.1.1** (Cosets)

Let  $H$  be a subgroup of  $G$  then for each  $a \in G$  we define the left coset of  $H$  as

$$aH = \{ah \in G | h \in H\}$$

We will find this definition usefull in our study of groups. First we prove basic propertys of cosets.

**Theorem 1.1.2** (Properties of Cosets )

1.  $aH = H$  iff  $a \in H$ .
2.  $aH = bH$  iff  $a \in bH$
3.  $o(aH) = o(H)$ .
4.  $aH = bH$  or  $aH \cap bH = \emptyset$ .

*Proof.* (1) Assume  $aH = H$ , then since  $e \in H$ ,  $ae = a \in aH$  thus  $a \in H$ . Conversely assume that  $a \in H$ , then for  $ah \in aH$  we have by closure  $ah \in H$ . Similarly if  $h \in H$  then  $ha^{-1} \in H$  and so  $h \in aH$ . For (2) The proof is similar. For (3) we want to find a bijection from  $aH$  to  $H$ . Define  $\phi : H \rightarrow aH$  as  $\phi(h) \mapsto ah$ . Then one may easily show this is a bijection. (4) Assume  $aH \neq bH$ , then for the sake of contradiction assume that  $aH \cap bH \neq \emptyset$ . We fix  $x$  in the intersection and note that  $x = ah_1$  and  $x = bh_2$ . Then  $b = ah_1h_2^{-1} \in aH$ , but then by (2)  $aH = bH$ ; a contradiction.  $\square$

Note that the last two parts of the theorem show that the cosets of a subgroup  $H$  partition the group  $G$ , later we will find out when it makes sense to define an operation on these equivalence classes.

## 1.2 Lagrange's Theorem

This is often called the A, B, Cs in finite group theory and one of the biggest results in MTH 316.

**Theorem 1.2.1** (Lagrange)

*Let  $H \leq G$  then  $o(H) | o(G)$ .*

*Proof.* Let  $H \leq G$ . Using the result from the last section we already know that the cosets of  $H$  partition  $G$ . Now consider the set of left cosets of  $H$  in  $G$ .

$$\{g_1H, g_2H, \dots, g_rH\}$$

Since  $a \in aH$  we know that each element of  $G$  is in at least one coset, then since the cosets are disjoint,

$$ro(H) = o(G)$$

□