MTH 316 Homework 2

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Question 1.

Let $|G| = \infty$, Prove G has infinitly many subgroups.

Proof. We consider the case where G contains at least one element of infinite order seperatly. So Assume $|G| = \infty$ and that all elements of G have finite order. Then we let

$$S = \{\langle a \rangle | a \in G\}$$

If S is infinite we have found an infinite collection of subgroups and are done. So assume S is finite. Then since S contains all cyclic subgroups of G the union $\bigcup_{x \in S} x = G$ must hold since $a \in \langle a \rangle$ and for all $a \in G$ either $\langle a \rangle$ is an element of S or it is equivalent to an element of S. But every element of G has finite order and thus all the cyclic subgroups have finite order. But then the finite union of finite sets must be finite, and since the union of S is equal to G this implys that G is finite; a contradiction. Hence S must be an infinite family of subgroups.

Now we consider the cases where G contains at least one element of infinite order. So let $h \in G$ and $|h| = \infty$. Now consider $\langle h \rangle$. It is clear the subgroup generated by h is both cyclic and infinite so it will sufice to show that h has infinite subgroups. Since the order of h is infinite we have $h^n \neq h^m$ for all $n \neq m$ since otherwise we would have $h^{n-m} = e$ implying the order of h is finite. Now let $n, m \in \mathbb{N}$ with $n < m$ and assume $\langle h^n \rangle = \langle h^m \rangle$. Then $h^n \in \langle h^m \rangle$ so $h^n = (h^m)^t$ for $t \in \mathbb{N}$. But this implys $m < n$ a contradiction. So then we must have $\langle h^n \rangle \neq \langle h^m \rangle$ for $0 < n < m$. Thus we can easily create infinitly many subgroups of $\langle h \rangle$ and it follows that G has infinitly many subgroups.

$\square$

Question 2.

Let G be a group such that the only subgroups of G are the trivial subgroup and G itself. Prove $|G|$ is prime

Proof. It is clear by the previous result that G must be finite, if it were infinite then it must have infinitly many subgroups. We first prove that G is cyclic and then we use the fundamental theorem of finite cyclic groups. Let $|G| = n$. Note for all non idenity elements $g \in G$ we must have $\langle g \rangle = G$, otherwise g would generate a proper subgroup of G. Thus G is cyclic. We then have by the FTFCG that there exists a uniqiue subgroup of order k for each $k \in \mathbb{N}$ such that $k|n$. Since the only subgroups of of G have orders 1 and n it follows that the only divisors of n are one and itself. Thus n is prime.

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