MTH 316 Notes

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Question 1.

Let H, G be groups. Show that $H \rtimes_\phi G$ is a group.

Proof. It is clear that the operation is closed since $g_1\phi_{h_1}(g_2) \in G$ and $h_1h_2 \in H$ which gives $(g_1\phi_{h_1}(g_2), h_1h_2) \in G \times H$. Simillarly it is clear that the opperation will be associative since G and H are groups. Now we prove (e_G, e_H) is the idenitity, note that ϕ_e must be the idenitity automorphism .

$$(e, e) \star (g, h) = (e\phi_e(g), eh) = (eg, h) = (g, h)$$

Now we must find the inverse of an arbitrary element $(g, h) \in G \rtimes_\phi H$. A calculation shows

$$(\phi_{h^{-1}}(g^{-1}), h^{-1}) \star (g, h) = (\phi_{h^{-1}}(g^{-1})\phi_{h^{-1}}(g), h^{-1}h)$$

Now we can use the properties of homomorphisms to get $\phi_{h^{-1}}(g^{-1})\phi_{h^{-1}}(g) = \phi_{h^{-1}}(e) = e$. So we get

$$(\phi_{h^{-1}}(g^{-1}), h^{-1}) \star (g, h) = (e, e)$$

And we are done.

□

Question 2.

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let $n \geq 2$ show $\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2 \cong D_n$. Where $\phi(0) = id$ and $\phi(1)$ is the automorphism $k \mapsto -k$.

I will use the presentation of D_n given in Dummit and Foote as the definiton of D_n. That is

$$D_n = \langle r, s | r^n = s^2 = e, sr = r^{-1}s \rangle$$

where r is a notation of $2\pi/n$ radians and s is the flip across the axis intersecting the vertcie labeled 1 and the origin. I note now that D_n is never cyclic since $s \neq r^i$ for any i, to see this notice that $s \neq e$ but it fixes 1 and every rotation does not fix 1.

Proof. Note is is clear $|\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2| = 2n = |D_n|$. We need to use the condition $n \geq 2$ to ensure that $|\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2|$ is at least 4, since D_n must be non-cyclic it can only be defined for $n \geq 2$ (since all groups of lower order are cyclic). Now we define the map

$$\psi : \mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2 \to D_n$$

$$\psi(m, n) = r^m s^n$$

Since D_n has n rotations r (counting id as a rotation) and since s is cyclic of order 2, it is clear this map must be surjective; then since we know $\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2$ and D_n have the same order, ψ must necessarily be injective as well, hence ψ is a bijection. Now to show that the mapping will preserve group structure we consider two cases.

Case I: Let $(a, 0), (c, d) \in \mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2$ then

$$\psi\big((a, 0) \star (c, d)\big) = \psi(a + c, d) = r^{a+c}s^d = r^a(r^c s^d) = \psi(a, 0)\psi(c, d)$$

Case II: Let $(a, 1), (c, d) \in \mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2$ then

$$\psi\big((a, 1) \star (c, d)\big) = \psi(a - c, 1 + d) = r^{a-c}s^{1+d} = r^a r^{-c} s^1 s^d =$$

$$(r^a s^1)(r^c s^d) = \psi(a, 1)\psi(c, d)$$

Thus $\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2 \cong D_n$ as desired. $\square$

Question 3.

If ϕ is not trivial, prove that $G \rtimes_\phi H$ is not abelian.

Proof. if G or H are not abelian then we are done. So assume they are both abelian. Then since ϕ is not trivial, we fix $h \in H$ such that $\phi_h \neq id$. Then for $g, g_1, h \notin \{e\}$,

$$(g, h) * (g_1, e) = (g\phi_h(g_1), h)$$

$$(g_1, e) * (g, h) = (g_1\phi_e(g), h) = (g_1 g, h)$$

so if $G \rtimes_\phi H$ is abelian $g\phi_h(g_1) = g_1 g$ thus ϕ_h is the idenitity automorphism, a contradiction.

$\square$