MTH 525: Topology

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Chapter 1

Topology and Basis

1.1 Topology

Definition 1.1.1
Let X be a set. A topology *on X is a collection of subsets T such that*

1. *X, $\varnothing \in \mathfrak{T}$.*
2. *Closed under arbitrary unions.*
3. *Closed under finite intersections.*

On a given set there may be many diffrerent topologies. That can be defined on that set. Let $X = \{1, 2, 3, 4, 5\}$, we will write down several examples of topologies on X.

1. $\mathfrak{T} = \{\varnothing, X\}$
2. $\mathfrak{T} = \mathfrak{p}(X)$
3. $\mathfrak{T} = \{\varnothing, X, \{1\}\}$
4. $\mathfrak{T} = \{\varnothing, X, \{1\}, \{2\}, \{1, 2\}\}$

are all topologies on X. The first is called the trivial topology or the indescrete topology and the second is the descrete topology. We will now give an example of another topology, the finite complement topology.

Definition 1.1.2
Finite complement Topology Let X be a set and let $\mathfrak{T}$ consist of all subsets $U \subset X$ such that the complemnet of U in X is finite or X.

Proposition 1.1.3
The finite complement topology is a topology

Proof. We must show all three conditions are true, first we show that X and $\varnothing$ are in $\mathfrak{T}$. Note, $X - \varnothing = X$ and $X - X = \varnothing$ and both of these satisfy are conditions on $\mathfrak{T}$. Now we show $\mathfrak{T}$ is closed under arbitrary unions. Let $\{U_\alpha\}_{\alpha \in J} \subset \mathfrak{T}$ and we want to show that $(\bigcup_\alpha U_\alpha)^c \in \mathfrak{T}$. We have by DeMorgans laws

$$(\bigcup_{\alpha \in J} U_\alpha)^c = \bigcap_{\alpha \in J} U_\alpha$$

and since each $U_i \in \mathfrak{T}$ we now that each set is finite, thus since the intersection of finite sets are finite we are done. Now we must show closure under finite intersections, so let $\{U_1, \cdots, U_n\}$ be a subset of $\mathfrak{T}$. Then we have

$$(\bigcap U_i)^c = \bigcup U_i$$

and since each U_i is finite, and we have a finite number of sets to union, the result is finite. Hence we have showed the finite complement topology is indeed a topology. □

We could replace the finite condidtion with countable and we would still have a topology since the union of countable sets is again countable.

Given two topologies on a set we can also compare them.

Definition 1.1.4
Let X be a set and let $\mathfrak{T}$ and $\mathfrak{T}'$ be topologies on X. We say $\mathfrak{T}$ is finer than $\mathfrak{T}'$ if $\mathfrak{T}' \subset \mathfrak{T}$. We say $\mathfrak{T}$ is corser in the reverse situation.

1.2 Basis for a Toplogy

Definition 1.2.1
Basis Let X be a set, we say $\mathfrak{B}$ is a Basis *if*

1. *For all $x \in X$ there exists $B \in \mathfrak{B}$ such that $x \in B$.*
2. *if B_1, $B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$ then there exists $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

We define the topology generated by $\mathfrak{B}$ as the collection $\mathfrak{T}$ such that for any $U \subset X$, if for all $x \in U$ there exists $B \in \mathfrak{B}$ such that $x \in B \subset U$ then $U \in \mathfrak{T}$. For any $U \in \mathfrak{T}$ we say that U is open.

Proposition 1.2.2
The topology generated by a basis is a topology.

Proof. The first condidtion for a basis gives us X as an open set and the empty set satisfys our condition vacuosly. Now we must prove closure under arbitrary unions. Let $\{U_\alpha\}_{\alpha \in J} \subset \mathfrak{T}$ be a collection of open sets and consider their union. Then for $x \in \bigcup U_\alpha$ we must have x apearing in some U_α since it is in the union, but U_α is by assumtion open so there exists $B \in \mathfrak{B}$ such that

$$x \in B \subset U_\alpha \subset \bigcup_{\alpha \in J} U_\alpha$$

as desired. Now we must show that the finite intersection of open sets is again open, for that let $\{U_1, \cdots, U_n\}$ be a collection of open sets. Then if x lies in their intersection it must lie in each U_i. Thus there exists a familiy of basis elements $\{B_1, \cdots, B_n\}$ such that $x \in B_1 \subset U_i$. It follows then that $x \in \bigcap_{i<n} B_i$. Now to see that this intersection must be a basis element, we use induction on the second part of the definition of a basis. □

We now look at another way to define the topology generated by a basis.

Lemma 1.2.3
Let X be a topological space and let $\mathfrak{B}$ be a basis for the topology on X. Then $\mathfrak{T}$ is equal to set containg all unions of elements of $\mathfrak{B}$

Proof. Let $U \in \mathfrak{T}$, we want to write U as a union of basis elements. By definition we know that for each $x \in U$ there exists $B_x \in \mathfrak{B}$ satisfying $x \in B_x \subset U$. Taking the union over all B_x gives us the desired result. Now since Basis elements are open, any union of them must be contained in $\mathfrak{T}$, by definition. □

It may be helpful to be able to check whether or not a given set of subsets forms a basis for the topology.

Lemma 1.2.4
Let X be a topological space. Let $\mathfrak{C}$ be a collection of open sets such that for all $U \in \mathfrak{T}$ and all $x \in U$ there exists $C \in \mathfrak{C}$ such that

$$x \in C \subset U$$

Then $\mathfrak{C}$ is a basis for the toplogy on X

Proof. The first condtion of a basis is satisfied by assumtion. Now suppose $x \in C_1 \cap C_2$ we must show there exists $C_3 \in \mathfrak{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. We may use the fact that $\mathfrak{C}$ is a collection of opensets together with our assumtion to produce such an element.

Now we must show that $\mathfrak{C}$ generates the correct topology. Let $\mathfrak{C}$ generate $\mathfrak{T}'$. If U is open in $\mathfrak{T}$ then by assumtion it is open in $\mathfrak{T}'$. If U is open in $\mathfrak{T}'$ then it is a union of elements of $\mathfrak{C}$, since $\mathfrak{C}$ is a collection of open sets of $\mathfrak{T}$, U must be open in $\mathfrak{T}$. $\square$