MTH 435: Analysis HW 1

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Question 1.

Determine all accumulation points of the following sets in $\mathbb{R}$ and decide if the sets are open or closed

(a) All numbers of the form $\frac{1}{n}$ for $n \in \mathbb{N}$

ans: There is only one accumulation point, 0. Since by the archimedean property, each neighborhood around 0 will contain a point of the form $\frac{1}{k}$ for a large enough choice of k. The set is not closed since it does not contain all its limit points, it is also not open since every point is isolated.

(b) All numbers of the form $2^{-n} + 5^{-m}$ for $n, m \in \mathbb{N}$.

Ans: Let $A = \{2^{-n} + 5^{-m} \,|\, n, m \in \mathbb{N}\}$. A has accumulation points, $s = \frac{1}{2^m}$ and $t = \frac{1}{5^n}$ for $n, m \in \mathbb{N}$ Consider a neighborhood $B(s, \epsilon)$ for $\epsilon > 0$ around $\frac{1}{2^{-m}}$, then there exists $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ by the archimedean property. Then $\frac{1}{5^n} < \epsilon$ for $n > N$. Hence $s + 5^{-n} \in B(s, \epsilon)$; so $s = \frac{1}{2^m}$ is a limit point for every choice of $m \in \mathbb{N}$.

Now fix a neighborhood $B(t, \epsilon)$, since $n > N \implies \frac{1}{n} < \epsilon$ we again have $\frac{1}{2^n} < \frac{1}{n} < \epsilon$ so $t + \frac{1}{2^n} \in B(t, \epsilon)$, so that t is a limit point.

This set does not contain all of its limit points so it cannot be closed. Further since $A \subset \mathbb{Q} \subset \mathbb{R}$, all of its points are isolated, so it is not open.

(c) $A = \{(-1)^n + \frac{1}{m} \,|\, n, m \in \mathbb{N}\}$.

Ans: A has limit points $1, -1$. Let $\epsilon > 0$, and consider $B(1, \epsilon)$. then by the archimedean property, we fix $M \in \mathbb{N}$ such that $m > M \implies \frac{1}{m} < \epsilon$. Then $1 + \frac{1}{m} \in B(1, \epsilon)$. Hence 1 is a limit point, the argument for -1 is the same.

This set also fails to contain its limit points so it cannot be closed and again it is a subset set of $\mathbb{R}$ completely contained in the rationals, so it cannot be open.

Question 2.

The same as Exercise 3l2 for the following sets in $\mathbb{R}^2$

(a) All complex numbers of the form $\frac{1}{n} + \frac{i}{m}$ for $n, m \in \mathbb{N}$

Ans: All limit points are of the form $\frac{1}{n}$ for any $n \in \mathbb{N}$ or $\frac{i}{m}$ for any $m \in \mathbb{N}$. For an element of the form $\frac{1}{n}$ fix an arbitrary epsilon neighborhood around it the by choosing m such that $\frac{1}{m} < \epsilon$ we will have $\frac{1}{n} + \imath\frac{1}{m}$ in the epsilon ball. And the argument for elements of the form $\frac{\imath}{m}$ is similar. Since the set of complex numbers of the desired form does not contain all of its limit points it is not closed. And again it is a subset of the rationals in the complex numbers so it is not open.

(b) All points (x, y) such that $x^2 - y^2 < 1$

Claim: Let $S = \{(x, y)|x^2 - y^2 < 1\}$. $T = \{(x, y)|x^2 + y^2 \leq 1\}$. Then $S' = T$

Proof. Let $t \in T$, $t = (x, y)$. Then fix $\epsilon > 0$ and consider $B(t, \epsilon)$. Then if $y > 0$, note that the point $(x, y + \frac{\sqrt{\epsilon}}{2}) \in B(t, \epsilon)$, since

$$\|(x, y) - (x, y + \frac{\sqrt{\epsilon}}{2})\| = \frac{\epsilon}{4} < \epsilon$$

and then

$$x^2 - (y + \frac{\sqrt{\epsilon}}{2})^2 = x^2 - y^2 - \sqrt{\epsilon}y - \frac{\sqrt{\epsilon}}{2}$$

$$\leq 1 - \sqrt{\epsilon}y - \frac{\sqrt{\epsilon}}{2} < 1$$

since $y > 0$. If $y < 0$, consider $(x, y - \frac{\sqrt{\epsilon}}{2})$. Then t is a limit point of S. Hence $T \subset S'$. Now if we take a limit point of S and suppose its not in T we will quickly get a contradiction since we are in $\mathbb{R}^2$, and $x^2 + y^2 > 1$ we can fix a ball around (x, y) that doesn't contain a point satisfying $x^2 + y^2 < 1$.

This set is not closed since it does not contain all its limit points. It is however an open set, since for every point in S, one can fix a ϵ ball such that all points in the ball satisfy our condition. Let (x, y) be in the set. Then let $\epsilon = \frac{1}{2} \inf \|(x, y) - v\|$ for all $v = (x_0, y_0)$ satisfying $x_0^2 + y_0^2 = 1$. Then any element in the ball $B((x, y), \epsilon)$ has distance less than ϵ from (x, y) and so also satisfies our condition. Thus we have an open neighborhood around x which is contained in the set. Hence the set is open.

□

(c) All points (x, y) such that $x > 0$.

Claim: limit points are all points such that $x \geq 0$.

Proof. Let $(x, y) \in \mathbb{R}^2$ such that $x \geq 0$, then let $\epsilon > 0$. Then $(x+\epsilon/2, y)$ is contained in the ϵ ball around (x, y) and satisfies the condition that the first coordinate be greater than 0. hence (x, y) is a limit point.

The set does not contain all its limit points so it is not closed. It is open, for (x, y) with $x > 0$, take $r = \frac{1}{2}\|x\|$ and consider the ball $B((x, y), r)$. This clearly, every point in this set satisfies our requirement. Then let $(a, b) \in B((x, y), r)$ and let $\|(a, b)-(x, y)\| = h$, then take $l = \frac{1}{2}(r-h)\|$. Then for any point $z \in B((a, b), l)$, by the triangle inequality we have

$$\begin{aligned} \|z - (x, y)\| &\leq \|z - (a, b)\| + \|(a, b) - (x, y)\| \\ &< r - h + h = r \end{aligned}$$

Hence $z \in B((x, y), r)$. Thus there is a neighborhood around (a, b) completely contained in $B((x, y), r)$, and thus (a, b) is an interior point. Since (a, b) was arbitrary it follows that every point is interior and so the ball is open. Then for every point (x, y) with $x > 0$, we have a ball with all points having the same property, and thus the original set in question is open. □

Question 3.

Prove that the interior of a set in $\mathbb{R}^n$ is open

Proof. let A° be the interior, We prove that A° is the union of all open sets contained in A. Indeed, consider an element of the union of all open sets contained in A, then since it is in the union it is in an open set contained in A, but this is the definition of being in the interior. Now conversely, consider an element in the interior of A. Then there exists a open set contained in A containing it. But then this element must appear in the union of all open sets contained in A. Hence $A^\circ = \bigcup U$ where U runs over all open sets $U \subset A$. Since the interior is an arbitrary union of open sets, it must be open. □

Question 4.

let S' denote the derived set and $\overline{S}$ the closure of a set S in $\mathbb{R}^n$. Prove the following

(a) S' is closed in $\mathbb{R}^n$; that is $(s')' \subset S'$

Proof. Let x be a limit point of the derived set of S. Then every neighborhood of x contains a limit point of S. Since a neighborhood is by definition open, there is a another neighborhood around each limit point of S contained in the neighborhood around x, these then must contain elements of S, and so every neighborhood around x contains points of S and as such $x \in S'$. As desired. □

(b) If $S \subset T$, then $S' \subset T'$

Proof. Suppose x is a limit point of S, then every neighborhood contains a point of S, since $S \subset T$, each neighborhood around x contains a point of T, so x is a limit point of T.

□

(c) $(S \cup T)' = S' \cup T'$

Proof. If $x \in (S \cup T)'$ then every neighborhood intersects $S \cup T$, then we prove that every neighborhood either intersects S or every neighborhood intersects T. Suppose not, then there exists U, V, open around x where U intersects S but not T is the reverse holds for V. Then taking the intersection $U \cap V$ gives a neighborhood around x which intersects $S \cup T$ nowhere, a contradiction. Then without loss of generality assume every neighborhood intersects S, then $x \in S' \cup T'$.

The converse is easy since if $x \in S' \cup T'$ then either x is a limit point of S or T. if x is a limit point of S, then every neighborhood of x must intersect $S \cup T$ so that $x \in (S \cup T)'$.

□

(d) $\overline{(S')} = S'$.

Proof. Per an earlier result (a), we know that S' contains all of its limit points. Hence it is closed. So since $\overline{S'} = S' \cup (S')'$ and $(S')' \subset S$, S' is equal to its closure. □

(e) $\overline{S}$ closed in $\mathbb{R}^n$

Proof. Taking an element in the complement $x \in \mathbb{R}^n \setminus \overline{S}$, we know that x cannot adhere to S, so there must exist a neighborhood of x which does not contain a point of S, And if a neighborhood of x contained a point of s', then we already know that would imply x is a limit point which would be a contradiction. Hence a neighborhood around x does not contain a point of $S \cup S' = \overline{S}$, and thus is an interior point. Since x was arbitrary, we have that $\mathbb{R}^n \setminus \overline{S}$ is open and so $\overline{S}$ is closed.

□

(f) Let x be in the intersection of all closed sets containing S. Now suppose that $x \notin S$ and that x is not a limit point, then fix a neighborhood U around x that doesn't intersect S. Then $\mathbb{R}^n \setminus U$ is a closed set containing S which does not contain x, a contradiction. Hence either $x \in S$ or x is a limit point and in either case x is in the closure of S.

Now let x be in the closure. Suppose there existed a closed set $C \supset A$ that did not contain x. then $\mathbb{R}^n \setminus C$ is an open neighborhood of x which does not intersect A so that x is not in the closure; a contradiction. hence x must be in every closed set containing A and then it follows that x will be in the intersection.

Proof. □

Question 5.

Prove that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and $A \cap \overline{B} \subset \overline{A \cap B}$ if A is open.

Proof. Let $x \in \overline{A \cap B}$ then every neighborhood of x, intersects A and B. Since every neighborhood intersects A, $x \in \overline{A}$ and since every neighborhood intersects B, $x \in \overline{B}$. Hence $x \in \overline{A} \cap \overline{B}$.

Let A be open and let $x \in A \cap \overline{B}$. Then there exists a neighborhood U of x completely contained in A, since A is open. Then for an arbitrary neighborhood V of A, taking $W = U \cap V \subset V$ is completely contained in A. But since $x \in \overline{B}$, W must also contain a point of B that lies in A. Hence W is a neighborhood around x that contains a point of $A \cap B$. Since $W \subset V$, it follows that the arbitrary neighborhood V also contains a point of $A \cap B$. Thus $x \in \overline{A \cap B}$.

$\square$