Question 1.

Use the IVT to prove that the equation

$$\frac{x-1}{x^2+1} = \frac{3-x}{x+1} \tag{1}$$

has a real solution.

Proof. some algebra shows (1) to be equivalent to the cubic polynomial $p(x) = -x^3 + 2x^2 - x + 4$. Then evaluating this at -1 gives a positive output since the odd power terms are negitive, the negitive sign will cancel out and we get $p(-1) = 1 + 2 + 1 + 4 = 8 > 0$. On the other hand, $p(3) = -27 + 18 - 3 + 4 = -8 < 0$. Hence by the IVT there must exist a point $c \in (-1, 3)$ such that $p(c) = 0$. □

Question 2.

(a) Prove that $\overline{S}$ is the intersection of all closed subsets of $\mathbb{R}^n$ containing S.

Proof. Let A denote the intersection of all closed sets which contain S. Then let $x \in A$, then it follows that x is in every closed set which contains S by definition; since the closure is one such set, we must have $x \in \overline{S}$.

Conversly, assume $x \in \overline{S}$ and let C be a closed set such that $S \subset C$. Then assume for the sake of contradiction that we have $x \notin C$. Then $X \setminus C$ is an open neighborhood of x, furthermore, this neighborhood is disjoint from S since $S \subset C$. Hence x is not an adherent point of S and as such is not in the closure $\overline{S}$, a contradiction. Hence $x \in C$, but since C was an arbitrary closed set containing S it follows that x is in every closed set which contains S and so it will be in the intersection. □

(b) Let S and T be subsets of $\mathbb{R}^n$. Prove that $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$ and that $S \cap \overline{T} \subset \overline{S \cap T}$.

Proof. First suppose that $x \in \overline{S \cap T}$. Then by part (a) we have that x is in every closed set containg $S \cap T$. Now consider any closed set C

containg S, since $S \cap T \subset S$, C contains $S \cap T$ and hence must contain x, then x is in every closed set containg S and by part (a) we have $x \in \overline{S}$. The argument to see the $x \in \overline{T}$ is similiar. Then since x is in the clousure of S and T, it is in their intersection $\overline{S} \cap \overline{T}$.

Now we answer the second part, assume that S is open and that $x \in S \cap \overline{T}$. Then let U be a neighborhood of x. Then $U \cap S$ is open and there exist V with $x \in V \subset U \cap S$. Since $x \in \overline{T}$ this neighborhood V must contain at least a point y of T, but then $V \subset S$ so $y \in S$, hence $y \in S \cap T$ so that V contains a point of the intersection. Since $V \subset U$ and U was arbitrary, it follows that every neighborhood of x contains a point of $\overline{S \cap T}$. □

Question 3.

Let X be non-empty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}$$

Give examples to show that each of these inequalities can be either equalities or strice inequalities.

Proof. Let $\alpha_1 = \sup\{f(x) : x \in X\}$ and let $\alpha_2 = \sup\{g(x) : x \in X\}$.Then fix $x_0 \in X$ and let $y = f(x_0) + g(x_0)$; it follows $f(x_0) \leq \alpha_1$ and $g(x_0) \leq \alpha_2$. Then adding these two inequalities gives

$$y = f(x_0) + g(x_0) \leq \alpha_1 + \alpha_2$$

Hence, $\alpha_1 + \alpha_2$ is an upper bound on the set $\{(f + g)(x) : x \in X\}$. From this the desired conclusion follows. To see an example where the inequality is strict consider $f(x) = \sin^2(x)$ and $g(x) = \cos^2(x)$, then

$$\sup\{f(x) + g(x) : x \in [0, 2\pi]\} = 1$$

but

$$\sup\{f(x)\} + \sup\{g(x)\} = 1 + 1 = 2$$

for $x \in [0, 2\pi]$. An example where they are equal comes from considering f, g as any two constant functions.

Now we prove the statement in terms of infiniums, the argument is very similar. Let $\ell_1 = \inf\{f(x)\}$ and $\ell_2 = \inf\{g(x)\}$. Then for any $y = f(x_0) + g(x_0)$ we have $\ell_1 \leq f(x_0)$ and $\ell_2 \leq g(x_0)$. Then once again adding these gives

$$\ell_1 + \ell_2 \leq f(x_0) + g(x_0) = y$$

which shows that $\ell_1 + \ell_2$ is a lower bound for the set $\{(f+g)(x) : x \in X\}$. Hence it must be less than or equal to the infimum of that set.

An example of a strict inequality again comes from considering $f(x) = \sin^2(x)$ and $g(x) = cos^2(x)$. we have

$$0 = \inf\{\sin^2(x)\} + \inf\{\cos^2(x)\} < 1 = \inf\{\sin^2(x) + \cos^2(x)\}$$

and they are again equal to each other if we consider f, g as constant functions. $\square$

Question 4.

Let $K \subset \mathbb{R}^n$ be compact and let $x \in \mathbb{R}^n$. Show that $x + K$ is compact.

Proof. Since the function $f : \mathbb{R}^n \to \mathbb{R}^n$, taking $y \to x + y$ is continuous, $x + K$ is simply the image of a compact set under a continuous function, hence it is compact. To see that f is in fact continuous, considier a basis element of $\mathbb{R}^n$, $B(x, r) = \{s \in \mathbb{R}^n | \|x - s\| < r\}$. Then the pre image, is the set $f^{-1}(B(x, r)) = -p + B(x, r) = \{s - p | \|x - s\| < r\} = \{s | \|(x - p) - s\| < r\} = B(x - p, r)$ which is open. $\square$

Question 5.

Assume f has finite derivative in (a, b) and is continuous on $[a, b]$ with $f(a) = f(b) = 0$. Prove that for every real λ there is some $c \in (a, b)$ such that $f'(c) = \lambda f(c)$.

Proof. Let $\lambda \in \mathbb{R}$. Then consider the function $e^{-\lambda x} f(x)$. Observe that this function is differentialble on (a, b) and continuous on $[a, b]$ and $e^{-\lambda a} f(a) =$

$e^{-\lambda a}0 = 0 = e^{-\lambda b}f(b)$. Hence we may apply rolles theorem to this function. Hence there exists $c \in (a, b)$ such that

$$(e^{-\lambda c}f(c))' = 0$$

$$(e^{-\lambda c})'f(c) + e^{-\lambda c}f'(c) = 0$$

equvialently,

$$f'(c) = -(-\lambda)\frac{e^{-\lambda c}}{e^{-\lambda c}}f(c)$$

$$f'(c) = \lambda f(c)$$

as desired. □

Question 6.

Let $f : S \to T$ be uniformly continuous. Prove that if (x_n) is cauchy, then $(f(x_n))$ is cauchy.

Proof. Let (x_n) be a cauchy sequence in S. Fix $\epsilon > 0$. We show that there exists $N \in \mathbb{N}$ such that for $n, m > N$ we have $d_T(f(x_n), f(x_m)) < \epsilon$. By assumption there exists $\delta > 0$ such that $d_S(a, b) < \delta$ implies $d_T(f(a), f(b)) < \epsilon$. Then fix $N \in \mathbb{N}$ such that for $n, m > N$ we have $d_S(x_n, x_m) < \delta$; then we must have $d_T(f(x_m), f(x_n)) < \epsilon$ for all $n, m > N$. Hence the sequence $(f(x_n))$ is cauchy. □