MTH 451 Workseet 5

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Question 1.

We need to solve $.9 = F(X)$ for x. We have

$$F(x) = 2\int_0^x e^{-2x}\,\mathrm{d}x = 2(\tfrac{-1}{2}e^{-2x} + \frac{1}{2}) = -e^{-2x} + 1$$

Then solving $F(x) = .9$ gives

$$\begin{aligned} .1 &= e^{2x} \\ x &= \frac{\ln(.1)}{2} \approx 1.15 \end{aligned}$$

Question 2.

We will make use of the short-cut formula $\sigma_{XY} = E(XY) - E(X)E(Y)$. We will ned the marginal distribution for X and Y so we calculate those first.

$$f_X(x) = \frac{4}{3}\int_0^1 1 - xy \,\mathrm{d}y = \frac{4}{3}[y - \frac{1}{2}xy^2|_0^1] = \frac{4}{3}[1 - \frac{1}{2}x]$$

and

$$f_Y(y) = \frac{4}{3}\int_0^1 1 - xy \,\mathrm{d}x = \frac{4}{3}[1 - \frac{1}{2}y^2]$$

Then we have that $E(X) = \int x f_X(x)\,\mathrm{d}x$ so we calculate the expected value for X and Y.

$$E(X) = \frac{1}{3}\int_0^1 4x - 2x^2 \,\mathrm{d}x = \frac{1}{3}(2x^2 - \frac{2}{3}x^3|0^1) = \frac{4}{9}$$

and

$$E(Y)\frac{1}{3}\int_0^1 4y - 2y^2 \,\mathrm{d}y = \frac{1}{3}(2y^2 - \frac{2}{3}y^3|0^1) = \frac{4}{9}$$

Then $E(XY)$ is given by $\iint_R xyf(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Computing this gives

$$\frac{4}{3}\int_0^1\int_0^1 xy - x^2y^2 \,\mathrm{d}x\,\mathrm{d}y = \frac{4}{3}\int_0^1 \frac{1}{2}x^2y - \frac{1}{3}x^3y^2|_0^1 \,\mathrm{d}y =$$
$$\frac{4}{3}\int_0^1 \frac{1}{2}y - \frac{1}{3}y^2 \,\mathrm{d}y = \frac{4}{3}(\frac{1}{4}y^2 - \frac{1}{9}y^3|_0^1) = \frac{5}{27}$$

Then we are ready to use $\sigma_{XY} = E(XY) - E(X)E(Y)$ and get

$$\sigma_{XY}\frac{5}{27} - \frac{16}{81} = \frac{-1}{81}$$

Since the covarience is non-zero they cannot be independant.

Question 3.

We first make use of the lineararity of $E(X)$

$$E(X+Y) = E(X) + E(Y) = 3.3$$

This gives use the expected value for the outer diamiter. Then using the short-cut formula for varience it is easy to see that

$$V(X+Y) = E((X+Y)^2) - E(X+Y)^2 =$$

$$E(X^2) + E(2XY) + E(Y^2) - E(X+Y)^2 =$$

$$E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2 + 2\big(E(XY) - E(X)E(Y)\big) = V(X) + V(Y)$$

Where the last step follows from the fact that X and Y are independant. Then we want the standard deviation so we must find the square root of the varience.

$$\sqrt{V(X+Y)} = \sqrt{.02^2 + .005^2} \approx .0206$$

Question 4.

We have that $f_{X|Y}(x|2) = \frac{f(x,2)}{f_Y(2)}$. We fist calculate the marginal distribution of Y.

$$f_Y(y) = \int_0^\infty e^{-x}e^{-\frac{y}{2}}\,\mathrm{d}x = -e^{-x}e^{-\frac{y}{2}}\big|_0^\infty = e^{-\frac{y}{2}}$$

then we have $f_Y(2) = e^{-1}$. Then

$$f_{X|Y}(x|2) = \frac{e^{-x}e^{-1}}{e^{-1}} = e^{-x}$$

and we are done.

Question 5.

We begin by finding the marginal distribution of Y.

$$f_Y(y) = 24\int_0^{1-y} y - xy - y^2 \,\mathrm{d}x = 24[xy - \frac{1}{2}x^2y - xy^2\big|_0^{1-y}] =$$
$$= 24[y(1-y) - \frac{1}{2}y(1-y)^2 - y^2(1-y)]$$

some algebra gives

$$f_Y(y) = 24y(1-y)(\frac{1}{2} - \frac{1}{2}y)$$

We then have that

$$f_{X|Y}(x|y) = \frac{1-x-y}{(1-y)(\frac{1}{2}-\frac{1}{2}y)}$$

as the PDF for X given $Y = y$. Now to calculate $P(X < \frac{1}{4}|y = \frac{1}{2})$ we need to calculate a CDF.

$$F_{X|Y}(\tfrac{1}{4}|\tfrac{1}{2}) = \int_0^{\frac{1}{4}} f_{X|Y}(x|\tfrac{1}{2}) = I$$
$$I = \int_0^{\frac{1}{4}} 4 - 8x \,\mathrm{d}x = 4x - 4x^2\big|_0^{\frac{1}{4}} = \frac{3}{4}$$

So we have $P(X < \frac{1}{4}|\frac{1}{2}) = \frac{3}{4}$.