MTH 316 Homework 1

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Question 1.

Prove that the function $\phi : G \to G$ defined by $\phi(g) = g^{-1}$ is an automorphism of G iff G is Abelian.

Proof. First Assume that G is Abelian, then define the function $\phi(g) = g^{-1}$ for all $g \in G$. Then it is clear that ϕ is a injection since if $\phi(g_1) = \phi(g_2)$ then $g_1^{-1} = g_2^{-1}$; thus $g_1 = g_2$. To see surjectivity let $h \in G$ and it is clear $\phi(h^{-1}) = h$. It then follows that ϕ is a bijection. We now must show that ϕ is an isomorphism. Observe

$$\phi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \phi(a)\phi(b).$$

Where the third equality is justified since G is abelian by assumption. Thus ϕ is an automorphism of G. Conversly, Assume that ϕ as defined above is an automorphism of G. Then for all $a, b \in G$ we have $\phi(ab) = \phi(a)\phi(b)$. Then consider

$$ab = \phi((ab)^{-1}) = \phi(b^{-1}a^{-1}) = \phi(b^{-1})\phi(a^{-1}) = ba$$

Thus G is Abelian and we are done.

$\square$

Question 2.

Let ϕ and ψ be isomorphisms from G to $\bar{G}$. Define

$$H = \{g \in G | \phi(g) = \psi(g)\}$$

show $H \leq G$.

Proof. We use the two step subgroup test. Notice that since all isomorphisms map the identity in G to the identity in $\bar{G}$, we know $e \in H$. Now suppose $a, b \in H$. Then

$$\phi(a) = \psi(a) \textit{ and } \phi(b) = \psi(b)$$

Then we must have

$$\phi(ab) = \phi(a)\phi(b) = \psi(a)\psi(b) = \psi(ab)$$

so $ab \in H$. Now

$$\phi(a^{-1}) = \phi(a)^{-1} = \psi(a)^{-1} = \psi(a^{-1})$$

Thus $a^{-1} \in h$ and this completes the proof.

□