MTH 436 Homework 4

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Morgan and I worked together on this homework.

Problem 1.

Proof. Let $\mathcal{A}$ be a set of closed subsets of $\mathbb{R}$ such that at least one element is bounded, say $C \in \mathcal{A}$. Then it follows from the Hine-Borel theorem that this set is compact. We prove the contrapositive, that is, suppose that for all $F_1, \dots, F_n \in \mathcal{A}$, we have the intersection $\bigcap^n F_i \neq \varnothing$. Then we prove theat $\bigcap_{F \in \mathcal{A}} F \neq \varnothing$. We do this by recalling the finite intersection propertery from 525. We know a space X is compact if and only if for all collections of closed sets having the finite intersection property, the intersection over the whole collection is non empty. First note that C (The compact set in $\mathcal{A}$ above) must have a non-trivial intersection with every other $A \in \mathcal{A}$, by assumption. Then consider the collection $\mathcal{B} = \{C \cap A \,|\, A \in \mathcal{A}\}$ considered as subsets of the compact metric space C. Then this is a collection of closed sets (in C) and it has the finite intersection property, since $\mathcal{A}$ does. Hence, by the theorem stated above $\bigcap_{B \in \mathcal{B}} B \neq \varnothing$. Which is to say, $\bigcap_{A \in \mathcal{A}} \Big(C \cap A\Big) = \bigcap_{A \in \mathcal{A}} A \neq \varnothing$ as desired. □

Problem 2.

Proof. (a) F is defined as the complement of a union of open sets and hence is closed

(b) Let I be an interval contained in F and $x, y \in I$; further suppose that $x < y$. Then by density there exists r_m with $x < r_m < y$, but then by definition F cannot contain the interval $(r_m - \frac{1}{2^m}, r_m + \frac{1}{2^m})$, hence neither can I, hence I cannot be an interval. It follows $x = y$.

(c) We have

$$|\bigcup^{\infty}(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k})| = \sum^{\infty} |(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k})| = \sum^{\infty} \frac{1}{2^{k-1}} = 2$$

Since $|\mathbb{R}| = \infty$, it follows that $|F| = \infty$.

□

Problem 3.

Proof. Consider the space $(\mathbb{R}, \{\varnothing, \mathbb{R}\})$. Then the only continuous functions into $\mathbb{R}$ with the usually sigma algebra of borel sets are the constant functions; If f takes more than one value so that $f(x_1) \neq f(x_2)$, let B be a borel set containing $f(x_1)$ but not $f(x_2)$. Then the preimage of B will be some proper subset of $\mathbb{R}$, hence, not in our sigma algbra. Construct the function,

$$f(x) = \begin{cases} a & x \in [0, \infty) \\ -a & x \in (-\infty, 0) \end{cases}$$

Then this will produce the desired result. It is not constant so it is not measurable, but $|f|$ is constant.

□

Problem 4.

Proof. (a) Let $E \in \mathcal{S}$, then $E \in \mathcal{T}$ and $E \subset X$ by assumption, then $E \cap X = E$. Conversly, it is clear that $F \cap X \subset X$, and a σ-algebra is closed under intersections, so $F \cap X \in \mathcal{T}$ hence it is also in $\mathcal{S}$.

(b) Let $\mathcal{S} = \{F \cap X | F \in \mathcal{T}\}$. $\varnothing \in S$ since $\varnothing \in \mathcal{T}$, Then $X \setminus (F \cap X) = (Y \setminus F) \cap X$, and $Y \setminus F \in \mathcal{T}$ since $\mathcal{T}$ is a σ-algebra hence this set is closed under complementation. Then given a collection of sets of the form $F \cap X$ for $F \in \mathcal{T}$. we have

$$\bigcup^{\infty} F_i \cap X = X \cap \Big(\bigcup^{\infty} F_i\Big) \in \mathcal{S}$$

□

Problem 5.

Proof. Let $f : B \to \mathbb{R}$ be a borel measurable function defined on a borel set B. Define g as in the problem. Let C be a borel set and write $C = (C \cap f(B)) \cup (C \cap \{0\})$. Then

$$g^{-1}(C) = g^{-1}(C \cap f(B)) \cup g^{-1}(C \cap \{0\})$$

Since $C \cap f(B) \subset f(B)$ and f is borel measurable, the first primage on the RHS must be a borel set. Then if $C \cap \{0\} = \varnothing$ the second priamge is trivial and if $C \cap \{0\} = \{0\}$ the second primage is $\mathbb{R} \setminus B$ wich is a borel set. Then since the union of borel sets is a borel set, g is borel measureable. $\square$

Problem 6.

Proof. Let $f(x) > 0$ for all x, then we can write

$$f(x)^{g(x)} = e^{\ln(f(x)^{g(x)})} = e^{g(x)\ln(f(x))}$$

Where the RHS is S measurable by the product and composistion results obtained in class. Note that $f(x) > 0$ is required so the $\ln(f(x))$ is defined. $\square$

Problem 7.

Proof. Let μ and ν be measures. Then $(\mu + \nu)(\varnothing) = 0 + 0 = 0$. let E be a collection of disjoint sets. We have

$$(\mu + \nu)(\bigcup E) = \mu(\bigcup E) + \nu(\bigcup E) = \sum^{\infty} \mu(E) + \sum^{\infty} \nu(E) = \sum (\mu + \nu)(E)$$

Where the last equality follows from basic facts about infinite series, hence $(\mu + \nu)$ is a measure. $\square$