MTH 525: Topology

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Question 1.

Show that a first countable T_1 space is G_δ.

Proof. Let X be first countable and T_1 and let $x \in X$. Then let $\mathcal{A}$ be a countable basis at x. Then let $B = \cap_{A \in \mathcal{A}} A$, clearly B is a countable intersection of open sets, we show that $B = \{x\}$. Clearly $x \in B$, now suppose that $y \neq x$ and $y \in B$, then note that since X is a T_1 space, $X \setminus \{y\}$, is an open set around x. Now there must exist a basis element containing x contained in $X \setminus \{y\}$, hence there is an open set $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$, so that $y \notin B$ a contradiction. Hence $B = \{x\}$. $\square$

For an example consider $\mathbb{R}^\omega$ in the box toplogy, this space is not first countable, since given a countable collection of open sets about a point x, $\{U_n\}_{n \in \mathbb{N}}$. We start by selection an open set $V_1 \subset \pi_1(U_1)$ such that $x_1 \in V_1$, then we select $V_2 \subset \pi_2(U_2)$ with $x_2 \in V_2$ and so on, then the open set $V = \bigtimes_{i=1}^{\infty} V_i$ is an open set but does not contain any element U_i since $\pi_i(V) \subset \pi_i(U_i)$. Hence $\mathbb{R}^\omega$ is not first countable. On the other hand given x, the sets $A_n = \bigtimes_{i=1}^{\infty}(x_i - \frac{1}{n}, x_i + \frac{1}{n})$. It is clear that this gives a countable collection and that x is the only element of the intersection.

Question 2.

Show that $\mathbb{R}_\ell$ and I_o^2 are not metrizable

Proof. Note that $\mathbb{R}_\ell$ is not second countable, since given any countable collection of open sets of the form $[a_i, b_i)$ we can find a reall number $\xi \notin \{x | x = a_i\}$ since $\mathbb{R}$ is uncountable. Then the set $[\xi, b)$ is open by the definition of lower limit topology, but there is no basis element containing ξ contained in this open set, a contradiction. However $\mathbb{R}_\ell$ does have a countable dense subset, $\mathbb{Q}$, not that for an arbitrary $x \in \mathbb{R}$, every nbhd of x, $[a, b)$ will contain rational points, hence every real number is a limit point of $\mathbb{Q}$, hence $\mathbb{Q}$ is dense.

We know that if a space is metrizable then second countable is equivalent to having a countably dense subset. Since this is not the case for $\mathbb{R}_\ell$, it must be the case that $\mathbb{R}_\ell$ is not metrizable.

We will emploly a similiar argument for the ordered squar I_o^2, first note that it cannot have a countable basis since $\{x\} \times (1/3, 2/3)$ is open for each $x \in [0,1]$ and is an uncountable disjoint collection, and so for each point $x \times \frac{1}{2} \in \{x\} \times (\frac{1}{3}, \frac{2}{3})$, there must exist a basis element $B_x \subset \{x\} \times (\frac{1}{3}, \frac{2}{3})$. Thus the basis must be uncountble. But again $\mathbb{Q}^2$ restricted to the ordered square will give a countable dense subset. Thus the space cannot be metrizable becasue second countable and the existence of a countable dense subset are not equivalent.

□

Question 3.

Which of the four countablility axioms does S_ω and $\overline{S_\omega}$ satisfy?

Proof. S_Ω is first countable, given $a \in S_\Omega$, Then S_a is countable and since S_Ω is totally ordered, there is an immeadiate sucessor of a, say b_1 then define the imeadiate sucessor recursivly as $b_1 < b_2 < b_3 < \ldots$ wich also gives a countable collection. Then considering all open sets (x, y) with $x \in S_a$ and $y = b_i$ is a countable collection of open sets about a, and given an arbitrary open set around a, it is clear it must contain a set of this form.

Now we show that this set is not second countable by showing that it has no countable dense subset, note that any countable set in S_Ω is bounded, but S_Ω itself is uncountable and has no maximal element, hence there cannot be a countable dense subset, since given any countble set B, there exists $c, d \in S_\Omega$ such that for all $b \in B$, $b < c < d$ hence d will not be a limit point.

Now we show that this space is not lindelof, consider the convering S_a for all $a \in S_\Omega$, and suppose a countble subcollection covers S_Ω, then we get that the countable union of countable sets convers S_Ω, which cannot be the case since S_Ω is uncountable and the countable union of countable sets is countable.

First note that $\overline{S_\Omega}$ is no longer first countable since we have included the point Ω, Given any countable collection of open sets around Ω, the set of lower bounds of these intervals forms a countable set and as such is bounded, then we may take an element larger than the upperbound and form an open set containg Ω that doesn't contain any element of our countable collect, hence $\overline{S_\Omega}$ fails to be first countable at the point Ω. Secondly, we know that $\overline{S_\Omega}$ is the one point compactification of S_Ω and as such it is compact. Thus it is trivially lindelof. The other conditions are the same as above. □

Question 4.

Let $P : X \to Y$ be closed continuous and surjective.

(a) Show that X Hausdorff implies the same for Y.

Proof. Let p be a closed continuous surjective map s.t. $p^{-1}(y)$ is compact for all $y \in Y$. Let $a_1, a_2 \in Y$, the since their pre images a disjoint compact sets, they can be seperated into disjoint open sets U and V. Then let $A = Y \setminus P(X \setminus U) \subset P(U)$. Note that A is open since U is open, $X \setminus U$ is closed and then its image is closed becasue p is a closed map, hence the complement in Y is open. We have $a_1 \in A$ and $A \cap P(V) = \varnothing$, since $p^{-1}(A) \subset U$ is disjoint from V. Then letting $A_2 = Y \setminus p(X \setminus V)$ gives a similiar open set about a_2, then we have that Y is Hausdorff.

□

(b) Same but for regularity

Proof. Assume that X is regular and let $a \in Y$. We show that every nbhd of a, U has a open V such that $\overline{V} \subset U$. Note that $p^{-1}(U)$ is open and contains the compact set $P^{-1}(a)$. For each $x \in p^{-1}(a)$, by regularity there exists a nbhd V_x such that $x \in V_x$ and $\overline{V_x} \subset p^{-1}(U)$. Then These V_x's form an open cover of $p^{-1}(a)$ and hence there exists a finite subcover, $V = \bigcup_{i=1}^n V_{x_i}$. Since the finite union of closed sets are closed we also have $\bigcup_{i=1}^n \overline{V}$, which is a closed set containd in the pre image of U, then its image is a closed set contained in U. and the set $Y \setminus P(X \setminus V) \subset p(V)$ is an open set containg a whose closure is in U.

□

(c) local compactness

Proof. Let X be locally compact and let $a \in Y$, then $p^{-1}(a)$ is compact and for all $x \in p^{-1}(a)$ there exists a compact C_x and an open U_x such that $U_x \subset C_x$. Then the U_x's form an open cover and hence a finite number of them must cover $p^{-1}(a)$. Then let $U = \bigcup_{i=1}^{n} U_i$ and $C = \bigcup_{i=1}^{n} C_i$, then $p(C)$ is compact since it is the image of a compact set and taking $A = Y \setminus p(X \setminus U)$ gives an open nbhd of a.

Thus Y is locally compact. □

(d) countable basis.

Proof. As in the given hint, let $\mathfrak{B}$ be a basis, and given a finite subset of $\mathfrak{B}$, J, let U_J be the union of all $p^{-1}(W)$ for W open in Y such that $p^{-1}(W) \subset \cup J$. Then we show that $p(U_J)$ is a basis for Y. Clearly the collection of $p(U_J)$ is countable since, since there are countably many finite subsets J of $\mathfrak{B}$. Now Let $V \subset Y$ be open. Then consider a open covering of $p^{-1}(V)$ by basis elements in $\mathfrak{B}$, for each $p^{-1}(a) \subset p^{-1}(V)$ we know that there exists a finite subcollection covering the compact set $p^{-1}(a)$, then unioning the finite subcover is a finite union of elements in $\mathfrak{B}$, call it B. Let U be the corrosponding open set consisting of the union of all $p^{-1}(W)$ where W is open and $p^{-1}(W)$ is contained in B. Then $p(U) \subset V$ since it is the union of open sets whose pre images lie in B and $B \subset p^{-1}(V)$. Hence by reapeating this process for each compact $p^{-1}(y)$ we can write V as the union of such open sets of the form $p(U)$, so that V is open in the topology generated by elements of the desired form. □

Question 5.

Topological groups.

Proof. First we prove normality, so assume that X is normal, Then by the given hint, we know that p is closed continuous and surjective. Let A_1, A_2 be disjoint closed sets in the quotient space X/G. Then since p is a continuous function, $p^{-1}(A_1), p^{-1}(A_2)$ are both closed and disjoint. By normality of X, they can be seperated by disjoint open sets U_1 and U_2 respectivily. Then using a similiar trick as above we define

$$V_1 = Y \setminus p(X \setminus U_1)$$

and

$$V_2 = Y \setminus p(X \setminus U_2)$$

Note that V_1 is open since U_1 is open, its complement is closed, then since p is a closed map the image of $X \setminus U_1$ is closed and hence its complement V_1 is open. We also have $A_1 \subset V_1$ Since $p^{-1}(A_1) \subset U_1$, equivalent statments hold for V_2, and since $p^{-1}(V_1) \subset U_1$ and $p^{-1}(V_2) \subset U_2$, V_1 and V_2 are disjoint. Hence X/G is normal.

Now to do the other ones we let $\overline{x} \in X/G$, then $p^{-1}(\overline{x}) = \alpha(G, x)$. Then since G is compact and α continuous, the image $\alpha(G, x)$ for fixed x is compact. That is the pre image of a fiber is compact. Also by the hint given we have that p is closed continuous and surjective. Thus it follows p is a perfect map and the above results in the previous question provide the proof. $\square$