MTH 316 Notes

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Chapter 1

Cosets and Lagrange's theorem

1.1 Cosets

Definition 1.1.1 (Cosets)
Let H be a subgroup of G then for each $a \in G$ we define the left coset of H as

$$aH = \{ah \in G | h \in H\}$$

We will find this definition usefull in our study of groups. First we prove basic propertys of cosets.

Theorem 1.1.2 (Properties of Cosets)

1. $aH = H$ *iff* $a \in H$.
2. $aH = bH$ *iff* $a \in bH$
3. $o(aH) = o(H)$.
4. $aH = bH$ *or* $aH \cap bH = \varnothing$.

Proof. (1) Assume $aH = H$, then since $e \in H$, $ae = a \in aH$ thus $a \in H$. Conversly assume that $a \in H$, then for $ah \in aH$ we have by closure $ah \in H$. Simillarly if $h \in H$ then $ha^{-1} \in H$ and so $h \in aH$. For (2) The proof is similar. For (3) we want to find a bijection from aH to H. Define $\phi : H \to aH$ as $\phi(h) \mapsto ah$. Then one may easily show this is a bijection. (4) Assume $aH \neq bH$, then for the sake of contradiction assume that $aH \cap bH \neq \varnothing$. We fix x in the intersection and note that $x = ah_1$ and $x = bh_2$. Then $b = ah_1h_2^{-1} \in aH$, but then by (2) $aH = bH$; a contradiction. □

Note that the last two parts of the theorem show that the cosets of a subgroup H partition the group G, later we will find out when it makes sense to define an operation on these equivalence classes.

1.2 Lagrange's Theorem

This is often called the A, B, Cs in finite group theory and one of the biggest results in MTH 316.

Theorem 1.2.1 (Lagrange)
Let $H \leq G$ then $o(H)|o(G)$.

Proof. Let $H \leq G$. Using the result from the last section we already know that the cosets of H partition G. Now consider the set of left cosets of H in G.

$$\{g_1 H, g_2 H, \dots \, g_r H\}$$

Since $a \in aH$ we know that each element of G is in atleast one coset, then since the cosets are disjoint,

$$ro(H) = o(G)$$

□

Theorem 1.2.2 (Orbit stabilizer)
Let G be a group acting on a set A. Then define for all $a \in A$ the set $\operatorname{stab}(a) = \{g \in G | g \cdot a = a\}$ and $orb(a) = \{b \in A | g \in G \text{ s.t. } g \cdot a = b\}$. Then $\operatorname{stab}(a)$ is a normal subgroup and $|G| = |\operatorname{stab}(a)||orb(a)|$.

Proof. Consider the map $\phi : G \backslash \operatorname{stab}(a) \to orb(a)$ wehre $g \operatorname{stab}(a) \mapsto g \cdot a$. So if $g \operatorname{stab}(a) = g' \operatorname{stab}(a)$. then $g^{-1}g' \in \operatorname{stab}(a)$ and hence $(g^{-1}g') \cdot a = a$ which implies $g' \cdot a = g^{-1} \cdot a$ and so our mapping is well defined. Now we show that ϕ is a isomorphism.

□