MTH 513 · LINEAR ALGEBRA

Problem Set 1

POSTED on Friday, 9 September, 2022.

DUE by 11:59pm on Sunday, 18 September 2022, via Brightspace.

SUBMISSION GUIDELINES / INSTRUCTIONS:

- Review *general submission guidelines* before submitting your assignment, in particular how to create a single pdf document from multiple handwritten pages, page numbering, problem statements, etc.
- Make this "cover page" the first page in your submitted pdf file.
- When you are done with your work, rename the document as specified below and submit it via Brightspace.

YOURLASTNAME–hw1–mth–513

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Name: ______________________________

1. Let $X, Y \in \mathbb{R}^{n\times n}$ be arbitrary and assume that a good-hearted oracle proved for you the fact that $(X \cdot Y)^T = Y^T \cdot X^T$. Use this fact to **prove** that if $A_i \in \mathbb{R}^{n\times n}$ for $i = 1, \ldots, k$, then for $k \geq 2$ the following equality holds

$$\Big(A_1 A_2 \cdots A_{k-1} A_k\Big)^T \;=\; A_k^T A_{k-1}^T \cdots A_2^T A_1^T \,. \tag{1}$$

Remark: This is essentially just asking you to set up *proof by induction* correctly. The "harder" part of this problem would be proving the base case, which you may assume is true for the purpose of this problem right now.

2.

> Recall that the set of complex numbers is defined as
>
> $$\mathbb{C} \;:=\; \big\{a + b\mathtt{i} \,\big|\, a, b \in \mathbb{R},\, \mathtt{i}^2 = -1\big\}\,. \tag{2}$$
>
> Moreover, the *addition* and the *multiplication*, $+ : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$, are defined as
>
> $$\big(a + b\mathtt{i}\big) + \big(c + d\mathtt{i}\big) \;:=\; \big(a + c\big) + \big(b + d\big)\mathtt{i}\,, \tag{3}$$
>
> $$\big(a + b\mathtt{i}\big) \cdot \big(c + d\mathtt{i}\big) \;:=\; \big(ac - bd\big) + \big(bc + ad\big)\mathtt{i}\,, \tag{4}$$
>
> for all complex numbers $a + b\mathtt{i}$ and $c + d\mathtt{i}$. Clearly, the set of complex numbers whose imaginary part is zero represents the set of real numbers, that is, $\mathbb{R}$ is a proper subset of $\mathbb{C}$.

Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 real matrices and consider function $\phi : \mathbb{C} \to \mathbb{R}^{2\times 2}$ given by

$$\phi(a + b\mathtt{i}) \;:=\; \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad \text{for all } \; a + b\mathtt{i} \in \mathbb{C}\,. \tag{5}$$

(a) Show that ϕ is an injective (or one-to-one) function.

(b) Describe the range/image of ϕ, that is, describe the set $\phi(\mathbb{C}) = \{\phi(z) \,|\, z \in \mathbb{C}\}$.

(c) Prove or disprove: $\phi(z_1 + z_2) \;=\; \phi(z_1) + \phi(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

(d) Prove or disprove: $\phi(z_1 \cdot z_2) \;=\; \phi(z_1)\cdot\phi(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

3. The "same" way the set of real numbers was extended into the set of complex numbers, one can also continue this process and look for an extension of complex numbers. To that end, let us consider the set

$$\mathbb{H} \;:=\; \big\{a + b\mathtt{i} + c\mathtt{j} + d\mathtt{k} \,\big|\, a, b, c, d \in \mathbb{R}\big\}\,, \tag{6}$$

where the following multiplication conditions are imposed:

(i) $\mathtt{i}^2 = \mathtt{j}^2 = \mathtt{k}^2 = -1$,

(ii) $\mathtt{ij} = \mathtt{k},\;\; \mathtt{ji} = -\mathtt{k},\;\; \mathtt{jk} = \mathtt{i},\;\; \mathtt{kj} = -\mathtt{i},\;\; \mathtt{ki} = \mathtt{j},\;\; \mathtt{ik} = -\mathtt{j}$,

(iii) every $a \in \mathbb{R}$ commutes with $\mathtt{i}, \mathtt{j}, \mathtt{k}$.

Now the addition of two elements in $\mathbb{H}$ can be defined analogous to complex numbers where we simply add the corresponding parts. The multiplication is slightly more difficult, but it can be completely defined given the conditions (i)-(iii) from above along with the distributive law.

Observation: $\mathbb{H}$ is *NOT* a field since multiplication is not commutative, for example, condition (ii) gives us $\mathtt{ij} = \mathtt{k} \neq -\mathtt{k} = \mathtt{ji}$. However, except the commutativity of multiplication $\mathbb{H}$ does satisfy all other conditions to be a field.

Example: Multiply $(\mathtt{i}+\mathtt{j})(\mathtt{i}-\mathtt{j})$ assuming the distribute law and the conditions (i)-(iii).

$$(\mathtt{i}+\mathtt{j})(\mathtt{i}-\mathtt{j}) \;=\; \mathtt{i}^2 - \mathtt{ij} + \mathtt{ji} - \mathtt{j}^2 \;=\; -1 - \mathtt{k} - \mathtt{k} - (-1) \;=\; -2\mathtt{k}\,,$$

while $\mathtt{i}^2 - \mathtt{j}^2 = -1 - (-1) = 0$.

(a) Let $q = a + b\mathtt{i} + c\mathtt{j} + d\mathtt{k}$ and $w = e + f\mathtt{i} + g\mathtt{j} + h\mathtt{k}$ be two arbitrary elements of $\mathbb{H}$. Write the product $q\,w$ in the form $z_0 + z_1\mathtt{i} + z_2\mathtt{j} + z_3\mathtt{k}$, where $z_1, z_2, z_3, z_4 \in \mathbb{R}$.

(b) Find $A \in \mathbb{R}^{4\times 4}$ and $\mathbf{b} \in \mathbb{R}^4$ (both obviously related to q and/or w) such that

$$A\,\mathbf{b} \;=\; \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}.$$

Remark: Part (b) may have to wait until the second homework, but we can discuss it in class on Tuesday.

4. Consider the following system of equations over the finite field $\mathbb{Z}_3$, that is, all coefficients and operations are done over $\mathbb{Z}_3$,

$$\begin{array}{ccccccc} x & + & 2y & + & z & = & 1 \\ x & + & & & z & = & 1 \\ x & + & y & + & z & = & 1 \end{array} \tag{7}$$

(a) What is the reduced row echelon form of the associated augmented matrix? Write down the sequence of operations you performed to obtain the reduced row echelon form.

(b) Clearly describe the solution set and state how many different solutions are there.