MTH 435: Analysis and Topology

Evan Fox

September 16, 2022

Contents

Chapter 1

Introduction

1.1 Algebraic and Order properties

The Real numbers, denoted $\mathbb{R}$ form a field, that is $\mathbb{R}$ is an Abelian group under addition and multiplication that has distinct identites and satisfies the distributive property. All important algebraic properties can be derivied from the fact that $\mathbb{R}$ is a field.

Definition 1.1.1
Let $P \neq \varnothing$ be a subset of $\mathbb{R}$ not containg 0. We say P is the set of positive real numbers and it satisfies

1. $a, b \in P \implies a + b \in P$
2. $a, b \in P \implies ab \in P$
3. $a \in P \implies a \in P \vee -a \in P \vee a = 0$

We are now in a posistion to define the ordering we wish to place on $\mathbb{R}$

Definition 1.1.2
For $a, b \in \mathbb{R}$ such that $b - a \in P$ then we say $a < b$. If $b - a \in P \cup \{0\}$ then $a \leq b$

It follows imeditly from tricotomy that exactly one of $a < b, a = b, a > b$ must hold. It is also clear that this is a total ordering on $\mathbb{R}$, but we must check that this turns $\mathbb{R}$ into an ordered field.

Lemma 1.1.3
The ordering defined above is a strict total order.

Proof. The order is irreflexive since $a - a = 0 \notin P$, then if $a - b \in P$ and $b - a \in P$ we can add to get $0 \in P$ a contradiction. Now we must prove the transitive property, suppose $a, b, b \in \mathbb{R}$ satisfy $a < b$ and $b < c$, then $b - a, c - b \in P$ thus so is their sum, $c - a \in P$ which implies $a < c$ as desired. $\square$

Lemma 1.1.4
The ordering defined above turns $\mathbb{R}$ into an ordered field, let $a, b, c \in \mathbb{R}$.

1. *if $a < b$ then $a + c < a + b$*
2. *if $c > 0$ and $a < b$ then $ac < bc$*
3. *if $c < 0$ and $a < b$ then $ac > bc$*

Proof. For the first item, we have $b - a > 0$ then by adding and subtracting c we get

$$0 < b - a - c + c$$

$$a + c < b + c$$

Next assume $c > 0$, then we have $0 < b - a$, since both of theses are positive, so is their product,

$$0 < c(b - a) \implies 0 < cb - ca \implies ca < cb.$$

Now let $c < 0$, then $-c > 0$ and the argument is the same as above. $\square$

Thus we have turned $\mathbb{R}$ into an ordered field.

Theorem 1.1.5
The natural numbers are all positive, we will prove this in the following steps.

1. *If $a \in \mathbb{R}$ with $a \neq 0$ then $a^2 > 0$*
2. $1 > 0$
3. $\mathbb{N} \subset P$

Proof. If $a > 0$ we are done, suppose $a < 0$, then $-a \in P$ and since $a^2 = (-a)(-a) \in P$ we have $a^2 \in P$. Now note $1^2 = 1$ so $1 \in P$, then since we definied the natural number n as $1 + \cdots + 1, n$ times, we see that all natural numbers are positive. $\square$

Theorem 1.1.6
If $a \in \mathbb{R}$ satisfies $0 \leq a < \epsilon$ for all $\epsilon > 0$, then $a = 0$

Proof. Suppose $a > 0$, then let $\epsilon_0 = a/2$, then

$$0 < \epsilon_0 < a < \epsilon$$

a contradiction. $\square$

Theorem 1.1.7
If $ab > 0$ then a, b are both posistive or both negitive.

Proof. Suppose $ab > 0$ and that at least one is negative, without loss of generality say $a < 0$, then if $b > 0$, we have $-ab > 0$ so $-(ab) \in P$ but we assumed $ab \in P$; a contradiction, thus b is negitive. $\square$

1.2 Absoulte Value

Next we define a function of great importance on $\mathbb{R}$.

$$|a| = \left\{ \begin{array}{ll} a, & \textit{if } a > 0 \\ 0, & \textit{if } a = 0 \\ -a, & \textit{if } a < 0 \end{array} \right\} \tag{1.1}$$

Now we prove some basic properties of the absoulte value function,

Theorem 1.2.1
Basic properties of the absoutle value.

1. $|ab| = |a||b|$
2. $|a|^2 = a^2$
3. *If* $c > 0$ *then* $|a| \leq c \Leftrightarrow -c \leq a \leq c$
4. $-|a| \leq a \leq |a|$

Proof. To prove (1) first note that if a or b is zero we are done. Then we just consider the four possible cases on the signs of a and b. For example if $a > 0$ and $b < 0$, we have $|ab| = -ab$ and $|a| = a, |b| = -b$ so $|a||b| = -ab$. The rest are left as an exercise. Proving (2) is simillar, if $a > 0$ we are done and if $a < 0$, then $|a|^2 = (-a)^2 = a^2$. Now suppose $c > 0$ and $|a| \leq c$, if $a \leq 0$, then the result is clear. If $a < 0$ we have $|a| = -a$, so $-a < c$, rearranging gives $-c < a$. So we are finished. Now suppose $-c \leq a \leq c$, then $-a \leq c$ and $a \leq c$. But the absoute value maps to a or $-a$ so in either case we are done. Now for (4) we let $c = |a|$ and apply (3) to get the result. $\square$

Theorem 1.2.2 (Triangle Inequality)
For all $a, b \in \mathbb{R}$

$$|a+b| \;\leq\; |a| + |b| \tag{1.2}$$

Proof. By the above, we have that

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

implies

$$|a+b| \leq |a| + |b|$$

as desired. □

The Triangle inequality is very important and equality hold only when a, b have the same sign.

Theorem 1.2.3
The following two inequalitys hold for all $a, b \in \mathbb{R}$

1. $|a-b| \leq |a| + |b|$
2. $||a| - |b|| \leq |a-b|$

Proof. Proof of (1) follows from subing $-b$ into the triangle inequality. To prove (2) start my applying the triangle inequality to $a = a - b + b$ to get $|a| \leq |a-b| + |b|$, and $b = b - a + a$ to get $|b| \leq |b-a| + |a|$, then subtracting gives

$$|a| - |b| \leq |a-b|$$

and

$$|b| - |a| \leq |a-b|.$$

We may multiply by -1 to get

$$-|b| + |a| \geq -|a-b|$$

and it follows,

$$-|a-b| \leq |a| - |b| \leq |a-b|.$$

Now we let $|a-b| = c$ and use the third result from theorem 1.2.1, to get

$$|a| - |b| \leq |a-b|$$

□

1.3 Archimeadian Property and Completness

The completness axiom is the last thing that we need in order to call are a complete ordered field.

Definition 1.3.1
A subset $A \subset \mathbb{R}$ is bounded is above if there exists $u \in \mathbb{R}$ such that for all $a \in A$ we have $a \leq u$, we say A is bounded below if the other inequality holds. We call u an upper bound or lower bound for the set A.

We say a set is bounded if it is bounded above and below.

Definition 1.3.2
We say that an upper bound $\alpha \in \mathbb{R}$ is a least upper bound if

1. *α is an upper bound.*
2. *For an arbitrary upper bound u, we have $\alpha \leq u$.*

Definition 1.3.3
Every subset $A \subset \mathbb{R}$ that is bounded above has a least upper bound.

We will see that this property is very important.

Theorem 1.3.4
Let $A \subset \mathbb{R}$, an upper bound $\alpha \in \mathbb{R}$ satisfies $\alpha = \sup(A)$ if and only if for all $\epsilon > 0$ there exists $a \in A$ such that $\alpha - \epsilon < a$

Proof. Let $\alpha = \sup(A)$ then for all $\epsilon > 0$, $\alpha - \epsilon < \alpha$ so it cannot be an upper bound. Conversly, let α be an upper bound with the desired property and let u be an upper bound, then if $u < \alpha$ we have $u = \alpha - \epsilon$ for some $\epsilon > 0$, but then by assumtion there is an $a \in A$ such that $u < a$; a contradiction. $\square$

Lemma 1.3.5
$\mathbb{N}$ is not bounded above in $\mathbb{R}$

Proof. Assume that $\mathbb{N}$ is bounded, then there exists a least upper bound, let $\alpha = \sup(A)$, then there exists $n \in \mathbb{N}$ such that $\alpha - 1 < n$, but then $\alpha < n + 1$; a contradiction. $\square$

Theorem 1.3.6
For all $\epsilon \in \mathbb{R}_{>0}$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. By the unboundedness of n, we may choose $n \in \mathbb{N}$ such that $\frac{1}{e} < n$, then $\frac{1}{n} < \epsilon$. $\square$

Now we may look at the algebraic properties of sup

Theorem 1.3.7
let S, A, B be sets and $r \in \mathbb{R}$

1. $\sup(r+S) = r + \sup(S)$
2. $\sup(A+B) = \sup(A) + \sup(B)$
3. *if $a \leq b$ for all $a \in A, b \in B$ then* $\sup(A) \leq \inf(B)$.

Proof. exercise. See HW1 $\square$

Definition 1.3.8
A function $f : D \to \mathbb{R}$ is bounded in $\mathbb{R}$ is there exists $M \in \mathbb{R}$ such that

$$-M \leq f(x) \leq M \, \forall x \in D$$

Lemma 1.3.9
if f, g are bounded functions such that $f(x) \leq g(x)$ then $\sup(f(x)) \leq \sup(g(x))$, *but* $\sup(f(x)) \nleq \inf(g(x))$ *in general.*

Proof. The first part is easy. To see that the second statement is false consider $f : [0,1] \to \mathbb{R}$ as $f(x) = x^2$ and $g : [0,1] \to \mathbb{R}$ as $f(x) = x$. Then $g(x) \geq f(x)$ but $\sup(f(x)) = 1 > 0 = \inf(g(x))$. $\square$

Theorem 1.3.10
For $x, y \in \mathbb{R}$ with $x < y$ there exists $m, n \in \mathbb{Z}$ such that

$$x < \frac{m}{n} < y$$

That is $\mathbb{Q}$ is dense *in $\mathbb{R}$.*

Proof. Let $x, y \in \mathbb{R}$ with x, y. By the archimeadian property fix $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. This gives

$$0 < 1 + nx < y$$

Now fix $m \in \mathbb{Z}$ such that $xn < m \leq xn + 1$. Then from $eq(1)$ it follows

$$nx < x < nx + 1 < ny \implies nx < m < ny \implies x < \frac{m}{n} < y.$$

$\square$