Analysis Chapter 1

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Question 1.

Show that there doesn't exist a rational number s such that $s^2 = 6$.

Proof. First we prove that $6|a^2 \implies 6|a$. Suppose that 6 does not devide a and let $a = p_1p_2 \dots p_n$ be the prime factorization of 6, then we know that 3 and 2 cannot both be common factors, But then since 2 and 3 are prime they cannot be the square of a prime number, hence 2 and 3 cannot both appear in $a^2 = p_1^2p_2^2 \dots p_n^2$, hence 6 does not devide a^2. It now follows by contrapositive that if $6|a^2$ then $6|a$.

Now assume there exists $a, b \in \mathbb{N}$ such that

$$6 = (\frac{a}{b})^2$$

If a and b have any common factors we may cancel them out, so we assume $(a, b) = 1$. Then $6b^2 = a^2$ implies $6|a$ so we may fix $m \in \mathbb{Z}$ such that $6m = a$. Then $6b^2 = (6m)^2$ implies $b^2 = 6m^2$ and $6|b$; thus a and b share a common factor, a contradiction.

□

Question 2.

If $a, b \in \mathbb{R}$ show that $|a + b| = |a| + |b|$ iff $ab \geq 0$.

Proof. For the forward direction we use contrapositive, assume $ab < 0$ then without loss of generality assume $a > 0$ and $b < 0$, we show that

$$|a + b| \neq |a| + |b|$$

we have by our assumtions on a and b that $|a| + |b| = a - b$. Now there are three cases (by tricotomy) for what $|a + b|$ can map to, if $|a + b| = 0$ we are done since $|a| + |b| > 0$. If $a + b$ is poisitve $|a + b| = a + b$, but then

$a+b=a-b$ would imply $b=0$, contradicting our assumtion on b. If $a+b$ is negative $|a+b|=-a-b$, but now $-a-b=a-b$ would force $a=0$, contradicting our assumtion on a. Hence, in every case equality does not hold. This proves the first direction.

Now assume $ab \geq 0$, if either is equal to zero we are done. If they are both posisitive it follows since $a+b$ will be posistive giving

$$|a+b| = a+b = |a|+|b|.$$

Then if they are both negative, their sum will be negitive so,

$$|a+b| = -a-b = |a|+|b|$$

completing the opposite direction. □

Question 3.

Find all $x \in \mathbb{R}$ that satisfy the inequality

$$4 < |x+2| + |x-1| < 5.$$

Proof. The terms $x+2$, $x-1$ have different signs only if $x \in (-2,1)$ but it is clear no such x is a solution, so assume $x \notin (-2,1)$, then the terms have the same sign and by the above we may add them to get

$$4 < |2x+1| < 5$$

which we may split into $4 < |2x+1|$ and $|2x+1| < 5$. For the first case we have

$$4 < |2x+1| \implies 4 < 2x+1 \vee 2x+1 < -4$$

which gives $3/2 < x$ and $x < -5/2$; written in interval notation as $(-\infty,-5/2)\cup(3/2,\infty)$. Now for

$$|2x+1| < 5 \implies -5 < 2x+1 < 5 \implies -6/2 < x < 4 = (-6/2, 4)$$

We need both conditions to be satisfied so we must take the intersection over our two solutions,

$$(-\infty,-5/2)\cup(3/2,\infty)\cap(-6/2,4) = (-6/2,-5/2)\cup(3/2,4)$$

□

Question 4.

place holder

(a) *Proof.* Assume without loss of generality that $b < a$, then $|a-b| = a-b$. Thus

$$\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = \frac{1}{2}(2a) = a$$

We also have

$$\frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = \frac{1}{2}(2b) = b$$

and we are done. □

(b) *Proof.* The minimum of a, b, c must be one of them, the minimum of a, b must be a or b, so

□

Question 5.

Let $S_4 = \{1 - (-1)^n/n : n \in \mathbb{N}\}$ find inf and sup

Proof. $\inf S_4 = \frac{1}{2}$, $\frac{1}{2}$ is an element of S_4 that occurs for $n = 2$, for $n \neq 2$, if n is odd we will be adding 1 which gives $\frac{1}{2} < 1 + 1/n$. If $n > 2$ is even we have $1/n < 1/2$ which implies $1 - 1/2 < 1 - 1/n$, so $1/2$ is the minimal element. It follows that if a set contains a minimal element it is the infium since we have $1/2 < x$ for all $x \in S_4$ it is a lower bound and since it is an element of S_4, given any lower bound l, we must have $l \leq 1/2$.

$\sup S_4 = 2$, I proceed with a very simillar argument as above, 2 appears in S_4 when $n = 1$, for any $n > 1$ either we are subtracting from 1, or adding a number smaller than 1 to 1, in either case we get somthing less than 2. Thus 2 is the maximal element of the set and hence it must be the supremum. □

Question 6.

Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B = \{a + b : a \in A, b\ inB\}$. Prove that $\sup(A + B) = \sup(a) + \sup(B)$

Proof. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. Then for all a, b we have $a \leq \alpha$ and $b \leq B$, thus

$$a + b \leq \alpha + \beta, \forall a \in A, b \in B.$$

So $\alpha + \beta$ is an upperbound. But then for $\epsilon > 0$ there exists $b_0 \in B$ such that $\beta - \frac{1}{2}\epsilon < b_0$ and $a_0 \in A$ such that $a - \frac{1}{2}\epsilon < a_0$, so $\alpha + \beta - \epsilon < a_0 + b_0$, Thus $\alpha + \beta = \sup(A + B)$. A very simillar argument works for infinium. Let $a = \inf(A)$ and $b = \inf(B)$. Then $a + b$ is a lower bound for $A + B$ for the same reasons stated above. But then given $\epsilon > 0$ I can find elements a_0, b_0 in A and B respectivly such that $a + b + \epsilon > a_0 + b_0$, so $a + b = \inf(A + B)$. □

Question 7.