MTH 513: Chapter 7, Part 1

Last updated: December 1, 2022

§7.1 – §7.2 Eigenvalues and Eigenvectors

Definition 1. *A nonzero vector $\mathbf{x} \in \mathbb{C}^n$ is a* **right eigenvector** *of $A \in \mathbb{C}^{n\times n}$ if there exists a scalar $\lambda \in \mathbb{C}$, called an* **eigenvalue***, such that*

$$A\mathbf{x} = \lambda \mathbf{x}\,.$$

Similarly, a nonzero vector $\mathbf{y} \in \mathbb{C}^n$ is a **left eigenvector** *corresponding to an eigenvalue μ if*

$$\mathbf{y}^* A = \mu\,\mathbf{y}^*\,.$$

The set of distinct *eigenvalues of A, denoted $\Lambda(A)$, is called the* **spectrum** *of A.*

$$\begin{aligned}
(\lambda, \mathbf{x}) \text{ is an eigenpair of } A \quad &\iff \quad A\mathbf{x} = \lambda\mathbf{x}\,, \qquad \mathbf{x} \neq \mathbf{0} \\
&\iff \quad (A - \lambda I)\mathbf{x} = \mathbf{0}\,, \qquad \mathbf{x} \neq \mathbf{0} \\
&\iff \quad (A - \lambda I) \text{ is singular (not invertible)} \\
&\iff \quad \det(A - \lambda I) = 0
\end{aligned}$$

(i) $\{\mathbf{x} \neq \mathbf{0}_n : \ \mathbf{x} \in \mathcal{N}(A - \lambda I)\}$ is the set of all eigenvectors associated with λ. From now on, we refer to the subspace $\mathcal{N}(A - \lambda I)$ as the **eigenspace** of A associated with eigenvalue λ and denote it by $E_\lambda(A)$ or E_λ.

(ii) Let λ is an eigenvalue of A and $\mathbf{x}$ is an associated eigenvector, then $(\lambda, \mathbf{x})$ is referred to as an **eigenpair** of A.

(iii) The **characteristic polynomial** for A is $\pi_A(\lambda) = \det(A - \lambda I)$. The degree of $\pi_A(\lambda)$ is n, and the leading term in $\pi(\lambda)$ is $(-1)^n\lambda^n$.

(iv) The **characteristic equation** for A is $\pi(\lambda) = 0$.

(v) The eigenvalues of A are the solutions of the characteristic equation, or equivalently, the roots of the characteristic polynomial.

(vi) Altogether, A has n eigenvalues, but some may be complex numbers (even if the entries of A are real numbers), and some eigenvalues may be repeated.

(vii) If A contains only real numbers, then its complex eigenvalues must occur in conjugate pairs – i.e., if $\lambda \in \Lambda(A)$, then $\overline{\lambda} \in \Lambda(A)$.

Discussion: If $A \in \mathbb{R}^{n\times n}$, then $\pi_A(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n with real coefficients. Consequently, if λ_0 is a root of $\pi_A(\lambda)$, then $\overline{\lambda_0}$ is also a root of $\pi_A(\lambda)$.[1] In other words, if $A \in \mathbb{R}^{n\times n}$, then complex eigenvalues come in conjugate pairs.

Example 2. *Let $D = \operatorname{diag}(d_1, \dots, d_n)$ be an $n \times n$ diagonal matrix. Determine its eigenvalues and associated eigenvectors.*

Solution: *We start by finding the characteristic polynomial of D which is given by*

$$\pi_D(\lambda) \;=\; \det(D - \lambda I) \;=\; \det \begin{bmatrix} d_1 - \lambda & & & \\ & d_2 - \lambda & & \\ & & \ddots & \\ & & & d_n - \lambda \end{bmatrix} = \prod_{i=1}^{n} (d_i - \lambda)\,. \tag{1}$$

From (1) it follows that $\pi_D(\lambda) = 0$ for all $\lambda = d_i$, i.e., eigenvalues of D are just diagonal entires d_i.

Now in order to determine an e-vector associated with an e-value $\lambda = d_i$ we look at $\mathcal{N}(D - d_i I)$.

$$\left[\begin{array}{c|c} D - d_i I & \mathbf{0} \end{array}\right] \;=\; \left[\begin{array}{ccccccc|c} d_1 - d_i & & & & & & & 0 \\ & \ddots & & & & & & \vdots \\ & & d_{i-1} - d_i & & & & & 0 \\ & & & 0 & & & & 0 \\ & & & & d_{i+1} - d_i & & & 0 \\ & & & & & \ddots & & \vdots \\ & & & & & & d_n - d_i & 0 \end{array}\right] \tag{2}$$

From (2) it is easy to see that for $\mathbf{x} = \mathbf{e}_i$ we have $(D - d_i I)\mathbf{x} \;=\; (D - d_i I)\mathbf{e}_i \;=\; \mathbf{0}$, that is, $\mathbf{x} = \mathbf{e}_i$ is an eigenvector of D associated with an eigenvalue $\lambda = d_i$.

Discussion: *Note that here we are not claiming that $\mathbf{x} = \mathbf{e}_i$, or just a scalar multiple of $\mathbf{e}_i$, is the only eigenvector of D associated with eigenvalue $\lambda = d_i$. This claim would be true if $d_i \neq d_j$ for all $i \neq j$, otherwise, there exist eigenvectors of D other than scalar multiples of $\mathbf{e}_i$.*

[1] This is a fact that you have probably encountered the first time in a high school algebra/precalculus course. A more rigorous result from complex analysis is often referred as “Complex Factor Theorem” of “Conjugate Pairs Theorem’.

Example 3. *Let T be an $n \times n$ upper-triangular matrix. Determine the eigenvalues of T.*

Solution: *Again, one starts with finding the characteristic polynomial of T, that is,*

$$\pi_T(\lambda) \;=\; \det(T-\lambda I) \;=\; \det\begin{bmatrix} t_{1,1}-\lambda & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ 0 & t_{2,2}-\lambda & t_{2,3} & \cdots & t_{2,n} \\ 0 & 0 & t_{3,3}-\lambda & \cdots & t_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & t_{n,n}-\lambda \end{bmatrix} = \prod_{i=1}^{n} (t_{ii}-\lambda)\,. \tag{3}$$

From (3) it follows that $\pi_T(\lambda) = 0$ when $\lambda = t_{ii}$, that is, eigenvalues of T are exactly the diagonal entries of T.

Theorem 4. *If T is a triangular matrix, lower or upper, then the eigenvalues of T are exactly the diagonal entries of T.*

Example 5. *Let A and B be two square matrices non necessarily of the same size. Show that the characteristic polynomial of $C = \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right]$ is the product of characteristic polynomials of A and B.*

Solution: *Suppose $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ so that $C \in \mathbb{C}^{(m+n)\times(m+n)}$. Then the characteristic polynomial of C is given by*

$$\begin{aligned} \pi_C(\lambda) \;=\; \det(C-\lambda I_{n+m}) \;&=\; \det\left(\left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right] - \lambda \left[\begin{array}{c|c} I_n & \\ \hline & I_m \end{array}\right]\right) \\ &=\; \det\left[\begin{array}{c|c} A-\lambda I_n & \\ \hline & B-\lambda_m \end{array}\right] \\ &=\; \det(A-\lambda I_n)\cdot\det(A-\lambda I_m) \\ &=\; \pi_A(\lambda)\cdot\pi_B(\lambda)\,. \end{aligned}$$

Theorem 6. *Let $D = \operatorname{diag}(A_1, A_2, \ldots, A_k)$ be a block-diagonal matrix with square block A_i, $i = 1, 2, \ldots, k$.*

(i) $\pi_D(\lambda) \;=\; \displaystyle\prod_{i=1}^{k} \pi_{A_i}(\lambda)\,.$

(ii) $\Lambda(D) \;=\; \displaystyle\bigcup_{i=1}^{k} \Lambda(A_i)\,.$ *(eigenvalues of D is just a union of eigenvalues of A_i)*

Example 7. *Let $J_k(\lambda_0)$ be a $k \times k$ matrix of the form*

$$J_k(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ & \lambda_0 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & & & \lambda_0 & 1 \\ & & & & \lambda_0 \end{bmatrix}_{k\times k}.$$

(a) What are the eigenvalues of $J_4(13)$?

Solution: Since $J_4(13)$ is an upper triangular matrix, by Example 3 we know its eigenvalues are exactly its diagonal entries, that is, $\lambda = 13$.

(b) What is the dimension of associated eigenspace(s)?

Solution: This is exactly one of your homework problems. More specifically, $J_4(13) - 13 \cdot I_4$ is given by

$$J_4(13) - 13 \cdot I_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\operatorname{rank}(J_4(13) - 13 \cdot I_4) = 4 - 1 = 3$. From the Rank-Nullity Theorem we know that $\dim \mathcal{N}(J_4(13) - 13 \cdot I_4) = 1$ and a basis for $\mathcal{N}(J_4(13) - 13 \cdot I_4)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(c) Determine the eigenvalues and dimensions of associated eigenspaces of the matrix

$$J = \begin{bmatrix} J_k(a) & & \\ & J_\ell(b) & \\ & & J_m(c) \end{bmatrix}.$$

Solution: From Example 5 it follows that the set of eigenvalues of J is just the union of eigenvalues of diagonal blocks, which from the previous part we know are exactly $\lambda = a$, $\lambda = b$, and $\lambda = c$. Finally, from our homework assignment we know that a basis for each of the associated eigenspaces are given by

$$\text{Basis for } E_{\lambda=a} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline \mathbf{0}_{\ell\times 1} \\ \hline \mathbf{0}_{m\times 1} \end{bmatrix}_{(k+\ell+m)\times 1} \right\}$$

$$\text{Basis for } E_{\lambda=b} = \left\{ \begin{bmatrix} \mathbf{0}_{k\times 1} \\ \hline 1 \\ 0 \\ \vdots \\ 0 \\ \hline \mathbf{0}_{m\times 1} \end{bmatrix}_{(k+\ell+m)\times 1} \right\}$$

$$\text{Basis for } E_{\lambda=c} = \left\{ \begin{bmatrix} \mathbf{0}_{k\times 1} \\ \hline \mathbf{0}_{\ell\times 1} \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(k+\ell+m)\times 1} \right\}$$

Example 8. *Compute eigenvalues and associated eigenspaces of* $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution: We start by computing the characteristic polynomial of A.

$$\begin{aligned}
\pi_A(\lambda) \;=\; \det\big(A - \lambda I\big) \;&=\; \det \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{bmatrix} \\
&=\; (-5-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \\
&=\; -(5+\lambda)\big((1-\lambda)^2 - 1\big) \\
&=\; -(5+\lambda)\big((1-\lambda-1)(1-\lambda+1)\big) \\
&=\; \lambda\,(5+\lambda)(2-\lambda) \qquad (4)
\end{aligned}$$

From (4) we see that the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = -5$, and $\lambda_3 = 2$. Next we look for the associated eigenvectors.

$\boxed{\lambda_1 = 0}$ We start by looking at $\mathcal{N}(A - \lambda_1 I) = \mathcal{N}(A)$

$$(A - 0\cdot I)\mathbf{x} = \mathbf{0} \qquad \Longleftrightarrow \qquad \left[\begin{array}{ccc|c} 1-0 & 2 & 1 & 0 \\ 0 & -5-0 & 0 & 0 \\ 1 & 8 & 1-0 & 0 \end{array}\right]$$

$$\xrightarrow{\texttt{rref}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And so a basis for the associated eigenspace $E_{\lambda_1=0}$ is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\boxed{\lambda_2 = -5}$ We start by looking at $\mathcal{N}(A - \lambda_2 I) = \mathcal{N}(A + 5I)$

$$(A + 5 \cdot I)\mathbf{x} = \mathbf{0} \qquad \Longleftrightarrow \qquad \left[\begin{array}{ccc|c} 1+5 & 2 & 1 & 0 \\ 0 & -5+5 & 0 & 0 \\ 1 & 8 & 1+5 & 0 \end{array}\right]$$

$$\xrightarrow{\texttt{rref}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{23} & 0 \\ 0 & 1 & \frac{35}{46} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And so a basis for the associated eigenspace $E_{\lambda_2=-5}$ is

$$\left\{ \begin{bmatrix} 4 \\ -35 \\ 46 \end{bmatrix} \right\}$$

$\boxed{\lambda_3 = 2}$ We start by looking at $\mathcal{N}(A - \lambda_3 I) = \mathcal{N}(A - 2I)$

$$(A - 2 \cdot I)\mathbf{x} = \mathbf{0} \qquad \Longleftrightarrow \qquad \left[\begin{array}{ccc|c} 1-2 & 2 & 1 & 0 \\ 0 & -5-2 & 0 & 0 \\ 1 & 8 & 1-2 & 0 \end{array}\right]$$

$$\xrightarrow{\texttt{rref}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And so a basis for the associated eigenspace $E_{\lambda_3=2}$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example 9. *Let A be an $n \times n$ matrix.*

(a) *Prove A and A^T have exactly the same characteristic polynomials.*

Proof. The desired conclusion follows directly from the following chain of equalities

$$\begin{aligned}
\pi_{A^T}(\lambda) &= \det(A^T - \lambda I) \\
&= \det\left(A^T - \lambda I^T\right) \\
&= \det\left((A - \lambda I)^T\right) \\
&= \det(A - \lambda I) \\
&= \pi_A(\lambda)
\end{aligned}$$

$\square$

(b) *Show that if λ is an eigenvalue of A, then $\overline{\lambda}$ is an eigenvalue of A^*.*

Proof. Assume that λ is an eigenvalue of A an $\mathbf{x}$ an associated eigenvector. Then

$$A\mathbf{x} = \lambda\mathbf{x}\,. \tag{5}$$

Taking a complex conjugate of equation (5) gives

$$\mathbf{x}^* A^* = \overline{\lambda}\mathbf{x}^*\,,$$

and so $\overline{\lambda}$ is an eigenvalue of A^* with an associated *left* eigenvector $\mathbf{x}$. $\square$

(c) *Show that if $(\lambda, \mathbf{x})$ is an eigenpair of A, then $(\lambda, c \cdot \mathbf{x})$ is an eigenpair of A, where c is an arbitrary* nonzero *constant.*

Proof. Let $(\lambda, \mathbf{x})$ be an eigenpair of A so that

$$A\mathbf{x} = \lambda\mathbf{x}\,. \tag{6}$$

Multiplying equation (6) by a nonzero scalar c we obtain

$$A(c \cdot \mathbf{x}) = \lambda(c \cdot \mathbf{x})\,. \tag{7}$$

Since $\mathbf{x}$ is an eigenvector by assumption it is nonzero. Further, since $c \neq 0$, it follows that $c\mathbf{x} \neq \mathbf{0}$. Therefore, it follows from (7) that $(\lambda, c \cdot \mathbf{x})$ is an eigenpair of A.

$\square$

Special Case: *If $\mathbf{x}$ is an eigenvector of A associated with an eigenvalue λ, then $\pm\dfrac{1}{||\mathbf{x}||_2}\mathbf{x}$ is also an eigenvector of A associated with λ. In other words, with any eigenvalue there are at least two eigenvectors of unit length (norm equals to one).*

(d) *Show that if* $(\lambda, \mathbf{x})$ *is an eigenpair of* A*, then* $(\lambda + \varepsilon, \mathbf{x})$ *is an eigenpair of* $A + \varepsilon I_n$*, where* ε *is an arbitrary constant.*

Proof. Assume that $(\lambda, \mathbf{x})$ is an eigenpair of A so that

$$A\mathbf{x} = \lambda\mathbf{x}\,. \tag{8}$$

Adding $\varepsilon\mathbf{x}$ to the both sides of equation (8) gives

$$\begin{aligned} A\mathbf{x} + \varepsilon\mathbf{x} &= \lambda\mathbf{x} + \varepsilon\mathbf{x} \\ (A + \varepsilon I)\mathbf{x} &= (\lambda + \varepsilon)\mathbf{x}\,, \end{aligned}$$

and so $(\lambda + \varepsilon, \mathbf{x})$ is an eigenpair of $A + \varepsilon I$. □

(e) *Assume that* A *is a nonsingular matrix. Show that if* $(\lambda, \mathbf{x})$ *is an eigenpair of* A*, then* $(\lambda^{-1}, \mathbf{x})$ *is an eigenpair of* A^{-1}*.*

Proof. Assume that $(\lambda, \mathbf{x})$ is an eigenpair of A so that

$$A\mathbf{x} = \lambda\mathbf{x}\,. \tag{9}$$

Multiplying from the left the both sides of equation (9) by A^{-1} gives

$$\mathbf{x} = \lambda A^{-1}\mathbf{x} \tag{10}$$

Furthermore, dividing by λ both sides of (10) (why can we do this?) gives

$$\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}\,. \tag{11}$$

Therefore, from (11) it follows that $(\frac{1}{\lambda}, \mathbf{x})$ is an eigenpair of A^{-1}. □

(f) *Prove that if* $(\lambda, \mathbf{x})$ *is an eigenpair of* A*, then* $(\lambda^k, \mathbf{x})$ *is an eigenpair of* A^k*.*

Proof. (by induction) Let $(\lambda, \mathbf{x})$ be an eigenpair of A, so that,

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{12}$$

Multiplying (12) on the left by A gives

$$A^2\mathbf{x} = A \cdot \big(\lambda\mathbf{x}\big) = \lambda \cdot A\mathbf{x} = \lambda \cdot \lambda\mathbf{x} = \lambda^2\mathbf{x}\,. \tag{13}$$

Now assume that $(\lambda^{k-1}, \mathbf{x})$ is an eigenpair of A^{k-1}, so that

$$A^{k-1}\mathbf{x} = \lambda^{k-1}\mathbf{x}\,. \tag{14}$$

Multiplying (14) on the left by A gives

$$A^k\mathbf{x} = A \cdot \big(\lambda^{k-1}\mathbf{x}\big) = \lambda^{k-1} \cdot A\mathbf{x} = \lambda^{k-1} \cdot \lambda\mathbf{x} = \lambda^k\mathbf{x}\,,$$

as desired. □

MTH 513: Chapter 7, Part 2

Last updated: December 1, 2022

---

§7.1 – §7.2 Eigenvalues and Eigenvectors

Definition 10. *Two $n \times n$ matrices A and B are said to be* **similar** *whenever there exists a nonsingular matrix P such that $P^{-1}AP = B$. The product $P^{-1}AP$ is called a* **similarity transformation** *of A.*

Example 11. *Let $B = P^{-1}AP$, for some nonsingular matrix P. Prove that characteristic polynomials of A and B are the same. Conclude that the eigenvalues of A and B are the same.*

Proof. We start by computing the characteristic polynomial of B

$$\begin{aligned}
\pi_B(\lambda) &= \det\big(B - \lambda I\big) \\
&= \det\big(P^{-1}AP - \lambda I\big) \\
&= \det\Big(P^{-1} \cdot \big(A - \lambda I\big) \cdot P\Big) \\
&= \det\big(P^{-1}\big) \cdot \det\big(A - \lambda I\big) \cdot \det\big(P\big) \\
&= \det\big(A - \lambda I\big) \cdot \det\big(P^{-1}\big) \cdot \det\big(P\big) \\
&= \pi_A(\lambda) \cdot \det\big(P^{-1} \cdot P\big) \\
&= \pi_A(\lambda) \cdot \det\big(I\big) \\
&= \pi_A(\lambda) \cdot 1 \\
&= \pi_A(\lambda)
\end{aligned}$$

Since the characteristic polynomials of A and B are identical, then their eigenvalues are also identical. □

Main Message: Similar matrices have identical eigenvalues. Hence one possible strategy for computing eigenvalues of a matrix is to transform it via similarity to a matrix whose eigenvalues are easy to compute, e.g., a diagonal or a triangular matrix.

Question: Can every matrix be transformed to a diagonal matrix via similarity?

Answer: **NO**. We have already seen this in Section 4.9. But here it is again. To see why that is the case let A be an $n \times n$ *nonzero* matrix such that $A^k = 0_{n\times n}$. Assume that there exists an invertible matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}.$$

Then one obtains

$$D^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix} = \underbrace{(P^{-1}AP)\cdot(P^{-1}AP)\cdots(P^{-1}AP)}_{k\,times} = P^{-1}A^kP = 0_{n\times n}\,.$$

Now it follows that for all $i = 1, 2, \dots, n$,

$$d_i^k = 0 \quad\Longrightarrow\quad d_i = 0 \quad\Longrightarrow\quad D = 0_{n\times n}\,.$$

Consequently, $A = PDP^{-1} = 0_{n\times n}$ which contradicts the assumption that A is nonzero.

Definition 12. *A square matrix A is said to be* **nilpotent** *if $A^k = 0$ for some positive integer k.*

Proposition 13. *Let A be an $n \times n$ matrix. The following statements are equivalent.*

(a) *A is nilpotent.*

(b) *All eigenvalues of A are zero.*

(c) *The characteristic polynomial of A is $\pi_A(\lambda) = \lambda^n$.*

Proof. $\underline{(a) \Longrightarrow (b)}$ Assume that A is a nilpotent matrix and so $A^k = 0_{n\times n}$ for some k. Now let $(\lambda, \mathbf{x})$ be an arbitrary eigenpair of A so that

$$A\mathbf{x} = \lambda\,\mathbf{x}\,. \tag{15}$$

Multiplying (15) by A^{n-1} on the left gives

$$A^n\mathbf{x} = \lambda^n\mathbf{x} \quad\Longleftrightarrow\quad \mathbf{0} = \lambda^n\mathbf{x}\,. \tag{16}$$

Since $\mathbf{x}$ is an eigenvector and is nonzero, it follows that (16) only holds if $\lambda^n = 0$, or equivalently, $\lambda = 0$.

$\underline{(b) \implies (c)}$ Assume that all eigenvalues of A are zero. Since $\det(A - \lambda I)$ is a polynomial of degree n whose roots are exactly eigenvalues of A, it must be that $\det(A - \lambda I) = \pm\lambda^n$.

$\underline{(c) \implies (a)}$ We will do this in a few pages. $\square$

Definition 14. *An $n \times n$ matrix A is said to be* **diagonalizable** *if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.*

Question: What matrices are diagonalizable?

Answer: Let $A \in \mathbb{C}^{n\times n}$ and assume that there exists an invertible matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \Longleftrightarrow \quad AP = PD\,. \tag{17}$$

Looking at $AP = PD$ column-wise we have that for each $i = 1, \ldots, n$ it holds that

$$A \cdot \begin{bmatrix} P_{*1} & P_{*2} & \cdots & P_{*n} \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} P_{*1} & P_{*2} & \cdots & P_{*n} \end{bmatrix}$$

$$\begin{bmatrix} AP_{*1} & AP_{*2} & \cdots & AP_{*n} \end{bmatrix} = \begin{bmatrix} \lambda_1 P_{*1} & \lambda_2 P_{*2} & \cdots & \lambda_n P_{*n} \end{bmatrix}$$

$$AP_{*i} = \lambda_i P_{*i}$$

that is, (λ_i, P_{*i}) is an eigenpair of A.

Diagonalizability

- A **complete set of eigenvectors of $A \in \mathbb{F}^{n\times n}$** is any set of n *linearly independent* eigenvectors for A. Not all matrices have a complete sets of eigenvectors, e.g., nilpotent matrices. Matrices that fail to posses complete sets of eigenvectors are sometimes called **deficient** or **defective** matrices.

- Matrix $A \in \mathbb{F}^{n\times n}$ is diagonalizable if and only if A possesses a complete set of eigenvectors. Moreover, $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ if and only if the columns of P constitute a complete set of eigenvectors and the λ_j's are the associated eigenvalues – that is, each (λ_j, P_{*j}) is an eigenpair for A.

`Make sure you do a few computational examples here!!!`

Example 15. (cont. of Ex. 8) *Determine if A is a diagonalizable matrix, where* $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution: From Example 8 we have

$$P^{-1}AP \;=\; D \;=\; \begin{bmatrix} 0 & & \\ & -5 & \\ & & 2 \end{bmatrix},$$

where

$$P \;=\; \begin{bmatrix} -1 & 4 & 1 \\ 0 & -35 & 0 \\ 1 & 46 & 1 \end{bmatrix}.$$

Remark 16. *Note that the eigenvalues in matrix D can be put in any order, as long as one re-orders the corresponding eigenvectors in P. For example, one can check that*

$$\widetilde{P}^{-1}\, A\, \widetilde{P} \;=\; \widetilde{D} \;=\; \begin{bmatrix} 2 & & \\ & 0 & \\ & & -5 \end{bmatrix},$$

where

$$\widetilde{P} \;=\; \begin{bmatrix} 1 & -1 & 4 \\ 0 & 0 & -35 \\ 1 & 1 & 46 \end{bmatrix}.$$

MTH 513: Chapter 7, Part 3

Last updated: December 1, 2022

§7.1 – §7.2 Schur's Theorem and Implications

As a preparation, we start by reviewing some basic facts.

Definition 17. *An $n \times n$ matrix A is said to be* **unitary** *if $A^*A = AA^* = I_n$. Equivalently, A is a unitary matrix if its columns (rows) form an orthonormal basis for $\mathbb{C}^n$.*

Lemma 18. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$ be unitary matrices. Then*

$$C = \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right]$$

is unitary.

Lemma 19. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be unitary matrices. Then $A \cdot B$ is also a unitary matrix.*

Theorem 20 (**Schur's Triangularization Theorem**)**.** *Every square matrix is unitarily similar to an upper-triangular matrix. That is, for each $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix U (not unique) and an upper-triangular matrix T (not unique) such that $U^*AU = T$ (or $A = UTU^*$), and the diagonal entries of T are the eigenvalues of A.*

Proof. We prove this statement by induction. For $n = 1$, this result holds trivially. Namely,

$$A = \begin{bmatrix} a_{11} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix}^* = U \cdot T \cdot U^* .$$

For the inductive hypothesis, we assume that all $(n-1)\times(n-1)$ matrices are unitarily similar to an upper triangular matrix.

Next we let A be $n \times n$ matrix and suppose that $(\lambda, \mathbf{x})$ is an eigenpair for A, i.e., $A\mathbf{x} = \lambda\mathbf{x}$. Without loss of generality, assume that $\mathbf{x}$ is of unit length, otherwise, $\frac{1}{||\mathbf{x}||_2}\mathbf{x}$ is one such eigenvector. Now let Q be a unitary matrix such that

$$Q = \left[\begin{array}{c|c} \mathbf{x} & \widetilde{Q} \end{array}\right],$$

where $\widetilde{Q} \in \mathbb{C}^{n\times(n-1)}$. (Why and how can this be done?) Then we have the following chain of equalities

$$
\begin{aligned}
Q^* A Q &= \left[\begin{array}{c} \mathbf{x}^* \\ \hline \widetilde{Q}^* \end{array}\right] A \left[\begin{array}{c|c} \mathbf{x} & \widetilde{Q} \end{array}\right] = \left[\begin{array}{c} \mathbf{x}^* \\ \hline \widetilde{Q}^* \end{array}\right] \left[\begin{array}{c|c} A\,\mathbf{x} & A\,\widetilde{Q} \end{array}\right] \\
&= \left[\begin{array}{c} \mathbf{x}^* \\ \hline \widetilde{Q}^* \end{array}\right] \left[\begin{array}{c|c} \lambda\,\mathbf{x} & A\,\widetilde{Q} \end{array}\right] = \left[\begin{array}{c|c} \lambda \mathbf{x}^*\mathbf{x} & \mathbf{x}^* A \widetilde{Q} \\ \hline \lambda\,\widetilde{Q}^*\mathbf{x} & \widetilde{Q}^* A \widetilde{Q} \end{array}\right] \\
&= \left[\begin{array}{c|c} \lambda & \mathbf{x}^* A \widetilde{Q} \\ \hline \mathbf{0}_{n-1} & A_2 \end{array}\right], \qquad (18)
\end{aligned}
$$

where (18) follows from the fact that $\mathbf{x}$ is a unit vector by assumption, columns of $\widetilde{Q}$ are orthogonal to $\mathbf{x}$, and $A_2 = \widetilde{Q}^* A \widetilde{Q}$.

Since A_2 is an $(n-1) \times (n-1)$ and the inductive hypothesis applies and so there exists an $(n-1) \times (n-1)$ unitary matrix $\widetilde{R}$ such that $\widetilde{R}^* A_2 \widetilde{R} = \widetilde{T}$, where $\widetilde{T}$ is an $(n-1) \times (n-1)$ upper triangular matrix. Consider a new matrix

$$
R = \left[\begin{array}{c|c} 1 & \\ \hline & \widetilde{R} \end{array}\right] \in \mathbb{C}^{n \times n},
$$

and observe that R is also unitary and

$$
R^* = \left[\begin{array}{c|c} 1 & \\ \hline & \widetilde{R}^* \end{array}\right] \in \mathbb{C}^{n \times n}.
$$

Going back to (18) we obtain the following chain of equalities

$$
\begin{aligned}
R^*(Q^* A Q) R &= \left[\begin{array}{c|c} 1 & \\ \hline & \widetilde{R}^* \end{array}\right] \cdot \left[\begin{array}{c|c} \lambda & \mathbf{x}^* A \widetilde{Q} \\ \hline \mathbf{0}_{n-1} & A_2 \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & \\ \hline & \widetilde{R} \end{array}\right] \\
(QR)^* A (QR) &= \left[\begin{array}{c|c} \lambda & (\star) \\ \hline & \widetilde{R}^* A_2 \widetilde{R} \end{array}\right] = \left[\begin{array}{c|c} \lambda & (\star) \\ \hline & \widetilde{T} \end{array}\right] =: T. \qquad (19)
\end{aligned}
$$

Clearly, $U = QR$ is a unitary matrix since the product of unitary matrices is again a unitary matrix, and T is an upper triangular matrix. Finally, from (19) the desired conclusion follows. □

Remark 21. *One can always arrange for eigenvalues of A to appear "consecutively" along the diagonal of T, that is, all repeating eigenvalues appear consequently, and so on. Furthermore, Schur's form is not unique, that is, neither T nor U is unique. Also, note that even if A is real, then T and U might be complex. Finally, there is something called "real Schur form" so if you would like to know more, I am happy to provide you with additional references.*

Theorem 22. *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$\operatorname{trace}(A) \;=\; \sum_{i=1}^{n} \lambda_i \qquad \text{and} \qquad \det(A) \;=\; \prod_{i=1}^{n} \lambda_i\,,$$

where λ_i are eigenvalues of A.

Remark 23. *If you have not reviewed properties of determinants, now is the time to go over determinant review.*

Proof. From the Schur's Triangularization Theorem we know that there exists an upper triangular matrix T whose diagonal entires are the eiganvalues of A and the unitary matrix U such that

$$T \;=\; U^* A U\,. \tag{20}$$

Taking trace of the both sides of (20), together with the facts that $\operatorname{trace}(XY) = \operatorname{trace}(YX)$ for all $X, Y \in \mathbb{C}^{n\times n}$ and $U^*U \;=\; U^*U = I_n$ for a unitary matrix U, imply that

$$\begin{aligned}
\operatorname{trace}(T) \;&=\; \operatorname{trace}(U^*AU) \;=\; \operatorname{trace}(UU^*A) \;=\; \operatorname{trace}(I_nA)\\
\sum_{i=1}^{n} \lambda_i \;&=\; \operatorname{trace}(A)\,.
\end{aligned}$$

On the other hand, taking det of the both sides of (20), together with the facts that $\det(XY) = \det(X)\det(Y)$ for all $X, Y \in \mathbb{C}^{n\times n}$, imply that

$$\begin{aligned}
\det(T) \;&=\; \det(U^*AU) \;=\; \det(U^*)\det(A)\det(U)\\
\prod_{i=1}^{n} \lambda_i \;&=\; \frac{1}{\det(U)}\;\det(A)\;\det(U) \;=\; \det(A)\,.
\end{aligned}$$

$\square$

Remark 24. *Note that nearly identical argument would work for square matrices that are diagonalizable. However, the argument via Schur's Theorem (or via Jordan Canonical Form - not covered yet!) hold for* all *matrices.*

Theorem 25. *Let $A \in \mathbb{C}^{n\times n}$ be a* **normal** *matrix, that is, $A^*A = AA^*$. Prove that A is* unitarily diagonalizable.

Proof. By the Schur's Triangularization Theorem we know that there exist a unitary matrix U such that $U^*AU = T_A$, where T_A is an upper triangular matrix. Equivalently, we have that

$$A = U\,T_A\,U^* \qquad \text{and} \qquad A^* = U\,T_A^*\,U^*\,. \tag{21}$$

Since A is normal we have the following chain of equalities

$$\begin{aligned} AA^* &= A^*A \\ \big(U\,T_A\,U^*\big)\big(U\,T_A^*\,U^*\big) &= \big(U\,T_A^*\,U^*\big)\big(U\,T_A\,U^*\big) \\ U\,T_A\,\big(U^*\,U\big)\,T_A^*\,U^* &= U\,T_A^*\,\big(U^*\,U\big)\,T_A\,U^* \\ U\,T_A\,T_A^*\,U^* &= U\,T_A^*\,T_A\,U^* \\ T_A\,T_A^* &= T_A^*\,T_A\,. \end{aligned} \tag{22}$$

From (22) it follows that T_A is also normal. Furthermore, from our take-home exam we know that since T_A is an upper triangular and a normal matrix, then it must be diagonal. □

Theorem 26 (**Spectral Theorem for Hermitian Matrices**)**.** *Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. The following statements are true.*

(i) *A is unitarily diagonalizable.*

(ii) *All eigenvalues of A are real.*

(iii) *Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.*

Proof. Assume A is Hermitian. Then $A^*A = AA = AA^*$ and so A is also normal. By Theorem 25 it follows that A is unitarily diagonalizable, that is, there exists a unitary matrix U and a diagonal matrix D such that

$$U^*\,A\,U = D\,. \tag{23}$$

Taking a complex conjugate of (23) implies that

$$\begin{aligned} U^*\,A^*\,(U^*)^* &= D^* \\ U^*\,A\,U &= D^* \\ D &= D^*\,. \end{aligned} \tag{24}$$

But (24) implies that D is also a Hermitian matrix and so

$$\overline{d_i} \;=\; d_i\,, \qquad\qquad i = 1, 2, \ldots, n\,. \tag{25}$$

But (25) implies that each d_i must be real and so all eigenvalues of A are real.

Finally, to prove (iii) let $(\lambda, \mathbf{x})$ and $(\mu, \mathbf{y})$ be eigenpairs of A where $\lambda \neq \mu$. Then

$$A\mathbf{x} \;=\; \lambda\mathbf{x} \qquad \text{and} \qquad A\mathbf{y} \;=\; \mu\mathbf{y}\,.$$

Then

$$\begin{aligned}
\mathbf{y}^* A\mathbf{x} &= \lambda\mathbf{y}^*\mathbf{x} \\
\mathbf{y}^* A^*\mathbf{x} &= \lambda\mathbf{y}^*\mathbf{x} \qquad (26) \\
\mu\mathbf{y}^*\mathbf{x} &= \lambda\mathbf{y}^*\mathbf{x} \\
(\lambda - \mu)\mathbf{y}^*\mathbf{x} &= 0\,, \qquad (27)
\end{aligned}$$

where (26) follows from the assumption that A is Hermitian. Finally, the assumption that $\lambda \neq \mu$, together with (26), imply that $\mathbf{y}^*\mathbf{x} = 0$, as desired. $\square$

Example 27. *Let* $A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Compute* A^2, A^3, A^4.

Solution:

$$A^2 = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 28. *Let J be a $k \times k$ upper-triangular matrix with diagonal entries equal to zero. Prove that $J^k = 0_{k\times k}$.*

Proof. `Easy exercise in induction; make sure you do it!` $\square$

Lemma 29. *Let k be a positive integer and let T_i be a $k \times k$ block matrices such that*

$$T_i \;=\; \begin{bmatrix} C_{11}^{(i)} & \star & \cdots & \star \\ & C_{22}^{(i)} & \cdots & \star \\ & & \ddots & \vdots \\ & & & C_{kk}^{(i)} \end{bmatrix}, \qquad i = 1, 2, \ldots, k\,.$$

Further, assume $C_{ii}^{(i)}$ is the zero matrix for each i. Then the product $T_1T_2\cdots T_k$ is the zero matrix of the appropriate size.

Proof. Exercise for you! □

Just to get a feel how Lemma 29 works, let us consider the case when $k = 3$. Then

$$T_1T_2T_3 \;=\; \begin{bmatrix} \boxed{0} & \star & \star \\ 0 & X & \star \\ 0 & 0 & Y \end{bmatrix} \begin{bmatrix} A & \star & \star \\ 0 & \boxed{0} & \star \\ 0 & 0 & B \end{bmatrix} \begin{bmatrix} P & \star & \star \\ 0 & Q & \star \\ 0 & 0 & \boxed{0} \end{bmatrix} \tag{28}$$

$$= \begin{bmatrix} 0 & 0 & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & YB \end{bmatrix} \begin{bmatrix} P & \star & \star \\ 0 & Q & \star \\ 0 & 0 & \boxed{0} \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

Definition 30. *Let $q(t)$ be a scalar polynomial such that*

$$q(t) \;=\; a_kt^k + a_{k-1}t^{k-1} + \cdots + a_1t + a_0\,.$$

If $A \in \mathbb{R}^{n\times n}$, then $q(A)$ is an $n \times n$ matrix such that

$$q(A) \;=\; a_kA^k + a_{k-1}A^{k-1} + \cdots + a_1A + a_0I\,.$$

Lemma 31. *Let $q(t)$ be an arbitrary scalar polynomial. Then for any matrix A and a nonsingular matrix S we have that*

$$q(S^{-1}AS) \;=\; S^{-1}q(A)S\,.$$

Proof. Let $q(t) \;=\; a_kt^k + a_{k-1}t^{k-1} + \cdots + a_1t + a_0$ so that

$$\begin{aligned} q(S^{-1}AS) \;&=\; a_k(S^{-1}AS)^k + a_{k-1}(S^{-1}AS)^{k-1} + \cdots + a_1(S^{-1}AS) + a_0(S^{-1}AS) \\ &=\; a_kS^{-1}A^kS + a_{k-1}S^{-1}A^{k-1}S + \cdots + a_1S^{-1}AS + a_0S^{-1}AS \\ &=\; S^{-1}\Big(a_kA^k + a_{k-1}A^{k-1} + \cdots + a_1A + a_0I\Big)S \\ &=\; S^{-1}q(A)S\,. \end{aligned}$$

□

Theorem 32 (**Cayley-Hamilton Theorem**). *Let $A \in \mathbb{C}^{n\times n}$. Then $p(A) = 0_{n\times n}$, that is, A satisfies its own characteristic equation.*

Proof. Let $\lambda_1, \ldots, \lambda_k$ be *distinct eigenvalues* of A with algebraic multiplciites $m_1, m_2, \ldots, m_k$, respectively. Then the characteristic polynomial of A is of the form

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; (\lambda - \lambda_1)^{m_1} \cdot (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}\,,$$

where $m_1 + \cdots + m_k = n$. By the Schur's Triangularization Theorem we know that there exists a unitary matrix U such that

$$U^*AU \;=\; T_A \;=\; \begin{bmatrix} T_1 & \star & \cdots & \star \\ & T_2 & \cdots & \star \\ & & \ddots & \vdots \\ & & & T_k \end{bmatrix}, \qquad \text{where} \qquad T_i \;=\; \begin{bmatrix} \lambda_i & \star & \cdots & \star \\ & \lambda_i & \cdots & \star \\ & & \ddots & \vdots \\ & & & \lambda_i \end{bmatrix}_{m_i\times m_i}$$

for $i = 1, \ldots, k$. By Lemma 28 we know have that $(T_i - \lambda_i I)^{m_i} = 0_{m_i\times m_i}$ and

$$(T_A - \lambda_i I)^{m_i} \;=\; \begin{bmatrix} \star & \cdots & \star & \cdots & \star \\ & \ddots & \vdots & & \vdots \\ & & \boxed{0} & \cdots & \star \\ & & & \ddots & \vdots \\ & & & & \star \end{bmatrix}$$

By Lemma 29 it follows that

$$(T_A - \lambda_1 I)^{m_1}(T_A - \lambda_2 I)^{m_2} \cdots (T_A - \lambda_k I)^{m_k} \;=\; 0_{n\times n}\,.$$

Finally, Lemma 31 implies that

$$U^*p(A)U \;=\; p(U^*AU) \;=\; p(T_A) \;=\; (T_A - \lambda_1 I)^{m_1}(T_A - \lambda_2 I)^{m_2} \cdots (T_A - \lambda_k I)^{m_k} \;=\; 0_{n\times n}\,,$$

and consequently, $p(A) \;=\; 0_{n\times n}$. $\square$

Multiplicities

For $\lambda \in \Lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we adopt the following definitions.

- The **algebraic multiplicity** of λ is the number of times it is repeated as a root of the characteristic polynomial. In other words, $alg\ mult_A(\lambda_i) = m_i$ if and only if $(x-\lambda_1)^{m_1} \cdot (x-\lambda_2)^{m_2} \cdot \ldots \cdot (x-\lambda_s)^{m_s} = 0$ is the characteristic equation for A.

- When $alg\ mult_A(\lambda_0) = 1$, then λ_0 is called a **simple eigenvalue**.

- The **geometric multiplicity** of λ_0 is $\dim \mathcal{N}(A - \lambda_0 I)$. In other words, $geo\ mult_A(\lambda_0)$ is the maximal number of linearly independent eigenvectors associated with λ_0.

- Eigenvalues such that $alg\ mult_A(\lambda_0) = geo\ mult_A(\lambda_0)$ are called **semi simple eigenvalues of A**.

Multiplicity Inequality

For every $A \in \mathbb{F}^{n\times n}$, and for each $\lambda \in \Lambda(A)$,

$$geo\ mult_A(\lambda) \le ald\ mult(\lambda)\,. \tag{31}$$

Proof. Suppose $alg\, mult_A(\lambda) = k$. By Schur's Triangularization Theorem we know that there exist a unitary matrix U such that

$$U^*AU = T_A = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline 0 & T_{22} \end{array}\right],$$

where T_{11} is a $k\times k$ upper triangular matrix with diagonal entires λ, and T_{22} is an $(n-k)\times(n-k)$ upper-triangular matrix such that $\lambda \notin \Lambda(T_{22})$. Consequently, $T_{22} - \lambda I_{n-k}$ is nonsingular, and

$$\begin{aligned} \operatorname{rank}(A-\lambda I) &= \operatorname{rank}\Big(U^*(A-\lambda I)U\Big) = \operatorname{rank}(U^*AU - \lambda I) = \operatorname{rank}(T_A - \lambda I) \\ &= \operatorname{rank}\left(\left[\begin{array}{c|c} T_{11}-\lambda I_k & T_{12} \\ \hline 0 & T_{22}-\lambda I_{n-k} \end{array}\right]\right) \ge \operatorname{rank}(T_{22}-\lambda I_{n-k}) = n-k\,. \end{aligned}$$

Now we have that

$$alg\ mult_A(\lambda) = k \ge n - \operatorname{rank}(A-\lambda I) = \dim \mathcal{N}(A-\lambda I) = geo\ mult_A(\lambda)\,.$$

$\square$

Independent Eigenvectors

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a set of *distinct* of A.

- If $\{(\lambda_1, \mathbf{x}_1), (\lambda_2, \mathbf{x}_2), \ldots, (\lambda_k, \mathbf{x}_k)\}$ is a set of eigenpairs for A, then $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a linearly independent set.

- If $\mathcal{B}_i$ is a basis for $\mathcal{N}(A - \lambda_i I)$, then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is a linearly independent set.

Proof. Read the proof on page 512. □

Diagonalizability and Multiplicities

A matrix $A \in \mathbb{C}^{n\times n}$ is diagonalizable if and only if

$$\textit{deo mult}_A(\lambda) \;=\; \textit{alg mult}_A(\lambda)$$

for each $\lambda \in \varLambda(A)$ – that is, if and only if every eigenvalues is semisimple.

Distinct Eigenvalues

If no eigenvalue of A is repeated, then A is diagonalizable. However, the converse is not true!

MTH 513: Chapter 7, Part 4

Last updated: December 1, 2022

A Few Remarks on the Jordan Canonical Form

Recall that for a positive integer k and an arbitrary scalar λ, we define the **Jordan block** as

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \lambda & 1 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}_{k\times k}.$$

Block-diagonal matrix consisting of Jordan blocks is called the **Jordan segment**, namely,

$$\mathcal{J}(\lambda) = \begin{bmatrix} J_1(\lambda) & & & \\ & J_2(\lambda) & & \\ & & \ddots & \\ & & & J_t(\lambda) \end{bmatrix}.$$

Theorem 33. *Let A be an $n \times n$ arbitrary matrix, and assume that $\lambda_1, \ldots, \lambda_r$ are the only distinct eigenvalues of A. Then there exists a nonsingular matrix P such that*

$$P^{-1}AP = J_A = \begin{bmatrix} \mathcal{J}(\lambda_1) & & & \\ & \mathcal{J}(\lambda_2) & & \\ & & \ddots & \\ & & & \mathcal{J}(\lambda_r) \end{bmatrix},$$

where J_A has ONE Jordan segment $\mathcal{J}(\lambda_j)$ for each distinct eigenvalue λ_j. Moreover, J_A is unique up to permutation of Jordan segments and Jordan blocks within each segment.

Example 34. *Let A be the matrix whose Jordan form is given by*

$$J_A = \begin{bmatrix}
2 & 1 & & & & & & & & & & & & & & \\
0 & 2 & & & & & & & & & & & & & & \\
 & & 2 & & & & & & & & & & & & & \\
 & & & -4 & & & & & & & & & & & & \\
 & & & & -4 & 1 & & & & & & & & & & \\
 & & & & & -4 & 1 & & & & & & & & & \\
 & & & & & & -4 & 1 & & & & & & & & \\
 & & & & & & & -4 & & & & & & & & \\
 & & & & & & & & 0 & 1 & & & & & & \\
 & & & & & & & & 0 & 0 & & & & & & \\
 & & & & & & & & & & 7 & 1 & & & & \\
 & & & & & & & & & & & 7 & 1 & & & \\
 & & & & & & & & & & & & 7 & & & \\
 & & & & & & & & & & & & & 7 & 1 & \\
 & & & & & & & & & & & & & 0 & 7 & \\
 & & & & & & & & & & & & & & & 7
\end{bmatrix}$$

(a) *What are the eigenvalues of A and their algebraic and geometric multiplicities?*

(b) *What is rank of A?*

(c) *What is determinant of A?*

(d) *What is trace of A?*

(e) *Find a polynomial $q(\lambda)$ of the minimal degree such that $q(A) = 0_{16\times 16}$.*