Home Work 1

Evan Fox

2/6/2023

Problem 1. *If u, v are the only vertices of a graph G with odd degree, prove that there exists a u, v path in G*

Proof. u, v must lie in the same connected component of G or else considering the induced subgraph on the vertices of the component containing u but not v, we would obtain a graph with an odd number of odd degree vertices, contradicting the handshaking lemma proved in class. Hence u, v lie in the same component and hence there exitsts a u, path. $\square$

Problem 2. *Show that $\delta(G) \geq 3$ implies the existence of an even cycle*

Proof. Let $x_1 x_2 \dots x_n$ be the longest path in G, then it follows that x_n must have at least two more neighbors on the path, x_k and x_l. Suppose that $k < l$. Then if the cylce $x_k x_{k+1} \dots x_n x_k$ is even, we are done; so suppose not. Then we have $|\{x_i \,|\, k \leq i < n\}|$ is an even integer. It follows that there is either an even number of vertices less than l or an even number greater than l (where I am ordering vertices by their index and not including x_n). In the first case $x_k \dots x_l x_n x_k$ is an even cycle and in the second we have $x_l, \dots x_n x_l$ will be an even cycle. $\square$

Problem 3. *Suppose that G has no isolated vertices and that no induced subgraph on G has exactly two edges. Show that G is complete.*

Proof. Let $u, v \notin E(G)$, since no vertex in G is isolated, let u', v' be adjaceint to u and v recpectivly. If $u' = v'$ then $G[u, v, u']$, is an induced subgraph with two edges, so we may assume that they are distinct. If either $uv', vu' \in E(G)$, then we may take $G[v', u, v]$ or $G[u', u, v]$ gives us a induced subgraph with two edges so again we may assume that this doesnt happen. Now if $u'v' \in E(G)$, then $G[u, u', v']$ again is a counter example. But now in the last case we only have edges uu' and vv' so taking induced subgraph on all four vertices gives a induced subgraph with two edges. Hence there must exits a u, v edge. $\square$

Problem 4. *Can there exist a function taking $k \in \mathbb{N}$ to the minimal degree $\delta(G)$ which insures G is k-connected?*

Proof. No, It is not hard to construct a graph G of arbitrarily large minimal degree that is not 2-connected. Take two disjoint graphs G_1, G_2 with minimal degree n, then add a vetex v and connectect it to every every vertex in G_i for $i = 1, 2$. G thus defined is not 2-connected since removing v results in two connected components. $\square$