# MTH535 Homework 1

Evan Fox

9/24

**Problem 1.**

*Proof.* Let $A = \{\bigcup_{i=1}^n E_i \,|\, n \in \mathbb{N},\, E_j \in S,\, E_i \cap E_j = \varnothing \text{ for } i \neq j\}$ We have $S \subset A$ since for any $X \in S$, we can write $X$ as a disjoint union of elements in $S$ in a trivial way. Further, any algebra containing $S$, by definition, contains all finite unions of elements of $S$, which is a superset of all finite unions of disjoint elements in $S$, thus the algebra contains $A$. All that remains is to see that $A$ is an algebra itself. We have $\varnothing \in S \subset A$. We need to prove that $A$ is closed under complements, take $X \in A$, then we have $X = \bigcup_{i=1}^n E_i$ for a finite collection of pairwise disjoint sets $E_i \in S$, $i = 1, \ldots, n$. Now since $E_i \in S$, we have by property (c), that $E_i^c = \bigcup_{j=1}^{n_i} R_{j,i}$. Then

$$X^c = \left(\bigcup_{i=1}^n E_i\right)^c = \bigcap_{i=1}^n E_i^c = \bigcap_{i=1}^n \left(\bigcup_{j=1}^{n_i} R_{j,i}\right) \tag{1}$$

let $N = \max\{n_i \,|\, i = 1, \ldots, n\}$ and set $R_{j,i} = \varnothing$ for $n_i < j \leq N$. Then 1 becomes

$$X^c = \bigcap_{i=1}^n \left(\bigcup_{j=1}^{N} R_{j,i}\right) = \bigcup_{j=1}^N \left(\bigcap_{i=1}^{n} R_{j,i}\right) \tag{2}$$

Since $E_1^c = \bigcup_{j=1}^{n_1} R_{j,1}$ is a disjoint union, we have for any $k_1 \neq k_2$, $R_{k_1,1} \cap R_{k_2,1} = \varnothing$. We also have that $\bigcap_{i=1}^n R_{k_1,i} \subset R_{k_1,1}$ and $\bigcap_{i=1}^n R_{k_2,i} \subset R_{k_2,1}$. Hence the intersections $\bigcap_{i=1}^n R_{j,i}$ are pairwise disjoint for all $j$. Since $S$ is closed under finite intersections, $\bigcap_{i=1}^n R_{j,i} \in S$. Thus, 2 expresses $X^c$ as a finite pairwise disjoint union of elements of $S$ and $X^c \in A$. Finially, we show $A$ is closed under finite interesections (closure of $A$ under finite unions then follows from DeMorgan's laws). Let $A_1, A_2 \in A$, then we have pairwise disjoint collections $\{E_1, \ldots, E_n\}$ and $\{F_1, \ldots, F_m\}$ such that

$$A_1 = \bigcup_{i=1}^n E_i$$

$$A_2 = \bigcup_{i=1}^m F_i$$

Then

$$A_1 \cap A_2 = \left[\bigcup_{i=1}^n E_i\right] \cap \left[\bigcup_{j=1}^m F_j\right] = \bigcup_{j=1}^m \left[\bigcup_{i=1}^n E_i\right] \cap F_j = \bigcup_{j=1}^m \bigcup_{i=1}^n E_i \cap F_j \tag{3}$$

if $n \neq m$ let $F_i = E_i$ for $\min\{n, m\} < i \leq \max\{n, m\}$, Then

$$A_1 \cap A_2 = \bigcup_{i,j}^{\max\{n,m\}} E_i \cap F_j \in S \tag{4}$$

since the union is finite and the sets $E_i \cap F_j$ are disjoint since $\{E_1, \ldots, E_n\}$ is. Since $A$ is closed under the intersection of any two of its elemetns, it follows from induction and associativity of intersections that it is closed under all finite intersections. Hence $A$ is an algebra containing $S$ and since any algebra containing $S$ also contains $A$, it follows that $A = \mathcal{A}(S)$. $\square$

**Problem 2.**

*Proof.* 1. Suppose not and let $\alpha : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ be any bijection. Let $\pi_n : \{0,1\}^{\mathbb{N}} \to \mathbb{Z}_2$ be the projection on to the $n^{th}$ component. Then consider $b_i = \pi_i(\alpha(i)) + 1$ and let $b = (b_1, b_2, \dots, b_k, \dots) \in \{0,1\}^{\mathbb{N}}$ then for all $n \in \mathbb{N}$, $b \neq \alpha(n)$ since they differ in the $n^{th}$ component by construction. Thus $b \notin \mathtt{Im}(\alpha)$, so $\alpha$ is not a surjection. a contradiction.

2. Define $\alpha : \mathcal{P}(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ by $\alpha(A) = \{\alpha_1, \alpha_2, \dots\}$ where $\alpha_n = 1$ if $n \in A$ and 0 otherwise. Then given $b \in \{0,1\}^{\mathbb{N}}$, $b = (b_1, b_2, \dots)$, we have $S = \{n \,|\, b_n = 1\} \in \mathcal{P}(\mathbb{N})$ with $\alpha(S) = b$. Hence $\alpha$ is a surjection and since by the above we know that $\{0,1\}^{\mathbb{N}}$ is uncountable, it follows that $\mathcal{P}(\mathbb{N})$ is uncountable as well.

$\square$

**Problem 3.**

*Proof.* From class we know that $m^*([0,1]) = 1$, since the outer measure of an interval is its length, but the outer measure of a countable set is 0(also proven in class), hence if we assume that $[0,1]$ is countable its an interval whose outermeasure is not its length a contradiction. $\square$

**Problem 4.**

*Proof.* Let $E$ have positive outer measure, we write $E$ as the union of countably many disjoint bounded sets, for example by considering the collection $b_k = (k, k+1]$ for all $k \in \mathbb{Z}$, and taking $A_k = b_k \cap E$. If $m^*(A_k) = 0$ for all $k \in \mathbb{Z}$, then by countable subadditivity of outer measure, we have $m^*(E) = m*(\bigcup_{n\in\mathbb{Z}} A_k) \leq \sum_{n\in\mathbb{Z}} m^*(A_n) = 0$ contradicting our assumption that $E$ has positive outer measure. $\square$

**Problem 5.**

*Proof.* Assume $E$ is measureable, Fix $\epsilon > 0$. By thm 2.11 parts 1 and 3 proved in class we may fix an open set $\mathcal{O}$ such that $E \subset \mathcal{O}$ and $m^*(\mathcal{O} \setminus E) < \frac{\epsilon}{2}$ and a closed set $\mathcal{C}$ such that $\mathcal{C} \subset E$ and $m^*(E \setminus \mathcal{C}) < \frac{\epsilon}{2}$ Then by measureability of $E$ and $\mathcal{C}$, we apply excision,

$$m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E) < \frac{\epsilon}{2}$$

$$m^*(E \setminus \mathcal{C}) = m^*(E) - m^*(\mathcal{C}) < \frac{\epsilon}{2}$$

Then adding these inequalities and applying excision on the result yeilds $m^*(\mathcal{O} \setminus \mathcal{C}) < \epsilon$. Conversly, suppose that for every $\epsilon > 0$, there exists $\mathcal{O} \supset E$ open and $\mathcal{C} \subset E$ closed such that $m^*(\mathcal{O} \setminus \mathcal{C}) < \epsilon$. Then by monotinicity, $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O} \setminus \mathcal{C}) < \epsilon$. so by thm 2.11 $E$ is measureable. $\square$

**Problem 6.**

*Proof.* if $E$ is not measurable then by the negation of 2.11 there exists $\epsilon_0 > 0$ such that for all $\mathcal{O}$ open sets containing $E$, $m^*(\mathcal{O} \setminus E) > \epsilon_0$. By definition of outer measure as an infimum, we may fix a countable collection of bounded open intervals $\{I_k\}_{k=1}^{\infty}$ convering $E$, such that $\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon_0$. Then let $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$, we have

$$m^*(\mathcal{O}) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon_0$$

so that

$$m^*(\mathcal{O}) - m^*(E) < \epsilon_0 < m^*(\mathcal{O} \setminus E)$$

‘ as desired. $\square$