Combinatorial Geometry for Undergraduate Students

Pablo Soberón

Baruch College, City University of New York, One Bernard Baruch Way, New York, NY 10010, United States
Email address: pablo.soberon-bravo@baruch.cuny.edu

The content of these notes was used three years in a short lecture series for a summer “Research Experience for Undergraduates” program at Baruch College. Each year, the participants I had the pleasure to mentor learned to topic, gave me substantial feedback, and subsequently proved some impressive results of their own. Their feedback was fundamental for the writing of these notes. The participants during the three years I prepared the first draft are (in chronological order) Sherry Sarkar, Alexander Xue, Travis Dillon, John A. Messina, Yaqian Tang, Ilani Axelrod-Freed, João Pedro Carvalho, and Yuki Takahashi.

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Contents

1. Great resources

The purpose of these notes is to gather selected tools and topics in combinatorial geometry to help you, an undergraduate student, get started in research. It contains some of the main results of this area, how they relate to each other, and some preliminary tools and techniques which are often helpful. In essence, these notes contain what I consider "basics of discrete geometry," with a sprinkle of my personal opinions.

The topology section describes how to apply some tools from algebraic topology to discrete geometry. This is a *black box topology* approach (i.e., you'll learn how to use topology rather than how to do topology).

There are several texts on related topics, many of which are masterpieces of mathematical writing. If you want to dive deeper, the following resources will help substantially:

- Keith Ball. An Elementary Introduction to Modern Convex Geometry. Flavors of Geometry, Vol 31. MSRI Publications, 1997.
- Jiří Matoušek. Using the Borsuk–Ulam theorem. Universitext. Springer 2003.
- Jiří Matoušek. Lectures on discrete geometry. Graduate texts in mathematics, Vol. 212. Springer Science & Business Media, 2013.
- Herbert Edelsbrunner and John L. Harer. Computational Topology, and introduction. American Mathematical Society, 2010.
- Imre Bárány. Combinatorial Convexity. University Lecture Series, Vol 77. American Mathematical Society, 2021

Part 1

Intersection patterns of convex sets

CHAPTER 1

Prerequisites and basic techniques

1. Linear Algebra

This book focuses on combinatorial geometry in real spaces and assumes the reader is familiar with basic linear algebra. The most common space we will work with is $\mathbb{R}^d$, the d-dimensional real space. For a finite set $v_1, \ldots, v_n$ of vectors in $\mathbb{R}^d$, a **linear combination** of $v_1, \ldots, v_n$ is any vector v that can be written as

$$v = \alpha_1 v_1 + \ldots + \alpha_n v_n,$$

where $\alpha_1, \ldots, \alpha_n$ are real numbers. Linear combinations lay the foundation of linear algebra and are used to define interesting structures such as subspaces of $\mathbb{R}^d$. Our first goal is to modify these notions by focusing on two particular types of linear combinations.

We say that a vector v is an **affine combination** of $v_1, \ldots, v_n$ if there exist real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\begin{aligned} v &= \alpha_1 v_1 + \ldots + \alpha_n v_n \quad \text{and} \\ 1 &= \alpha_1 + \ldots + \alpha_n. \end{aligned}$$

We say that a vector v is a **convex combination** of $v_1, \ldots, v_n$ if there exist real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\begin{aligned} v &= \alpha_1 v_1 + \ldots + \alpha_n v_n, \\ 1 &= \alpha_1 + \ldots + \alpha_n, \quad \text{and} \\ 0 &\le \alpha_1, 0 \le \alpha_2, \ldots, 0 \le \alpha_n. \end{aligned}$$

Each of these types of combinations is more restrictive than the previous one. Just as we use linear combinations to define the span of a set of vectors, we can extend this notion to our new types of linear combinations. For a set $V \subset \mathbb{R}^d$ we define

$$\begin{aligned} \operatorname{span}(V) &= \{v : v \text{ is a linear combination of a finite subset of } V\} \\ \operatorname{aff}(V) &= \{v : v \text{ is an affine combination of a finite subset of } V\} \\ \operatorname{conv}(V) &= \{v : v \text{ is a convex combination of a finite subset of } V\}. \end{aligned}$$

We call these the **linear span**, **affine span**, and **convex hull** of V, respectively. One of the first observations is that

$$\operatorname{conv}(V) \subset \operatorname{aff}(V) \subset \operatorname{span}(V).$$

We say that a set $H \subset \mathbb{R}^d$ is a **hyperplane** if there exists $v \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$H = \{x : \langle x, v \rangle = \alpha\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard dot product. A hyperplane H as described above defines two **closed half-spaces**

$$\begin{aligned} H^+ &= \{x : \langle x, v \rangle \ge \alpha\}, \\ H^- &= \{x : \langle x, v \rangle \le \alpha\}. \end{aligned}$$

Exercise 1.1 Let V be a subset of $\mathbb{R}^d$. If $V \subset H$ for some hyperplane H, then $\operatorname{aff}(V) \subset H$.

Exercise 1.2 Let V be a subset of $\mathbb{R}^d$. If $V \subset H^+$ for some closed half-space H^+, then $\operatorname{conv}(V) \subset H^+$.

Affine hulls and linear spans are very similar. We can embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ by appending a coordinate 1 to each vector in $\mathbb{R}^d$,

$$\begin{aligned} f : \mathbb{R}^d &\to \mathbb{R}^{d+1} \\ v &\mapsto (v, 1). \end{aligned}$$

Then, the set $H = \{f(v) : v \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ is a hyperplane. We can see that for $V \subset \mathbb{R}^d$ we have

$$H \cap \operatorname{span}(f(V)) = \operatorname{aff}(f(V)).$$

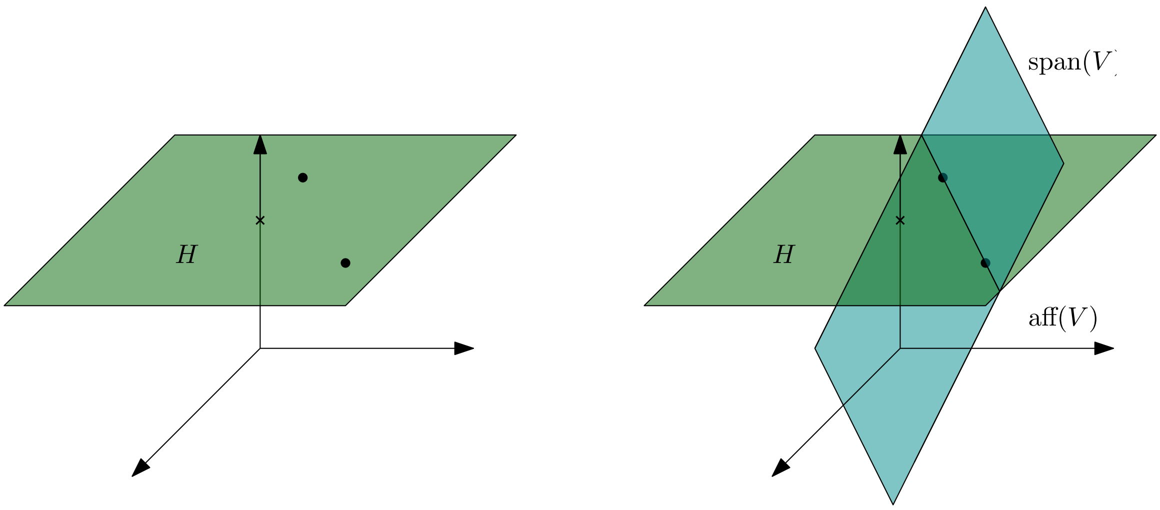


FIGURE 1. For two non-parallel vectors in $\mathbb{R}^d$, their affine hull is the line going through them, and their linear span is the 2-dimensional plane going through them and the origin. The convex hull is the segment between them.

Convex hulls are more interesting. For two points x, y, their convex hull is the set of points of the form $\alpha x + (1-\alpha)y$ for $\alpha \in [0,1]$. We can rewrite this as $x + \alpha(y - x)$. Notice that for $\alpha = 0$ this point is x, and for $\alpha = 1$ this point is y. Therefore, the convex hull of two points is the segment between them.

Exercise 1.3 [matroid property for affine hulls] Let $A \subset \mathbb{R}^d$ and $x, y \in \mathbb{R}^d$. Show that if $x \notin \operatorname{aff}(A)$ and $x \in \operatorname{aff}(A \cup \{y\})$ then $y \in \operatorname{aff}(A \cup \{x\})$.

Exercise 1.4 [anti-matroid property for convex hulls] Let $A \subset \mathbb{R}^d$ and $x, y \in \mathbb{R}^d$. Show that if $x \notin \operatorname{conv}(A)$ and $x \in \operatorname{conv}(A \cup \{y\})$ then $y \notin \operatorname{conv}(A \cup \{x\})$.

1.1. Dependence and independence. Among linear combinations, linear dependences play a central role. A **linear dependence** of a set of vectors $v_1, \dots, v_n$ in $\mathbb{R}^d$ is a set of coefficients $\alpha_1, \dots, \alpha_n$ such that

$$\bar{0} = \alpha_1 v_1 + \ldots + \alpha_n v_n.$$

A set of vectors is called **linearly independent** if the only linear dependence they have is given by $\alpha_1 = \ldots = \alpha_n = 0$. A basic fact of linear algebra is that if $v_1, \dots, v_n$ are linearly independent, then $n \leq d$. We now extend these notions to affine combinations.

An **affine dependence** of a set of vectors $v_1, \dots, v_n$ in $\mathbb{R}^d$ is a set of coefficients $\alpha_1, \dots, \alpha_n$ such that

$$\begin{aligned} \bar{0} &= \alpha_1 v_1 + \ldots + \alpha_n v_n. \\ 0 &= \alpha_1 + \ldots + \alpha_n. \end{aligned}$$

We say that $v_1, \dots, v_n$ are **affine independent** if their only affine dependence is given by $\alpha_1 = \ldots = \alpha_n = 0$, which we also call the **trivial linear dependence**. The second condition may seem counterintuitive at first glance. Didn't affine combinations require coefficients to sum to 1? The reason for this change is the following lemma.

LEMMA 1.1.1. *Let $v_1, \dots, v_n$ be vectors in $\mathbb{R}^d$ that are affine independent. Then, any vector in* $\operatorname{aff}\{v_1, \dots, v_n\}$ *can be expressed as an affine combination of $v_1, \dots, v_n$ in a unique way.*

PROOF. Let $v \in \operatorname{aff}\{v_1, \dots, v_n\}$. Suppose that we can express v as an affine combination of $v_1, \dots, v_n$ in two ways:

$$\begin{aligned} v &= \alpha_1 v_1 + \ldots + \alpha_n v_n \\ v &= \beta_1 v_1 + \ldots + \beta_n v_n. \end{aligned}$$

Then, for $\gamma_i = \alpha_i - \beta_i$, we have that $\gamma_1, \dots, \gamma_n$ is an affine dependence of $v_1, \dots, v_n$. By the affine independence, we have $0 = \gamma_i = \alpha_i - \beta_i$ for all i. This implies $\alpha_i = \beta_i$ for all i, as we wanted. $\square$

Affine independence can be directly linked to linear dependence using either of the two following exercises.

Exercise 1.5 Let $v_0, v_1, \dots, v_n$ be $n+1$ vectors in $\mathbb{R}^d$. Prove that they are affine independent if and only if the n vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent in $\mathbb{R}^d$.

Exercise 1.6 Let $v_1, \dots, v_n$ be n vectors in $\mathbb{R}^d$. For each $i = 1, \dots, n$, let $w_i = (v_i, 1) \in \mathbb{R}^{d+1}$. Prove that the vectors $v_1, \dots, v_n$ are affine independent if and only if the vectors $w_1, \dots, w_d$ are linearly independent in $\mathbb{R}^{d+1}$.

Using either of these characterizations, it becomes clear that for n vectors to be linearly independent in $\mathbb{R}^d$, it's necessary that $n \leq d + 1$.

2. Convexity

We say that a set $V \subset \mathbb{R}^d$ is **convex** if for every $x, y \in V$, the segment between x, y is contained in V. In other words, for $x, y \in V$ we also have $\text{conv}(\{x, y\}) \subset V$. By vacuity, we consider the empty set to be convex.

Exercise 1.7 Let B be the unit ball in $\mathbb{R}^d$,

$$B = \{(x_1, \ldots, x_d) : x_1^2 + \ldots + x_d^2 \leq 1\}.$$

Prove that B is convex.

Exercise 1.8 Prove that a closed half-space in $\mathbb{R}^d$ is convex.

Exercise 1.9 Let $A \subset V$ be sets in $\mathbb{R}^d$. If V is convex, show that $\text{conv}(A) \subset V$.

Exercise 1.10 Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. Show that $\bigcap \mathcal{F}$ is also convex.

Exercise 1.11 Let $A \subset \mathbb{R}^d$. Prove that $\text{conv}(A)$ is equal to the intersection of all convex sets containing A.

Exercise 1.12 Prove that a set $V \subset \mathbb{R}^d$ is convex if and only if $V = \text{conv}(V)$.

Exercise 1.13 Let $f : \mathbb{R}^d \to \mathbb{R}^l$ be a linear function, and $V \subset \mathbb{R}^d$. Prove that $f(V)$ is a connvex set in $\mathbb{R}^l$.

An important property of convex sets is the hyperplane separation theorem.

THEOREM 2.0.1. *Let A, B be two closed convex sets in $\mathbb{R}^d$. If $A \cap B = \emptyset$, then there exists a hyperplane H separating the two sets. In other words, one of the open half-spaces defined by H contains A and the other contains B.*

Exercise 1.14 Let A be a closed set in $\mathbb{R}^d$. Prove that $\text{conv}(A)$ is equal to the intersection of all closed half-spaces that contain A.

Exercise 1.15 Let A be a closed convex set in $\mathbb{R}^d$, and b a point in $\mathbb{R}^d$ such that $b \notin A$. Show that there exists a unique point $a \in A$ such that $\text{dist}(a, b)$ is minimized.

3. Birkhoff's theorem

Let us look at an example of convexity in linear algebra. We say that an $m \times m$ matrix A is *doubly stochastic* if

- each entry is non-negative,
- the sum of each of its rows is one, and
- the sum of each of its columns is one.

An example of a doubly stochastic matrix is a *permutation matrix.* A permutation matrix P is an $m \times m$ matrix whose entries are only 0 or 1 and such that it has exactly one 1 in each row and in each column. We can think of placing m rooks in an $m \times m$ chessboard so that none can attack another. Alternatively, a permutation matrix corresponds to a permutation $\pi : [m] \to [m]$ so that its entry p_{ji} is given by

$$p_{ji} = \begin{cases} 1 & \text{if } i = \pi(j) \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that you only need to know permutation matrices if you want to generate any doubly stochastic matrix.

THEOREM 3.0.1 (Birkhoff 1946 [**Bir46**]). *Let A be an $m \times m$ doubly stochastic matrix. Then, A is a convex combination of permutation matrices.*

PROOF. We show a graph-theoretic proof. We make a bipartite graph G with vertex set $[m] \times [m]$ and whose edges have positive weights. We distinguish the left component as $[m]_1$ and the right component as $[m]_2$. If $a_{ji} > 0$, we place an edge (j, i) and give it $a_j i$.

Let $R \subset [m]_1$. The sum s_1 of the weights incident to R is at most the sum s_2 of the weights of all edges incident to $N(R) \subset [m]_2$, the neighborhood of R. Notice that, by the conditions of the problem, $s_1 = |R|$ and $s_2 = |N(R)|$. Therefore, $|R| \leq |N(R)|$. By Hall's marriage theorem, there exists a perfect matching in this bipartite graph. In other words, there is a permutation $\pi[m] \to [m]$ such that $a_{j\pi(j)} > 0$.

$$\alpha = \min_{j \in [m]} a_{j\pi(j)} > 0$$

Let P be the permutation matrix induced by the permutation π. Then, $A - \alpha P$ is a matrix that

- only has non-negative entries,
- its column sums and row sums are constant, and
- it has at least one more zero than A.

Therefore we can repeat the argument and continue to substract scaled copies of permutation matrices until we get the zero matrix. At this points,

$$A - \alpha_1 P_1 - \alpha_2 P_2 - \ldots + \alpha_k P_k = 0,$$

so we have

$$A = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_k P_k$$

where all the coefficients $\alpha_1, \ldots, \alpha_k$ are non-negative. Their sum is the sum of columns of the right side, which must be equal to 1 as A is doubly stochastic. Therefore, the right side is a convex combination.

□

4. Minkowski sum

The Minkowski sum is a way to add two sets A, B in $\mathbb{R}^d$. The definition is simple

$$A \oplus B = \{a + b : a \in A, b \in B\}.$$

This sum satisfies some nice properties. It is commutative and it is closed for many families of sets in $\mathbb{R}^d$. One intuitive way to think about this sum is to consider the set A as a brush and B as a drawing we want to trace. We are effectively tracing B with a big brush A. For example, if $A \subset \mathbb{R}^2$ is a disk an B is a full rectangle, then $A \otimes B$ is a larger rectangle with rounded corners.

Exercise 1.16 Show that the Minkowski sum of convex sets is convex. Show that the Minkowski sum of centrally symmetric sets is centrally symmetric.

Exercise 1.17 Let $B(x)_r \subset \mathbb{R}^d$ be the ball centered at x with radius r. Show that $B(x)_r \oplus B(y)_s = B(x+y)_{r+s}$.

5. Two products

In $\mathbb{R}^d$, we have many different interpretations of what it means to multiply two vectors. The standard inner product $\langle \cdot, \cdot \rangle$ is the first one we usually find. Now we define two additional products.

5.1. Tensor product. The tensor product is a product that can be used to multiply vectors of different dimensions. Given two vectors $x, y \in \mathbb{R}^d$, we can think of them as $d \times 1$ matrices. Then, we can write their inner product as a matrix multiplication:

$$\langle x, y \rangle = x^T y.$$

By reversing the order, we can also define their tensor product as

$$x \otimes y = xy^T.$$

Notice that $x \otimes y$ is a $d \times d$ matrix, so we can think of it as a vector in $\mathbb{R}^{d^2}$. Moreover, we can perform the same operation if $x \in \mathbb{R}^a$ and $y \in \mathbb{R}^b$, even for $a \neq b$. Since we define the tensor product as a product of matrices, many properties of this product are immediate. First, let us think of $\cdot \otimes \cdot$ as a function:

$$\begin{aligned} \cdot \otimes \cdot : \mathbb{R}^a \times \mathbb{R}^b &\to \mathbb{R}^{ab} \\ (x, y) &\mapsto xy^T. \end{aligned}$$

From the definition, it is clear that $\otimes$ is a bilinear function (it is linear as a function of x and as a function of y). The linearity will be useful for us. The first important observation is that we can distribute sums as usual:

$$\left(\sum_{i=1}^{n} \alpha_i x_i \right) \otimes y = \sum_{i=1}^{n} \alpha_i \left(x_i \otimes y \right).$$

Exercise 1.18 Show that, if $x_1, \ldots, x_k$ are linearly dependent vectors in $\mathbb{R}^a$, and $y \in \mathbb{R}^b$, then the vectors $x_1 \otimes y, \ldots, x_k \otimes y$ are linearly dependent.

Exercise 1.19 Let $y \in \mathbb{R}^b$ be a non-zero vector. Show that, the vectors $x_1, \ldots, x_k$ in $\mathbb{R}^a$ are in convex position if and only if the vectors, $x_1 \otimes y, \ldots, x_k \otimes y$ are in convex position.

Exercise 1.20 Let $y \in \mathbb{R}^b$ be a non-zero vector. Show that the set $\{x \otimes y : x \in \mathbb{R}^a\}$ is an a-dimensional subspace of $\mathbb{R}^{ab}$.

6. Problems of Chapter 1

Problem 1 Let $\mathcal{F} = \{K_1, \dots, K_r\}$ be a family of r closed convex sets in $\mathbb{R}^d$. Prove that $\bigcap \mathcal{F} = \emptyset$ if and only if there exists a family $\mathcal{H} = \{H_1, \dots, H_r\}$ of r closed half-spaces such that

- $\bigcap \mathcal{H} = \emptyset$ and
- $K_i \subset H_i$ for $i = 1, \dots, r$.

Problem 2 Consider the moment curve γ in $\mathbb{R}^d$. This curve is given by a map $\gamma : \mathbb{R} \to \mathbb{R}^d$ such that $\gamma(t) = (t, t^2, \dots, t^d)$. Prove that a hyperplane in $\mathbb{R}^d$ cuts the moment curve in at most d different points.

CHAPTER 2

Three Pillars of Combinatorial Geometry

It is six in the morning
The house is asleep
Nice music is playing
I prove and conjecture

Letter from Paul Erdős to Vera Sós

1. Prelude

The 20th century marked the beginning of combinatorial geometry. It was in the early 1900s that mathematicians realized that convex sets had very rich intersection properties. Convex sets are structured enough to have intersection properties that we do not expect for families of sets in general, as we discuss in this chapter.

The three results presented here lay the ground of a variety of results. This chapter covers the initial results on top of which a significant part of combinatorial geometry is built. These are the building blocks of the area. In recent decades, the study of intersection patterns of convex sets has accelerated considerably. I attribute this growth to three reasons.

First, many new variants were discovered, such as colorful and fractional versions of the theorems below. In addition to being appealing on their own, these new versions allowed us to solve longstanding problems. Many results are aesthetically pleasing and easy to explain[1].

Second, combinatorial geometry is at the crossroads of topology and combinatorics. The geometry of the spaces involved provides ample opportunities to use topological machinery, and often yields stronger results than the linear-algebraic approaches. The pursuit of answering the question *"Is this result a consequence of linear algebra or topology?"* has fueled research in this area.

Third, the 20th century also witnessed the arrival of computers in everyday life. The study of computer science and the analysis of complexity classes and algorithms moved from being an academic exercise to a pragmatic necessity. Computational geometry quickly became a hot topic. The importance of convexity cannot be overstated in linear programming and (convex) optimization. The combinatorial properties of points in $\mathbb{R}^d$ are constantly used in the development of clustering algorithms and the solution to geometric range queries. Many of the results in combinatorial geometry are *existence results*. We can often prove the existence of certain partitions of families of points or the existence of families of sets with some intersection property. Computational geometers are the first to ask the following follow-up question: *how do you find such an object?*

[1]Some of my comments should be taken with a grain of salt. Would you trust a shoe-maker if he said that shoes are the defining trait of an advanced civilization?

2. Radon's lemma

2.1. Theorem and proof. Radon's theorem shows an unexpected principle in mathematics: if you want a result to be remembered, call it a lemma. The result of this section was only called a theorem after geometers realized that there were many generalizations and extensions of this particular result. Radon proved it in passing when he was proving Helly's theorem, without giving it much importance.

THEOREM 2.1.1 (Radon 1921 [**Rad21**]). *For any set of $d+2$ points in $\mathbb{R}^d$, there exists a partition of them into two sets A, B such that $\operatorname{conv}(A), \operatorname{conv}(B)$ have a non-empty intersection.*

PROOF. Let $v_1, \dots, v_{d+2}$ be our $d+2$ vectors. Since we have more than $d+1$ vectors in $\mathbb{R}^d$, they cannot be affinely independent. This means we can find coefficients $\alpha_1, \dots, \alpha_n$, not all zero, such that

$$0 = \sum_{i=1}^{d+1} \alpha_i v_i \qquad \text{and}$$
$$0 = \sum_{i=1}^{d+1} \alpha_i.$$

Since the sum of the coefficients is zero and they are not all equal to zero, some must be positive and some must be negative. Let $I_1 = \{i : \alpha_i \geq 0\}$, $I_2 = \{i : \alpha_i < 0\}$. For $i \in I_2$, let $\beta_i = -\alpha_i$. Then, we have

$$\bar{0} = \sum_{i=1}^{d+1} \alpha_i v_i = \sum_{i \in I_1} \alpha_i v_i + \sum_{i \in I_2} \alpha_i v_i \qquad \text{and}$$
$$0 = \sum_{i=1}^{d+1} \alpha_i = \sum_{i \in I_1} \alpha_i + \sum_{i \in I_2} \alpha_i$$

If we move every term with index in I_2 to the left-hand side, we have

$$\sum_{i \in I_2} \beta_i v_i = \sum_{i \in I_2} (-\alpha_i) v_i = \sum_{i \in I_1} \alpha_i v_i$$
$$\sum_{i \in I_2} \beta_i = \sum_{i \in I_2} (-\alpha_i) = \sum_{i \in I_1} \alpha_i.$$

We denote $S = \sum_{i \in I_2} \beta_i = \sum_{i \in I_1} \alpha_i$. Since not all coefficients were zero, we have $S > 0$. Therefore, by dividing by S the first equation, we have

$$\sum_{i \in I_2} \left(\frac{\beta_i}{S} \right) v_i = \sum_{i \in I_1} \left(\frac{\alpha_i}{S} \right) v_i.$$

The coefficients used on each side are non-negative and add to one, so we have two convex combinations. In other words, if we define $A = \{v_i : i \in I_1\}, B = \{v_i : i \in I_2\}$, then we have shown that $\operatorname{conv}(A) \cap \operatorname{conv}(B)$ is not empty. $\square$

A partition as the theorem indicates is called a **Radon partition**. For $d = 2$ it's easy to interpret this result. Any four points in the plane look like a triangle with a point inside, or like the vertices of a convex quadrilateral. In the first case, a partition has A being the three vertices of the convex hull and B the interior point. In the second case, each of A and B are the vertices of one of the diagonals of the quadrilateral.

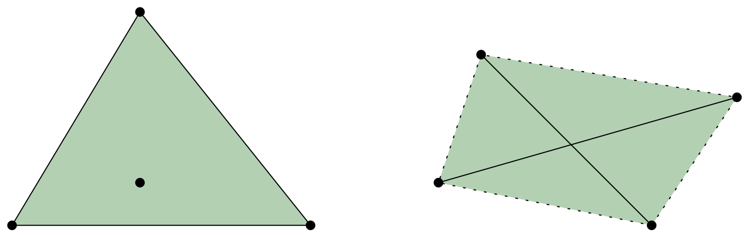


FIGURE 1. Two combinatorially distinct ways to place four points on the plane.

Exercise 2.21 Prove that the number of points in Radon's lemma is optimal. In other words, find an example of $d+1$ points in $\mathbb{R}^d$ without a Radon partition.

2.2. Colorful Radon. Lovász proved a colorful variant of Radon's theorem around 1992. His proof appeared in a paper by Bárány and Larman [**BL92**]. Lovász's proof is topological, using the Borsuk-Ulam theorem. There is a newer, simple linear algebraic proof that uses tools similar to the previous section [**Sob15**]. We will revisit this theorem when we discuss applications of topology to discrete geometry, but we present the linear algebraic proof here.

THEOREM 2.2.1 (Colorful Radon). *Let $F_1, \ldots, F_{d+1}$ be pairs of points in $\mathbb{R}^d$. Then, we can find two disjoint sets A, B, such that each contains exactly one point from each F_i and their convex hulls intersect.*

The reason why we call this result "colorful" is the interpretation where each F_i is a pair of points of a certain color. Then, the sets A, B are rainbow sets (using one point of each color).

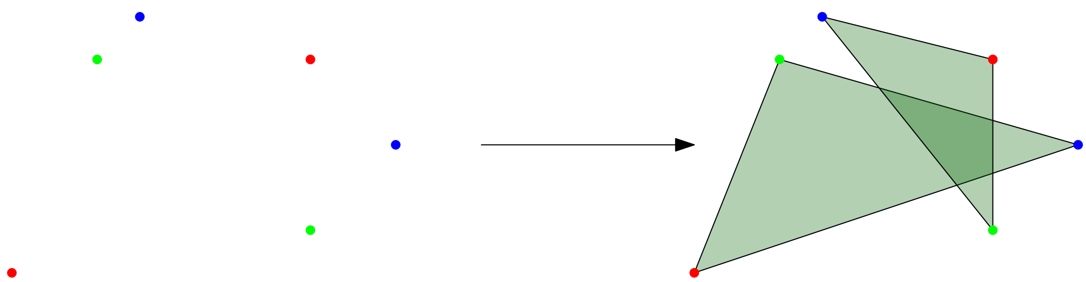


FIGURE 2. Colorful Radon in the plane.

PROOF. We denote the sets $F_i = \{x_i, y_i\}$ arbitrarily and consider the $d+1$ vectors $v_i = x_i - y_i$. Since we have $d+1$ vectors in $\mathbb{R}^d$, they have a non-trivial linear dependence

$$0 = \sum_{i=1}^{d+1} \alpha_i (x_i - y_i).$$

If any coefficient α_i is negative, we can swap the names of x_i and y_i and replace the coefficient by $-\alpha_i$. This preserves the linear dependence and reduces the number of negative coefficients. Therefore, after several relabelings, we can assume that $\alpha_i \geq 0$ for all i.

Since not all coefficients were zero, the sum $\sum_{i=1}^{d+1} \alpha_i$ is positive. By scaling all α_i by the same positive constant, we may assume that

$$\begin{aligned} 0 &= \sum_{i=1}^{d+1} \alpha_i (x_i - y_i), \\ 0 &\leq \alpha_i \qquad \text{for all } i, \text{ and} \\ 1 &= \sum_{i=1}^{d+1} \alpha_i. \end{aligned}$$

Finally, we can notice that this means

$$\sum_{i=1}^{d+1} \alpha_i x_i = \sum_{i=1}^{d+1} \alpha_i y_i,$$

and both sides are clearly convex combinations. Therefore, by taking $A = \{x_i : i = 1, \ldots, d+1\}, B = \{y_i : i = 1, \ldots, d+1\}$ we have the sets we were looking for. □

3. Helly's theorem

3.1. History. Helly's theorem is one of the most widely known results in combinatorial geometry, and rightfully so. If you look for the date when this theorem was proved, you may find conflicting answers: 1913, 1921, and 1923. In 1913 Eduard Helly discovered his theorem and told Johann Radon about it in a letter. In 1914 he went to fight in World War I in the Austrian army. He was taken as prisoner and sent to Siberia, which is why we did not return to Vienna until 1922. In 1920, Johann Radon proved Helly's theorem [**Rad21**] using Radon's theorem (Radon's theorem, subsequently, took life of its own). The first proof of Helly's theorem published by Helly appeared in 1923 [**Hel23**]. Helly found yet another proof of his theorem in 1930 [**Hel30**], which is strong enough to prove topological a version that does not need convexity.

3.2. Theorem and proof. Helly's theorem is one of the first results concerning the intersection structure of convex sets.

Theorem 3.2.1 (Helly's theorem). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d+1$ or fewer sets in $\mathcal{F}$ have non-empty intersection, then all the sets in $\mathcal{F}$ have non-empty intersection.*

Proof. We proceed by induction on $|\mathcal{F}|$. If $|\mathcal{F}| \leq d+1$, by the conditions of the theorem, we are done. Assume that the statement holds for every family $\mathcal{F}$ such that $|\mathcal{F}| = n$, and we wish to prove it for a family $\mathcal{G}$ such that $|\mathcal{G}| = n+1$, for some integer $n \geq d+1$. Let $\mathcal{G} = \{K_1, \ldots, K_{n+1}\}$. For each $i = 1, \ldots, n+1$, we define $\mathcal{G}_i = \mathcal{G} \setminus \{K_i\}$. We know that each $\mathcal{G}_i$ has n elements and satisfies the condition of Helly's theorem. Therefore, we can find a point $p_i \in \bigcap \mathcal{G}_i$.

Notice that $p_i \in K_j$ for all $j \neq i$. Since $p_1, \ldots, p_{n+1}$ is a collection of at least $d+2$ elements, by Radon's lemma there exists a partition of them into two sets A, B whose convex hulls intersect. Let $p \in A \cap B$. We claim that $p \in K_i$ for all i, which would finish the proof.

Let K_i be any of the sets in $\mathcal{G}$. The corresponding point p_i is either in A or B. We assume without loss of generality that $p_i \in A$. Let us show that $\operatorname{conv} B \subset K_i$. For each $p_l \in B$, we know $l \neq i$, and so $p_l \in K_i$. Therefore, $B \subset K_i$, and the convexity of K_i implies $\operatorname{conv} B \subset K_i$. Therefore, $p \in \operatorname{conv} B \subset K_i$, as we wanted. □

The size of the subfamilies we have to check is optimal. For example, consider any finite family of hyperplanes in general position. Any d of them intersect, but no $d+1$ of them intersect.

3.3. A quick application: the Centerpoint theorem. Given a finite set of real numbers or a probability distribution in $\mathbb{R}$, the median is an important parameter to compute. Unlike the average, this parameter is resistant to outliers (i.e., if there is an error in your data collection and one point was replaced by an absurdly large quantity, the median would remain mostly unchanged but the average would not). *What should a high-dimensional version of the median be?*

One way to extend the median to high dimensions is by using half-spaces. In $\mathbb{R}$, the median of X has $|X|/2$ points of X in either side. For a finite set $X \subset \mathbb{R}^d$, we say that p is a **centerpoint** of X if every closed half-space H^+ such that $p \in H^+$ has at least $|X|/(d+1)$ points of X.

THEOREM 3.3.1. *Every finite subset of $\mathbb{R}^d$ has at least one centerpoint.*

PROOF. Let $X \subset \mathbb{R}^d$ be a finite set. We construct a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$.

$$\mathcal{F} = \left\{ \text{conv}(Y) : Y \subset X, \ |Y| > \left(\frac{d}{d+1}\right)|X| \right\}.$$

Let us show that $\cap\mathcal{F}$ is not empty. By Helly's theorem, it suffices to show that any $d+1$ or fewer sets of $\mathcal{F}$ have a non-empty intersection. Let[2] $\text{conv}(Y_1), \dots, \text{conv}(Y_{d+1})$ be sets in $\mathcal{F}$. We know that

$$|Y_i| > \left(\frac{d}{d+1}\right)|X|.$$

Therefore,

$$|Y_1| + \ldots + |Y_{d+1}| > d|X|$$

By the pigeonhole principle, there exists an element of X was counted more than d times in $|Y_1| + \ldots + |Y_{d+1}|$. This means it was counted at least $d+1$ times, which can only happen if it is in every single Y_i. Therefore,

$$\bigcap_{i=1}^{d+1} Y_i \neq \emptyset.$$

We conclude by noticing that

$$\emptyset \neq \bigcap_{i=1}^{d+1} Y_i \subset \bigcap_{i=1}^{d+1} \text{conv}(Y_i).$$

We claim that any point $p \in \cap\mathcal{F}$ is a centerpoint of X. If that's not the case, then there exists a closed half-space H^+ such that $p \in H^+$ and $|X \cap H^+| < \frac{|X|}{d+1}$. Let us assume we have such a half-space and look for a contradiction. The complementary open half-space H^- satisfies $p \notin H^-$ and $|H^- \cap X| > \left(\frac{d}{d+1}\right)|X|$. Let $Y = H^- \cap X$. By construction, we have $\text{conv}(Y) \in \mathcal{F}$, so $p \in \text{conv}(Y)$. However, $\text{conv}(Y) \subset H^-$, so $p \notin \text{conv}(Y)$. This is the contradiction we were looking for. $\square$

[2] In this argument I'm taking $d+1$ sets, what happens if I take fewer?

3.4. A useful lemma. Helly's theorem is relevant in optimization, particularly in linear programming. A linear program is a special type of optimization problem. The first part of the input in a linear program is a set of n vectors in $\mathbb{R}^d$ and n constants:

$$a_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1d} \end{bmatrix}, \ldots, a_n = \begin{bmatrix} a_{n1} \\ \vdots \\ a_{nd} \end{bmatrix} \qquad b_1, \ldots, b_n.$$

These vectors and constants define the *feasibility region*, which is the set of vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$ that satisfy the inequalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1d}x_d &\leq b_1 \\ &\vdots \\ a_{n1}x_1 + a_{12}x_2 + \ldots + a_{nd}x_d &\leq b_n. \end{aligned}$$

If we denote by A the $n \times d$ matrix whose i-th row is a_i^T and consider $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ the n inequalities above are often described as the set of vectors x satisfying the inequality

$$Ax \leq b.$$

We consider an inequality between two vectors in $\mathbb{R}^n$ satisfied if it is satisfied for every coordinate. Each linear inequality defines a closed half-space, so the feasibility region is the intersection of a family of closed half-spaces; it is convex. Moreover, Helly's theorem gives us an easy way to check if this region is not empty: it suffices that any $d+1$ of those inequalities have a common solution.

We can aim for more. Notice we did not finish defining a linear program. A linear program is a problem where we want to find an extreme point of a feasibility region. In other words, given a non-zero vector $v \in \mathbb{R}^d$ we want to:

- maximize $\langle x, v \rangle$
- subject to $Ax \leq b$

where $\langle \cdot, \cdot \rangle$ stands for the dot product. If we swipe with the hyperplane orthogonal to v, Figure 3 shows a geometric illustration of the solution.

Helly's theorem implies that the solution to a linear program with n constraints actually only needs d constraints. More precisely:

LEMMA 3.4.1 (Useful lemma, Wegner 1975 [**Weg75**]). *Let $\mathcal{F}$ be a finite family of n convex sets in $\mathbb{R}^d$ with non-empty intersection, and let $v \in \mathbb{R}^d$ be a non-zero vector. There exists a subfamily $\mathcal{G} \subset \mathcal{F}$ of at most d sets such that*

$$\max_{x \in \bigcap \mathcal{F}} \langle x, v \rangle = \max_{x \in \bigcap \mathcal{G}} \langle x, v \rangle$$

Note: we are assuming that $\{\langle x, v \rangle : x \in \bigcap \mathcal{F}\}$ is bounded above, or otherwise $\{\langle x, v \rangle : x \in \bigcap \mathcal{G}\}$ would also be unbounded above for any subfamily $\mathcal{G}$ and we would be done.

PROOF. We denote by $\lambda = \max\limits_{x \in \bigcap \mathcal{F}} \langle x, v \rangle$. Consider the open half-space

$$H^- = \{x \in \mathbb{R}^d : \langle x, v \rangle > \lambda\}.$$

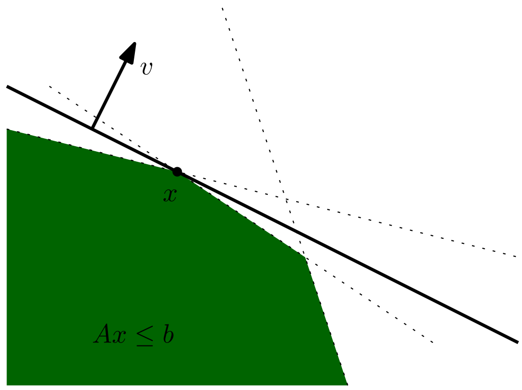


Figure 3. A solution to a linear program. The dotted lines are the constraints for the feasibility region. Notice that the optimal solution is in the intersection of two constraints in this figure.

By definition,

$$\bigcap(\mathcal{F} \cup \{H^-\}) = \emptyset.$$

By Helly's theorem, there must be a subfamily of $\mathcal{F} \cup \{H^-\}$ of at most $d+1$ elements whose intersection is empty. Since $\bigcap \mathcal{F} \neq \emptyset$ by hypothesis, this subfamily must contain H^-. In other words, it must include H^- and at most d sets of $\mathcal{F}$. Let $\mathcal{G}$ be this subset of $\mathcal{F}$. We have

$$\bigcap(\mathcal{G} \cup \{H^-\}) = \emptyset.$$

This means that $\max\limits_{x \in \bigcap \mathcal{G}} \langle x, v\rangle \leq \lambda$. Since $\mathcal{G} \subset \mathcal{F}$, we also have $\max_{x \in \bigcap \mathcal{G}} \langle x, v\rangle \geq \lambda$. Therefore,

$$\max_{x \in \bigcap \mathcal{G}} \langle x, v\rangle = \lambda = \max_{x \in \bigcap \mathcal{F}} \langle x, v\rangle,$$

as we wanted to prove. $\square$

Therefore, we can solve a linear program with n constraints by solving $\binom{n}{d}$ linear programs with d constraints each. Even though this is polynomial in n when d is fixed, one can usually do much better. We won't dive into linear programming in these notes. If you are interested in their connection with Helly's theorem, I recommend [**ALS17**, Section 3]. The "useful lemma" will be used repeatedly in these notes.

4. Carathéodory's theorem

Suppose you are given a finite (but very large) set of points X in $\mathbb{R}^d$ and a point p in $\mathbb{R}^d$. You want to determine if $p \in \operatorname{conv}(X)$ or not. This means solving a system of equations with an extremely large number of variables (one for every element of X), and then seeing if at least one of those solutions is made only of non-negative numbers[3]. Carathéodory's theorem tells us that this problem can be simplified significantly: it is enough to check the $(d+1)$-tuples of elements of X, rather than the whole set. Carathéodory's original proof works for finite sets X, but we include a proof that covers the case when $|X|$ is infinite.

Theorem 4.0.1. *Let X be a set of points in $\mathbb{R}^d$, and let $p \in \operatorname{conv}(X)$. There exists a subset $C \subset X$ of at most $d+1$ points such that $p \in \operatorname{conv}(C)$.*

[3]The fact that the coefficients must add to one can be included as an additional equation in the system.

PROOF. Since $p \in \operatorname{conv} X$, then p can be written as a convex combination of a finite number n of points of X,

$$p = \lambda_1 x_1 + \ldots + \lambda_n x_n.$$

If $n \leq d+1$, we are done. Otherwise, $n \geq d+2$. We will prove that we can write p as a convex combination of at most $n-1$ points of $x_1, \ldots, x_n$. If any of $\lambda_i = 0$, then we can remove x_i and we are done. Otherwise, $\lambda_i > 0$ for all i. Since $n \geq d+2$, there is an affine dependence of the points $x_1, \ldots, x_n$ so that

$$\begin{aligned} 0 &= \alpha_1 x_1 + \ldots + \alpha_n x_n \\ 0 &= \alpha_1 + \ldots + \alpha_n \end{aligned}$$

and not all α_i are zero. Let $I = \{i \in [n] : \alpha_i > 0\}$. Let $\varepsilon = \max_{i \in I} \frac{\lambda_i}{\alpha_i}$ and notice the following (in)equalities:

$$\begin{aligned} p &= (\lambda_1 - \varepsilon\alpha_1)x_1 + \ldots + (\lambda_n - \varepsilon\alpha_n)x_n \\ 1 &= (\lambda_1 - \varepsilon\alpha_1) + \ldots + (\lambda_n - \varepsilon\alpha_n) \\ 0 &\leq \lambda_i - \varepsilon\alpha_i \qquad \text{for all } i = 1, \ldots, n \\ 0 &= \lambda_i - \varepsilon\alpha_i \qquad \text{for at least one value of } i. \end{aligned}$$

Therefore, p is a convex combination of fewer elements of $x_1, \ldots, x_n$. We repeat this argument until we use $d+1$ or fewer points of X.

□

5. Problems of Chapter 2

Problem 3 Let X be a set of non-zero vectors in the plane and p a point in the plane. Suppose that $|\angle xpy| < 2\pi/3$ for any $x, y \in X$. Prove that $p \notin \operatorname{conv} X$.

Problem 4 Let d be a positive integer. Prove that there does not exist a set of $d+2$ non-zero vectors in $\mathbb{R}^d$ that pairwise have negative dot product.

Problem 5 Let X be a finite set of n elements. Let $U_1, \ldots, U_{n+1}$ be non-empty subsets of X. Prove that there exist two non-empty disjoint sets $A, B \subset \{1, \ldots, n\}$ such that

$$\bigcup_{i \in A} U_i = \bigcup_{i \in B} U_i$$

Problem 6 Let K be a subset of $\mathbb{R}^d$. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Prove that if the intersection of any $d+1$ or fewer sets of $\mathcal{F}$ contains a translate of K, then $\cap\mathcal{F}$ contains a translate of K.

Problem 7 Let $\mathcal{F}$ be a finite family of axis-parallel boxes in $\mathbb{R}^d$. Prove that if any two sets in $\mathcal{F}$ intersect, then $\cap\mathcal{F} \neq \emptyset$.

Problem 8 (Russia Mathematics Olympiad 1972) Let $\mathcal{F}$ be a family of 50 closed intervals in $\mathbb{R}$. Prove that there either exist eight intervals that are pairwise disjoint, or there exist eight intervals whose intersection is not empty.

Problem 9 Let F be a finite set of points in $\mathbb{R}^2$ so that no two points of F are at distance greater than one. Prove that there exists a closed disk B of radius $\frac{\sqrt{3}}{2}$ such that $A \subset B$.

Problem 10 Let $\mathcal{F}$ be a finite family of vertical segments in $\mathbb{R}^2$. Prove that if any three or fewer sets of $\mathcal{F}$ have a transversal line, then $\mathcal{F}$ has a transversal line.

Problem 11 Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^2$. Prove that if the intersection of any five or fewer sets in $\mathcal{F}$ contains an axis-parallel rectangle of area 1, then $\cap\mathcal{F}$ contains an axis-parallel rectangle of area one.

Problem 12 Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^2$. Prove that if the intersection of any four or fewer sets in $\mathcal{F}$ contains an axis-parallel rectangle of area 1, then $\cap\mathcal{F}$ contains an axis-parallel rectangle of area one.

Problem 13 Let X be a finite set in $\mathbb{R}^d$. Prove that X is in convex position if and only if every subset of size $d+2$ of X is in convex position.

Problem 14 **(Kirchberger theorem, 1908)** Suppose we have a finite number of wolves and a finite number of sheep in $\mathbb{R}^d$. You know it is impossible to build a straight fence (a hyperplane) to separate the wolves from the sheep. Prove that there is a subset of $d+2$ animals for which we cannot separate the wolves from the sheep by a straight fence.

Problem 15 Let $x_1, \ldots, x_{d+1}$ be points in $\mathbb{R}^d$, and let $p \in \mathbb{R}^d$. Suppose $p \notin \text{conv}\{x_1, \ldots, x_{d+1}\}$. Let q be the point closest to p in $\text{conv}\{x_1, \ldots, x_{d+1}\}$. Show that q can be written as the convex combination of at most d points of $\{x_1, \ldots, x_{d+1}\}$.

CHAPTER 3

A second wave of theorems

Work it harder. Make it better. Do it faster. Makes us stronger.

- Daft Punk

1. The three pillars, revisited.

Even though Carathéodory, Helly, and Radon's theorems were proved at the start of the twentieth century, it wasn't until the eighties and nineties that their best known generalizations were discovered. These generalizations uncovered nuanced properties of convex sets in euclidean spaces, and paved the way for topological, algebraic, and probabilistic methods to be applied.

This second wave was also fueled by the appearance of new proofs techniques (topological combinatorics was growing quickly) and new developments in extremal combinatorics. The early 1900s' were the time when combinatorial geometry came as a surprise; mathematicians figured that these topics were worth studying. The late 1900s' mark a solidification of the area. Computational geometry becomes important in this time, which gives mathematicians a new perspective on old problems.

How about a third wave? As we approach more recent dates, this is harder to determine objectively (i.e., is a new result fundamentally important, or am I biased to think so because it is so recent?). Jiří Matoušek considered 2010 as yet another miraculous year [**Mat11**], in which plenty of new techniques were discovered, and groundbreaking progress in old, stagnant problems was achieved. Despite the recent progress in discrete geometry, the variations listed here are still mark the standard of quality for a good result in cominatorial geometry. In other words, if you prove a new theorem, can you make it colorful? Can you make it fractional?

2. Colorful versions

2.1. Colorful Helly. We have already proved the colorful Radon theorem. You might have noticed that colorful Radon doesn't immediately imply Radon's theorem. In contrast, the colorful Helly theorem generalizes Helly's theorem.

THEOREM 2.1.1 (Colorful Helly; Lovász 1982 [**Bár82**]). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. Suppose that for every choice $F_1 \in \mathcal{F}_1, \ldots, F_{d+1} \in \mathcal{F}_{d+1}$ we know that*

$$\bigcap_{i=1}^{d+1} F_i \text{ is not empty.}$$

Then, there is an index i such that $\cap \mathcal{F}_i$ is not empty.

The reason we call this result *colorful* is the interpretation where each $\mathcal{F}_i$ is a family of sets of a particular color. Then, if every colorful $(d+1)$-tuple intersect, there is at least one color that has non-empty intersection. Helly's theorem is recovered if $\mathcal{F}_1 = \ldots = \mathcal{F}_{d+1}$. This result first appeared in a paper by Imre Bárány, where he included it with a proof by László Lovász. We present a proof that's very different from the one we used to prove Helly's theorem. If you consider $\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F}_{d+1}$, this is the second proof of Helly's theorem in this book. The ideas are similar to those from Lemma 3.4.1.

For a finite family $\mathcal{G}$ of compact convex sets in $\mathbb{R}^d$ and a non-zero vector $v \in \mathbb{R}^d$, we say a point $p \in \bigcap \mathcal{G}$ is a v-directional minimum if

$$\langle v, p \rangle = \min_{x \in \bigcap \mathcal{G}} \langle v, x \rangle.$$

Proof. Let v be a unit vector in $\mathbb{R}^d$ such that every the intersection of every d sets in $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{d+1}$ has a unique v-directional minimum. For each colorful d-tuple, record its v-directional minimum. We can choose a d-tuple whose v-directional minimum p is maximal, i.e., $\langle p, v \rangle$ is maximized among any v-directional minimum of a colorful d-tuple.

Assume without loss of generality that p is the v-directional minimum of $K_1 \cap \ldots K_d$ so that $K_i \in \mathcal{F}_i$ for each i. We will show that $p \in K_{d+1}$ for each $K_{d+1} \in \mathcal{F}_{d+1}$, finishing the proof. Let K_{d+1} be any set in $\mathcal{F}_{d+1}$, and H be the hyperplane

$$H = \{x : \langle x, v \rangle = \langle p, v \rangle\}$$

Let $\mathcal{A} = \{K_1, \ldots, K_d, K_{d+1}\}$. We know that $\bigcap \mathcal{A}$ is not empty, so it must have a point p^*. Since $\bigcap \mathcal{A} \subset \bigcap (\mathcal{A} \setminus \{K_{d+1}\})$, we know that $\langle p^*, v \rangle \geq \langle p, v \rangle$ by the minimality of $\langle p, v \rangle$. For each $i \in \{1, \ldots, d+1\}$, let q_i be the v-directional minimum of $\bigcap (\mathcal{A} \setminus \{K_i\})$. We know $\langle q_i, v \rangle \leq \langle p, v \rangle$ by the maximality of $\langle p, v \rangle$. Since both p^* and q_i are in $\bigcap (\mathcal{A} \setminus \{K_i\})$, by convexity, there must be a point $p_i \in \operatorname{conv}\left(\bigcap (\mathcal{A} \setminus \{K_i\})\right)$ such that $\langle p_i, v \rangle = \langle p, v \rangle$. Notice that $p_{d+1} = p$, since p is the unique v-directional minimum of $\bigcap (\mathcal{A} \setminus \{K_{d+1}\})$.

These conditions tell us that $p_1, \ldots, p_{d+1}$ are points in H, which has dimension $d-1$. By Radon's lemma, they can be split into two sets B, C such that their convex hulls intersect. Just as in our original proof of Helly, a point in $\operatorname{conv}(B) \cap \operatorname{conv}(C)$ is in each set of $\mathcal{A}$. In particular, it must be in $\bigcap (\mathcal{A} \setminus \{K_{d+1}\})$, so it must be equal to p. It must also be in K_{d+1}, so $p \in K_{d+1}$, as we wanted. □

2.2. Colorful Carathéodory. The colorful Carathéodory theorem is similar to the colorful Helly theorem. Now, instead of having a set $X \subset \mathbb{R}^d$ and a point $p \in \operatorname{conv}(X)$ we will have $d+1$ sets $X_1, \ldots, X_{d+1}$ (the color classes), and $p \in \operatorname{conv}(X_1) \cap \ldots \cap \operatorname{conv}(X_{d+1})$. Instead of writing p as a convex hull of a subset of $d+1$ points of X, we want to write p as a convex combination of a *colorful* set (i.e., it has at most one point of each color class).

Theorem 2.2.1 (Colorful Carathéodory, Bárány 1982 [**Bár82**]). *Let d be a positive integer and $X_1, X_2, \ldots, X_{d+1}$ be finite sets in $\mathbb{R}^d$. Let p be a point such that $p \in \operatorname{conv}(X_i)$ for $i = 1, \ldots, d+1$. Then, we there exist $x_1 \in X_1, \ldots, x_{d+1} \in X_{d+1}$ such that*

$$p \in \operatorname{conv}\{x_1, \ldots, x_{d+1}\}.$$

Of course, the original theorem by Carathéodory is recovered if we take $X_1 = X_2 = \ldots = X_{d+1}$. This is why many people consider the "simplest" case of the colorful Carathéodory theorem to be when the color classes coincide. I consider the simplest case when the color classes are each of two points. Indeed, this was the motivation that led Imre Bárány to this theorem. He needed the version when $|X_i| = 2$ for all i while working on another problem and figured that even if the color classes have a larger size the theorem might still hold. The proof of this case is similar to the one we presented of Theorem 2.2.1.

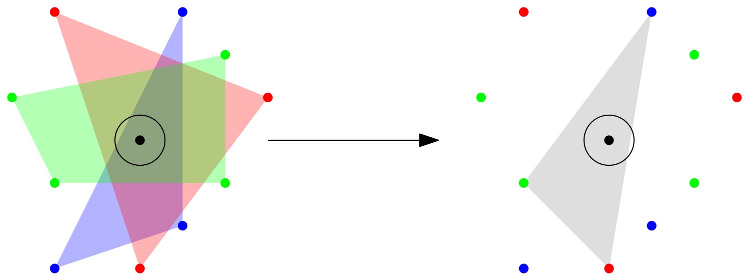


FIGURE 1. A colorful Caratheodory selection in the plane.

PROOF OF THE CASE $|X_1| = \ldots = |X_{d+1}| = 2$. We can assume without loss of generality that $p = 0$. Therefore, if x_i is one of the two points in X_i, we can assume without loss of generality[1] that the other is $-x_i$. Now consider the $d+1$ vectors $x_1, \ldots, x_{d+1}$. Since we have $d+1$ vectors in $\mathbb{R}^d$, they have a non-trivial linear dependence.

$$0 = \alpha_1 x_1 + \ldots + \alpha_{d+1} x_{d+1}$$

where not all α_i are zero. Now, if any α_i is negative, we can swap the names of x_i and $-x_i$ and the sign of α_i. Therefore, we may assume that $\alpha_i \geq 0$ for all i. We can divide both sides by $\alpha_1 + \ldots + \alpha_{d+1}$ (which is a positive constant) to write 0 as a convex combination of $x_1, \ldots, x_{d+1}$. $\square$

Notice that we can also multiply this convex combination by -1 to write 0 as a convex combination of $-x_1, \ldots, -x_{d+1}$, which finishes the proof.

PROOF OF THE GENERAL CASE. Let $x_1, \ldots, x_{d+1}$ be the colorful choice such that

$$\mu = \text{dist}(p, \text{conv}\{x_1, \ldots, x_{d+1}\})$$

is minimal. We want to show that $\mu = 0$. If it is not zero, let $q \in \text{conv}\{x_1, \ldots, x_{d+1}\}$ be the point closest to p. By Problem 2.15, q can be written as a convex combination of at most d elements of $\{x_1, \ldots, x_{d+1}\}$. We may assume without loss of generality that we are not using x_{d+1}. Let H^+ be the closed half-space defined by

$$H^+ = \{x \in \mathbb{R}^d : \langle x, p - q \rangle \geq \langle p, p - q \rangle\}.$$

Since $p \in \text{conv}(X_{d+1})$, we know that there exists $x^*_{d+1} \in X_{d+1} \cap H^+$. Now consider the set $\text{conv}\{x_1, \ldots, x_d, x^*_{d+1}\}$. Since it contains $x_1, \ldots, x_d$, it must contain q. However, the triangle $\triangle pqx^*_{d+1}$ has an obtuse angle at p, so its side pq is larger than its height from q. Therefore, $\text{dist}(p, \text{conv}\{x_1, \ldots, x^*_{d+1}\}) < \mu$, which contradicts the minimality of μ.

$\square$

3. Fractional versions

An intuitive way to think about fractional versions is that they show the robustness of a result. For example, Helly's theorem tells us what happens if every $(d+1)$-tuple of a family of convex sets is intersecting. Can we still get interesting consequences if *almost all* $(d+1)$-tuples intersect? Fortunately, the answer to this question is positive.

THEOREM 3.0.1 (Katchalski, Liu 1979 [**KL79**]). *Given $\alpha \in (0,1)$ and a positive integer d, there exists a positive constant $\beta = \beta(\alpha, d)$ for which the following statement holds. For any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if at least $\alpha\binom{|\mathcal{F}|}{d+1}$ of*

[1] As usual, don't trust me, verify.

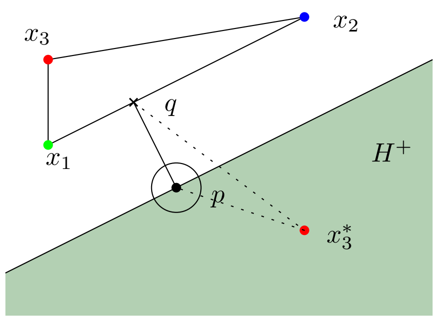


FIGURE 2. Construction for the proof of colorful Carathéodory.

the possible $(d+1)$*-tuples of* $\mathcal{F}$ *intersect, then there is a subfamily* $\mathcal{G} \subset \mathcal{F}$ *such that* $|\mathcal{G}| \geq \beta|\mathcal{F}|$ *whose intersection is not empty.*

The bound that Katchalski and Liu prove is $\beta \geq \frac{\alpha}{d+1}$. This has a simple proof, based on Lemma 3.4.1, which we present below. The optimal bound $\beta = 1-(1-\alpha)^{1/(d+1)}$ was proved independently by Kalai [**Kal84**, **Kal86**] and Eckhoff [**Eck85**].

PROOF. We assume without loss of generality that the sets in $\mathcal{F}$ are compact. Let v be a vector such that every d-tuple of sets in $\mathcal{F}$ attains its v-directional maximum at a single point. By Lemma 3.4.1, we know that for each $(d+1)$-tuple $\mathcal{F}' \subset \mathcal{F}$ with non-empty intersection, there exists a d-tuple $\mathcal{G} \subset \mathcal{F}'$ such that

$$\max_{x \in \bigcap \mathcal{G}} \langle x, v\rangle = \max_{x \in \bigcap \mathcal{F}'} \langle x, v\rangle.$$

Let $\mathcal{A} \subset \binom{\mathcal{F}}{d+1}$ the set of $(d+1)$-tuples of $\mathcal{F}$ with non-empty intersection. This assignment creates a function

$$\mathcal{A} \to \binom{\mathcal{F}}{d}.$$

By the pigeonhole principle, there must be a d-tuple $\mathcal{G}_0$ that was assigned to at least

$$\frac{|\mathcal{A}|}{\binom{|\mathcal{F}|}{d}}$$

different $(d+1)$-tuples. Notice that

$$\frac{|\mathcal{A}|}{\binom{|\mathcal{F}|}{d}} \geq \frac{\alpha\binom{|\mathcal{F}|}{d+1}}{\binom{|\mathcal{F}|}{d}} = \frac{\alpha\left(|\mathcal{F}|-d\right)}{d+1} > \left(\frac{\alpha}{d+1}\right)|\mathcal{F}| - 1$$

Let p be the unique point of $\bigcap \mathcal{G}_0$ that attains the v-directional maximum of this intersection. Every $(d+1)$-tuple of $\mathcal{F}$ assigned to $\mathcal{G}_0$ represents one additional set of $\mathcal{F}$ that contains[2] p. Since every set in $\mathcal{G}_0$ also contains p, we have found at least $\left(\frac{\alpha}{d+1}\right)|\mathcal{F}|$ sets in $\mathcal{F}$ with non-empty intersection. □

4. Tverberg's theorem

4.1. Birch's theorem on the plane. While he was an undergraduate at Trinity College in Cambdrige University, Bryan Birch proved the following theorem

THEOREM 4.1.1 (Birch 1959 [**Bir59**]). *Given* $3r$ *points on the plane, there is a partition of them into* r *triples whose convex hulls intersect.*

---

[2] The reason why the additional set must contain p is the same as in the proof of the colorful Helly theorem. Don't trust me, verify it.

The proof gives a glimpse of why combinatorial geometry is often much easier on dimension two.

PROOF. Given a set of $3r$ points, let p be a centerpoint of the set. Then, we order them clockwise as seen from p as $x_1, \dots, x_{3r}$. The r triples are going to be of the sets of the form

$$A_i = \{x_i, x_{i+r}, x_{i+2r}\} \text{ for } i = 1, \dots, r.$$

Let us prove that $p \in \operatorname{conv} A_i$ for each i. If that's not the case, then $\operatorname{conv}(A_i)$ and p may be separated by a line ℓ. Let H^+ be the closed half-plane of ℓ that contains p. Since none of x_i, x_{i+r}, x_{i+2r} are in H^+, exactly one of the clockwise angles $x_i p x_{i+r}$, $x_{i+r} p x_{i+2r}$, $x_{i+2r} p x_i$ contains all the points in our set in H^+. This means that there are at most $r-1$ such points (by the ordering of the x_j's). However, since p is a centerpoint, there must be at least r such points, a contradiction. □

Birch noticed that he could improve the result, and $3r-2$ points were sufficient. In this case, the partition is simply a partition into r sets, which need not be triples. He conjectured a high-dimensional generalization, which Tverberg proved to be true. However, if you hear someone talking about Birch's conjecture, it's much more likely the context is elliptic curves.

4.2. Brief history. Tverberg had heard of Birch's conjecture, and traveled in 1963 to Manchester to discuss his ideas with Birch. He was staying at a hotel where he needed to put coins in the heater to get it to work. ~~Unfortunately~~, he was out of shillings one night and was unable to sleep due to the cold. He decided to keep thinking about the problem, and the solution (as he describes it) "dawned on him".

THEOREM 4.2.1 (Tverberg 1966 [**Tve66**]). *Given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there exists a partition of them into r sets whose convex hulls intersect.*

Notice that the case $r = 2$ is precisely Radon's lemma.

Tverberg's proof of his theorem uses a clever argument about points moving in $\mathbb{R}^d$. Essentially, he proves the result for a simple set of points. Then, he shows that if a single points moves in a fixed directions, at any moment that the current partition stops working we can swap points and avoid any problems. This way, we can arrive to any other set of $(r-1)(d+1)+1$ points.

The original proof is not very transparent. A "wishful thinking" among combinatorial geometers was that we would find a proof similar to Radon's proof of his lemma. In Radon's proof, the existence of a partition is reduced to an affine dependence, and the sign of the coefficients provides this partition. *Would it be possible to extend such a proof to use coefficients of r types instead of just two?*

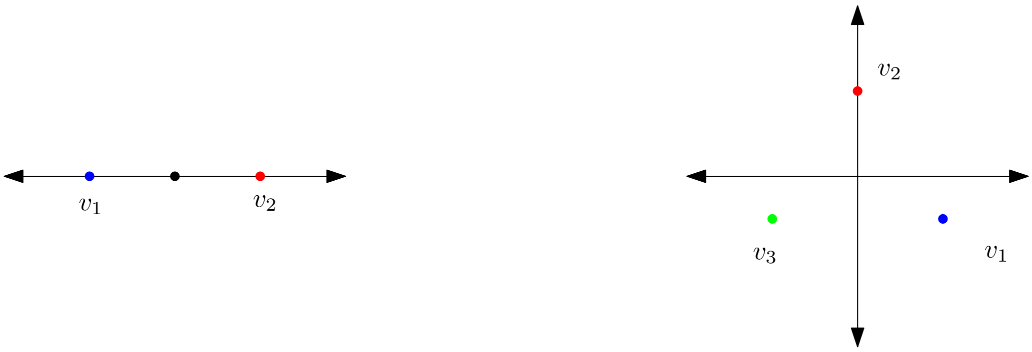
Surprisingly, this wishful thinking turned out to be correct. Karanbir Sarkaria found such a proof in 1992 [**Sar92**]. The proof we present here is a simplified version by Bárány and Onn [**BO95**].

4.3. Proof of Tverberg's theorem. This proof relies on the tensor product, described in Section 5.1. Let $n = (r-1)(d+1)$. Suppose we are given $n+1$ points $a_1, a_2, \dots, a_{n+1}$ in $\mathbb{R}^d$. We first embed these points in $\mathbb{R}^{d+1}$ by defining

$$\begin{aligned} \mathbb{R}^d &\to \mathbb{R}^{d+1} \\ a_i &\mapsto b_i = (a_i, 1) \end{aligned}$$

Additionally, we consider r vectors $v_1, \dots, v_r \in \mathbb{R}^{r-1}$ whose only linear dependence, up to scalar multiples, is

$$v_1 + \ldots + v_r = 0.$$



$r = 2$ $r = 3$

FIGURE 3. For the case $r = 2$, we can take $v_1 = -1, v_2 = 1$. For the case $r = 3$ we can take

In other words, if $\beta_1 v_1 + \ldots \beta_r v_r = 0$ then we have $\beta_1 = \ldots = \beta_r$. We can think of these vectors as the vertices of a regular simplex centered at the origin in $\mathbb{R}^{r-1}$, see Figure 3. We aim to use the vectors $v_1, \ldots, v_r$ as coefficients.

Now, for $i \in [n+1], j \in [r]$, we construct the vectors

$$u_{i,j} = b_i \otimes v_j \in \mathbb{R}^{(d+1)(r-1)} = \mathbb{R}^n.$$

If i is fixed, the convex hull of the family $X_i = \{u_{i,1}, \ldots, u_{i,r}\}$ contains the zero vector, since the sum of its elements is the zero vector. Notice that we have $n+1$ such families in $\mathbb{R}^n$. By the colorful Carathéodory theorem (Theorem 2.2.1), we can choose one element of each such that the resulting set contains the origin.

In other words, for each $i \in [n]$ we can choose $j(i) \in [r]$ such that

$$0 \in \operatorname{conv}\{u_{1,j(1)}, \ldots, u_{n+1,j(n+1)}\}.$$

There must be coefficients $\alpha_1, \ldots, \alpha_{n+1}$ of a convex combination such that

$$0 = \sum_{i=1}^{n+1} \alpha_i u_{i,j(i)} = \sum_{i=1}^{n+1} \alpha_i \left(b_i \otimes v_{j(i)}\right).$$

Let us group the sum above by the value of $j(i)$. In other words, we define the sets $I_1, \ldots, I_r$ by $I_j = \{i \in [n+1] : j(i) = j\}$. Then,

$$\begin{aligned} 0 = \sum_{i=1}^{n+1} \alpha_i \left(b_i \otimes v_{j(i)}\right) &= \left(\sum_{i \in I_1} \alpha_i \left(b_i \otimes v_{j(i)}\right)\right) + \ldots + \left(\sum_{i \in I_r} \alpha_i \left(b_i \otimes v_{j(i)}\right)\right) \\ &= \left(\sum_{i \in I_1} \alpha_i \left(b_i \otimes v_1\right)\right) + \ldots + \left(\sum_{i \in I_r} \alpha_i \left(b_i \otimes v_r\right)\right) \\ &= \left(\sum_{i \in I_1} \alpha_i b_i\right) \otimes v_1 + \ldots + \left(\sum_{i \in I_r} \alpha_i b_i\right) \otimes v_r \end{aligned}$$

The fact that $v_1, \ldots, v_r$ only have a linear dependence implies that

$$\sum_{i \in I_1} \alpha_i b_i = \sum_{i \in I_2} \alpha_i b_i = \ldots = \sum_{i \in I_r} \alpha_i b_i$$

Let $\beta_i = r\alpha_i$. Then, we also have

$$\sum_{i\in I_1} \beta_i b_i = \sum_{i\in I_2} \beta_i b_i = \ldots = \sum_{i\in I_r} \beta_i b_i.$$

The last coordinate of each b_i is equal to 1, so

$$\sum_{i\in I_1} \beta_i = \sum_{i\in I_2} \beta_i = \ldots = \sum_{i\in I_r} \beta_i.$$

Moreover, $\sum_{i=1}^{n+1} \beta_i = r\sum_{i=1}^{n+1} \alpha_i = r$. Therefore, for each $j \in [r]$, the coefficients $\{\beta_i : i \in I_j\}$ are the coefficients of a convex combination. Now let us look at the first d coordinates of b_i, which is precisely the vector a_i. This implies that

$$\sum_{i\in I_1} \beta_i a_i = \sum_{i\in I_2} \beta_i a_i = \ldots = \sum_{i\in I_r} \beta_i a_i.$$

Each of these is a convex combination and they are all equal. Therefore, the sets $A_j = \{a_i : i \in I_j\}$ form the partition we were looking for.

4.4. The selection lemma. A neat application of Tverberg's theorem is the *selection lemma*. The idea behind this lemma is, given a finite set $S \subset \mathbb{R}^d$, to find a point p which is "very deep" within S. We encountered a result of this kind with the centerpoint theorem in Section 3.3. The centerpoint measures the depth of p by determining the smallest number of points of S that a half-space that contains p can have. Now, we want to measure the depth of p by the number of simplices with vertices on S that contain p. The total possible number of simplices with vertices in S is $\binom{|S|}{d+1}$. It turns out that there are points p contained in a fixed proportion of the total amount of simplices.

THEOREM 4.4.1 (Selection Lemma; Bárány 1982 [**Bár82**]). *Let d be a positive integer. There exists a constant $c_d > 0$, that depends only on d, such that the following statement holds. For any finite set $S \subset \mathbb{R}^d$, there exists a point p contained in at least $c_d\binom{|S|}{d+1}$ simplices with vertices in S.*

The estimate we prove gives $c_d \sim \frac{1}{(d+1)^{d+1}}$, although there are currently better bounds [**Gro10**, **Kar11**]. The only value of c_d which is known exactly is $c_2 = 2/9$.

PROOF. Let $r = \left\lfloor \frac{|S|}{d+1} \right\rfloor$. Then, $|S| \geq (r-1)(d+1)+1$, so by Tverberg's theorem S can be split into r sets $A_1, \ldots, A_r$ whose convex hulls intersect. Let p be the point of intersection. Color each A_i of a different color. If we take any $d+1$ color classes, by the colorful Carathéodory theorem there exists a colorful simplex that contains p. Therefore, we obtain $\binom{r}{d+1}$ different simplices with this property. Finally, observe that

$$\binom{r}{d+1} = \binom{\left\lfloor \frac{|S|}{d+1} \right\rfloor}{d+1} \sim \frac{1}{(d+1)^{d+1}}\binom{|S|}{d+1}.$$

□

5. The integer lattice

In 1973 Jean-Paul Doignon proved an integer version of Helly's theorem [**Doi73**]. In this result, instead of guaranteeing a point in the intersection of a family of convex sets, we guarantee a point *with integer coordinates* in the intersection of a family of convex sets. An interesting part of the history of this result is that it was rediscovered a couple of times [**Bel76**, **Sca77**] since it has applications in integer programming and economics.

THEOREM 5.0.1 (Doignon, 1973). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every 2^d or fewer sets in $\mathcal{F}$ contains a point of $\mathbb{Z}^d$, then $\bigcap \mathcal{F}$ contains a point of $\mathbb{Z}^d$.*

Bell's proof is probably the simplest. We show Hoffman's approach [**Hof79**], as it has led to more generalizations. Hoffman's argument show how the number 2^d is related to the maximum possible number of integer points in convex position whose convex hull does not contain any additional integer point.

LEMMA 5.0.2. *For any set P of $2^d + 1$ or more points in $\mathbb{Z}^d$, the convex hull of P contains an integer point that is not a vertex of* $\operatorname{conv} P$.

PROOF. Given a point in $\mathbb{Z}^d$, look at its entries modulo 2. There are 2^d possibilities. By the pigeonhole principle, there are two points x, y of P whose corresponding entries have the same parity. Therefore, $(x + y)/2$ is in $\mathbb{Z}^d$ and in $\operatorname{conv} P$. Moreover, $(x + y)/2$ cannot be a vertex of $\operatorname{conv} P$. $\square$

The following lemma is the core of the proof of Doignon's theorem.

LEMMA 5.0.3. *Let $P \subset \mathbb{R}^d$ be a point of more than 2^d points in $\mathbb{Z}^d$. Then,*

$$\bigcap_{p \in P} \operatorname{conv}(P \setminus \{p\})$$

has at least one point of $\mathbb{Z}^d$.

PROOF. Notice that it is enough to prove the result above for sets P that have exactly $2^d + 1$ elements. We prove this theorem by induction. We can consider the family of all subsets of size $2d + 1$ of $\mathbb{Z}^d$, ordered by inclusion of their convex hull. In other words, for two subset P', P of $2^d + 1$ points of $\mathbb{Z}^d$ each, we say that $P' \le P$ if $\operatorname{conv} P' \subset \operatorname{conv} P$.

If P is a set of $2^d + 1$ elements of $\mathbb{Z}^d$, it has a finite number of predecessors. We are going to show that *If the conclusion of the theorem holds for any $P' \le P$, then it holds for P.* If P has no predecessors in this partial order, the conditions is met by vacuity so the result will hold for P. The finite number of predecessor implies that this induction will eventually go through every set P we want to consider.

Let P be a set of $2^d + 1$ points of $\mathbb{Z}^d$. We may assume without loss of generality that every $p \in P$ is a vertex of $\operatorname{conv} P$. Otherwise, if some $p_0 \in P$ is not a vertex of $\operatorname{conv} P$ we have that $p_0 \in \operatorname{conv}(P \setminus \{p_0\})$, so $p_0 \in \bigcap_{p \in P} \operatorname{conv}(P \setminus \{p\})$ and we are done.

By Lemma 5.0.2 we know that $(\operatorname{conv} P) \setminus P$ contains at least one integer point. For each integer point q in $(\operatorname{conv} P) \setminus P$, let $m(q)$ be the number of elements $p \in P$ such that $q \notin \operatorname{conv}(P \setminus \{p\})$. Let q_0 be an integer point such that $m(q_0)$ is minimal. If $m(q_0) = 0$, we are done.

Otherwise, assume that $m(q_0) > 0$ and we look for a contradiction. There is at least one $p' \in P$ such that $q_0 \notin \operatorname{conv}(P \setminus \{p'\})$. This means that the set $P' = P \cup \{q_0\} \setminus \{p\}$ is a strict predecessor of P. By the induction hypothesis, $\bigcap_{p \in P'} \operatorname{conv}(P' \setminus \{p\})$ contains an integer point r.

Now, notice that if $q_0 \in \operatorname{conv}(P \setminus \{p\})$ for some $p \in P$, we have

$$r \in \operatorname{conv}(P' \setminus \{p\}) \subset \operatorname{conv}(P \setminus \{p\}).$$

The last containment holds because we can write q_0 as a convex combination of the elements of $P \setminus \{p\}$, and therefore any convex combination of the elements of $P' \setminus \{p\}$ is also a convex combination of the elements of $P \setminus \{p\}$ (we only substitute q_0 if it was actually used).

Additionally

$$r \in \operatorname{conv}(P' \setminus \{q_0\}) = \operatorname{conv}(P \setminus \{p'\}).$$

This means that $r \in \operatorname{conv} P$, it is not an element of P, and $m(r) < m(q_0)$, contradicting the minimality of $m(q_0)$. □

Now we are ready to prove Doignon's theorem.

Proof of Theorem 5.0.1. We prove this result by induction on $n = |\mathcal{F}|$. If $|\mathcal{F}| = 2^d$, we havee nothing to prove. Therefore, we may assume that $n > 2^d$ and that the result holds for all families of $n-1$ sets. We denote the elements of $\mathcal{F}$ as $\mathcal{F} = \{F_1, \dots, F_n\}$.

By the induction hypthesis, for each $i = 1, \dots, n$ we can find an integer point in the intersection of $\mathcal{F} \setminus \{F_i\}$. Let $P = \{p_1, \dots, p_n\}$. By Lemma 5.0.3, we can find an integer point r in

$$\bigcap_{i=1}^{n} \operatorname{conv}(P \setminus \{p_i\}).$$

Notice that for each $i = 1, \dots, n$ we have $P \setminus \{p_i\} \subset F_i$, so we have

$$r \in \bigcap_{i=1}^{n} \operatorname{conv}(P \setminus \{p_i\}) \subset \bigcap_{i=1}^{n} F_i = \bigcap \mathcal{F},$$

as we wanted to prove. □

6. Problems of Chapter 3

Problem 16 Let S be a set of $2r$ points in the plane. For any set of points $X \subset \mathbb{R}^2$, let $\operatorname{Rec}(X)$ be the smallest axis-parallel rectangle in the plane that contains X. Prove that there exist at least $r^2/4$ different partitions of S into r pairs $A_1, \dots, A_r$ such that

$$\bigcap_{j=1}^{r} \operatorname{Rec}(A_j) \neq \emptyset.$$

• (challenge) Show that for sets of points in $\mathbb{R}^3$ and axis-parallel boxes there may not be a single partition of $2r$ points into r pairs whose induced axis-parallel boxes intersect.

Problem 17 *Not all Helly-type properties have a colorful version.* Consider the following claim: *A finite set S in $\mathbb{R}^d$ is in convex position if and only if every subset of $d+2$ points is in convex position.*

- Prove this claim.
- State what the colorful version of this claim would say.
- Disprove the colorful version.

Problem 18 *Weak Tverberg.* The following weaker version of Tverberg's theorem can be proved without tensor products. Prove it using the centerpoint theorem. Given a set of $(r-1)d(d+1)+1$ points in $\mathbb{R}^d$ there exists a partition of them into r sets whose convex hulls intersect.

Problem 19 For fixed positive integers k, n, d, such that $k \leq d$, find a family of n convex sets in $\mathbb{R}^d$ such that the largest intersecting subfamily has $k+d$ sets and the number of intersecting $(d+1)$-tuples is

$$\binom{n}{d+1} - \binom{n-k}{d+1}.$$

Why does this construction show that Kalai's bound on the fractional Helly theorem is optimal?

Problem 20 (Linström 1972 [**Lin72**]) Let X be a finite set of n elements. Let $U_1, \dots, U_{(r-1)n+1}$ be non-empty subsets of X. Prove that there exist r non-empty, pairwise disjoint sets $A_1, \dots, A_r \subset \{1, \dots, n\}$ such that

$$\bigcup_{i \in A_1} U_i = \dots = \bigcup_{i \in A_r} U_i$$

Problem 21 Construct a finite family $\mathcal{F}$ of convex sets such that the intersection of any $2^d - 1$ or fewer sets in $\mathcal{F}$ contains a point in $\mathbb{Z}^d$ but $\bigcap \mathcal{F}$ does not contain a point in $\mathbb{Z}^d$.

Problem 22 (De Loera, La Haye, Oliveros, Roldán-Pensado 2015 [**LRPOLH15**]) State and prove a colorful version of Doignon's theorem.

Problem 23 **(r-fold Kirchberger, Pór 1998)** Let $A_1, \dots, A_r$ be sets in $\mathbb{R}^d$ such that $\bigcap_{j=1}^r \operatorname{conv}(A_j) \neq \emptyset$. Prove that there exists a set $C \subset \bigcup_{j=1}^r A_j$ such that $|C| \leq (r-1)(d+1)+1$ and

$$\bigcap_{j=1}^{r} \operatorname{conv}(A_j \cap C) \neq \emptyset.$$

CHAPTER 4

The analytic side of convexity

Simplify, simplify.

- Henry David Thoreau[1]

Combinatorial geometry in the twentieth century (and today!) largely focused on intersection patterns of convex sets. As mentioned in Chapter 2, the theorems of Radon, Helly, and Carathéodory provided a new connection between linear algebra and combinatorics, which fueled research and interest in the area.

However, convex sets had been studied extensively by geometers, well before mathematicians were interested in their intersection patterns. The point of view was analytic, and the goal was to understand the general *shape* of high-dimensional convex sets. The following questions may help build some intuition regarding what type of questions are interesting in this context: *do you think that a general convex set more similar to a sphere or to a hypercube? Is the volume concentrated near the boundary or near its center? How do the volume and the surface area relate to each other?*

In this chapter, we introduce some classic results by linking them to different versions of Helly's theorem. If you are interested in this side of convexity, you will enjoy reading expository notes dedicated to that subject. The best reference is Keith Ball's introduction to modern convex geometry [**Bal97**].

1. Quantitative versions of Helly's theorem

Helly's theorem characterizes families of convex sets whose intersection is not empty. However, we may want to guarantee that the intersection is large. For example, consider the following volumetric version of Helly's theorem.

Theorem 1.0.1 (Bárány, Katchalski, Pach 1982 [**BKP82**]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every $2d$ or fewer sets in $\mathcal{F}$ has volume greater than or equal to 1, then $\bigcap \mathcal{F}$ has volume greater than or equal to d^{-2d^2}.*

There have been many improvements on this result since the 80's. It exemplifies how we might want extend Helly's theorem. We might change the function to optimize, the size of the subfamilies to check, or the guarantee we get in the conclusion. You can see that in the result above the guarantee of the volume in $\bigcap \mathcal{F}$ is smaller than the condition on $2d$-tuples. We call the theorem above a *non-exact* quantitative Helly theorem. Compare this with an *exact* quantitative Helly theorem, such as the following.

Theorem 1.0.2 (Sarkar, Xue, Soberón 2021+ [**SXS21**]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every $2d$ or fewer sets in $\mathcal{F}$*

[1] If Thoreau had been a discrete geometer instead of a naturalist, he probably would have said "Parametrize, parametrize" instead.

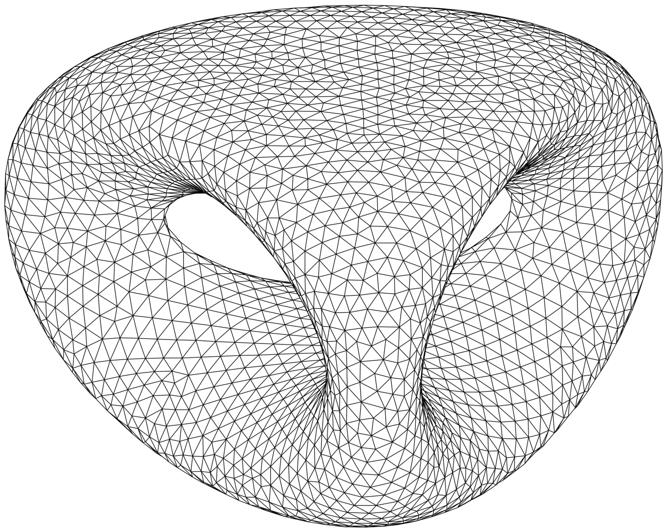


FIGURE 1. An example of a mesh on a surface of genus two. You may want to minimize the number of triangles or vertices in the mesh to optimize the rendering speed of the surface. Image credit: Wikipedia CC BY-SA 4.0.

contains an axis-parallel box of volume 1*, then* $\bigcap \mathcal{F}$ *contains an axis-parallel box of volume* 1*.*

An interesting question is to decide which functions of convex sets have an associated exact quantitative Helly theorem, which ones have a non-exact quantitative Helly theorem, and which ones have neither.

2. Approximation of convex sets by polytopes

From a combinatorial point of view, the simplex convex sets are polytopes. We can measure the complexity of a polytope by looking at some simple parameters, such as the number of vertices or the number of facets of the polytope. If you want to venture further, then there are more elaborate parameters that mathematicians care about, such as the f-vector and the h-vector of the polytope.

Given a convex body K in $\mathbb{R}^d$ (a convex body is a compact convex set with non-empty interior), we may want to approximately as closely as possible with a polytope of low complexity. Approximating general mathematical objects with low complexity objects is commonplace in mathematics (for example, approximating continuous functions with polynomials). In the case of convex sets and polytopes, it has an additional motivation: computer graphics.

Suppose you want to make a program that renders a sphere or a particular surface. A common approach is to make a mesh on the surface and render the surface as a union of piece-wise linear parts (see Figure 1). What is the simplest mesh you need to approximate the surface efficiently?

The theory of approximating convex sets by polytopes is quite rich [**Gru93a**, **Bro07**]. Given a convex set K in $\mathbb{R}^d$, we often seek a polytope P such that

- $P \subset K$
- P has few vertices
- P is very close to K

or such that

- $K \subset P$
- P has few facets
- P is very close to K

How "close" P and K are is often measured by the Hausdorff metric. For this section, we use results regarding the Nikodym metric

$$\Delta(K,P) = \mathrm{vol}((K \cup P) \setminus (K \cap P))$$

The main result we will use is the following[2]:

THEOREM 2.0.1 (Gruber 1993, Böröcyzky 2000 [**Gru93b**, **Bör00**]). *Let $\varepsilon > 0$ and d be a positive integer. There is a constant $k = \Theta_d(\varepsilon^{-(d-1)/2})$ such that for any convex set $K \subset \mathbb{R}^d$ there exists a polytope P such that*

- *$K \subset P$,*
- *P has at most k facets, and*
- *$\mathrm{vol}(P) \leq (1+\varepsilon)\,\mathrm{vol}(K)$.*

The quantity $\varepsilon^{(-d-1)/2}$ frequently appears in this kind of result. Just like $d+1$ is "the magic number" in combinatorial geometry, $\varepsilon^{(-d-1)/2}$ is the "magic asymptotic behavior" in analytic convexity. It turns out that $\varepsilon^{(-d-1)/2}$ is asymptotically optimal. If K is a sphere, then it can be shown that this many facets are needed. Using the theorem above, we can get a Helly-type theorem for the volume.

THEOREM 2.0.2. *Let $\varepsilon > 0$, d be a positive integer, and $k = k(\varepsilon, d)$ be the parameter of Theorem 2.0.1. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of every kd or fewer sets in $\mathcal{F}$ has volume at least one, then $\bigcap \mathcal{F}$ has volume at least $(1+\varepsilon)^{-1}$.*

PROOF. Let $K = \bigcap \mathcal{F}$. By Steinitz's theorem (Problem 28), K has non-empty interior. If K is not bounded, then $\mathrm{vol}(K) = \infty$. Otherwise, by Theorem 2.0.1 we can find a polytope P with k facets, that contains K and whose volume is at most $(1+\varepsilon)\,\mathrm{vol}(K)$. We can write P as the intersection of k half-spaces $H_1, \dots, H_k$. For each H_i, as $\bigcap \mathcal{F} \subset H_i$, by Lemma 3.4.1 we can find d sets $F_{i1}, \dots, F_{id} \in \mathcal{F}$ such that

$$\bigcap_{j=1}^{d} F_{ij} \subset H_i$$

Consider the family $\mathcal{G} = \{F_{ij} : 1 \leq i \leq k,\ 1 \leq j \leq d\}$. We have

$$\bigcap \mathcal{G} = \bigcap_{i=1}^{k} \left(\bigcap_{j=1}^{d} F_{ij} \right) \subset \bigcap_{i=1}^{k} H_i = P$$

Since $\mathcal{G}$ has kd elements, its intersection has volume at least 1. This implies that

$$1 \leq \mathrm{vol}\left(\bigcap \mathcal{G}\right) \leq \mathrm{vol}(P) \leq (1+\varepsilon)\,\mathrm{vol}(K),$$

which implies the desired conclusion. □

3. Concentration of volume of the sphere

3.1. Classic results. Volume in high dimensions is tricky, and our low-dimensional intuition often works against us. For example, how large is the radius of the d-dimensional ball of volume one? We follow Keith Ball's exposition in this section [**Bal97**] (so you know where to look if this is skipping too many details).

[2]The notation $\Theta_d(\cdot)$ means that there are some multiplicative factors that depend on d which are not included.

This radius, r_d is roughly $\sqrt{\frac{d}{2\pi e}}$. One way to figure this out is to consider the function

$$f(x_1, \ldots, x_d) = e^{-(x_1^2 + \ldots + x_d^2)/2}$$

and compute

$$\int_{\mathbb{R}^d} f(x)$$

in two different ways[3]. How about the $(d-1)$-volume of a slice through the center of the volume one sphere. If r_{d-1} is the radius of the volume one $(d-1)$-dimensional sphere, then this volume is

$$\left(\frac{r_d}{r_{d-1}}\right)^{d-1} \sim \left(\frac{d}{d-1}\right)^{(d-1)/2} \sim \sqrt{e}.$$

The parallel slices at distance x from the center have radius $\sqrt{r_d^2 - x^2}$, so they have $(d-1)$ volume

$$\begin{aligned}\left(\frac{\sqrt{r_d^2 - x^2}}{r_{d-1}}\right)^{d-1} &\sim \sqrt{e}\left(\frac{\sqrt{r_d^2 - x^2}}{r_d}\right)^{d-1} = \sqrt{e}\left(1 - \frac{x^2}{r_d^2}\right)^{(d-1)/2} \\ &\sim \sqrt{e}\left(1 - \frac{2\pi e x^2}{d}\right)^{(d-1)/2} \sim \sqrt{e}\left(e^{-\pi e x^2}\right).\end{aligned}$$

The big surprise here is that the last expression does *not* depend on the dimension. So, if x is fixed, the total volume of the unit volume ball between two parallel hyperplanes at (a fixed) distance x from the center converges as d goes to infinity. There are many nice consequences of this result. One, that you'll prove in Problem 26, is that for any n vectors in $\mathbb{R}^d$ there is a fixed proportion of them that are pointing roughly in the same direction.

3.2. A fractional Helly theorem for the diameter. Let us use this to prove a Helly-type theorem for the diameter [**DS20**]. We'll assume that you have completed Problem 26.

Given a compact convex set $K \subset \mathbb{R}^d$, and a unit vector v, we denote the v-directional width as

$$\max_{x,y \in K} \langle v \cdot (x - y) \rangle$$

THEOREM 3.2.1 (Dillon, Soberón 2020+ [**DS20**]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ and let $c > 0$. Let $\beta = 1 - (1 - \gamma(c))^{1/2d}$, where $\gamma(\cdot)$ is the function from Problem 26. Suppose that the intersection of any $2d$ or fewer sets in $\mathbb{R}^d$ is not empty and has diameter greater than or equal to one. Then, there is a subfamily $\mathcal{G} \subset \mathcal{F}$ such that $|\mathcal{G}| \geq \beta|\mathcal{F}|$ and whose intersection has diameter greater than or equal to $cd^{-1/2}$.*

An interesting bit here is that if we take $c \to 0$, then $\beta \to 1$, so we get close to proving an old conjecture of Bárány, Katchalski, and Pach. They conjecture that there exists some absolute constant $c > 0$ for which we can take $\beta = 1$ (i.e., $\mathcal{G} = \mathcal{F}$).

PROOF. Given a convex set $K \in \mathbb{R}^d$, a unit vector $v \in \mathbb{R}^d$, and a constant $c > 0$ we'll consider a set $S(K) \subset \mathbb{R}^{2d}$ as

$$S_v(K) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \langle v, x - y \rangle = cd^{-1/2}\}$$

[3] One is by noticing that f is invariant over rotations, so the integral is related to the volume of spheres. The other is to notice that $f(x_1, \ldots, x_d) = e^{-x_1^2/2} \cdot \ldots \cdot e^{-x_d^2/2}$, so the integral can be computed explicitly.

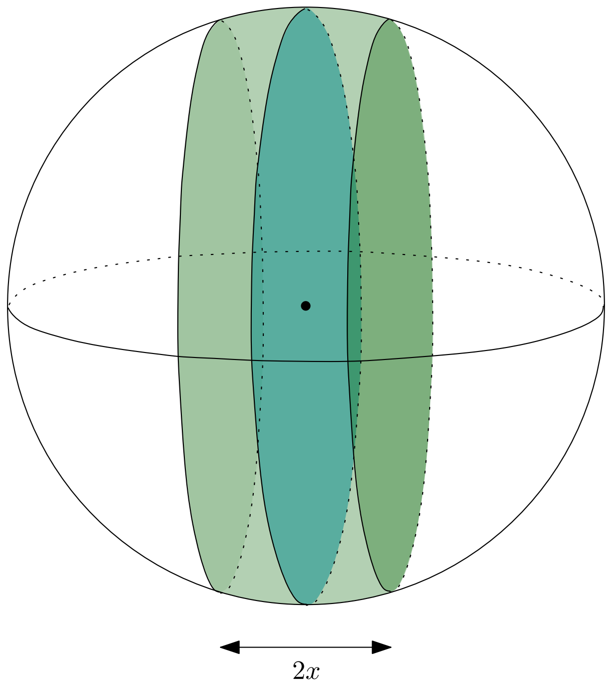


FIGURE 2. The region between two parallel hyperplanes at distance x from the origin.

Since $S(K)$ is defined by a linear equation on $\mathbb{R}^{2d}$, it is really in a $(2d-1)$-dimensional affine space. Notice that if K is convex, so is $S(K)$.

For each $2d$-tuple in $\mathcal{F}$, consider a unit vector contained in its intersection. By Problem 20, we can find a unit vector v whose dot product with at least $\gamma(c)\binom{|\mathcal{F}|}{2d}$ of these vectors has an absolute value greater than or equal to $cd^{-1/2}$.

In other words, for this particular v if we consider the family

$$\mathcal{F}' = \{S_v(K) : K \in \mathcal{F}\},$$

we know that at least $\gamma(c)\binom{|\mathcal{F}'|}{2d}$ of its $2d$-tuples have non-empty intersection. If we apply the fractional Helly theorem (Theorem 3.0.1) with Kalai's bounds, we obtain the desired conclusion. □

3.3. Tarski's plank problem. In dimension two, we say that a plank of width w is the region of the plane between two parallel lines at distance w. In 1932 Tarski conjecture the following

THEOREM 3.3.1. *If a unit disk in the plane is covered by a finite number of planks, the sum of their widths must be at least* 2.

This was proved in 1951 by Bang [**Ban51**], and the solution is quite clever.

PROOF. Consider the two-dimensional plane Π as embedded in $\mathbb{R}^d$ by considering $(x, y) \mapsto (x, y, 0)$. Then, the unit disk is the intersection of a unit ball with Π. A two-dimensional plank can be extended to three dimensions using the direction vector $(0, 0, 1)$. In other words, if we solve Tarski's problem in dimension three, it would imply the result in dimension two (considering a plank of width w in $\mathbb{R}^3$ as the region between two parallel planes at distance w).

So far, it seems we just made our task harder. However, if a family of planks in $\mathbb{R}^3$ covers a unit ball, they also cover its surface. If the widths of the planks are $w_1, \dots, w_n$, by Problem 24 they cover at most a $(\sum_{i=1}^{n}(w_i/2))$-fraction of the surface area. Therefore, the sum of the widths must be at least 2. $\square$

Tarski's plank problem holds in higher dimensions and for other ways of measuring the width of the planks (if we use a Minkowski metric).

4. The Brunn-Minkowski inequality

4.1. Main result. Let's motivate this section with a math competition problem.

Problem (Putnam 2003, A2). *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be non-negative real numbers. Prove that*

$$\Big((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\Big)^{1/n} \geq (a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n}.$$

Solution. If $a_i + b_i = 0$ for some i, then $a_i = b_i = 0$ and both sides of the inequality are zero. Otherwise, notice that we can replace a_i, b_i by $\alpha a_i, \alpha b_i$ for any positive α and both sides of the inequality are just multiplied by $\alpha^{1/n}$. Therefore, we may assume without loss of generality that $a_i + b_i = 1$ for all i. Now we can apply the AM-GM inequality and obtain

$$(a_1a_2\dots a_n)^{1/n} \leq \frac{a_1+a_2+\dots+a_n}{n}$$
$$(b_1b_2\dots b_n)^{1/n} \leq \frac{b_1+b_2+\dots+b_n}{n}$$

We can add the two inequalities and get

$$(a_1a_2\dots a_n)^{1/n} + (b_1b_2\dots b_n)^{1/n} \leq \frac{(a_1+a_2+\dots+a_n)+(b_1+b_2+\dots+b_n)}{n}$$
$$(a_1a_2\dots a_n)^{1/n} + (b_1b_2\dots b_n)^{1/n} \leq \frac{(a_1+b_1)+(a_2+b_2)+\dots+(a_n+b_n)}{n} = \frac{n}{n}$$
$$(a_1a_2\dots a_n)^{1/n} + (b_1b_2\dots b_n)^{1/n} \leq 1 = \Big((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\Big)^{1/n}.$$

$\square$

There are a few other solutions to this problem, and it's one of my favorite Putnam inequalities. Here is the shortest solution I know:

Solution. Apply the Brunn-Minkowski inequality to two axis-parallel boxes in $\mathbb{R}^n$. $\square$

The Brunn-Minkowski inequality is a broad inequality about high-dimensional volume. There are now many generalizations of it, so we'll stick with one of the first versions

Theorem 4.1.1. *Let A, B be two compact convex sets in $\mathbb{R}^d$. Then,*

$$\operatorname{vol}(A \oplus B)^{1/d} \geq \operatorname{vol}(A)^{1/d} + \operatorname{vol}(B)^{1/d}$$

A simple consequence is the following. Given $\lambda \in [0,1]$, then

$$\operatorname{vol}((\lambda A) \oplus ((1-\lambda)B)^{1/d} \geq \operatorname{vol}(\lambda A)^{1/d} + \operatorname{vol}((1-\lambda)B)^{1/d} = \lambda \operatorname{vol}(A)^{1/d} + (1-\lambda)\operatorname{vol}(B)^{1/d}$$
$$\geq \operatorname{vol}(A)^{\lambda/d}\operatorname{vol}(B)^{(1-\lambda)/d}$$

We can raise both sides to the d-th power and obtain a version where the dimension is no longer explicitly present:

$$\mathrm{vol}((\lambda A) \oplus (1-\lambda)B)) \geq \mathrm{vol}(A)^{\lambda}\,\mathrm{vol}(B)^{1-\lambda}.$$

In other words, the volume is a **log-concave** function.

We won't prove the theorem in these notes. If you are curious about it, there are two nice ways to approach this problem. The first one it to try to prove the Prékopa–Leindler inequality; this one of those rare cases where proving something more general is easier if you have the right setup. The second approach is to prove the statement when A, B are finite unions of pairwise disjoint axis-parallel boxes, and use induction on the total number of boxes. The base of induction is the case we did above[4]. The application we show below only uses the case for boxes.

4.2. A volume Helly theorem for boxes. We mentioned that we cannot hope to get rid of the loss of volume in Theorem 1.0.1. However, we can do it if we change the problem slightly.

THEOREM 4.2.1 (Sarkar, Xue, Soberón 2021 [**SXS21**]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Suppose that the intersection of any* $2d$ *or fewer sets in $\mathcal{F}$ is not empty and contains an axis-parallel box of volume* 1*. Then, the intersection of the whole family contains an axis-parallel box of volume one.*

By replacing the volume of a set with the volume of the largest axis-parallel box contained in it, we suddenly have an exact quantitative Helly theorem. Moreover, the size of the subfamilies we need to check is exactly the same as in the theorem of Bárány, Katchaslki, and Pach.

PROOF. Given two vectors $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in $\mathbb{R}^d$ such that $x_i \leq y_i$ for all i (which we summarize as $x \leq y$), we denote by $\mathrm{box}(x, y)$ the axis-parallel box for which x, y are two opposite corners.

Given a set M in $\mathbb{R}^d$, we now define a set

$$S(M) = \left\{ (x_1, \ldots, x_d, y_1, \ldots, y_{d-1}) \in \mathbb{R}^{2d-1} : \begin{array}{c} \text{There exists } y_d \in \mathbb{R} \text{ such that for} \\ x = (x_1, \ldots, x_d),\ y = (y_1, \ldots, y_d) \text{ we} \\ \text{have } x \leq y, \quad \mathrm{vol}(\mathrm{box}(x, y)) = 1, \\ \text{and } \mathrm{box}(x, y) \subset M. \end{array} \right\}.$$

The key observation is that if $M \subset \mathbb{R}^d$ is convex, then $S(M) \subset \mathbb{R}^{2d-1}$ is also convex.

If we have two points $z_1, z_2 \in S(M)$, it means that for each of them we can find coordinates $y_{1,d}, y_{2,d}$ such that $(z_1, y_{1,d})$ and $(z_2, y_{2,d})$ each represent axis-parallel boxes of volume one contained in M. A convex combination $\lambda((z_1, y_{1,d}) + (1 - \lambda)(z_2, y_{2,d})$ represents a box contained in M, which has volume at least 1 by the Brunn-Minkowski inequality (or simply by the 2003 Putnam problem). Therefore there is a value $y_{\lambda,d} \leq \lambda y_{1,d} + (1-\lambda) y_{2,d}$ that completes $\lambda z_1 + (1-\lambda) z_2$ to an axis-parallel box contained in M of volume exactly one.

Now, let $\mathcal{G} = \{S(F) : F \in \mathcal{F}\}$. If we apply Helly's theorem to $\mathcal{G}$, we obtain exactly what we wanted. □

You may notice that you can apply any Helly-type theorem. In particular, an application of the colorful Helly theorem immediately gives a colorful version of Theorem 4.2.1. There is no equivalent colorful version of Theorem 1.0.1 known.

[4]The unfortunate consequence is that we have to choose between the short solution to the Putnam problem being valid and not needing to know Prékopa–Leindler to prove Brunn–Minkowski.

5. John's theorem

5.1. Main theorem. Ellipsoids, affine images of balls, are a common convex set to encounter in convexity. Given a compact convex set $K \subset \mathbb{R}^d$, John's theorem describes the largest volume ellipsoid contain in K.

THEOREM 5.1.1. *Let K be a compact convex set K in $\mathbb{R}^d$ such that the unit ball B_d centered at the origin is the maximal volume ellipsoid contained in K. Then, we can find contact points $u_1, u_2, \ldots, u_k$ of B_d and K and positive real numbers $c_1, \ldots, c_k$ such that for all $x \in \mathbb{R}^d$*

$$||x||^2 = \sum_{i=1}^{k} c_i \langle x, u_i \rangle^2$$

$$0 = \sum_{i=1}^{k} c_i u_i$$

The first condition is equivalent to the following two conditions:

$$x = \sum_{i=1}^{k} c_i \langle x, u_i \rangle u_i \qquad \text{for all } x \in \mathbb{R}^d, \text{ or}$$

$$I_{d\times d} = \sum_{i=1}^{k} c_i u_i \otimes u_i$$

The second equation is obtained by looking at each of the two sides of the first equation as linear functions on $\mathbb{R}^d$ and comparing their associated matrices. This immediately tells us that the number of *essential* contact points is bounded.

THEOREM 5.1.2. *Let K be a compact convex body such that B_d is maximal volume ellipsoid. Then, we can find a set S of $d(d+3)/2$ or fewer contact points of K and B_d such that the polytope*

$$K_S = \{x : \langle x, s \rangle \le 1 \text{ for all } s \in S\}$$

does not contain an ellipsoid of volume larger than B_d.

PROOF. By John's theorem, we can pick contact points $u_1, \ldots, u_k$ of K and B_d and positive real numbers $c_1, \ldots, c_k$ such that

$$I_{d\times d} = \sum_{i=1}^{k} c_i u_i \otimes u_i.$$

If we look at the trace of both sides, we can use the fact that $\mathrm{tr}(u_i \otimes u_i) = ||u_i||^2 = 1$. Therefore,

$$d = \sum_{i=1}^{d} c_i.$$

John's condition can be written as

$$\frac{1}{d} \in \mathrm{conv}\{u_i \otimes u_i : i = 1, \ldots, k\}.$$

Notice that each of $u_i \otimes u_i$ is a symmetric $d \times d$ matrix. The dimension of this space is $d(d+3)/2$. However, they also all have trace 1, so they are in an affine subspace of dimension $\left(\frac{d(d+3)}{2}\right) - 1$.

By Carathéodory's theorem, we can find $C \subset \{1, \ldots, k\}$ of cardinality at most $d(d+3)/2$ such that

$$\frac{1}{d} \in \mathrm{conv}\{u_i \otimes u_i : i \in C\}.$$

This means that B_d is the John ellipsoid of the polytope K_S with $\{u_i : i \in C\}$, as we wanted. $\square$

One impressive thing about John's ellipsoid is that, if K is a compact convex set and $\mathcal{E}$ is its John ellipsoid, then

$$\mathcal{E} \subset K \subset d(\mathcal{E} - c) + c,$$

where c is the center of $\mathcal{E}$. In other words, if we blow up $\mathcal{E}$ by a factor of d from its center, it now contains K.

5.2. A volume Helly for ellipsoids. Theorem 4.2.1 shows how we can obtain exact volume Helly theorems if we measure the volume of certain certain subsets of a convex set, rather than the volume of the whole set. Fortunately, this holds for ellipsoids as well as axis-parallel boxes.

Theorem 5.2.1 (Damásdi, 2017, [**Dam17**]). *Let $\mathcal{F}$ be a finite family of compact convex sets in $\mathbb{R}^d$. Suppose that the intersection of any $d(d+3)/2$ or fewer contains an ellipsoid of volume one. Then, $\bigcap \mathcal{F}$ contains an ellipsoid of volume one.*

The first proof of this result was by Gábor Damásdi in his master's thesis. A very different proof can be obtained using the methods from Section 4.2 (see [**SXS21**]). We include Damásdi's proof here.

Proof. Let $\mathcal{F}$ be a finite family of compact convex sets such that the intersection of any $d(d+3)/2$ or fewer contains an ellipsoid of volume one. By Problem 28, $\bigcap \mathcal{F}$ has positive volume, so it contains an ellipsoid of positive volume. Therefore, the problem is equivalent to the following: assume that the John ellipsoid of $\bigcap \mathcal{F}$ has volume λ. Then, there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$ of $d(d+3)/2$ or fewer sets whose John ellipsoid has volume at most λ.

Using a linear transformation, we may assume without loss of generality that B_d is the John ellipsoid of $\bigcap \mathcal{F}$. We will go a bit further than what the problem requires and show that there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$ whose John ellipsoid is also B_d.

By John's theorem, there are contact points $u_1, \ldots, u_k$ in $S_d \cap \partial\left(\bigcap \mathcal{F}\right)$ that form a John decomposition of unity. Using Carathéodory's theorem as in the proof of Theorem 5.1.2, we can assume $k \le d(d+3)/2$. For each u_i, since $u_i \in \partial\left(\bigcap \mathcal{F}\right)$, we can find at least one set $K_i \in \mathcal{F}$ such that $u_i \in \partial K_i$. The family

$$\mathcal{F}' = \{K_i : i = 1, \ldots, k\}$$

satisfies that $u_i \in \partial\left(\bigcap \mathcal{F}'\right)$ for each i. Since these vectors form a John decomposition of unity, it means that B_d is the maximal volume ellipsoid in $\partial\left(\bigcap \mathcal{F}'\right)$, as we wanted. $\square$

This result allows us to obtain a simple volumetric Helly without any witness sets.

Corollary 5.2.2. *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Suppose that the intersection of any $d(d+3)/2$ or fewer sets in $\mathcal{F}$ has volume at least 1. Then, the volume of $\bigcap \mathcal{F}$ is at least d^{-d}.*

Proof. Let $\mathcal{F}' \subset \mathcal{F}$ be a subfamily of $d(d+3)/2$ or fewer sets of $\mathcal{F}$. By the conditions given, the volume of its intersection must be at least 1. Then, the volume of the John ellipsoid of $\bigcap \mathcal{F}'$ has volume at least d^{-d}. By Damásdi's theorem, $\bigcap \mathcal{F}$ contains an ellipsoid of volume d^{-d}, which must be a lower bound for its volume. $\square$

6. Problems of Chapter 4

Problem 24 Let S^2 be the unit sphere in $\mathbb{R}^3$. Prove that a plank of width w can cover at most a $(w/2)$-fraction of the surface area of S^2 (in general, it will cover exactly a $(w/2)$-fraction of the surface area if the two bounding hyperplanes intersect the sphere).

Problem 25 (Putnam 2019, A4) Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x, y, z)$ over the surface of S equals 0. Must $f(x, y, z)$ be identically 0?

Problem 26 Prove that there exists a function

$$\gamma : (0, 1) \to (0, 1)$$

such that the following happens. For any n unit vectors $v_1, \dots, v_n$ in $\mathbb{R}^d$. There exists a unit vector v such that

$$|\langle v, v_i\rangle| > cd^{-1/2}$$

for at least $\gamma(c)n$ indices i. Moreover, $\gamma(c) \to 1$ as $c \to 0$.

Problem 27 Prove that the Helly number in Theorem 1.0.1 is optimal. In other words, for each $\varepsilon > 0$, construct a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ such that the intersection of any $2d - 1$ or fewer sets of $\mathcal{F}$ has volume at least 1 but the volume of $\bigcap \mathcal{F}$ is at most ε.

Problem 28 (Steinitz, 1913, 1914, 1916 [**Ste13**,**Ste14**,**Ste16**][5]) Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Suppose that the intersection of any $2d$ or fewer sets of $\mathcal{F}$ has non-empty interior. Show that $\bigcap \mathcal{F}$ has non-empty interior.

Problem 29 Let C be a compact convex set in $\mathbb{R}^d$. Let Δ be a simplex contained in C of maximal volume, chose center of gravity is p. Prove that

$$C \subset -d(S - p) + p.$$

Problem 30 (Putnam 2016, B3) Let X be a finite set of points in the plane such that the area of any triangle with vertices in X is at most one. Show that there exists a triangle of area four that covers X.

Problem 31 For each positive integer n show that there exists a positive real number $\varepsilon = \varepsilon(n)$ such that the following holds. Any convex polygon in the plane of at most n sides that contains a circle of radius 1 must also contain a square of area $2 + \varepsilon$.

[5] What's the story here? Steinitz wrote a 3-part paper which was published as he built up on previous work. I think the 1914 one contains the theorem of this problem. An interesting bit: all three appeared before the first published proof of Helly's theorem by Radon in 1921.

Part 2

Topological Combinatorics

CHAPTER 5

Mass partitions

Groups, as individuals, will be known by their actions.

- *Guillermo Moreno*[1]

1. Basics of topological combinatorics

The first impression you may have of topological combinatorics might be that this sounds like a strange mix. I have yet to find a satisfying definition of combinatorics[2], but everyone would agree that it primarily involves discrete objects. We constantly study, count, and hit our heads against mathematical objects which are finite, distinct, and discrete. The results of counting and measuring in combinatorics are most likely non-negative integers.

Topology is the opposite in this regard. We want to study continuity and stretch that definition as much as possible. You'll see elaborate spaces, knots, and functions which push (or break) whatever intuition we could have. As we dive into the realm of advanced topological concepts, some combinatorial tools pop up by themselves. Homology groups, winding numbers, the degree of a function, and so on help us distinguish topological spaces. That way, combinatorial topology is quite natural. It's relatively easy to defend why combinatorics should be useful for a topologist, but why should topology be useful for a combinatorialist?

Well, how do you usually solve a combinatorics problem? You look for an invariant. Pick a random combinatorics problem from a math olympiad. Finding an invariant is most likely a very good strategy. Many surprising applications of other fields in combinatorics rely on using invariants from that field. If we assign linear spaces to our objects, their dimension might be interesting. If we assign them probability distributions, maybe their expected values help us. Then, if we assign them topological spaces or functions, the invariants we talked about (homology groups, winding numbers, degrees, etc...) are useful again.

Discrete geometry is an excellent sandbox to test this approach. The geometry of the spaces provides a natural way to construct topological objects associated to our problems. Yet, the resulting problems (such as topological extensions of Tverberg's theorem) turn into fantastic challenges for topologists.

2. Mass partitions

2.1. History of the ham sandwich theorem. Finding fair partitions of sets is an extremely old problem, and geometric versions are natural. Of course, one has to agree on a suitable definition of fairness. One of the earliest algorithmic solutions to split a set fairly between two persons dates back to the Bible[3], one

[1]The original phrasing had "men" instead of "individuals" (and was in Spanish), but we can adjust it.

[2]My current favorite is Daniel Kleitman's: "I do mathematics I like, and call the results combinatorics."

[3]It's Abraham and Lot's dispute in Genesis.

person divides the set in a way they consider fair and the other one chooses one of the two parts.

However, if we want to split a region in the plane, we can also impose geometric conditions on how we cut this region. It's fairly easy to see that we can halve any two sets with a straight line (we discuss this in the next section), but halving any three sets in space with a plane becomes more challenging.

This was the original conjecture of Steinhaus, now known as the ham sandwich theorem. The name comes from thinking of each set as an ingredient (say, bread, ham, and cheese), and we want to make sandwiches for two hungry guests[4]. The proof in high (i.e., three or more) dimensions uses topological tools. Steinhaus [**Ste38**] attributes the proof to Banach, who used the Borsuk–Ulam theorem.

The ham sandwich problem is also listed as problem 123 in the Scottish book. The Scottish book contains many problems and solutions that the mathematicians of the Lwów school of mathematics would discuss in their meetings at the "Scottish Café" (which is explains the name, even though the meetings were in Lwów, Poland). As you probably guessed, Hugo Steinhaus, Stanisław Ulam, and Stefan Banach were among the members of this school of mathematics.

The first version of the ham sandwich theorem that uses measures instead of volumes is by Stone and Tukey [**ST42**]. Steinhaus also published an updated proof of the ham sandwich theorem and some extensions shortly after [**Ste45**], which collects the work he did while in hiding during World War II.

2.2. Two proofs of the ham sandwich theorem in the plane. The ham sandwich theorem is a result about diving measures fairly in $\mathbb{R}^d$. If you are not familiar with measures, you can replace them with "d-dimensional volume of some shape" or "finite sets of points". The results for finite sets of points require a slightly more delicate treatment, but the overall intuition is the same. Let's start with the main result in dimension two.

Theorem 2.2.1 (Ham sandwich in the plane). *Let μ_1, μ_2 be two finite measures in $\mathbb{R}^2$ so that every line has measure zero in each of μ_1 and μ_2. Then, there exists a line splitting the plane into two closed half-planes H_1, H_2 so that*

$$\mu_1(H_1) = \mu_1(H_2)$$
$$\mu_2(H_1) = \mu_2(H_2).$$

The name follows since we can think of μ_1 as the ham and μ_2 as the bread. Then, the theorem tells us that we can make two fair sandwiches by making a single straight cut, regardless of how the ingredients were set on the table. We discuss two different ways to approach this problem. The first one uses the intermediate value theorem.

Proof. We denote by S^1 the set of unit vectors in $\mathbb{R}^2$. For each $v \in S^1$, consider a the line ℓ perpendicular to v. This line can be expressed as

$$\ell = \{x \in \mathbb{R}^d : \langle x, v\rangle = \lambda\}$$

for some $\lambda \in \mathbb{R}$. It defines two half-planes

$$H_\ell^+ = \{x \in \mathbb{R}^d : \langle x, v\rangle \geq \lambda\}$$
$$H_\ell^- = \{x \in \mathbb{R}^d : \langle x, v\rangle \leq \lambda\}$$

Notice that $\mu_1(H_\ell^+)$ is a decreasing function of λ, and $\mu_1(H_\ell^-)$ is an increasing function of λ. There must be an interval in which $\mu_1(H_\ell^+) = \mu_1(H_\ell^-)$. Given v,

[4]Steinhaus' original interpretation showed a much more carnivorous side. He wanted to cut a piece of meat so that the two halves had the same amount of meat, bone, and skin.

we choose λ to be the midpoint of this interval. This guarantees that ℓ is bisecting μ_1 and that if we replace v by $-v$ the line ℓ remains the same (the two half-spaces H_ℓ^+, H_ℓ^- would swap names).

Now consider the function

$$\begin{aligned} f : S^1 &\to \mathbb{R} \\ v &\mapsto \mu_2(H_\ell^+) - \mu_2(H_\ell^-). \end{aligned}$$

By the conditions on the measures, f is a continuous function. Moreover, we have $f(v) = f(-v)$. Pick a particular vector v. If $f(v) = 0$, then ℓ bisects both measures. Otherwise, as we move along S^1 from v to $-v$, f changes sign. By the intermediate value theorem, there is a vector v_0 such that $f(v_0) = 0$. This means that its corresponding line ℓ_0 halves both measures simultaneously. □

The second proof involves parametrizing the set of half-planes in $\mathbb{R}^2$. For any half-plane H^+ we can find a unique unit vector $v \in S^1$ and a real number λ such that

$$H^+ = \{x \in \mathbb{R}^2 : \langle v, x \rangle \geq \lambda\}$$

Therefore we can parametrize the set of half-planes with $S^1 \times \mathbb{R}$. We can consider every element $(v, \lambda) \in S^1 \times \mathbb{R}$ as a subset of $\mathbb{R}^3$, which looks like a cylinder. Therefore, we can construct a the function

$$\begin{aligned} g : S^1 \times \mathbb{R} &\to S^2 \\ (v, \lambda) &\mapsto \frac{1}{\|(v, \lambda)\|}(v, \lambda). \end{aligned}$$

This function is injective. Its image only misses the north pole and the south pole of S^2. We can assign a half-space to almost every point in S^2.

However, for any $v \in S^1$, the hyperplane defined by (v, λ) moves away from the origin as $\lambda \to \infty$. Similarly, H starts to cover the whole plane as $\lambda \to -\infty$. By convention, we associate to $(0, 0, 1)$ the set $\emptyset$ and to $(0, 0, -1)$ the set $\mathbb{R}^2$. We consider these as "degenerate" half-planes. This gives us a way of assigning a subset of $\mathbb{R}^2$ to each point of S^2. We assign a closed half-plane to most points of S^2, but the two poles get these special sets.

Every closed half-plane has a bounding line. We say that the bounding line of the two degenerate hyperplanes is the "line at infinity".[5]

Second proof of the ham sandwich theorem. Assign to every point $u \in S^2$ a (possibly degenerate) half-plane H^+ in $\mathbb{R}^2$ as above. Notice that the half-planes assigned to u and $-u$ are complementary, i.e., they share the same bounding line. Now we construct a function

$$\begin{aligned} f : S^2 &\to \mathbb{R}^2 \\ u &\mapsto \left(\mu_1(H^+) - \frac{1}{2}\mu_1(\mathbb{R}^2), \mu_2(H^+) - \frac{1}{2}\mu_2(\mathbb{R}^2)\right). \end{aligned}$$

The construction guarantees two things:

- the function f is continuous and
- for every $u \in S^2$ we have $f(u) = f(-u)$.

Fortunately, the Borsuk–Ulam theorem guarantees that this function will have a zero. This zero corresponds to a half-plane H^+ that has half of each measure, so its complement must have the other half. □

[5] This convention is helpful when you learn projective geometry. We do not need to dive into that topic in these notes.

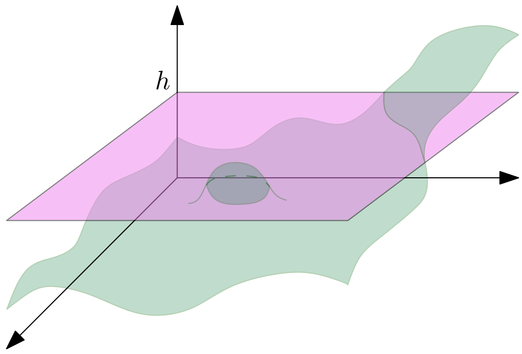


FIGURE 1. If h is a regular value of g, then at height h we see connected components homeomorphic to circles or lines.

The second proof might seem much more involved than the first one, so what's the point of including it? In this case, we can extract a bit more. For example, the second proof works for *charges*, which are more general than measures (they can assign negative values to sets).

3. The Borsuk–Ulam theorem

Most of this book is "black box topology", as we don't intend to build the theory nor prove the most technical tools. However, since the Borsuk–Ulam theorem is so prevalent in topological combinatorics, we must make an exception.

THEOREM 3.0.1 (Borsuk–Ulam). *Let $f : S^d \to \mathbb{R}^d$ be a continuous odd function. Then, there exists $x \in S^d$ such that $f(x) = 0$.*

There are plenty of proofs of this result, depending on how much algebraic topology you want to throw at it. Since we are not building too much theory here, we will follow Imre Bárány's geometric proof [**Bár80**].

First, we build our intuition. Consider a generic continuous smooth map $g : \mathbb{R}^d \to \mathbb{R}^d$. By generic, we mean that the Jacobian is non-singular at any point. Then, for any $p \in \mathbb{R}^d$, the set $f^{-1}(p)$ consists of a discrete (possibly infinite) set of points in $\mathbb{R}^d$. The same holds for generic maps $g : M \to N$ where M and N are d-dimensional manifolds.

Now, let's consider the case $g : M \to N$ where M is $(d+1)$-dimensional and N is d-dimensional. We can take as an example $M = \mathbb{R}^2, N = \mathbb{R}$. If $g : \mathbb{R}^2 \to \mathbb{R}$, we can construct the graph of this function as the set

$$\{(x, y, g(x, y)) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3,$$

which looks like a mountain range. If we intersect this mountain range with a horizontal plane at height h, we see $g^{-1}(h)$. Unless we are very unlucky, this is generally the union of sets homeomorphic to circles or infinite lines, as in Figure 1.

Now let's think of the case of maps between smooth manifolds M, of dimension $d+1$, and N, of dimension d. If M has a boundary, the set $g^{-1}(p)$ for some $p \in N$ can be a bit more complicated, but not much. If g is generic enough, then $g^{-1}(p)$ is a 1-dimensional manifold. Its connected components are either homeomorphic to a circle or a segment with endpoints in ∂M.

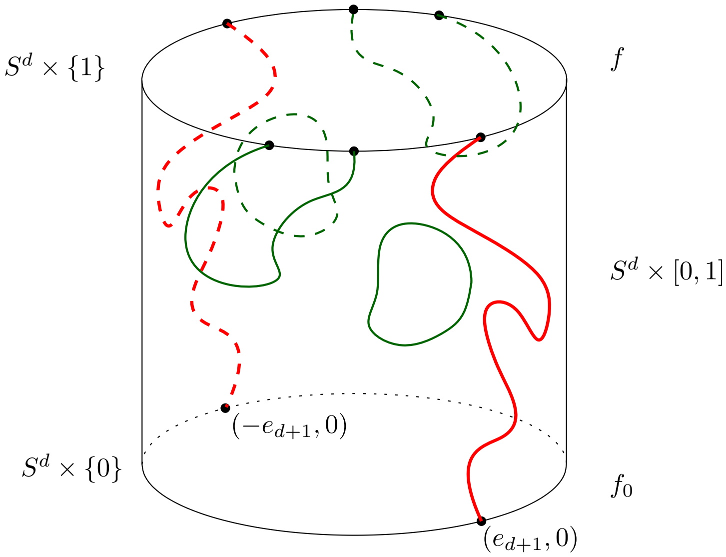


FIGURE 2. An illustration of Bárány's proof of the Borsuk–Ulam theorem. In the cylinder we highlight the set of zeroes of h. The pair of zeroes in the bottom sphere must be connected to zeroes on the top sphere. Antipodal pairs of zeroes of f cancel each other in pairs as shown.

PROOF OF THE BORSUK–ULAM THEOREM. Suppose we have a continuous odd map $f : S^d \to \mathbb{R}^d$. We construct a second odd map

$$\begin{aligned} f_0 : S^d &\to \mathbb{R}^d \\ (x_1, \ldots, x_{d+1}) &\mapsto (x_1, \ldots, x_d). \end{aligned}$$

This map has a single pair of antipodal points going to zero, $f_0^{-1}(0) = \pm e_{d+1}$. Now we define a map on a high-dimensional cylinder $S^d \times [0,1]$.

$$\begin{aligned} h : S^d \times [0,1] &\to \mathbb{R}^d \\ (x,t) &\mapsto tf(x) + (1-t)f_0(x). \end{aligned}$$

We have $h(x,0) = f_0(x)$ and $h(x,1) = f(x)$ for all $x \in S^d$, so h interpolates between our two maps. Moreover, h is odd on the first variable: $h(-x,t) = -h(x,t)$ for all $(x,t) \in S^d \times [0,1]$. If h is generic enough, then $Z = h^{-1}(0) \subset S^d \times [0,1]$ is a 1-dimensional manifold. Its connected components are either homeomorphic to circles (we don't care about those) or to segments. The segments must have their endpoints in $\partial(S^d \times [0,1]) = S^d \times \{0,1\}$. See Figure 2 for an illustration. Consider the function $z : S^d \times [0,1] \to S^d \times [0,1]$ defined by $z(x,t) = (-x,t)$. Notice $z : Z \to Z$. Moreover, z must map connected components of Z to connected components of Z. In particular, it sends the connected components homeomorphic to segments to connected components homeomorphic to segments. If I is one of those components, the $z(I) \neq I$, as any continuous map from a segment to itself must have a fixed point.

Therefore, the only antipodal pair of zeros in $S^d \times \{0\}$ must be connected in Z to another pair of zeros in $S^d \times \{1\}$, implying that f has zeros. $\square$

The proof assumes that h is generic enough, which is not always true. However, h can be approximated by generic maps which are odd in the first variable and do not change in $S^d \times \{0\}$ [**Tho54**]. An alternative way is to use simplicial complexes and piece-wise linear maps instead of differential geometry, as Musin did recently [**Mus12**].

4. High-dimensional mass partitions

Now we can prove the ham sandwich theorem in $\mathbb{R}^d$. Fortunately, there are no surprises.

THEOREM 4.0.1. *Let $\mu_1, \dots, \mu_d$ be finite measures in $\mathbb{R}^d$. Assume that every hyperplane has measure zero in each of $\mu_1, \dots, \mu_d$. Then, there exists a hyperplane such that its two complementary closed half-spaces H^+, H^- satisfy*

$$\mu_i(H^+) = \mu_i(H^-) \qquad \text{for } i = 1, \dots, d.$$

PROOF. Let $S^{d-1} \subset \mathbb{R}^d$ be the set of unit vectors in $\mathbb{R}^d$. For $v \in S^{d-1}$, let H_v be the hyperplane orthogonal to v that splits μ_1 into two equal parts. If there is a range of hyperplanes that do this, we pick one at the midpoint of this range. Let

$$\begin{aligned} H_v^+ &= \{x + \alpha v : x \in H_v,\ \alpha \geq 0\} \\ H_v^- &= \{x + \alpha v : x \in H_v,\ \alpha \leq 0\}. \end{aligned}$$

Now we define a function

$$\begin{aligned} f : S^{d-1} &\to \mathbb{R}^{d-1} \\ v &\mapsto \Big(\mu_2(H_v^+) - \mu_2(H_v^-), \dots, \mu_d(H_v^+) - \mu_d(H_v^-)\Big). \end{aligned}$$

The function f is continuous. Moreover, since $H_v^+ = H_{-v}^-$ and $H_v^- = H_{-v}^+$, we know $f(v) = -f(-v)$. Therefore, by the Borsuk–Ulam theorem, f must have a zero v_0. The hyperplane H_{v_0} simultaneously halves all measures. □

We have to be careful about the conditions on the measures or the result may not hold. However, since every measure in $\mathbb{R}^d$ can be approximated by nice measures[6] we can use the following ham sandwich theorem for general measures.

THEOREM 4.0.2 (General ham sandwich theorem). *Let $\mu_1, \dots, \mu_d$ be finite measures in $\mathbb{R}^d$. Then, there exists a hyperplane such that its two complementary closed half-spaces H^+, H^- satisfy*

$$\begin{aligned} \mu_i(H^+) &\geq \frac{1}{2}\mu_i(\mathbb{R}^d) \qquad \text{and} \\ \mu_i(H^-) &\geq \frac{1}{2}\mu_i(\mathbb{R}^d) \end{aligned}$$

for all $i = 1, \dots, d$.

One way to think about hyperplanes is as the set of zeros of a multivariate polynomial of degree one. If we increase the complexity of the polynomial, we can simultaneously halve more measures.

THEOREM 4.0.3 (Stone, Tukey 1942 [**ST42**]). *Let k, d be positive integers. For any $\binom{d+k}{k} - 1$ finite measures in $\mathbb{R}^d$ such that any hyperplane has measure zero in each measure, there exists a multi-variate polynomial $P : \mathbb{R}^d \to \mathbb{R}$ of degree at most k such that*

$$\mu(\{x \in \mathbb{R}^d : P(x) \geq 0\}) = \mu(\{x \in \mathbb{R}^d : P(x) \leq 0\})$$

for all given measures μ.

---

[6]Legalese: the set of measures in $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure is dense in the space of all possible measures, equipped with the weak topology.

PROOF. We consider a Veronese map $V : \mathbb{R}^d \to \mathbb{R}^{\binom{d+k}{k}-1} = \mathbb{R}^N$ that has one dimension for each non-constant monomial on d variables of degree at most k. For example, for $d = 3, k = 2$ we have

$$V(x, y, z) = (x^2, y^2, z^2, xy, xz, yz, x, y, z) \in \mathbb{R}^{\binom{5}{2}-1} = \mathbb{R}^9.$$

When we lift each measure of the N measures to $\mathbb{R}^N$, we obtain a measure in the higher-dimensional space. We can now apply the ham sandwich theorem in $\mathbb{R}^N$. The halving hyperplane in $\mathbb{R}^N$ corresponds to the polynomial we were looking for. □

Of course, if we restrict the Veronese map to lift $\mathbb{R}^d$ to a lower-dimensional space, we obtain interesting consequences.

COROLLARY 4.0.4. *Let k be a positive integer. Suppose we are given $k+1$ finite measures $\mu_1, \mu_2, \ldots, \mu_{k+1}$ in $\mathbb{R}^2$. Then, there exists a polynomial P of degree at most k whose graph splits each measure into two equal parts.*

In particular, even though we can split two measures in the plane with a line, we can split any three measures with a vertical parabola.

It's worth our time to play a bit with these lifting arguments. Consider the following example. We say a *wedge* in $\mathbb{R}^d$ is the intersection of two closed half-spaces. In 2001, Bárány and Matoušek proved that for any three measures in $\mathbb{R}^2$ there exists a wedge containing exactly half of each measure [**BM01**]. This was extended by Schnider to $d + 1$ measures in $\mathbb{R}^d$ [**Sch19**]. Here is a recent simpler proof [**ST21**]. We require that each hyperplane has measure zero in each measure.

PROOF. Let $v \in S^{d-1}$ be a fixed direction. By Problem 36, there either exists a halving hyperplane H for all measures (in which case we are done), or there exists a hyperplane H such that each side has strictly less than half of some measure. Now embed $\mathbb{R}^d \hookrightarrow \mathbb{R}^{d+1}$ by appending a coordinate 0, namely $x \mapsto (x, 0)$. We can lift the measures to a surface $S(H)$ by sending $x \mapsto (x, \operatorname{dist}(x, H))$. Notice that the surface $S(H)$ is contained in the union of two hyperplanes (see Figure 3).

We now have $d + 1$ measures in $\mathbb{R}^{d+1}$. The measures may not satisfy the conditions to use the ham sandwich theorem, but we can certainly use the ham sandwich theorem for general measures. By the construction of H, it's easy to show that the halving hyperplane H' is not one of the two flat components of $S(H)$. Therefore the measure of H' is zero in each of the lifted measures. When we project $H' \cap S(H)$ down to $\mathbb{R}^d$, we obtain the wedge we were looking for. □

5. Beyond Borsuk–Ulam

The scope of these notes is limited to applications of Borsuk–Ulam, but there are instances of problems in topological combinatorics in which you may need more than that. This is connected to *equivariant topology*, which is an exciting topic by itself.

As you've noticed, in the previous applications our go-to space is the d-dimensional sphere S^d. In each construction, the antipodal map $x \mapsto -x$ in the sphere is crucial. This is an example of a group action. Given a non-trivial group G and a topological space X, we say that G acts on X if for each $g \in G$ we have a homeomorphism

$$\begin{aligned} f_g : X &\to X \\ x &\mapsto f_g(x) \end{aligned}$$

we often denote $f_g(x)$ simply as gx for convenience. This family of homeomorphisms has to satisfy the following properties:

- The neutral element $e \in G$ satisfies $ex = x$ for all $x \in X$

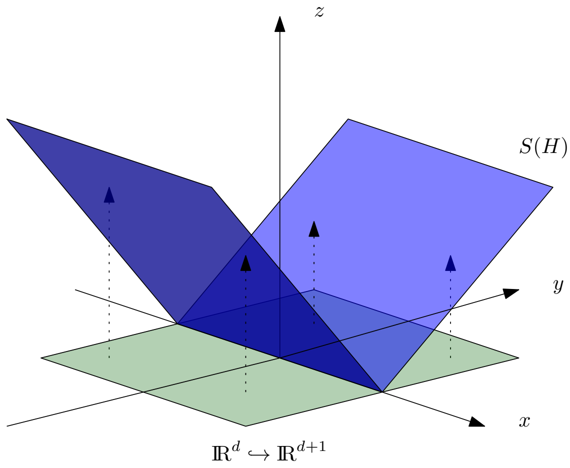


FIGURE 3. An illustration of $S(H)$ for $d = 2$

- For any two elements f, g in G, $f(gx) = (fg)x$.

You may see other texts ask for additional properties, but they can be deduced from the two above. The antipodal map in S^d is an action of $\mathbb{Z}_2$ in S^d.

If X, Y are two spaces with an action of a group G, we say that a continuous map $f : X \to Y$ is *equivariant* (or G-equivariant) if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. In other words, the map f "gets along" with the group action. We often write $f : X \to_G Y$ to make this explicit. A reformulation of the Borsuk–Ulam theorem is the following.

THEOREM 5.0.1. *Let n, m be positive integers. If $n > m$, there does not exist a $\mathbb{Z}_2$-equivariant map $f : S^n \to_{\mathbb{Z}_2} S^m$.*

there are many Borsuk–Ulam-type theorems to uncover as we change the group. Reducing mass partition problems to the study of equivariant functions is a very productive way to approach these problems, which is called the *test map scheme*. The goal is to follow the following process.

- First, parametrize the space of partitions by a topological space X. We call X the configuration space.
- Then, construct a space Y that tells you how you split the measures. This induces a natural map $f : X \to Y$.
- Ideally, there is a group G of symmetries in the problem. This should induce actions of G on X and Y so that f is equivariant.
- We study the properties of equivariant functions $f : X \to_G Y$.

The last step is a purely topological problem, so we can now focus on that part of the problem.

5.1. Examples of configuration spaces. It might be instructive to showcase how other groups appear in the test map scheme. Consider for example the space X of partitions of $\mathbb{R}^2$ into three convex sets.

The only way to partition $\mathbb{R}^2$ into three convex sets is to use two parallel lines or three rays coming from some point p. If we label the parts (C_1, C_2, C_3), we have an action of S_3 in X. If $\pi : [3] \to [3]$ is a permutation, we can define

$$\pi(C_1, C_2, C_3) = \left(C_{\pi(1)}, C_{\pi(2)}, C_{\pi(3)}\right)$$

We can also consider an action of S_3 in $\mathbb{R}^2$. If we consider

$$\mathbb{R}^2 \cong \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 : \alpha_1 + \alpha_2 + \alpha_3 = 1\}$$

Then we have a similar action of S_3 in $\mathbb{R}^2$. Consider the following problem.

PROBLEM. *What is the largest number k so that for any k finite probability measures in $\mathbb{R}^2$ there exists a partition of $\mathbb{R}^2$ into three convex sets that splits each measure into equal parts?*

If we use the test map scheme to solve it, the analysis boils down to the following purely topological problem.

PROBLEM. *What is the largest number k so that for any S_3-equivariant map $f : X \to_{S_3} \left(\mathbb{R}^2\right)^k$ there exists $x \in X$ such that $f(x) = (1/3, 1/3, 1/3)^k$?*

Of course, you have to define the action of S_3 on $(\mathbb{R}^2)^k$ and define Y properly to do this. The answer to the problem above is $k = 2$ [**IUY98**, **BKS00**, **Sak02**], although some work is done in advance to simplify the topological tools needed.

Mass partition problems showcase applications of topological results and motivate research on their extensions. If you find this topic interesting, it's worth looking at a comprehensive survey to see more applications of the test map scheme [**RPS21**]. A curious consequence of the test map scheme is that we know much more results when the group involved is $\mathbb{Z}_p$ or $(\mathbb{Z}_p)^k$ for some prime p. The topological tools work better. This phenomenon often translates to conditions on discrete geometry results in which some of the parameters have to be prime numbers or powers of prime numbers. Such conditions are unexpected, and it can be a tough problem to determine if they are necessary.

6. Problems of Chapter 5

Problem 32 Show that the set of closed half-spaces in $\mathbb{R}^d$, together with the empty set and $\mathbb{R}^d$ itself, can be parametrized with a d-dimensional sphere.

Problem 33 (Stone, Tukey 1942 [**ST42**], Steinhaus 1945 [**Ste45**]) Let μ_1, μ_2, μ_3 be smooth finite measures in $\mathbb{R}^2$. Prove that there exists a circle that contains exactly half of each measure.

In this problem, by a smooth measure we mean a measure in which every line and every circle is assigned measure zero.

Problem 34 (Buck, Buck 1949 [**BB49**]) Let A be a convex set in $\mathbb{R}^2$. Prove that there exist three concurrent lines that split A into six parts of the same area.

Problem 35 (Uno, Kaneko, Kano 2009 [**UKK09**]) Let μ_1, μ_2 be two finite measures such that each line has size zero in each measure. Show that there exists a point p and two infinite rays starting at p (one vertical, one horizontal) such that the resulting shape splits both measures by half. The vertical ray may go up or down and the horizontal ray may go left or right.

Problem 36 Let $\mu_1, \dots, \mu_n$ be finite measures in $\mathbb{R}^d$ and $v \in S^{d-1}$. Assume that every hyperplane has measure zero in each μ_i. Then,

- there either exists a hyperplane H orthogonal to v that halves each of the n measures simultaneously or
- there exist a hyperplane H orthogonal to v and indices i', i in $[n]$ such that for the two closed half-spaces H^+ and H^- defined by H we have

$$\mu_i(H^+) < \frac{1}{2}\mu_i(\mathbb{R}^d) \qquad \text{and}$$
$$\mu_{i'}(H^-) < \frac{1}{2}\mu_i(\mathbb{R}^d).$$

CHAPTER 6

There and back again

"Where did you go to, if I may ask?' said Thorin to Gandalf as they rode along.
To look ahead,' said he.
And what brought you back in the nick of time?'
Looking behind,' said he."

— *J.R.R. Tolkien, The Hobbit*

1. About topological proofs

Our first introduction to combinatorial geometry was with our three pillars: Carathéodory, Helly, and Radon. In this chapter we revisit two, Helly and Radon, from a topological point of view. Carathéodory's theorem does have topological extensions, but they are a bit too elaborate for the scope of this book [**Hol16**].

One interesting historical remark is that the topological side of combinatorial geometry grew organically along the linear algebraic side. Even Eduard Helly himself was already proving topological versions of his theorem back in the 1930's [**Hel30**]. The topological proof of the colorful Radon theorem [**Bár82**][1] was found several decades earlier than the linear algebraic proof we presented in Chapter 2 [**Sob15**].

Perhaps the most interesting cases are those results for which we only know a topological proof, but everything in the problem is *completely linear-algebraic.* The colorful version of Tverberg's theorem is an example, or the colorful versions of Helly for matroids [**KM05**].

2. Colorful Radon revisited.

Let's recall the statement of the colorful Radon theorem from Chapter 2.

Theorem 2.2.1. *Let $F_1, \dots, F_{d+1}$ be pairs of points in $\mathbb{R}^d$. Then, we can find two disjoint sets A, B, such that each contains exactly one point from each F_i and their convex hulls intersect.*

Second proof. We first represent every colorful simplex in $\mathbb{R}^d$ with a d-dimensional simplex in $\mathbb{R}^{d+1}$. Let $e_1, \dots, e_{d+1}$ be the canonical basis of $\mathbb{R}^{d+1}$. We assign the points of F_i to $\{-e_i, e_i\}$ arbitrarily for each i. Then, each colorful simplex in $\mathbb{R}^d$ corresponds to a colorful d-dimensional simplex in $\mathbb{R}^{d+1}$. If we take the union of all these colorful simplices in $\mathbb{R}^{d+1}$, they form the boundary $\mathbb{O}^d$ of the octahedron

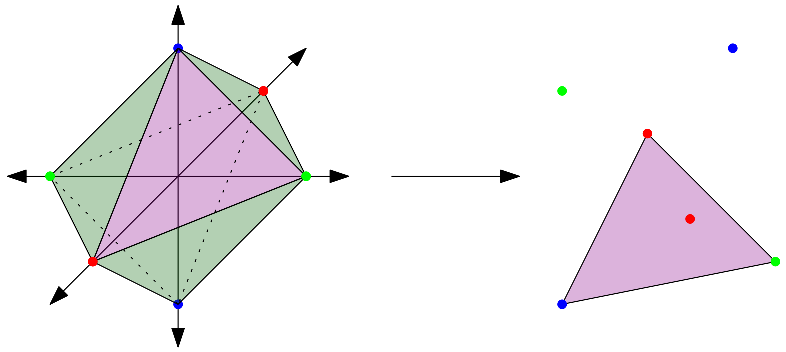
$$\operatorname{conv}(-e_1, e_1, \dots, -e_{d+1}, e_{d+1}).$$

We can map every facet of $\mathbb{O}^d$ to $\mathbb{R}^d$ by extending the maps of the vertices linearly. This gives us a continuous function $f : \mathbb{O}^d \to \mathbb{R}^d$, as depicted in Figure 1.

However, the map

$$\begin{aligned} g : S^d &\to \mathbb{O}^d \\ x &\mapsto x/\|x\|_1 \end{aligned}$$

[1]The reference is a paper by Imre Bárány, where he included a proof of Lóvász of this result.



$\mathbb{R}^{d+1}$ $\mathbb{R}^d$

FIGURE 1. We can see how each facet of the octahedron corresponds to a rainbow simplex in $\mathbb{R}^2$.

where $\|x\|_1$ denotes the 1-norm of x is a homeomorphism (the inverse g^{-1} looks like "inflating" the octahedron to the sphere). Therefore $f \circ g : S^d \to \mathbb{R}^d$ is a continuous function. By the Borsuk–Ulam theorem, there exists $x \in S^d$ such that $f(g(x)) = f(g(-x)) = f(-g(x))$. The points $g(x)$ and $-g(x)$ are antipodal points of the octahedron which are sent to the same point under f. They are sustained by opposite facets of $\mathbb{O}^d$, which correspond to the two vertex-disjoint colorful simplices in $\mathbb{R}^d$ whose convex hulls intersect in $f(g(x))$. $\square$

3. The topological Helly theorem

3.1. Motivation. Helly's theorem exhibits an interesting property of intersection patterns of convex sets. However, one can ask if convexity is absolutely necessary. It turns out that Helly's theorem works for more general families of sets, such as *good covers* of $\mathbb{R}^d$.

We say that a family $\mathcal{G}$ of sets in $\mathbb{R}^d$ is a good cover if

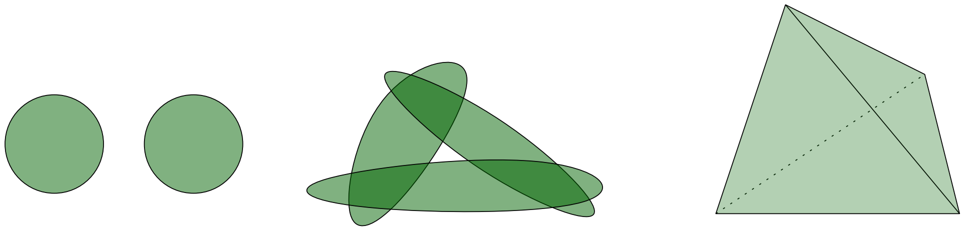
- $\bigcup \mathcal{G} = \mathbb{R}^d$,
- every set in $\mathcal{G}$ is non-empty and contractible, and
- every non-empty intersection of sets in $\mathcal{G}$ is contractible.

The family of all non-empty convex sets in $\mathbb{R}^d$ is a good cover. Every convex set is contractible. The intersection of convex sets is convex, so if it is non-empty, it is contractible.

The gist of the proof below is to look at critical non-intersecting families of sets. Consider the case of having $\lambda + 1$ convex sets in $\mathbb{R}^d$ so that any λ have a non-empty intersection, but they don't all have a point of intersection. What can you say about the union of these sets?

As we can see in Figure 2, the unions of each of these families look like spheres. For $\lambda = 1$ we have two disjoint convex sets, which look like S^0 (two disjoint points). For $\lambda = 2$, our three convex sets form a cycle, so it looks like S^1 (a circle). If we keep in mind the last part of Figure 2, the facets of a λ-dimensional simplex satisfy those properties, and this boundary is homeomorphic to $S^{\lambda-1}$.

A slightly simpler case is to look at sets $A_1, \dots, A_n$ in a good cover $\mathcal{G}$ with non-empty intersection. It is a simple induction exercise to show that $\bigcup_{i=1}^n A_i$ is also contractible. To see this for $n = 2$, it suffices to contract $A_1 \cup A_2$, and then see that the result is the cusp of two contractible sets, which is contractible as well.



$\lambda = 1$ $\lambda = 2$ $\lambda = 3$

FIGURE 2. Three cases of families of $\lambda + 1$ convex sets whose intersection is empty but every λ of them have a non-empty intersection.

3.2. Interlude: Mayer–Vietoris. Suppose we have two topological spaces A, B. If we can describe A, B, and $A \cap B$, it makes sense we can describe $A \cup B$. The Mayer–Vietoris exact sequence does precisely this if we want to describe the reduced homology groups of these spaces. If you have not seen homology[2] before, think of $\tilde{H}_k(X)$ as a group describing the k-dimensional holes of a space X. For the d-dimensional sphere S^d we have

$$\tilde{H}_k(S^d) = \begin{cases} 0 & \text{if } k \neq d \\ \mathbb{Z} & \text{if } k = d. \end{cases}$$

As we deal with groups, we need to recall the definition of an exact sequence. Given group H, K, and L and homomorphisms $f : H \to K$ and $g : K \to L$, we say that the sequence

$$H \xrightarrow{f} K \xrightarrow{g} L$$

is exact if $\ker g = \operatorname{im} f$. Longer sequences of homomorphisms are exact if every subsequence of three consecutive terms is exact. One particular case is that

$$0 \to H \xrightarrow{f} K \to 0$$

is exact if and only if f is an isomorphism.

The Mayer-Vietoris exact sequence says that the long sequence

$$\ldots \to \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to \tilde{H}_n(A \cup B) \to \tilde{H}_{n-1}(A \cap B) \to \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B) \to \ldots$$

is exact.

EXAMPLE 3.2.1. *Consider the top and bottom halves of a d-dimensional sphere,*

$$A = \{x \in S^d : x_{d+1} \geq 0\}$$
$$B = \{x \in S^d : x_{d+1} \leq 0\}.$$

We clearly have $A \cap B \cong S^{d-1}$. Each of A, B is contractible, so they have trivial homology groups. Therefore $H_n(A) \oplus H_n(B) = 0$. By Mayer–Vietoris, this means that the sequence

$$0 \to \tilde{H}_n(A \cup B) \to \tilde{H}_{n-1}(A \cap B) \to 0$$

is exact. Therefore, the two groups $\tilde{H}_n(A \cup B)$ and $\tilde{H}_{n-1}(A \cap B)$ are isomorphic. Therefore $\tilde{H}_{n-1}(S^{d-1}) \cong \tilde{H}_n(S^d)$ for all n. This way we can compute the homology of S^d inductively on d.

[2]The legalese: we work with singular homology over $\mathbb{Z}$. There are a few reasons why working over $\mathbb{Z}_2$ would simplify our life.

3.3. The topological Helly theorem.

LEMMA 3.3.1. *Let $A_1, \ldots, A_{\lambda+1}$ be a family of λ non-empty open subsets of $\mathbb{R}^d$ such that the intersection of any subset of them is either empty or contractible. If the intersection of $\{A_1, \ldots, A_{\lambda+1}\}$ is empty and the intersection of any λ sets in $\{A_1, \ldots, A_{\lambda+1}\}$ is not empty, then $A = \bigcup_{i=1}^{\lambda+1} A_i$ has the same reduced homology groups as $S^{\lambda-1}$.*

PROOF. We prove this by induction on λ. For $\lambda = 1$, the result is clear. Assume that the statement is true for $\lambda - 1$ and we want to prove it for λ. Let $A = \bigcup_{i=1}^{\lambda} A_i$, $B = A_\lambda$. For $i = 1, \ldots, \lambda$ let $B_i = A_i \cap A_{\lambda+1}$.

The sets $B_1, \ldots, B_\lambda$ satisfy the conditions of the lemma for $\lambda - 1$, so $\bigcup_{i=1}^{\lambda} B_i$ has the same reduced homology as $S^{\lambda-2}$. Moreover, $A \cap B = \bigcup_{i=1}^{\lambda} B_i$. The set B is contractible. Since $\bigcap_{i=1}^{\lambda} A_i \neq \emptyset$, each A_i is contractible and their non-empty intersections are contractible, then A is contractible.

In other words, A, B are two contractible sets whose intersection has the homology of $S^{\lambda-2}$. By Example 3.2.1, $A \cup B$ has the same homology groups as $S^{\lambda-1}$. □

We are now ready to prove a topological version of Helly's theorem.

THEOREM 3.3.2. *Let $\mathcal{G}$ be a good cover of $\mathbb{R}^d$. Let $\mathcal{F} \subset \mathcal{G}$ be a finite non-empty subset. If every $d+1$ or fewer sets in $\mathcal{F}$ have non-empty intersection, so does $\mathcal{F}$.*

PROOF. Assume for the sake of a contradiction that the statement is false. Let $\mathcal{F}'$ be the smallest subfamily of $\mathcal{F}$ such that $\bigcap \mathcal{F}' = \emptyset$, and let $\lambda + 1 = |\mathcal{F}'|$. We denote by $A_1, \ldots, A_{\lambda+1}$ the sets in $\mathcal{F}'$. By the conditions of the theorem, we know that $\lambda \geq d + 1$.

By Lemma 3.3.1, $\bigcup \mathcal{F}'$ has the homology of $S^{\lambda-1}$. Therefore, its homology group of dimension $\lambda - 1$ is non-zero. However, since it's a subset of $\mathbb{R}^d$, its homology of dimensions d and greater is trivial, giving us the desired contradiction. □

4. The topological Radon theorem

4.1. Two proofs of the topological Radon theorem. Consider a set of $d+2$ points in $\mathbb{R}^2$. Each of these points can be identified with a vertex of Δ^{d+1}, a $(d+1)$-dimensional simplex. Therefore, the set of points in $\mathbb{R}^d$ is basically the image of a map

$$f_0 : \text{vertices}(\Delta^{d+1}) \to \mathbb{R}^d.$$

We can extend f_0 linearly to be a map

$$f : \Delta^{d+1} \to \mathbb{R}^d.$$

Observe that for any face A of Δ^{d+1}, we have $f(A) = \text{conv}\{f(v) : v \text{ is a vertex of} A\}$. This means that we can rewrite Radon's lemma as follows.

THEOREM 4.1.1 (Radon's lemma, disguised). *For any linear map $f : \Delta^{d+1} \to \mathbb{R}^d$, there are two vertex-disjoint faces A, B of Δ^{d+1} such that $f(A) \cap f(B) \neq \emptyset$.*

For example, in the case $d = 2$, this is saying that there are essentially two ways of drawing a tetrahedron in the plane: there is either a vertex that is in front (or behind) the drawing of its opposite face or there are two opposite edges whose images intersect.

In 1976, Imre Bárány conjectured that, with this interpretation of Radon's lemma, maybe the linearity condition is not necessary; continuous functions should suffice.

This is indeed the case, as he proved a few years later in collaboration with Bajmóczy.

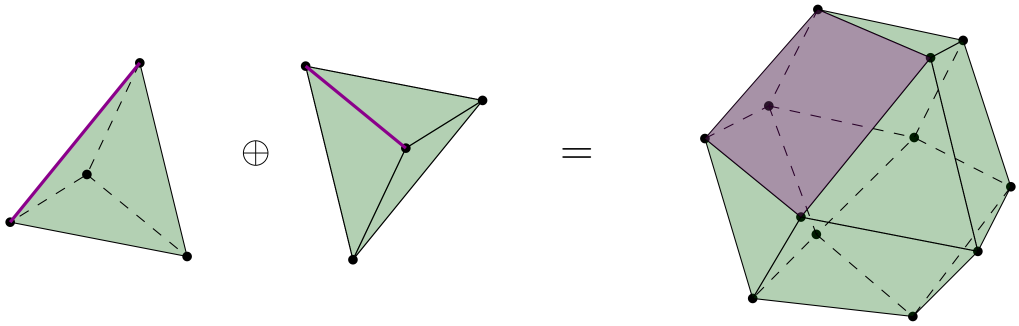


FIGURE 3. The Minkowski sum of a simplex K and $-K$ in $\mathbb{R}^3$ is a cuboctahedron. The facets of the cuboctahedron correspond to pairs of facets of K. Square faces are the Minkowski sum of two edges (highlighted in the figure), triangular faces corresponds to a facet in one summand and a vertex in another.

THEOREM 4.1.2 (Topological Radon; Bajmóczy, Bárány 1979 [**BB79**]). *For any linear map $f : \Delta^{d+1} \to \mathbb{R}^d$, there are two vertex-disjoint faces A, B of Δ^{d+1} such that $f(A) \cap f(B) \neq \emptyset$.*

We present two proofs of this result.

PROOF. Let K be a non-degenerate simplex in $\mathbb{R}^{d+1}$, which we use to represent Δ^{d+1}. Consider the polytope

$$P = K \oplus (-K),$$

formed by the Minkowski sum of K and $-K$. See Figure 3 for an illustration in $\mathbb{R}^3$.

Every point in the boundary ∂P can be written uniquely as $x - y$ where $x, y \in \Delta^{d+1}$ are points in pairwise-disjoint faces of K (Problem 37).

Therefore, we can consider the function

$$\begin{aligned} g : \partial(P) &\to \mathbb{R}^d \\ x - y &\mapsto f(x) - f(y). \end{aligned}$$

By Problem 37, the function g is well-defined and continuous. Moreover, since ∂P is homeomorphic to S^d, and g is an odd function, by the Borsuk–Ulam theorem[3] we there exists a point $p \in \partial P$ such that $g(p) = 0$. If $p = x - y$ for x, y in pairwise disjoint faces of K, then $f(x) = f(y)$, which is the conclusion we wanted.

□

In the proof above, we look at pairs of opposite faces and notice that we can glue them together to obtain a cuboctahedron, which is homeomorphic to a sphere. Alternatively, we can start with a sphere (or, in this case, an octahedron) and see how we can use that to parametrize pairs of points in opposite faces of K. That is the main idea behind the second proof.

SECOND PROOF. Let $\mathbb{O}^{d+1} \subset \mathbb{R}^{d+2}$ be the boundary of a particular octahedron centered at the origin. Formally,

$$\mathbb{O}^{d+1} = \{(x_1, \ldots, x_{d+2} : |x_1| + \ldots + |x_{d+2}| = 2\}$$

The reason why we choose the sum to be 2 instead of 1 will become apparent later in the proof, even though it does not matter much. Let $v_1, \ldots, v_{d+2}$ be the vertices of Δ^{d+1}. For $x \in \mathbb{O}^{d+1}$ we define

[3] Careful, it's important that the homeomorphism between S^d and ∂P preserves antipodality.

$$I(x) = \{i \in [d+2] : x_i > 0\}$$
$$J(x) = \{i \in [d+2] : x_i < 0\}.$$

Now we construct two points $A(x), B(x)$ in $\mathbb{R}^{d+1}$ defined as

$$\alpha(x) = \sum_{i \in I(x)} x_i$$
$$\beta(x) = -\sum_{i \in J(x)} x_i$$

The number $\alpha(x), \beta(x)$ are non-negative and their sum is 2. Now, we take the points

$$A(x) = \begin{cases} 0 \in \mathbb{R}^{d+1} & \text{if } I(x) = \emptyset \\ \left(\alpha(x) f\left(\frac{\sum_{i \in I(x)} x_i v_i}{\alpha(x)}\right), \alpha(x)\right) & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} 0 \in \mathbb{R}^{d+1} & \text{if } J(x) = \emptyset \\ \left(\beta(x) f\left(\frac{\sum_{i \in J(x)} -x_i v_i}{\beta(x)}\right), \beta(x)\right) & \text{otherwise} \end{cases}$$

This way, $A(x) = B(-x)$ and $A(-x) = B(x)$. These points are continuous functions of x.

Finally, consider

$$g : \mathbb{O}^{d+1} \to \mathbb{R}^{d+1}$$
$$x \mapsto A(x) - B(x).$$

This is a continuous antipodal map, so there must be a point for which $A(x) = B(x)$. This implies that $\alpha(x) = \beta(x) = 1$, so

$$f\left(\sum_{i \in I(x)} x_i v_i\right) = f\left(\sum_{i \in J(x)} (-x_i) v_i\right).$$

This is the overlap between two vertex-disjoint faces that we were looking for. □

4.2. The topological Tverberg theorem. Just as we wrote Radon's lemma in terms of linear maps from a simplex to $\mathbb{R}^d$, the same can be done with Tverberg's theorem.

THEOREM 4.2.1 (Tverberg's theorem, disguised). *Let $\Delta^{(r-1)(d+1)}$ be a simplex of dimension $(r-1)(d+1)$. Then, for any linear map $f : \Delta^{(r-1)(d+1)} \to \mathbb{R}^d$ there exist r pairwise disjoint faces of $\Delta^{(r-1)(d+1)}$ whose images intersect.*

This was the result that Imre Bárány actually conjectured to hold even if the linear map was replaced by a continuous map. Surprisingly, the result only holds for prime powers.

THEOREM 4.2.2 (the topological Tverberg theorem). *Let r be a prime power and $\Delta^{(r-1)(d+1)}$ be a simplex of dimension $(r-1)(d+1)$. Then, for any continuous map $f : \Delta^{(r-1)(d+1)} \to \mathbb{R}^d$ there exist r pairwise disjoint faces of $\Delta^{(r-1)(d+1)}$ whose images intersect.*

The case r prime was proved first [**BSS81**]. The version for r a prime power was proved a few years later [**Öza87**, **Vol96**]. The reason for this is that the topological tools, extensions of the Borsuk–Ulam theorem, require such conditions to work nicely (similar to what happens in mass partitions). It turns out that these conditions are necessary, as the counterexamples constructed by Frick show [**MW14**, **Fri15**].

The tools and techniques related to the topological Tverberg theorem exceed the scope of these notes, but knowing about this problem is important in this area. Out of the two proofs presented for the topological Radon theorem, it is the second that is usually generalized to prove the positive cases of the topological Tverberg theorem.

5. Problems of Chapter 6

Problem 37 Let $K \subset \mathbb{R}^d$ be a simplex and $P = K \oplus (-K)$. Prove that every point in $\partial(P)$ can be written as $x - y$ in a unique way for some $x, y \in K$ in pairwise-disjoint faces.

Problem 38 Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Suppose that each set in $\mathcal{F}$ is colored using one of $d+1$ possible colors. We know that any two sets of the same color intersect. Prove that there exists a hyperplane intersecting every set in $\mathcal{F}$.

CHAPTER 7

Fair divisions

One way to interpret the ham sandwich theorem is in the context of mathematical economy. We have several players, and we want do distribute some goods among them. Each player has a subjective opinion on value. For example, we can split $\mathbb{R}^d$, and player i measures the value of what they receive with a measure μ_i. Formally, the value they assign to a set $A \subset \mathbb{R}^d$ is $\mu_i(A)$. Players can disagree on the value of a set since they each have their own measure.

Then, the ham sandwich theorem says that we can find a hyperplane so that each of d different players agrees that both parts are equally valuable. This is an example of a fair division problem. We aim to split some resources among players and make such division as fair or agreeable as possible. Most mass partition problems are fair division problems, but the set of fair division problems is much larger. The preferences do not have to be given by measures.

1. Splitting necklaces

Let's start by looking at fair divisions in dimension one. In this section, we have two players (thieves in this interpretation) dividing the interval $[0, 1]$. We think of the interval as a necklace that our two thieves just stole. There are m kinds of pearls in the necklace, each with an even number of pearls. The thieves want to cut the necklace into segments. Then, they will distribute those segments among themselves so that each gets the same number of pearls of each kind, as in Figure 1.

A simple way of achieving this is to cut each pearl from the necklace and then do a fair distribution. This solution seems wasteful. What is the minimum number of cuts that the thieves require? It is possible that we first have all the pearls from type 1, then all the pearls of type 2, and so on until we get all the pearls of type m. We need at least one cut in each interval formed by pearls of a single type, so in this necklace, at least m cuts are needed. Surprisingly, this is the worst-case scenario: every necklace with m kinds of pearls can be distributed fairly with m cuts.

The solution to this problem is topological, which is striking given that the problem is completely combinatorial. The cuts should only depend on the number

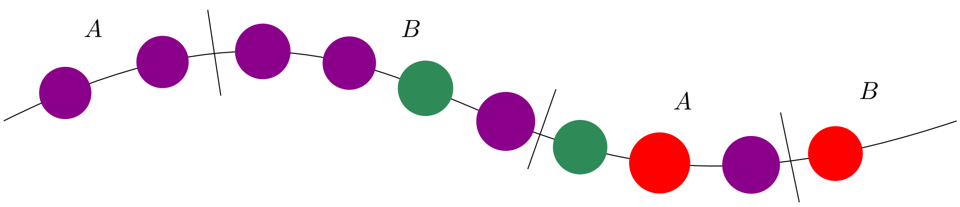


FIGURE 1. An example of a necklace with six pearls of the first kind, two of the second kind and two of the third kind. It can be divided fairly using three cuts among thieves A and B.

of pearls of each kind and their order. In this section, we solve the continuous version, in which we have m measures in $[0,1]$. The measures should assign a value 0 to any single point in $[0,1]$. The two versions are equivalent[1], so no generality is lost. The continuous version was first proved by Hobby and Rice [**HR65**], and the discrete version by Goldberg and West [**GW85**, **AW86**]. Alon later extended the results for r thieves [**Alo87**], showing that $(r-1)m$ cuts are always sufficient.

THEOREM 1.0.1 (Hobby, Rice 1965 [**HR65**]). *Let m be a positive integer and $\mu_1, \dots, \mu_m$ be m finite measures in $[0,1]$. Assume that $\mu_i(\{x\}) = 0$ for every $i \in [m]$ and $x \in [0,1]$. Then, we can split $[0,1]$ into $m+1$ or fewer intervals and distribute them among two sets A, B such that*

$$\mu_i(A) = \mu_i(B)$$

for all $i \in [m]$

FIRST PROOF. As usual, we look for a way to parametrize the partitions. Given a partition of the interval $[0,1]$ into $m+1$ intervals $I_1, \dots, I_{m+1}$ (in this order) distributed among the two sets let $x = (x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1}$ be a vector where $|x_k|$ is the length of I_k and

$$\operatorname{sign}(x_k) = \begin{cases} + & \text{if } I_k \in A \\ - & \text{if } I_k \in B \end{cases}$$

The sign is ambiguous for intervals of length zero, but we do not mind those. Then, $|x_1| + \ldots + |x_k| = 1$, so $x \in \mathbb{O}^m$, the boundary of the $(m+1)$-dimensional octahedron. Also, every point in this boundary corresponds to a distribution. If x corresponds to a distribution, then $-x$ corresponds to the same distribution where we just swap the names of A and B. Now we construct a map

$$\begin{aligned} f : \mathbb{O}^m &\to \mathbb{R}^m \\ x &\mapsto (\mu_1(A) - \mu_1(B), \dots, \mu_m(A) - \mu_m(B)). \end{aligned}$$

Note that $f(-x) = -f(x)$ and that this map is continuous. Since $\mathbb{O}^m$ is homeomorphic to an m-dimensional sphere, by the Borsuk–Ulam theorem this map must have a zero. The zero corresponds to a fair distribution. ☐

SECOND PROOF. We can embed $[0,1]$ in the moment curve in $\mathbb{R}^m$. The moment curve in $\mathbb{R}^m$ is a given by the embedding $\gamma : \mathbb{R} \to \mathbb{R}^m$ such that $\gamma(t) = (t, t^2, \dots, t^m)$. Our m measures in $\mathbb{R}^1$ now give us m measures in $\mathbb{R}^m$, so we can apply the ham sandwich theorem and find a hyperplane H simultaneously halving each of them. By Problem 2, $H \cap \gamma$ has at most m different points, so it cuts the curve into at most $m+1$ different intervals. The intervals on one side of H go to set A, and the others go to set B. ☐

2. Splitting cakes

2.1. Covers of simplices. Let us assume that we have to divide the interval $[0,1]$ among m players instead of two, but we are only allowed to make $m-1$ cuts (so each receives one interval). Partitions as in the previous section, where everyone agrees that everyone gets the same value, usually do not exist. However, we might be able to achieve *envy-free* distributions. In these distributions, everyone thinks they received the best interval, although they may not believe everyone receives the same value.

These are often called cake partitions. We think of the interval $[0,1]$ as a cake to be distributed among the m guests. Since we only seek envy-free partitions,

[1]It takes a formal argument to show this.

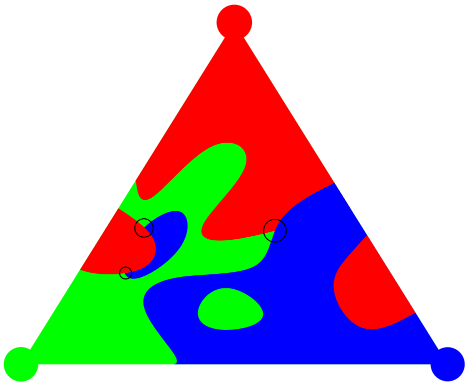


FIGURE 2. An example of a KKM cover of Δ^2. Notice that each edge only uses the colors of its vertices. Three points with all possible colors are highlighted.

the preferences of the players can be much more general than those induced by measures. We need the following conditions:

- Given a partition of the cake into r intervals, every player selects at least one non-empty piece as favorite. Players are allowed to have more than one favorite piece.
- The preferences are closed. In other words, suppose we have a sequence of partitions into r intervals, which converges. If player i always lists the j-th piece as a favorite, then they must do so as well in the limit partition.

The way we parametrize partitions is similar to the process in the previous section. If x_i is the length of the i-th piece, then the vector $x = (x_1, \ldots, x_m)$ only has non-negative entries whose sum is one. In other words, $x \in \Delta^{m-1}$, an $(m-1)$-dimensional simplex. For example, in $\mathbb{R}^3$, it's the intersection of the hyperplane $x+y+z=1$ with the positive octant, so it's an equilateral triangle. Let $v_1, \ldots, v_m$ be the vertices of Δ^{m-1}. We also denote by F_j the facet of Δ^{m-1} opposite to v_j. In other words, it's the set of points of $\Delta^{m-1} \subset \mathbb{R}^m$ whose j-th coordinate is zero.

Now consider the preferences of a particular guest, Alice. Given a point $x \in \Delta^{m-1}$, Alice can tell us which are her favorite pieces. For $j = 1, \ldots, m$, we denote by $A_j \subset \Delta^{m-1}$ the set of partitions in which Alice prefers the j-th piece. The conditions of the preference translate to the following properties of the sets $A_1, \ldots, A_m$:

- for $j = 1, \ldots, m$, the set A_j is closed,
- for $j = 1, \ldots, m$, we know $A_j \subset \Delta^{m-1} \setminus F_j$, and
- $\bigcup_{j=1}^m A_j = \Delta^{m-1}$.

The second condition says that Alice always prefers non-empty pieces, and the third is that Alice is hungry (she always wants at least one non-empty piece of cake). An m-tuple $(A_1, \ldots, A_m)$ as above is called a Knaster–Kuratowski–Mazurkiewicz (KKM) cover of Δ^{m-1}. An example is shown in Figure 2. The combinatorial properties of these covers were first studied in the early 20th century. Here is the first important result on KKM covers.

THEOREM 2.1.1 (Knaster, Kuratowski, Mazuriewicz 1929 [**KKM29**]). *Let m be a positive integer and $(A_1, \ldots, A_m)$ be a KKM cover of Δ^{m-1}. Then,*

$$\bigcap_{j=1}^m A_j \neq \emptyset.$$

An interpretation of this result is that for any hungry person, there is a partition in which they do not mind which pieces they receive. As you already know about colorful results in discrete geometry, let's jump ahead and state the colorful version of the theorem above.

THEOREM 2.1.2 (Gale 1984 [**Gal84**]). *Let m be a positive integer. For $i = 1, \dots, m$, let $(A_1^i, \dots, A_m^i)$ be m KKM covers of Δ^{m-1}. Then, there exists a permutation $\pi : [m] \to [m]$ such that*

$$\bigcap_{j=1}^{m} A_j^{\pi(j)} \neq \emptyset.$$

If all KKM covers are the same, then we obtain Theorem 2.1.1. The interpretation of this result is that, given m hungry guests, we can always find a partition of the cake into m intervals and a way to distribute the pieces so that everyone receives one of their favorite pieces.

2.2. Interlude: Borsuk–Ulam, homotopic maps, and Brouwer's fixed point theorem. Homotopy is an important concept in topology. We have encountered and used it a couple times in this book without mentioning it explicitly. Given topological spaces X, Y, we say that two continuous maps

$$f_0 : X \to Y$$
$$f_1 : X \to Y$$

are *homotopic* if there exists a continuous map $h : X \times [0,1] \to Y$ such that for all $x \in X$ we have $h(x,0) = f_0(x)$ and $h(x,1) = f_1(x)$. It's a standard exercise to show that homotopy induces an equivalence relation on the set of maps between two topological spaces, and homotopy preserves a significant number of topological properties.

We say that f is null-homotopic if it is homotopy equivalent to a map that sends all of X to a single point of Y. If Y is path-connected, we do not care to which point X is being sent.

The goal of this subsection is to show a particular version of the Borsuk–Ulam theorem. The Borsuk–Ulam theorem has a large number of equivalent formulations. It's quite useful to be familiar with a few if you are working on a problem in topological combinatorics so you don't need to waste time adapting the theorem to your particular needs. We denote by $B^d \subset \mathbb{R}^d$ the d-dimensional ball of radius one centered a the origin. Its boundary is S^{d-1}.

THEOREM 2.2.1. *Let d be a positive integer and $f : B^d \to B^d$ be a continuous map such that $f|_{S^{d-1}} : S^{d-1} \to S^{d-1}$. If $f|_{S^{d-1}}$ is homotopic to an odd map, then f is surjective.*

PROOF. We assume that the conclusion fails and seek a contradiction. Let f be such a map that is not surjective. We may assume without loss of generality that $f(x) \neq 0$ for all $x \in B^d$ (double-check this argument, you may need to use a homotopy to prove it!). We may also assume that $f|_{S^{d-1}}$ is an odd map, rather than homotopic to one. You can do this by considering an odd map $g : S^{d-1} \to S^{d-1}$ homotopic to $f|_{S^{d-1}}$. Using the homotopy, we can define a continuous smap in the region between S^{d-1} and $\frac{1}{2}S^{d-1}$ that is equal to g on S^{d-1} and to $f(2x)$ for $x \in \frac{1}{2}S^{d-1}$. Then, we extend this to a map on B^d by mapping x to $f(2x)$ for $x \in \frac{1}{2}B^d$.

Therefore, we can assume without loss of generality that $f : B^d \to \mathbb{R}^d \setminus \{0\}$ and that $f|_{S^{d-1}} : S^{d-1} \to S^{d-1}$ is odd. Now we define a map $\tilde{f} : S^d \to \mathbb{R}^d$. Recall that $S^d \subset \mathbb{R}^{d+1}$. For $x = (x_1, \dots, x_{d+1}) \in S^d$ we define

$$\tilde{f}(x) = \begin{cases} f(x_1, \dots, x_d) & \text{if } x_{d+1} \geq 0 \\ -f(-x_1, \dots, -x_d) & \text{if } x_{d+1} \leq 0. \end{cases}$$

Since $f|_{S^{d-1}}$ is odd, the function above is well defined (we just need to check that no problems arise if $x_{d+1} = 0$). Moreover, it would be a continuous odd function $f : S^d \to \mathbb{R}^d$ without a zero, contradicting the Borsuk–Ulam theorem. $\square$

We can dive a bit more into this kind of results. The key condition for us to extend a map from S^{d-1} to S^{d-1} to a non-surjective map from B^d to B^d is whether it is or not null-homotopic (see Problem 40). Therefore, one of the consequences of this section is the following corollary.

Corollary 2.2.2. *Let $f : S^d \to S^d$ be a continuous odd map. Then, f is not null-homotopic.*

Of course, if you see the Borsuk–Ulam theorem in a standard topology course, you probably know enough about the topological degree for the corollary above to be trivial. We can also prove yet another classic topological result: Brouwer's fixed point theorem.

Theorem 2.2.3. *Let $f : B^d \to B^d$ be a continuous map. Then, there exists $x \in B^d$ such that $f(x) = x$.*

Proof. Assume for a contradiction that there exists a continuous map $f : B^d \to B^d$ without fixed points. We construct a new map $g : B^d \to S^{d-1}$ as follows. For each $x \in B^d$, consider the open ray starting at $f(x)$ that goes through x. Since x and $f(x)$ are in B^d, the open ray must intersect S^{d-1} in exactly one point. Let $g(x)$ be this point. Note that for $x \in S^{d-1}$, $g(x) = x$. The map $g|_{S^{d-1}}$ is the identity, which is odd. Therefore, g would contradict Theorem 2.2.1. $\square$

2.3. Gale's theorem. Now we prove Theorem 2.1.2. Recall that a KKM cover of Δ^{m-1} is an m-tuple of sets $(A_1, \dots, A_m)$ such that

- for $j = 1, \dots, m$, the set A_j is closed,
- for $j = 1, \dots, m$, we know $A_j \subset \Delta^{m-1} \setminus F_j$, and
- $\bigcup_{j=1}^m A_j = \Delta^{m-1}$.

We say that $(A_1, \dots, A_m)$ is an *open cover* of Δ^{m-1} if each A_j is open instead of closed. If $(A_1, \dots, A_m)$ is a KKM cover of Δ^{m-1} and $\delta > 0$, we can define

$$A'_j = \{x \in \Delta^{m-1} : \operatorname{dist}(x, A_j) < \delta\}.$$

Then, if δ is sufficiently small, the m-tuple $(A'_1, \dots, A'_m)$ is an open KKM cover of Δ^{m-1}. We prove a version of Theorem 2.1.2 for open KKM covers of Δ^{m-1}. We leave it to the reader[2] to verify that, with the argument presented above, the version for open KKM covers implies the version for closed KKM covers.

Theorem 2.3.1 (Open version of Gale's theorem). *Let m be a positive integer. For $i = 1, \dots, m$, let $(A^i_1, \dots, A^i_m)$ be m open KKM covers of Δ^{m-1}. Then, there exists a permutation $\pi : [m] \to [m]$ such that*

$$\bigcap_{j=1}^m A_j^{\pi(j)} \neq \emptyset.$$

[2]As I write *we leave it to the reader,* the following quote echoes in my mind.
"You've become the very thing you swore to destroy." - Obi-Wan Kenobi to Anakin Skywalker. *The Revenge of the Sith.*

Proof. For $i, j \in [m]$, let $B^i_j = \Delta^{m-1} \setminus A^i_j$, which is a closed set. For $i, j \in [m]$ and $x \in \Delta^{m-1}$, let

$$f_{i,j}(x) = \frac{1}{\sum_{h=1}^m \mathrm{dist}(x, B^i_h)} \mathrm{dist}(x, B^i_j).$$

Since $(A^i_1, \ldots, A^i_m)$ is an open KKM cover, the function above is well defined and continuous. Now we construct an $m \times m$ matrix $M(x)$ such that its entry m_{ji} is $f_{i,j}(x)$. The entries of $M(x)$ are non-negative and the sum of each of its columns is 1. Let $u = (1, \ldots, 1)^T \in \mathbb{R}^m$ the vector with all entries equal to one. We define

$$\begin{aligned} f : \Delta^{m-1} &\to \Delta^{m-1} \\ x &\mapsto \frac{1}{m} M(x) u. \end{aligned}$$

In other words, the j-th entry of $f(x)$ is the average of the j-th row of $M(x)$. By the conditions on the covers, for each face $\sigma \subset \Delta^{m-1}$, we have $f(\sigma) \subset \sigma$. By Problem 41, the map $f|_{\partial \Delta^{m-1}}$ is homotopic to the identity. Therefore, by Theorem 2.2.1, the map f must be surjective. There exists an x such that $f(x)$ is the barycenter of Δ^{m-1}, the point with all coordinates equal to $1/m$.

For this value of x, the matrix $M(x)$ is doubly stochastic. By Birkhoff's theorem (Theorem 3.0.1 in Part 1, Chapter 1), it is a convex combination of permutation matrices. In particular, there exists a permutation $\pi : [m] \to [m]$ such that $m_{j\pi(j)} > 0$ for all $j \in [m]$. This means that for all $j \in [m]$ we have $x \notin B^{\pi(j)}_j$, so $x \in A^{\pi(j)}_j$. In other words, x is a point in the intersection of all sets $A^{\pi(j)}_j$, as we wanted to prove. □

3. Sperner's lemma

There is an alternate way to prove Theorem 2.1.2, which does not require Birkhoff's theorem. It is also interesting since it shows that it is worth thinking about the discrete versions of these problems.

We say that a *triangulation* T of Δ^{m-1} is a finite set of $(m-1)$-dimensional simplices such that

- their union is Δ^{m-1} and
- For any two simplices τ_1, τ_2 in T, their intersection is either empty or a face of each of τ_1 and τ_2.

The vertices of a triangulation T is simply the union of the vertices of each simplex in T. Now, assume that we have a color for each of the m vertices of Δ^{m-1} and a triangulation T of Δ^{m-1}. A *Sperner coloring* of T is a way to assign a color to each vertex v of T such that if v is in a face σ of Δ^{m-1}, then it has one of the colors corresponding to a vertex of σ.

A KKM cover induces a Sperner triangulation. If we have a KKM cover $(A_1, \ldots, A_m)$ of Δ^{m-1} and take a vertex v of T, we can assign color j to v if $v \in A_j$ (if v is in multiple sets of the cover, choose one of them). We can see that this is indeed a KKM cover.

Lemma 3.0.1 (Sperner's lemma). *Let m be a positive integer, T be a triangulation of Δ^{m-1} with a Sperner coloring. Then, the number of simplices σ of T whose vertices all have different colors is odd.*

If we want to show that there is at least one simplex with all colors, we can proceed in a similar way to how we proved the KKM theorem, avoiding Birkhoff's theorem entirely[3], but we aim for a simpler proof here.

[3]Note that Birkhoff is not needed to prove Theorem 2.1.1. The technical bit for this approach: a Sperner coloring induces a piecewise-linear map from Δ^{m-1} to Δ^{m-1}. Why is it surjective?

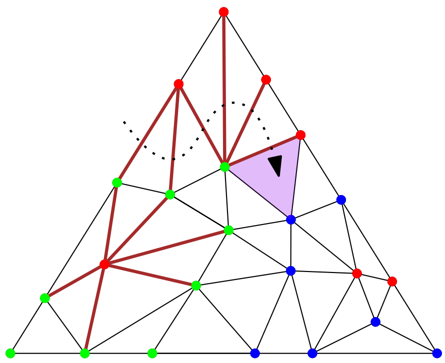


FIGURE 3. An example of a Sperner coloring of a triangulation of Δ^2. This coloring is induced by the KKM cover in Figure 2, as each vertex has the color it is on in that KKM cover. If we think of every green-red segment as a door, the idea behind the proof of Sperner's lemma is evident: rooms with a single door up and the exterior are paired up by "walking until you get stuck"

PROOF. We prove by induction on m that the number of colorful simplices is odd. For $m = 2$, let v_1, v_2 be the two vertices of the segment Δ^1. Since we start with color v_1 and end with color v_2, the triangulation T must have an odd number of segments in which we change colors.

Now, assume the result holds for Δ^{m-2}. Let $v_1, \ldots, v_m$ be the vertices of Δ^{m-1}. Consider the pairs (σ, σ') where

- σ is an $(m-1)$-dimensional simplex in T,
- σ' is a facet of σ whose $m-1$ vertices all have different colors and only misses the color corresponding to v_m.

We show that there is an odd number of these pairs. First, if σ' is in the boundary of Δ^{m-1}, it must be in F_m, the facet opposite to v_m. The triangulation $T \cap F_m$ is a triangulation of F_m (which is an $(m-2)$-dimensional simplex). Moreover, the Sperner coloring of T induces a Sperner coloring on $T \cap F_m$. By induction, there is an odd number of $(m-2)$-dimensional fully colored simplices on $T \cap F_m$.

For each $(m-2)$-dimensional simplex σ' missing the color of v_m there are two possibilities:

- we have $\sigma' \subset F_m$, in which case there is a single simplex σ such that (σ, σ') is a valid pair,
- we have $\sigma' \not\subset F_m$, in which case there are exactly two simplices σ such that (σ, σ') is a valid pair.

Since there is an odd number of pairs of the first kind and an even number of pairs of the second kind, the number of valid pairs is odd. Now, if we take an $(m-1)$-dimensional simplex σ, the following scenarios can happen

- the colors of the vertices of σ do not contain the set $\{v_1, \ldots, v_{m-1}\}$, so there are zero valid pairs of the form (σ, σ'),
- the colors of the vertices of σ form exactly the set $\{v_1, \ldots, v_{m-1}\}$, so there are exactly two valid pairs of the form (σ, σ') as exactly one color is repeated two times, or
- the colors of the vertices of σ form exactly the set $\{v_1, \ldots, v_{m-1}, v_m\}$, so there is exactly one valid pair of the form (σ, σ').

Therefore, there is an odd number of simplices of the third kind, which is the number of fully colored simplices. $\square$

We need one last tool in this section: the barycentric subdivision. Let P be an $(m-1)$-dimensional polytope. A full flag of faces of P is a sequence $Q_0 \subset Q_1 \subset Q_2 \ldots \subset Q_{m-2} \subset Q_{m-1} = P$ of m faces of P such that the dimension of Q_j is j for all j. So Q_0 is a vertex, Q_1 is an edge, Q_2 is a two-dimensional face, and so on. If $P = \Delta^{m-1}$, it's a simple exercise that there are exactly $m!$ full flags of faces of P.

For a full flag $Q_0 \subset Q_1 \subset Q_2 \ldots \subset Q_{m-2} \subset Q_{m-1} = P$ of P, consider the simplex

$$\operatorname{conv}\{\operatorname{bar}(Q_0), \operatorname{bar}(Q_1), \ldots, \operatorname{bar}(Q_{m-1})\} \subset P,$$

where $\operatorname{bar}(X)$ denotes the barycenter of X. If we take all simplices induced by full flags of P, it induces a triangulation $B(P)$ of P, which we call the barycentric subdivision. Every vertex of $B(P)$ corresponds to a face of P, so we can assign as a label the dimension of that face. This means that every full-dimensional simplex of $B(P)$ has all labels from 0 to $m-1$.

Given a triangulation T of Δ^{m-1}, we denote $B(T)$ the barycentric subdivision of T (we take the barycentric subdivision of every simplex in T). This is a triangulation of Δ^{m-1} in which every vertex has a labeled from 0 to $m-1$ and each full-dimensional simplex has all labels.

Second proof of Theorem 2.1.2. Let $v_1, \ldots, v_m$ be the vertices of Δ^{m-1}. We introduce m colors, one for each of the of Δ^{m-1}. Suppose that for all $i \in [m]$ we have a KKM cover $(A^i_1, \ldots, A^i_m)$. Let T be a triangulation of Δ^{m-1}. We take its barycentric subdivision $B(T)$. For a vertex v of $B(T)$ with label $i-1$, we assign it the color of v_j if $v \in A^i_j$ (if there are multiple choices for j, we choose one arbitrarily). This is a Sperner coloring of $B(T)$.

Therefore, there must be a full-dimensional simplex σ of $B(T)$ with all the colors. Since its vertices must have all the labels, this means that we can denotes its m vertices $u_1, \ldots, u_m$ so that $u_j \in A^{\pi(j)}_j$ for all $j \in [m]$. We do this argument for any triangulation T. Therefore, if we take a sequence of triangulations so that the diameter of the simplices converges to zero, we obtain the desired conclusion[4]. □

4. Problems of Chapter 7

Problem 39 You are given a collection of n baskets. Inside each basket there is some amount of apples, some amount of oranges, and some amount of bananas. Show that, regardless of how much fruit is in each basket, it is always possible to pick $\left\lfloor \frac{n+3}{2} \right\rfloor$ baskets so that you get at least half the total amount of each kind of fruit.

Problem 40 • Let $g : S^{d-1} \to S^{d-1}$ be a null-homotopic continuous map. Show that there exists a map $f : B^d \to B^d$ such that $f|_{S^{d-1}} = g$ and f is not surjective.

• Let $g : B^d \to B^d$ be a continuous map such that $g|_{S^{d-1}} : S^{d-1} \to S^{d-1}$. If g is not surjective, show that $g|_{S^{d-1}}$ is null-homotopic.

Problem 41 Let Δ^{m-1} be a simplex with m vertices and $f : \partial\Delta^{m-1} \to \partial\Delta^{m-1}$ be a continuous function on its boundary. Suppose that for every face σ of Δ^{m-1} we have $f(\sigma) \subset \sigma$. Prove that f is homotopic to the identity map.

[4] Do you trust me on this? It sounds like you should double-check this last argument!

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