MTH 535 Homework 2

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Problem 1.

Proof. Let $b_k = (k, \infty)$, then $m^*(b_k) = \infty$ for all k, so $\lim_{k\to\infty} m^*(b_k) = \infty$ but $B = \bigcap_{k=1}^{\infty} b_k = \varnothing$ so $m^*(B) = 0$. Therfore the assumption that b_1 have finite outer measure is critical for continuity of measure to hold (for intersections). $\square$

Problem 2.

Proof. Let $A \subset \mathbb{R}$ and let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint measurable sets. By proposistion 6 we have that for $n \in \mathbb{N}$,

$$m^*(A \cap \bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

Now monotinicity of outer measure and the fact that $A \cap \bigcup_{k=1}^{n} E_k \subseteq A \cap \bigcup_{k=1}^{\infty} E_k$ together with the above yeilds

$$\sum_{k=1}^{n} m^*(A \cap E_k) \leq m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \tag{1}$$

Since the above holds for all n, taking the limit as $n \to \infty$ we get

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \leq m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \tag{2}$$

The reverse inequality follows from sub additivity of outer measure,

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k) \tag{3}$$

Thus, $\sum_{k=1}^{\infty} m^*(A \cap E_k) = m^*(A \cap \bigcup_{k=1}^{\infty} E_k)$. $\square$

Problem 3.

Proof. Let $E \subseteq \mathbb{R}$ with $m^*(E) > 0$, by Vitali's theorem any given choice set C_E is nonmeasurable. Since any countable set has measure 0 and is thus measurable, C_E must be uncountable. $\square$

Problem 4.

Proof. Let E be non-measurable with finite outer measure, then by the negation of 2.11 (part 1) there exists $\epsilon_0 > 0$ such that for all open sets $O \supseteq E$, $m^*(O \setminus E) \geq \epsilon_0$. Since outer measure is defined as an infimum and $m^*(E) < \infty$, for each $n \in \mathbb{N}$ we finc a collection of disjoint bounded open sets $\{I_{n,k}\}_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} \ell(I_{n,k}) < m^*(E) + \frac{1}{n}$. Now define $G_n = \bigcup_{k=1}^{\infty} I_{n,k}$ so

$$m^*(G_n) \leq \sum_{k=1}^{\infty} \ell(I_{n,k}) < m^*(E) + \frac{1}{n}$$

Thus $m^*(G_n) - m^*(E) < \frac{1}{n}$. Let $G = \bigcap_{n=1}^{\infty} G_n$, Since $E \subseteq G_n$, $E \subseteq G$ and $m^*(E) \leq m^*(G)$. Conversly $G \subseteq G_n$ and $m^*(G_n) \leq m^*(E) + \frac{1}{n}$ for every n, hence $m^*(E) = m^*(G)$. G is G_δ by definition, and

$$m^*(G_n \setminus E) > \epsilon_0$$

$$\implies m^*(G \cap E^c) = m^*(\bigcap_{n=1}^{\infty} G_n \cap E^c) = \lim_{n \to \infty} m^*(G_n \cap E^c) > \epsilon_0$$

where in the last step I am using continuity of measure since $m^*(G_1) < m^*(E) + \frac{1}{n}$ implies G_1 has finite outer measure. □

Problem 5.

Proof. Let $F = \bigcap_{k=1}^{\infty} F_k$ where F_k is the subset of $[0,1]$ remaining after k steps of the generalized cantor removal process. Since at each step we are removing open sets, F_k is closed, and thus as a intersection of closed sets, F is closed. Now we prove that that $[0,1] \setminus F$ is dense in $[0,1]$. Let $x \in [0,1]$, We show that very open ball around x of radius $\epsilon > 0$ contains a point of $[0,1] \setminus F$. If $x \in [0,1] \setminus F$ we are done since $[0,1] \setminus F$ is a union of open intervals. So we assume that $x \in F$. Notice that at the n^{th} step of the removal process, F_n is the union of 2^n disjoint closed intervals of equal length, so that each closed interval has length less than $\frac{1}{2^n}$. This says that for $x \in F_n$, there exists a point u_n in $[0,1] \setminus F_n$ such that $|x - u_n| < \frac{1}{2^n}$. Since $x \in F$ implies $x \in F_n$ for all n, and since $\frac{1}{2^n} \to 0$ as $n \to \infty$, it follows that for any open ball of radius ϵ, say $B_\epsilon(x)$ there exits $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$, so that $u_n \in B_\epsilon(x)$. Thus since every open ball around $x \in [0,1]$ contains elements $u \in [0,1] \setminus F$, Thus x is a limit point of $[0,1] \setminus F$ and hence is contained in the clousure.

To compute $m^*(F)$, note that at step n we are removing 2^{n-1} intervals of length $\alpha \frac{1}{3^n}$, So

$$m^*([0,1] \setminus F_n) = \alpha \sum_{k=1}^{n} \frac{2^{k-1}}{3^k}$$

Thus

$$m^*([0,1] \setminus F) = \alpha \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \alpha$$

Then since F is measurable

$$m^*([0,1]) - m^*(F) = \alpha \implies m^*(F) = 1 - \alpha$$

□

Problem 6.

Proof. let f be continuous. Let E be the collection of subsets such that $e \in E$ if $f^{-1}(e)$ is Borel. Since f is continuous, the preimage of an open set is open, since open sets are Borel the preimage of an open set under f is borel, thus E contains all open sets. Further, since preimages are well behaved with respect to unions, intersections, and complements, for any countable collection $\{A_n\}_{n=1}^{\infty} \subseteq E$

$$f^{-1}(\bigcap_{n=1}^{\infty} A) = \bigcap_{n=1}^{\infty} f^{-1}(A)$$

$$f^{-1}(\bigcup_{n=1}^{\infty} A) = \bigcup_{n=1}^{\infty} f^{-1}(A)$$

$$f^{-1}(A_n^c) = f^{-1}(A_n)^c \text{ for all } n \in \mathbb{N}$$

That its, the preimage of any union of elements of E is a union of preimages of elements of E, and hence Borel (Borel sets are a σ-algebra); likewise, the preimage of a complement will be a complement of a preimage; and it follows the same is true for intersections. Thus, E is a σ-algebra which contains all the open sets, then it follows by defintion that all Borel sets are contained in E, that is, the preimage of a Borel set is Borel. □

Problem 7.

Proof. Let g be the continuous inverse of f, so $f \circ g = g \circ f = \texttt{id}$. By the above, for any Borel set B in the range of g, $g^{-1}(B)$ is Borel, since $g = f^{-1}$ we have that for any Borel set B, $(f^{-1})^{-1}(B) = f(B)$ is Borel, hence f maps Borel sets to Borel sets. $\square$