Homework 4

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Morgan Prior and I worked together

Problem 1. *Proof.* We prove that the smaller partite set of $G = X\dot{\cup}Y$ is at least $e(G)/\Delta(G)$, then it follows that X is a minimal vertex cover and since G is bipartite, this is also the largest matching.

We proceed with induction. If $n = 2$ the result clearly holds since the max degree is 1 and the number of edges is 1 we get $e(G)/\Delta(G) = 1$ which is equal to the size of the largest matching.

Now suppose that for $n \geq 2$ all bipartite graphs B have a matching of size at least $\frac{e(B)}{\Delta(B)}$. and let $G = X\dot{\cup}Y$ be bipartite. Without loss of generality, assume that $|X| \leq |Y|$. Then we remove a vertex v from X such that removing v does not lower the maximal degree of the graph (This is always possible unless $|X| = 1$, in which case the graph is a star and we have $\frac{e(G)}{\Delta G} = \frac{n-1}{n-1} = 1$ which is equal to the size of the largest matching). Now consider $G' = G \setminus v$. Then this is a graph on n vertices show that by our induction hypothesis, we have $|X \setminus v| = \frac{e(G')}{\Delta(G)}$. Then we have

$$\frac{G'}{\Delta(G)} \leq \frac{e(G)}{\Delta(G)} \leq \frac{e(G' + \Delta(G)}{\Delta(G)} \leq \frac{e(G')}{\Delta(G)} + 1$$

So that adding back v increases the ration by at most 1, but increases the size of X by exactly 1, hence we have $|X| \geq \frac{e(G)}{\Delta(G)}$.

We have established that in any bipartite graph, the smaller of the two partite sets is at least $\frac{e(G)}{\Delta(G)}$, since taking X gives a minimial vertex cover, we may apply the theorem proved in class for bipartite graphs, namely, that maximum size of a matching is the same as the minimum size of a vertex cover. Size a minimum vertex cover is at least $\frac{e(G)}{\Delta(G)}$, the size of a max matching is also at least as big.

For the example, since we know that there are more than $(k-1)n$ vertices and that the complete bipartite graph has max degree n, we have that

$$\alpha'(G) > \frac{(k-1)n}{n} = (k-1)$$

so it is at least k (because the inequality is strict and $\alpha'(G)$ is restricted to integers). $\square$

Problem 2. *Proof.* Let G be a group and let M be a matching of max size. Then let S be the set of all vertices in G that are contained in an edge of M. Note that $S = 2\alpha'(G)$. We prove that S is a vertex covering. Indeed, we have by maximality of M that for every edge e in G, either $e \in M$ or e is incident to an edge in M. In the first case e is covered by the vertices it connects, in the second case e is covered by the verties of the edge it is incident to. Hence S is a vertex covereing of the graph G. Then it follows

$$\beta(G) \leq S = 2\alpha'(G)$$

as desired.

For the second part, one can consider a graph G with k components, such that each component is a 3 cycle. It is clear that the three cycle has a max matching of 1 and a minimum vertex cover of size 2, then repeating this component k times gives the result. □

Problem 3. *Proof.* The max matching is 8 as pictured because there exists a vertex cover of size 8 □

Problem 4. *Proof.* Assume that T have a perfect matching, then by tuttes thm we have that $q(T \setminus v) \leq |\{v\}| = 1$. Now since T is a tree and have a perfect matching it cannot be on an odd number of vertices. If v is a leaft then clearly $T \setminus v$ has exactly one odd component. In the case that v is not a leaf, removing v splits the tree into two components (since any two vertices have exactly one path in a tree), whose odders must add to an odd number (so that adding v back gives T on an even number of vertices). Hence exactly one of the compents will be odd.

Now we must proceed in the opposite direction. Assume that T does not have a perfect matching, we will show that the condidtion given in the problem is violated. As in my last homework, I define a leaf edge to be an edge that contains a leaf. We begin to build a matching by first taking M to be the collections of all leaf edges in T, then we look at the subgraph T_1 wich is induced on all vertices not contained in a leaf edge of T. Inductivly, we define T_n to be the induced subgraph on all vertices not contained in a leaf edge of T_{n-1} and we identify T with T_0. Now for some $k \in \{0, 1 \dots\}$ we must have that this process fails, that is at some k, we will be unable to add all leaf edges of T_k to M without a contradiction (because we assumed there is no perfect matching.) At this step, it must be the case that there are two (or more) leaf edges e_1, e_2 which share a common vertice, say, w. We will show that $q(T \setminus w) \geq 2$. Since we are assuming that the process fails at step k, it then is succesful for the first $k-1$ steps (I deal with $k = 0$ below). The leaves in T_k then connect to vertices which themselves can be thought of as roots of a tree that contains a perfect matching, and hence contains an even number of vertices. Then when we consider $T \setminus w$, we get one componet with each tree plus the edge e_1 or e_2 (or more if thats the case), since the tree is on an even number of vertices, these are two odd components. The above doesnt quite apply to the case $k = 0$, which I deal with now, if on the first step we are unable to add all leaf edges to M, then as above this must be because at least two leaf edges both contain the same vertice, w, then $T \setminus w$, splits into two componets which contain exactly 1 point and the rest of the tree, hence this has more than one odd component. $\square$