MTH 436: Analysis HW 1

Evan Fox (efox20@uri.edu)

February 8, 2023

Question 1.

Show that if f is lipschitz with $\alpha > 1$ then f is constant and show that if $\alpha = 1$ then f is of bounded variation.

Proof. Let $\alpha > 1$. Then we have that for all $x, y \in [a, b]$,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M|x - y|^{\alpha - 1} \tag{1}$$

where $\alpha - 1 > 0$ by our assumption on α. Now let $c \in (a, b)$, f is differentiable at c if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ is finite for all c. Since the absoulute value of this limit is bounted above by $M|x - y|^{\alpha - 1}$, a term which goes to 0 as $x \to y$, $f'(c) = 0$ for all $c \in (a, b)$. Then it is a consequence of the mean value theorem that a function whose deriviative is zero must be constant.

Now if $\alpha = 1$, equation 1 becomes

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$

and a simillar argument to above will show that f' is bounded and f is clearly continuous. Thus, by a theorem proved in class will be of bounded variation.

□

Question 2.

(a) (a)

Proof. First $v(x) \leq p(x) + n(x)$ is clear, since for any partition P of $[a, b]$, Then we have

$$v(x) = \sum_{k=1}^{n} \Delta f_k = \sum_{k \in A(p)} \Delta f_k + \sum_{k \in B(P)} |\Delta f_k|$$

$$\leq \sup\{ \sum_{k \in A(P)} \Delta f_k | P \in \mathcal{P}[a, b]\} + \sup\{ \sum_{k \in B(P)} |\Delta f_k| | P \in \mathcal{P}[a, b]\}$$

$$= p(x) + n(x)$$

Now we must prove the opposite inequality, first note that for $P_1, P_2 \in \mathcal{P}[a, b]$ we have

$$\sum_{k \in A(P_1)} \Delta f_k \leq \sum_{k \in A(P_1 \cup P_2)} \Delta f_k \tag{2}$$

and

$$\sum_{k \in B(P_2)} |\Delta f_k| \leq \sum_{k \in B(P_1 \cup P_2)} |\Delta f_k| \tag{3}$$

(2) follows since if I evaluate the posistive variation at some partition $P = \{x_0, \ldots, x_1\}$, then adding a single point u to P lies in some interval $[x_k, x_{k+1}]$. If $f(x_k) < f(u) < f(x_{k+1})$ Then there is nothing to prove. If $f(u) < f(x_k)$, then $f(x_{k-1}) - f(u) > f(u) - f(x_k)$ so that the positive variation increases and the same is true if $f(u) > f(x_{k+1})$. Hence by adding a point to a partition P, I cannot decrease the positve variation, then by induction, I can add any finite number of points to the parition and I can only increase the positive variation. (3) is similliar.

Now we want to prove $p(x) + n(x) < v(x)$, so we prove that for any partitions P_1 and P_2, there exists a parition P such that

$$\sum_{k \in A(P_1)} \Delta f_k + \sum_{k \in B(P_2)} |\Delta f_k| \leq \sum_{k=1}^{n} |\Delta f_k| \tag{4}$$

Where the term on the right of the inequality is being evaluated at the partition P. Namely, choose $P = P_1 \cup P_2$, then using (2) and (3) and noticing that I can add the two terms on the right of the inequality when they are being evaluated at the same partition and the sum is the same as just evaluating f at the partition, the desired result follows. $\square$

(b) (d)

Proof. First note that this is clear intuitively, it is saying that the value of $f(x)$ for $x \in [a, b]$ is just the initial value at the start of the interval $f(a)$ plus the positive variation from a to x and minus the negitive varition (we subtract since negitive variation is defined with an absoulute value so that it is positive). We prove that $p(x) - n(x) = f(x) - f(a) = \sum_{k=1}^{n} \Delta f_k$. Let P_1 be the partition which maximizes the positive variation and let P_2 maximize the negitive variation on $[a, x]$. Then let $P = P_1 \cup P_2$. Then

$$p(x) - n(x) = \sum_{k \in A(P_1 \cup P_2)} \Delta f_k + \sum_{k \in B(P_1 \cup P_2)} -|\Delta f_k|$$
$$= \sum_{k=1}^{n} \Delta f_k = f(x) - f(a)$$

Where we dont need an absoulte value n the second line since we reintroduced the negitive to all the terms Δf_k with $k \in B(P_1 \cup P_2)$. ☐

(c) (e)

Proof. we have

$$f(x) = f(a) + p(x) - n(x)$$
$$f(x) = f(a) + p(x) - p(x) - n(x)$$
$$f(x) = f(a) + 2p(x) - v(x)$$
$$f(x) - f(a) + v(x) = 2p(x)$$

where the seconed step follows from (a). The other equation follows from adding and subtracting $n(x)$ rather than $p(x)$ ☐

(d) (f)

Proof. We proved in classes that if $x \in [a, b]$ is a point of continuity for f then it is also a point of continuity for v. Then using the results of the last section, we can see that $p(x) = \frac{1}{2}(f(x) + v(x) - f(a)$ and since the sum of continuous functions is continuous, we are done. The same argument applies to $n(x)$. ☐

Question 3.

10

(a) First let $g_1 = Im(H)$ and $g_2 = Re(H)$, these are just analogs to first and seconed projection if we considered H as a paramterized cure in $\mathbb{R}^2$.

g_1 is continuous on $[a, 2b-a]$ by the pasting lemma. We have $g_1(t) = f(t)$ for $t \in [a,b]$ and $g_1(t) = g(2b-t)$ for $t \in [b, ab-a]$. Given a partition $P \in \mathcal{P}[a, 2b-a]$, add b to P if it is not already there. Then

$$\sum_{k=1}^{n} |\Delta g_{1_k}| = \sum_{k=1}^{m} |\Delta f_k| + \sum_{k=m}^{n} |\Delta g_k|$$

where $x_m = b$. Clearly this is bounded by our assumptions of f and g. Further, since $V_{g_1}[a,b] = V_f[a,b]$ and $V_{g_1}[b, 2b-a] = V_{g(x)}[a,b]$ we have by the additive property of total variation, adding the previous two numbers gives the total variation of g_1 on the interval $[a, 2b-1]$. A similiar story is going to hold for g_2, hence since the components of H are of bounded variation, H defines a rectifiable curve.

(b) Look at last page.

(c) *Proof.* S is a closed set, so its boundry are all points $S \setminus intS$. For any point (x,y) not on Γ if $a < x < b$ and $f(x) < y < g(x)$ then we can find a open set of (x,y) contained in S, so it is not in the boundry. Hence the only points are the points in S that do not satisfy the above, i.e they must lie on the curve Γ. □

(d) I think in the second line defining H, they mean 2b-t not t.

Proof. $f - g$ is continuous since f and g are. Then the same approach as above will show that $Im(H)$ has a well defined total variation on $[a,b]$ and on $[b, 2b-1]$ since H is continuous on the union of these intervals, The total variation will be given by the additive property of variation.

again S defines a closed region in $\mathbb{R}^2$, so its interior is all points $S \setminus intS$. But then any point not on Γ_0 but in S, will be in the interior since its coordinates are given by strict inequalities $a < x < b$ and -1/2$(f-g) <$ $y < 1/2f - g$ □

(e) This is easy to see since the imaginary component of H is the y-axis, and in the first half of the interval $[a,b]$ the imaginary component of H is minus its value in the second half $[b, 2b-a]$, so it is fliping the curve over the x-axis. also note that it is zero at a and b, since g and f agree there.

(f) *Proof.* The curve Γ_0 (or Γ) is traced out by the imaginary component. When computing $\Lambda_{\Gamma_0}(P) = \sum_{k=1}^{n} \|(g-f)(x_k) - (g-f)(x_{k-1})\|$ at some partition $P \in \mathcal{P}[a, 2b-a]$, we see that

$$\Lambda_{\Gamma_0}(P) = \sum_{k=1}^{n} \|(g-f)(x_k) - (g-f)(x_{k-1})\|$$

$$= \sum_{k=1}^{n} \|g(x_k) - g(x_{k-1})\| + \sum_{k=1}^{n} \|f(x_k) - f(x_{k-1})\|$$

but this is the same as the arclength of Γ execpt I dont get to choose two different partitions for g and f. Hence by choosing the same partition for g and f we see that Λ_{Γ_0} must be obtained by some partition for Λ_Γ $\square$

Question 4.

11

Proof. Let f be absolutely continuous, and let $\epsilon > 0$. Then using the definition for $n = 1$, we see that it says given a subinterval (a_1, b_1) such that $b_1 - a_1 < \delta$ then $|f(b_1) - f(a_2)| < \epsilon$. So for any point $x \in [a, b]$, if $|x - y| < \delta$ then we cany take the one subinterval to be (x, y) if $x < y$ or (y, x) if $y < x$.

Now we prove that f is of bounded variation. Let $P \in \mathcal{P}[a, b]$ such that $\|P\| < \delta$ and $P = \{x_0, \ldots x_m\}$. Then on each interval $[x_{k-1}, x_k]$ consider a partition P_k, by grouping the points of the parition as disjoint subintervals we get $\sum_{P_k} |\Delta f| < \epsilon$ since the sum of the lengths of disjoint subintervals cannont exceed $x_k - x_{k-1} < \delta$ so that f is clearly of bounded variation on each subinterval. Then let $P' = P \cup \bigcup_{k=1}^{m} P_k$ and we have

$$\sum_{p'} |\Delta f_k| < m\epsilon$$

Since $P_k \in \mathcal{P}[x_{k-1}, x_k]$ was arbitrary and since $\bigcup_{k=1}^{m} P_k \in \mathcal{P}[a, b]$, we see that f is of bounded variation. $\square$

Question 5.

12

Proof. Fix M such that $|f(x)-f(y)| \leq M|x-y|$ then given $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$. Then Given (a_k, b_k) satisfing $\sum b_k - a_k < \delta$ we have

$$\sum |f(b_k) - f(a_k)| \leq M \sum |b_k - a_k| < \epsilon$$

where the first inequality followed by our assumption on f. $\square$