

MTH 525: Topology

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Question 1.

TYhis is something Which of the 9 topologies in the example given in the chapter are comparable. Numbering from top left to right,

Number	Finer than	Corser than	Not comparable with
1	none	1-9	none
2	8,9	1,7	3,4,5,6
3	1,4,7	6,9	2,5,8
4	1	3,8,9	2,5,6,7
5	1	9	2,3,4,6,7,8
6	1,3,4,7	9	2,5 ,8
7	1	2,3,6,8,9	4,5
8	1,2,4,7	9	3,5,6
9	1-9	none	none

Question 2.

Show that \mathbb{R}_K and \mathbb{R}_I are not comparable.

Proof. Let $x = 0$, and condisder $X = (-1, 1) - K \in \mathbb{R}_K$. For any $[a, b) \subset \mathbb{R}_I$, with $0 \in [a, b)$, then $0 < b$ and we may use the archimedian property of \mathbb{R} to fix $N \in \mathbb{N}$ such that $n \geq N$ implies $\frac{1}{n} < b$, so $\frac{1}{n} \in [a, b)$ since $\frac{1}{n} \notin X$, there is no basis element of \mathbb{R}_I containg zero that is a subset of X . Now consider the basis element of \mathbb{R}_I , $[0, b)$. We must show there is no basis element of \mathbb{R}_K containg 0 that is a subset of $[0, b)$. In order for 0 to be in a basis element of \mathbb{R}_K . we must have $a < 0$, but then $a \notin [0, b)$. \square

Question 3.

Show that $\mathfrak{B} = \{(a, b) | a, b \in \mathbb{Q}\}$ generates the standard topology on \mathbb{R} . Show that the this statement if false with the lower limit topology.

Proof. We have by earlier result that \mathfrak{B} is a basis if for every open set U and for all $x \in U$ there exists an element of \mathfrak{B} containing x that is a subset of U . So let $U \subset \mathbb{R}$ be open, for $x \in U$, x must appear in some basis element, say $x \in (a, b)$ for $a, b \in \mathbb{R}$. Now by the density of the rationals in \mathbb{R} , there exists $s, t \in \mathbb{Q}$ such that

$$a < s < x$$

and

$$x < t < b$$

Thus $x \in (s, t) \subset (a, b) \subset U$; it follows that \mathfrak{B} generates the topology on \mathbb{R} .

Now consider \mathbb{R}_l . We have that $[e, 3)$ is an open subset of \mathbb{R}_l , but there is no element of $\{[a, b) \mid a, b \in \mathbb{Q}\}$ containing e which is a subset of $[e, 3)$ since we must choose a as a rational if $e < a$ then we have a contradiction and if $e > a$ then $[a, b)$ is not a subset of $[e, 3)$. Hence, $\{[a, b) \mid a, b \in \mathbb{Q}\}$ does not form a basis for the lower limit topology on \mathbb{R} . \square

Question 4.

If A is a basis for the topology on X then the topology generated by A equals the intersection of all topologies containing A

Proof. Let A be a basis for the topology on X , \mathfrak{T} . Let $\{\mathfrak{D}_\alpha\}$ be the collection of all topologies containing A . We want to show that $\mathfrak{T} = \bigcap \mathfrak{D}_\alpha$. If $U \in \mathfrak{T}$ then U is equal to a union of elements of A , since A is contained in each \mathfrak{D}_α so is U , since each \mathfrak{D}_α is a topology. Now if $U \in \bigcap \mathfrak{D}_\alpha$, then it is in every topology containing A , since \mathfrak{T} is one such topology we have $U \in \mathfrak{T}$. In the case that A is a subbasis generating the topology \mathfrak{T}_A . If $U \in \mathfrak{T}_A$, then U is a union of finite intersections of elements of A , then since $A \in \mathfrak{D}_\alpha$ it follows from the definition of a topology that U will be in each \mathfrak{D}_α hence it is in the intersection. The second part of the argument is a repeat of the above.

□

Question 5.

is the finite complement topology true if we replace finite with infinite?

Ans: No, under \mathbb{Z} we have a collection of open sets

$$\mathfrak{U} = \{\dots, [-4, -3], [-2, -1], [1, 2], [3, 4], \dots\}$$

whose union equals $(-\infty, -1] \cup [1, \infty)$. Then it is clear the complement is not infinite.

Question 6.

Let $\{\mathfrak{T}_\alpha\}$ be a collection of topologies on X prove that the union and intersection of this set are topologies on X .

Proof. It is clear that $X, \emptyset \in \bigcap \{\mathfrak{T}_\alpha\}$. Now suppose that $\{U_\beta\}_{\beta \in J} \subset \bigcap \{\mathfrak{T}_\alpha\}$. Since each U_β is in the intersection, $\{U_\beta\} \subset \mathfrak{T}_\alpha$ for all α . Then since \mathfrak{T}_α is a topology, the union is in each \mathfrak{T}_α and thus is contained in the intersection. Now let U_1, \dots, U_n be a finite collection of elements in $\bigcap \{\mathfrak{T}_\alpha\}$. Then each U_i is in all \mathfrak{T}_α and again by the definition of topology the finite intersection over U_i is in each \mathfrak{T}_α and thus contained in the intersection.

Now in general the union of two topologies need not be a topology. However, the union of topologies on X will form a subbasis for a topology on X .

and this topology is the smallest one containing each of the topologies in the union. To see that the union is not always a topology just consider a case where the topologies are not comparable. The fact the the union forms a subbasis is clear since the elements of any one topology for a subbasis ($X \in \mathfrak{T}$ makes T a subbasis). It is also clear that the topology generated by the subbasis contains all topologies in the union. Now suppose there exists a topology \mathfrak{T} such that $\mathfrak{T}_\alpha \subset \mathfrak{T}$ for all α . Then if U is open in the topology generated by the subbasis $\cup\{\mathfrak{T}_\alpha\}$, U is a union of finite intersections of elements of $\cup\{\mathfrak{T}_\alpha\}$, hence by definition of topology U is open in \mathfrak{T} .

□