

# MTH 435: Analysis HW 1

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## Question 1.

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Show that there doesn't exist a rational number  $s$  such that  $s^2 = 6$ .

*Proof.* First we prove that  $6|a^2 \implies 6|a$ . Suppose that 6 does not divide  $a$  and let  $a = p_1 p_2 \dots p_n$  be the prime factorization of  $a$ , then we know that 3 and 2 cannot both be common factors, But then since 2 and 3 are prime they cannot be the square of a prime number, hence 2 and 3 cannot both appear in  $a^2 = p_1^2 p_2^2 \dots p_n^2$ , hence 6 does not divide  $a^2$ . It now follows by contrapositive that if  $6|a^2$  then  $6|a$ .

Now assume there exists  $a, b \in \mathbb{Z}$  such that

$$6 = \left(\frac{a}{b}\right)^2$$

If  $a$  and  $b$  have any common factors we may cancel them out, so we assume  $(a, b) = 1$ . Then  $6b^2 = a^2$  implies  $6|a$  so we may fix  $m \in \mathbb{Z}$  such that  $6m = a$ . Then  $6b^2 = (6m)^2$  implies  $b^2 = 6m^2$  and  $6|b$ ; thus  $a$  and  $b$  share a common factor, a contradiction.

□

## Question 2.

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If  $a, b \in \mathbb{R}$  show that  $|a + b| = |a| + |b|$  iff  $ab \geq 0$ .

*Proof.* For the forward direction we use contrapositive, assume  $ab < 0$  then without loss of generality assume  $a > 0$  and  $b < 0$ , we show that

$$|a + b| \neq |a| + |b|$$

we have by our assumptions on  $a$  and  $b$  that  $|a| + |b| = a - b$ . Now there are three cases (by tricotomy) for what  $|a + b|$  can map to, if  $|a + b| = 0$  we

are done since  $|a| + |b| > 0$ . If  $a + b$  is positive  $|a + b| = a + b$ , but then  $a + b = a - b$  would imply  $b = 0$ , contradicting our assumption on  $b$ . If  $a + b$  is negative  $|a + b| = -a - b$ , but now  $-a - b = a - b$  would force  $a = 0$ , contradicting our assumption on  $a$ . Hence, in every case equality does not hold. This proves the first direction.

Now assume  $ab \geq 0$ , if either is equal to zero we are done. If they are both positive it follows since  $a + b$  will be positive giving

$$|a + b| = a + b = |a| + |b|.$$

Then if they are both negative, their sum will be negative so,

$$|a + b| = -a - b = |a| + |b|$$

completing the opposite direction.  $\square$

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**Question 3.**

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Find all  $x \in \mathbb{R}$  that satisfy the inequality

$$4 < |x + 2| + |x - 1| < 5.$$

*Proof.* The terms  $x + 2$ ,  $x - 1$  have different signs only if  $x \in (-2, 1)$  but it is clear no such  $x$  is a solution, so assume  $x \notin (-2, 1)$ , then the terms have the same sign and by the above we may add them to get

$$4 < |2x + 1| < 5$$

which we may split into  $4 < |2x + 1|$  and  $|2x + 1| < 5$ . For the first case we have

$$4 < |2x + 1| \implies 4 < 2x + 1 \vee 2x + 1 < -4$$

which gives  $3/2 < x$  and  $x < -5/2$ ; written in interval notation as  $(-\infty, -5/2) \cup (3/2, \infty)$ . Now for

$$|2x + 1| < 5 \implies -5 < 2x + 1 < 5 \implies -6/2 < x < 4 = (-3, 2)$$

We need both conditions to be satisfied so we must take the intersection over our two solutions,

$$(-\infty, -5/2) \cup (3/2, \infty) \cap (-3, 2) = (-3, -5/2) \cup (3/2, 2)$$

$\square$

**Question 4.**

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- (a)
- Proof.*
- Assume without loss of generality that
- $b < a$
- , then
- $|a - b| = a - b$
- .

Thus

$$\frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = \frac{1}{2}(2a) = a$$

We also have

$$\frac{1}{2}(a + b - |a - b|) = \frac{1}{2}(a + b - (a - b)) = \frac{1}{2}(2b) = b$$

and we are done.  $\square$ 

- (b) Prove
- $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$

*Proof.* Note  $\min\{a, b\}$  is either  $a$  or  $b$ , then if  $c$  where such that  $c < a$  and  $c < b$ , it is clear. Now suppose  $a < b$  and  $a < c$ . then

$$\begin{aligned}\min\{a, b, c\} &= a = \min\{a, c\} \\ &= \min\{\min\{a, b\}, c\}.\end{aligned}$$

Then  $b < a$  and  $b < c$  is the same as above.  $\square$ **Question 5.**

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*Proof.*  $\inf S_4 = \frac{1}{2}$ ,  $\frac{1}{2}$  is an element of  $S_4$  that occurs for  $n = 2$ , for  $n \neq 2$ , if  $n$  is odd we will be adding to 1 which gives  $\frac{1}{2} < 1 + 1/n$ . If  $n > 2$  is even we have  $1/n < 1/2$  which implies  $1 - 1/2 < 1 - 1/n$ , so  $1/2$  is the minimal element of  $S_4$ . It follows that if a set contains a minimal element it is the infimum since we have  $1/2 < x$  for all  $x \in S_4$  it is a lower bound and since it is an element of  $S_4$ , given any lower bound  $l$ , we must have  $l \leq 1/2$ .

$\sup S_4 = 2$ , I proceed with a very similar argument as above, 2 appears in  $S_4$  when  $n = 1$ , for any  $n > 1$  either we are subtracting from 1, or adding a number smaller than 1 to 1, in either case we get something less than 2. Thus 2 is the maximal element of the set and hence it must be the supremum.  $\square$

**Question 6.**

Let  $A$  and  $B$  be bounded nonempty subsets of  $\mathbb{R}$ , and let  $A + B = \{a + b : a \in A, b \in B\}$ . Prove that  $\sup(A + B) = \sup(A) + \sup(B)$

*Proof.* Let  $\alpha = \sup(A)$  and  $\beta = \sup(B)$ . Then for all  $a, b$  we have  $a \leq \alpha$  and  $b \leq \beta$ , thus

$$a + b \leq \alpha + \beta, \forall a \in A, b \in B.$$

So  $\alpha + \beta$  is an upper bound for the set  $A + B$ . But then for  $\epsilon > 0$  there exists  $b_0 \in B$  such that  $\beta - \frac{1}{2}\epsilon < b_0$  and  $a_0 \in A$  such that  $a - \frac{1}{2}\epsilon < a_0$ , so  $\alpha + \beta - \epsilon < a_0 + b_0$ . Thus  $\alpha + \beta = \sup(A + B)$ . A very similar argument works for infimum. Let  $a = \inf(A)$  and  $b = \inf(B)$ . Then  $a + b$  is a lower bound for  $A + B$  for the same reasons stated above. But then given  $\epsilon > 0$ , I can find elements  $a_0, b_0$  in  $A$  and  $B$  respectively such that  $a + b + \epsilon > a_0 + b_0$ , so  $a + b = \inf(A + B)$ .  $\square$

**Question 7.**

I first reprove a result from class. If every element of a set  $B$  is an upper bound for a set  $A$ , then  $\inf(B)$  is an upper bound for  $A$ .

*Proof.* Assume every element of  $B$  is an upper bound for  $A$ , then if  $\inf(B) < a$  for some  $a \in A$ , there exists  $\epsilon = a - \inf(B) > 0$  such that  $\inf(B) + \epsilon = a$ , but by the epsilon formulation of infimum, we have there exists an element of  $B$  with  $b < a$ , a contradiction.  $\square$

*Proof.* We prove that every element of  $F = \inf\{f(x) | x \in X\}$  is an upper bound for the set  $G = \{g(y) | y \in Y\}$ , then it will follow  $\sup(G) \leq \inf(F)$ . Let  $y_0 \in Y$  be arbitrary, then for all  $x \in X$  we have

$$g(y_0) \leq f(x, y_0) \leq f(x).$$

The first inequality holds since  $g(y)$  is the infimum over all choices of  $x$  and the second holds since for each  $x \in X$  we have defined  $f(x)$  to be the supremum over  $y \in Y$ . Then for each  $g(y) \in G$ , we have  $g(y) \leq f(x)$  for all  $x \in X$ , thus each element of  $F$  is an upper bound for  $G$  and by the above, the result follows.  $\square$

**Question 8.**

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If  $u > 0$  and  $x < y$  show there exists a rational number  $x < ru < y$ .

*Proof.* Let  $x, y \in \mathbb{R}$  with  $x < y$ . Then fix  $n \in \mathbb{N}$  such that  $\frac{1}{n} < y - x$ , from this we immediately get  $0 < 1 + xn < yn$ . Now choose  $m \in \mathbb{N}$  to be the integer such that  $xn < m \leq xn + 1$ . Then we have

$$xn < m \leq xn + 1 < yn \implies xn < m < yn \implies x < \frac{m}{n} < y.$$

Now note  $\frac{x}{u}, \frac{y}{u} \in \mathbb{R}$  so there exists  $s, t \in \mathbb{Z}$  such that

$$\frac{x}{u} < \frac{s}{t} < \frac{y}{u} \implies x < \frac{su}{t} < y$$

and we are done. □