MTH 513

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Question 1.

Proof. Let X and Y be subspaces and assume that $X \cap Y = \{0_v\}$. Let $v \in V$, by assumption we have that $X + Y = V$, thus v can be represented as a linear combiniation of a vector in X plus a vector in Y. Now suppose that $v = x_1 + y_1$ and $v = x_2 + y_2$. Then $v - v = 0_V$, so $(x_1 - x_2) + (y_1 - y_2) = 0_V$. Since X and Y are subspaces $x_1 - x_2 \in X$ and $y_1 - y_2 \in Y$. Since $x_1 - x_2 = y_1 - y_2 \in X \cap Y$ we have $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$ so $y_1 = y_2$ and $x_1 = x_2$. Hence the representation is unique.

Conversly assume that every vector in V can be expressed uniquely as a sum of a vector in X and a vector in Y. Then let $w \in X \cap Y$. Since $w \in X$ we have $w = w + 0$ where $w \in X$ and $0 \in Y$ and since $w \in Y$ we have $w = 0 + w$ where $0 \in X$ and $w \in Y$. By assumption we know that the representation is unique; then we must thave $w = 0$, since if not we have found two different ways of writting w as a sum of a vector in X and a vector in Y. Hence $w = 0$ and since w was arbitrary, $X \cap Y = \{0_V\}$. $\square$

Question 2.

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Question 3.

(a) *Proof.* By definition, we have

$$\texttt{trace}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ik}b_{ki} = S \tag{1}$$

Then since $a_{ik}, b_{ki} \in \mathbb{F}$, they commute.

$$S = \sum_{i=1}^{n}\sum_{j=1}^{k} b_{ki}a_{ik} =$$

$$\sum_{i=1}^{n}(b_{1i}a_{i1} + \cdots + b_{ni}a_{ni}) =$$

$$(b_{11}a_{11} + \dots b_{n1}a_{1n}) + \cdots + (b_{1n}a_{1n} + \cdots + b_{nn}a_{nn})$$

Then using associativity and communitivity we may write,

$$S = (b_{11}a_{11} \dots b_{1n}a_{n1}) + \cdots + (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn})$$

$$= \sum_{i=1}^{n} b_{1i}a_{1i} + \cdots + \sum_{i=1}^{n} b_{ni}a_{ni}$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ki} = \texttt{trace}(BA)$$

as desired. ☐

(b) *Proof.* Note that by assumptions on S and K we have

$$(SK)^T = K^T S^T = -KS$$

Then $\texttt{trace}(SK) = \texttt{trace}((SK)^T) = \texttt{trace}(-KS) = -\texttt{trace}(KS) = \texttt{trace}(SK)$, by part (a) and the fact that trace is linear. Thus $2\texttt{trace}(SK) = 0$ and so $\texttt{trace}(SK) = 0$. ☐

Question 4.

Proof. Let $v \in \texttt{span}(E)$, then $v = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$ then by substituting in the definitions of w_i we get

$$\alpha_1(x + y + z) + \alpha_2(x - y + z) + \alpha_3(x - y - z)$$

and after distributing and re arranging, it is clear we get a linear combinition in terms of x, y, z.

$$v = (\alpha_1 + \alpha_2 + \alpha_3)x + (\alpha_1 - \alpha_2 + \alpha_3)y + (\alpha_1 - \alpha_2 - \alpha_3)z$$

hence $v \in \texttt{span}(S)$. Conversly let $v \in \texttt{span}(S)$ and let $v = \beta_1 x + \beta_2 y + \beta_3 z$, we want to find $\alpha_1, \alpha_2, \alpha_3$ such that $v = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$. Equivalently,

$$v = \alpha_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \alpha_2 \begin{bmatrix} x \\ -y \\ z \end{bmatrix} + \alpha_3 \begin{bmatrix} x \\ -y \\ -z \end{bmatrix} = \begin{bmatrix} \beta_1 x \\ \beta_2 y \\ \beta_3 z \end{bmatrix} \tag{2}$$

So we may solve the augmented system system

$$\begin{bmatrix} 1 & 1 & 1 & \beta_1 \\ 1 & -1 & -1 & \beta_2 \\ 1 & 1 & -1 & \beta_3 \end{bmatrix} \tag{3}$$

By adding -1 times row 1 to rows one and two, we see that the echelon form of this matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -2 \end{bmatrix} \tag{4}$$

which is consistent for any choice of β's. Hence $v \in \texttt{span}(E)$. $\square$