

# Analysis Chapter 1

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## Question 1.

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Show that there doesn't exist a rational number  $s$  such that  $s^2 = 6$ .

*Proof.* First we prove that  $6|a^2 \implies 6|a$ . Suppose that 6 does not divide  $a$  and let  $a = p_1 p_2 \dots p_n$  be the prime factorization of  $a$ , then we know that 3 and 2 cannot both be common factors, But then since 2 and 3 are prime they cannot be the square of a prime number, hence 2 and 3 cannot both appear in  $a^2 = p_1^2 p_2^2 \dots p_n^2$ , hence 6 does not divide  $a^2$ . It now follows by contrapositive that if  $6|a^2$  then  $6|a$ .

Now assume there exists  $a, b \in \mathbb{N}$  such that

$$6 = \left(\frac{a}{b}\right)^2$$

If  $a$  and  $b$  have any common factors we may cancel them out, so we assume  $(a, b) = 1$ . Then  $6b^2 = a^2$  implies  $6|a$  so we may fix  $m \in \mathbb{Z}$  such that  $6m = a$ . Then  $6b^2 = (6m)^2$  implies  $b^2 = 6m^2$  and  $6|b$ ; thus  $a$  and  $b$  share a common factor, a contradiction.

□

## Question 2.

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If  $a, b \in \mathbb{R}$  show that  $|a + b| = |a| + |b|$  iff  $ab \geq 0$ .

*Proof.* For the forward direction we use contrapositive, assume  $ab < 0$  then without loss of generality assume  $a > 0$  and  $b < 0$ , we show that

$$|a + b| \neq |a| + |b|$$

we have by our assumptions on  $a$  and  $b$  that  $|a| + |b| = a - b$ . Now there are three cases (by tricotomy) for what  $|a + b|$  can map to, if  $|a + b| = 0$  we are done since  $|a| + |b| > 0$ . If  $a + b$  is positive  $|a + b| = a + b$ , but then

$a + b = a - b$  would imply  $b = 0$ , contradicting our assumption on  $b$ . If  $a + b$  is negative  $|a + b| = -a - b$ , but now  $-a - b = a - b$  would force  $a = 0$ , contradicting our assumption on  $a$ . Hence, in every case equality does not hold. This proves the first direction.

Now assume  $ab \geq 0$ , if either is equal to zero we are done. If they are both positive it follows since  $a + b$  will be positive giving

$$|a + b| = a + b = |a| + |b|.$$

Then if they are both negative, their sum will be negative so,

$$|a + b| = -a - b = |a| + |b|$$

completing the opposite direction. □

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### Question 3.

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Find all  $x \in \mathbb{R}$  that satisfy the inequality

$$4 < |x + 2| + |x - 1| < 5.$$

*Proof.* The terms  $x + 2$ ,  $x - 1$  have different signs only if  $x \in (-2, 1)$  but it is clear no such  $x$  is a solution, so assume  $x \notin (-2, 1)$ , then the terms have the same sign and by the above we may add them to get

$$4 < |2x + 1| < 5$$

which we may split into  $4 < |2x + 1|$  and  $|2x + 1| < 5$ . For the first case we have

$$4 < |2x + 1| \implies 4 < 2x + 1 \vee 2x + 1 < -4$$

which gives  $3/2 < x$  and  $x < -5/2$ ; written in interval notation as  $(-\infty, -5/2) \cup (3/2, \infty)$ . Now for

$$|2x + 1| < 5 \implies -5 < 2x + 1 < 5 \implies -6/2 < x < 4 = (-6/2, 4)$$

We need both conditions to be satisfied so we must take the intersection over our two solutions,

$$(-\infty, -5/2) \cup (3/2, \infty) \cap (-6/2, 4) = (-6/2, -5/2) \cup (3/2, 4)$$

□

**Question 4.**

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place holder

- (a) *Proof.* Assume without loss of generality that  $b < a$ , then  $|a-b| = a-b$ .

Thus

$$\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = \frac{1}{2}(2a) = a$$

We also have

$$\frac{1}{2}(a+b-|a-b|) = \frac{1}{2}(a+b-(a-b)) = \frac{1}{2}(2b) = b$$

and we are done.  $\square$

- (b) *Proof.* The minimum of  $a, b, c$  must be one of them, the minimum of  $a, b$  must be  $a$  or  $b$ , so

$\square$

**Question 5.**

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Let  $S_4 = \{1 - (-1)^n/n : n \in \mathbb{N}\}$  find inf and sup

*Proof.*  $\inf S_4 = \frac{1}{2}$ ,  $\frac{1}{2}$  is an element of  $S_4$  that occurs for  $n = 2$ , for  $n \neq 2$ , if  $n$  is odd we will be adding 1 which gives  $\frac{1}{2} < 1 + 1/n$ . If  $n > 2$  is even we have  $1/n < 1/2$  which implies  $1 - 1/2 < 1 - 1/n$ , so  $1/2$  is the minimal element. It follows that if a set contains a minimal element it is the infimum since we have  $1/2 < x$  for all  $x \in S_4$  it is a lower bound and since it is an element of  $S_4$ , given any lower bound  $l$ , we must have  $l \leq 1/2$ .

$\sup S_4 = 2$ , I proceed with a very similar argument as above, 2 appears in  $S_4$  when  $n = 1$ , for any  $n > 1$  either we are subtracting from 1, or adding a number smaller than 1 to 1, in either case we get something less than 2. Thus 2 is the maximal element of the set and hence it must be the supremum.  $\square$

**Question 6.**

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Let