

MTH535 Homework 1

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Problem 1.

Proof. Let $A = \{\bigcup_{i=1}^n E_i \mid n \in \mathbb{N}, E_j \in S, E_i \cap E_j = \emptyset \text{ for } i \neq j\}$. We have $S \subset A$ since for any $X \in S$, we can write X as a disjoint union of elements in S in a trivial way. Further, any algebra containing S , by definition, contains all finite unions of elements of S , which is a superset of all finite unions of disjoint elements in S , thus the algebra contains A . All that remains is to see that A is an algebra itself. We have $\emptyset \in S \subset A$. We need to prove that A is closed under complements, take $X \in A$, then we have $X = \bigcup_{i=1}^n E_i$ for a finite collection of pairwise disjoint sets $E_i \in S$, $i = 1, \dots, n$. Now since $E_i \in S$, we have by property (c), that $E_i^c = \bigcup_{j=1}^{n_i} R_{j,i}$. Then

$$X^c = \left(\bigcup_{i=1}^n E_i \right)^c = \bigcap_{i=1}^n E_i^c = \bigcap_{i=1}^n \left(\bigcup_{j=1}^{n_i} R_{j,i} \right) \quad (1)$$

let $N = \max\{n_i \mid i = 1, \dots, n\}$ and set $R_{j,i} = \emptyset$ for $n_i < j \leq N$. Then 1 becomes

$$X^c = \bigcap_{i=1}^n \left(\bigcup_{j=1}^N R_{j,i} \right) = \bigcup_{j=1}^N \left(\bigcap_{i=1}^n R_{j,i} \right) \quad (2)$$

Since $E_1^c = \bigcup_{j=1}^{n_1} R_{j,1}$ is a disjoint union, we have for any $k_1 \neq k_2$, $R_{k_1,1} \cap R_{k_2,1} = \emptyset$. We also have that $\bigcap_{i=1}^n R_{k_1,i} \subset R_{k_1,1}$ and $\bigcap_{i=1}^n R_{k_2,i} \subset R_{k_2,1}$. Hence the intersections $\bigcap_{i=1}^n R_{j,i}$ are pairwise disjoint for all j . Since S is closed under finite intersections, $\bigcap_{i=1}^n R_{j,i} \in S$. Thus, 2 expresses X^c as a finite pairwise disjoint union of elements of S and $X^c \in A$. Finally, we show A is closed under finite intersections (closure of A under finite unions then follows from DeMorgan's laws). Let $A_1, A_2 \in A$, then we have pairwise disjoint collections $\{E_1, \dots, E_n\}$ and $\{F_1, \dots, F_m\}$ such that

$$A_1 = \bigcup_{i=1}^n E_i$$

$$A_2 = \bigcup_{i=1}^m F_i$$

Then

$$A_1 \cap A_2 = \left[\bigcup_{i=1}^n E_i \right] \cap \left[\bigcup_{j=1}^m F_j \right] = \bigcup_{j=1}^m \left[\bigcup_{i=1}^n E_i \right] \cap F_j = \bigcup_{j=1}^m \bigcup_{i=1}^n E_i \cap F_j \quad (3)$$

if $n \neq m$ let $F_i = E_i$ for $\min\{n, m\} < i \leq \max\{n, m\}$, Then

$$A_1 \cap A_2 = \bigcup_{i,j}^{\max\{n,m\}} E_i \cap F_j \in S \quad (4)$$

since the union is finite and the sets $E_i \cap F_j$ are disjoint since $\{E_1, \dots, E_n\}$ is. Since A is closed under the intersection of any two of its elements, it follows from induction and associativity of intersections that it is closed under all finite intersections. Hence A is an algebra containing S and since any algebra containing S also contains A , it follows that $A = \mathcal{A}(S)$. \square

Problem 2.

- Proof.* 1. Suppose not and let $\alpha : \mathbb{N} \rightarrow \{0, 1\}^{\mathbb{N}}$ be any bijection. Let $\pi_n : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{Z}_2$ be the projection on to the n^{th} component. Then consider $b_i = \pi_i(\alpha(i)) + 1$ and let $b = (b_1, b_2, \dots, b_k, \dots) \in \{0, 1\}^{\mathbb{N}}$ then for all $n \in \mathbb{N}$, $b \neq \alpha(n)$ since they differ in the n^{th} component by construction. Thus $b \notin \text{Im}(\alpha)$, so α is not a surjection. a contradiction.
2. Define $\alpha : \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\}^{\mathbb{N}}$ by $\alpha(A) = \{\alpha_1, \alpha_2, \dots\}$ where $\alpha_n = 1$ if $n \in A$ and 0 otherwise. Then given $b \in \{0, 1\}^{\mathbb{N}}$, $b = (b_1, b_2, \dots)$, we have $S = \{n \mid b_n = 1\} \in \mathcal{P}(\mathbb{N})$ with $\alpha(S) = b$. Hence α is a surjection. \square

Problem 3.

Proof. From class we know that $m^*([0, 1]) = 1$, since the outer measure of an interval is its length, but the outer measure of a countable set is 0 (also proven in class), hence if we assume that $[0, 1]$ is countable its an interval whose outermeasure is not its length a contradiction. \square

Problem 4.

Proof. Let E have positive outer measure, we write E as the union of countably many disjoint bounded sets, for example by considering the collection $b_k = (k, k+1]$ for all $k \in \mathbb{Z}$, and taking $A_k = b_k \cap E$. If $m^*(A_k) = 0$ for all $k \in \mathbb{Z}$, then by countable subadditivity of outer measure, we have $m^*(E) = m^*(\bigcup_{n \in \mathbb{Z}} A_k) \leq \sum_{n \in \mathbb{Z}} m^*(A_n) = 0$ contradicting our assumption that E has positive outer measure. \square

Problem 5.

Proof. Assume E is measurable, Fix $\epsilon > 0$. By thm 2.11 parts 1 and 3 proved in class we may fix an open set \mathcal{O} such that $E \subset \mathcal{O}$ and $m^*(\mathcal{O} \setminus E) < \frac{\epsilon}{2}$ and a closed set \mathcal{C} such that $\mathcal{C} \subset E$ and $m^*(E \setminus \mathcal{C}) < \frac{\epsilon}{2}$. Then by measureability of E and \mathcal{C} , we apply excision,

$$m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E) < \frac{\epsilon}{2}$$

$$m^*(E \setminus \mathcal{C}) = m^*(E) - m^*(\mathcal{C}) < \frac{\epsilon}{2}$$

Then adding these inequalities and applying excision on the result yields $m^*(\mathcal{O} \setminus \mathcal{C}) < \epsilon$. Conversely, suppose that for every $\epsilon > 0$, there exists $\mathcal{O} \supset E$ open and $\mathcal{C} \subset E$ closed such that $m^*(\mathcal{O} \setminus \mathcal{C}) < \epsilon$. Then by monotonicity, $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O} \setminus \mathcal{C}) < \epsilon$. so by thm 2.11 E is measurable. \square

Problem 6.

Proof. if E is not measurable then by the negation of 2.11 there exists $\epsilon_0 > 0$ such that for all \mathcal{O} open sets containing E , $m^*(\mathcal{O} \setminus E) > \epsilon_0$. By definition of outer measure as an infimum, we may fix a countable collection of bounded open intervals $\{I_k\}_{k=1}^{\infty}$ converging E , such that $\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon_0$. Then let $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$, we have

$$m^*(\mathcal{O}) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon_0$$

so that

$$m^*(\mathcal{O}) - m^*(E) < \epsilon_0 < m^*(\mathcal{O} \setminus E)$$

‘ as desired. \square