

MTH 435: Analysis and Topology

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Chapter 1

Introduction

1.1 Algebraic and Order properties

The Real numbers, denoted \mathbb{R} form a field, that is \mathbb{R} is an Abelian group under addition and multiplication that has distinct identities and satisfies the distributive property. All important algebraic properties can be derived from the fact that \mathbb{R} is a field.

Definition 1.1.1

Let $P \neq \emptyset$ be a subset of \mathbb{R} not containing 0. We say P is the set of positive real numbers and it satisfies

1. $a, b \in P \implies a + b \in P$
2. $a, b \in P \implies ab \in P$
3. $a \in P \implies a \in P \vee -a \in P \vee a = 0$

We are now in a position to define the ordering we wish to place on \mathbb{R}

Definition 1.1.2

For $a, b \in \mathbb{R}$ such that $b - a \in P$ then we say $a < b$. If $b - a \in P \cup \{0\}$ then $a \leq b$

It follows immediately from trichotomy that exactly one of $a < b, a = b, a > b$ must hold. It is also clear that this is a total ordering on \mathbb{R} , but we must check that this turns \mathbb{R} into an ordered field.

Lemma 1.1.3

The ordering defined above is a strict total order.

Proof. The order is irreflexive since $a - a = 0 \notin P$, then if $a - b \in P$ and $b - a \in P$ we can add to get $0 \in P$ a contradiction. Now we must prove the transitive property, suppose $a, b, c \in \mathbb{R}$ satisfy $a < b$ and $b < c$, then $b - a, c - b \in P$ thus so is their sum, $c - a \in P$ which implies $a < c$ as desired. \square

Lemma 1.1.4

The ordering defined above turns \mathbb{R} into an ordered field, let $a, b, c \in \mathbb{R}$.

1. if $a < b$ then $a + c < a + b$
2. if $c > 0$ and $a < b$ then $ac < bc$
3. if $c < 0$ and $a < b$ then $ac > bc$

Proof. For the first item, we have $b - a > 0$ then by adding and subtracting c we get

$$\begin{aligned} 0 &< b - a - c + c \\ a + c &< b + c \end{aligned}$$

Next assume $c > 0$, then we have $0 < b - a$, since both of these are positive, so is their product,

$$0 < c(b - a) \implies 0 < cb - ca \implies ca < cb.$$

Now let $c < 0$, then $-c > 0$ and the argument is the same as above. \square

Thus we have turned \mathbb{R} into an ordered field.

Theorem 1.1.5

The natural numbers are all positive, we will prove this in the following steps.

1. If $a \in \mathbb{R}$ with $a \neq 0$ then $a^2 > 0$
2. $1 > 0$
3. $\mathbb{N} \subset P$

Proof. If $a > 0$ we are done, suppose $a < 0$, then $-a \in P$ and since $a^2 = (-a)(-a) \in P$ we have $a^2 \in P$. Now note $1^2 = 1$ so $1 \in P$, then since we defined the natural number n as $1 + \dots + 1$, n times, we see that all natural numbers are positive. \square

Theorem 1.1.6

If $a \in \mathbb{R}$ satisfies $0 \leq a < \epsilon$ for all $\epsilon > 0$, then $a = 0$

Proof. Suppose $a > 0$, then let $\epsilon_0 = a/2$, then

$$0 < \epsilon_0 < a < \epsilon$$

a contradiction. □

Theorem 1.1.7

If $ab > 0$ then a, b are both positive or both negative.

Proof. Suppose $ab > 0$ and that at least one is negative, without loss of generality say $a < 0$, then if $b > 0$, we have $-ab > 0$ so $-(ab) \in P$ but we assumed $ab \in P$; a contradiction, thus b is negative. □

1.2 Absoulte Value

Next we define a function of great importance on \mathbb{R} .

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -a, & \text{if } a < 0 \end{cases} \quad (1.1)$$

Now we prove some basic properties of the absoulte value function,

Theorem 1.2.1

Basic properties of the absoulte value.

1. $|ab| = |a||b|$
2. $|a|^2 = a^2$
3. If $c > 0$ then $|a| \leq c \Leftrightarrow -c \leq a \leq c$
4. $-|a| \leq a \leq |a|$

Proof. To prove (1) first note that if a or b is zero we are done. Then we just consider the four possible cases on the signs of a and b . For example if $a > 0$ and $b < 0$, we have $|ab| = -ab$ and $|a| = a, |b| = -b$ so $|a||b| = -ab$. The rest are left as an exercise. Proving (2) is simillar, if $a > 0$ we are done and if $a < 0$, then $|a|^2 = (-a)^2 = a^2$. Now suppose $c > 0$ and $|a| \leq c$, if $a \leq 0$, then the result is clear. If $a < 0$ we have $|a| = -a$, so $-a < c$, rearranging gives $-c < a$. So we are finished. Now suppose $-c \leq a \leq c$, then $-a \leq c$ and $a \leq c$. But the absoulte value maps to a or $-a$ so in either case we are done. Now for (4) we let $c = |a|$ and apply (3) to get the result. □

Theorem 1.2.2 (Triangle Inequality)

For all $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b| \quad (1.2)$$

Proof. By the above, we have that

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

implies

$$|a + b| \leq |a| + |b|$$

as desired. \square

The Triangle inequality is very important and equality hold only when a, b have the same sign.

Theorem 1.2.3

The following two inequalitys hold for all $a, b \in \mathbb{R}$

1. $|a - b| \leq |a| + |b|$
2. $||a| - |b|| \leq |a - b|$

Proof. Proof of (1) follows from subing $-b$ into the triangle inequality. To prove (2) start my applying the triangle inequality to $a = a - b + b$ to get $|a| \leq |a - b| + |b|$, and $b = b - a + a$ to get $|b| \leq |b - a| + |a|$, then subtracting gives

$$|a| - |b| \leq |a - b|$$

and

$$|b| - |a| \leq |a - b|.$$

We may multiply by -1 to get

$$-|b| + |a| \geq -|a - b|$$

and it follows,

$$-|a - b| \leq |a| - |b| \leq |a - b|.$$

Now we let $|a - b| = c$ and use the third result from theorem 1.2.1, to get

$$|a| - |b| \leq |a - b|$$

\square

1.3 Archimedean Property and Completeness

The completeness axiom is the last thing that we need in order to call a complete ordered field.

Definition 1.3.1

A subset $A \subset \mathbb{R}$ is bounded above if there exists $u \in \mathbb{R}$ such that for all $a \in A$ we have $a \leq u$, we say A is bounded below if the other inequality holds. We call u an upper bound or lower bound for the set A .

We say a set is bounded if it is bounded above and below.

Definition 1.3.2

We say that an upper bound $\alpha \in \mathbb{R}$ is a least upper bound if

1. α is an upper bound.
2. For an arbitrary upper bound u , we have $\alpha \leq u$.

Definition 1.3.3

Every subset $A \subset \mathbb{R}$ that is bounded above has a least upper bound.

We will see that this property is very important.

Theorem 1.3.4

Let $A \subset \mathbb{R}$, an upper bound $\alpha \in \mathbb{R}$ satisfies $\alpha = \sup(A)$ if and only if for all $\epsilon > 0$ there exists $a \in A$ such that $\alpha - \epsilon < a$

Proof. Let $\alpha = \sup(A)$ then for all $\epsilon > 0$, $\alpha - \epsilon < \alpha$ so it cannot be an upper bound. Conversely, let α be an upper bound with the desired property and let u be an upper bound, then if $u < \alpha$ we have $u = \alpha - \epsilon$ for some $\epsilon > 0$, but then by assumption there is an $a \in A$ such that $u < a$; a contradiction. \square

Lemma 1.3.5

\mathbb{N} is not bounded above in \mathbb{R}

Proof. Assume that \mathbb{N} is bounded, then there exists a least upper bound, let $\alpha = \sup(\mathbb{N})$, then there exists $n \in \mathbb{N}$ such that $\alpha - 1 < n$, but then $\alpha < n + 1$; a contradiction. \square

Theorem 1.3.6

For all $\epsilon \in \mathbb{R}_{>0}$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. By the unboundedness of n , we may choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, then $\frac{1}{n} < \epsilon$. \square

Now we may look at the algebraic properties of sup

Theorem 1.3.7

let S, A, B be sets and $r \in \mathbb{R}$

1. $\sup(r + S) = r + \sup(S)$
2. $\sup(A + B) = \sup(A) + \sup(B)$
3. if $a \leq b$ for all $a \in A, b \in B$ then $\sup(A) \leq \inf(B)$.

Proof. exercise. See HW1 □

Definition 1.3.8

A function $f : D \rightarrow \mathbb{R}$ is bounded in \mathbb{R} if there exists $M \in \mathbb{R}$ such that

$$-M \leq f(x) \leq M \forall x \in D$$

Lemma 1.3.9

if f, g are bounded functions such that $f(x) \leq g(x)$ then $\sup(f(x)) \leq \sup(g(x))$, but $\sup(f(x)) \not\leq \inf(g(x))$ in general.

Proof. The first part is easy. To see that the second statement is false consider $f : [0, 1] \rightarrow \mathbb{R}$ as $f(x) = x^2$ and $g : [0, 1] \rightarrow \mathbb{R}$ as $f(x) = x$. Then $g(x) \geq f(x)$ but $\sup(f(x)) = 1 > 0 = \inf(g(x))$. □

Now we turn to a very important fact about \mathbb{R} , That you can write it as the disjoint union of rationals and irrationals, and that the closure of $\mathbb{Q} = \mathbb{R}$.

Theorem 1.3.10

For $x, y \in \mathbb{R}$ with $x < y$ there exists $m, n \in \mathbb{Z}$ such that

$$x < \frac{m}{n} < y$$

That is \mathbb{Q} is dense in \mathbb{R} .

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. By the archimedean property fix $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. This gives

$$0 < 1 + nx < y$$

Now fix $m \in \mathbb{Z}$ such that $xn < m \leq xn + 1$. Then from eq(1) it follows

$$nx < x \leq nx + 1 < ny \implies nx < m < ny \implies x < \frac{m}{n} < y.$$

□