MTH 435: Analysis HW 1

Evan Fox (efox20@uri.edu)

December 8, 2022

Question 1.

Let $a_n = \frac{1}{2}(a_{n-1} - a_{n-2})$ find the limit of a_n

Proof. Let $b_n = a_n - a_{n-1}$ be the nth difference in the sequence. Then observe that by the defining property of the sequence we have

$$b_n = \frac{-1}{2} b_{n-1} = \cdots = (-1)^{n-1} \frac{1}{2^{n-1}} b_2 \tag{1}$$

Now it follows that

$$a_n = a_n - a_{n-1} + a_{n-1} = b_n + a_{n-1}$$

We can then apply the same step to a_{n-1} to get $a_{n-1} = b_{n-1} + a_{n-2}$ so repeating $n-1$ times gives

$$a_n = b_n + b_{n-1} + \cdots + b_2 + a_1$$

Now using eq(1) we have

$$a_n = \sum_{k=1} (-1)^{k-1} \frac{1}{2^{k-1}} b_2 + a_1$$

which is equivalent to

$$a_n = (a_2 - a_1)(1 - \frac{1 - \frac{1}{4^n}}{3}) + a_1$$

taking a limit of a_n gives

$$\lim_{n \to \infty} a_n = (a_2 - a_1)\frac{2}{3} + a_1$$

as desired. ∎

Question 2.

Proof. let (S, d) be compact, then let (x_n) be a sequence. If (x_n) is eventually constant then it converges so suppose that it is not eventually constant. Then the set of all distinct values of the sequence is an infinite subset of S and as such it has a limit point a. We prove that there is a subsequence in (x_n) which converges to a. To see this take nbhds around a of the form $B(a, \frac{1}{n})$ then in each nbhd take a point of the sequence (x_n), such a point will exist since a is a limit point. then it is clear that this gives a subsequence of (x_n) which converges. □

Question 3.

Proof. We prove that a $\implies$ b. Assume that $\lim_{h\to 0} |f(x+h) - f(x)| = 0$ Then by letting $x = x - h$ we have $\lim_{h\to 0} |f(x) - f(x-h)| = 0$. Now compute

$$\lim_{h\to 0} |f(x+h) - f(x-h)| \leq \lim_{h\to 0} |f(x+h) - f(x)| + \lim_{h\to 0} |f(x) - f(x-h)| = 0$$

but since the limit is in absolute value bars it cannot be less than 0, hence the limit must be zero. To see that the converse is false, consider the function

$$f(x) = \begin{matrix} \texttt{sin}(|\frac{1}{x}|) & x \neq 0 \\ 0 & x = 0 \end{matrix} \tag{2}$$

Then at $x = 0$, $\lim_{h\to 0} |f(0+h) - f(0-h)| = 0$ since $f(h) - f(-h) = 0$ for all h. But $lim_{h\to 0}|f(h) - 0| = \lim_{h\to 0} |sin(|\frac{1}{h}|) \neq 0$.

□

Question 4.

Proof. We write $\lim_{x\to a}[\lim_{y\to b} f(x, y)]$ as a sequence,

$$L_n = (\lim_{y\to b} f(x_1, y), \lim_{y\to b} f(x_2, y), \dots)$$

where (x_n) converges to a. Each limit in this sequence exists by assumption. Now let $\epsilon > 0$ And fix $N \in \mathbb{N}$ such that $m \geq N$ satisfies $f(x_m, y_m) - L < \epsilon$. Such an N exists by assumption that $\lim_{(x,y)\to(a,b)} f(x,y) = L$. Then choose $x_k \in (x_n)$ such that $x_m < x_k < a$. Then

$$L_k - L = \lim_{y \to b} f(x_k, y) - L < \epsilon$$

. The other case is similar.

□

a) The first to limits exist for function (a) and the last on does not. The first limit ($y \to 0$ then x) is 1, and the second limit is -1, since these disagree the last limit cannot exist by the result proved above.

d)none converge for example if you take the limit in the y direction with fixed x, as $y \to 0$ we seem to be approaching $x \sin(\frac{1}{x})$ but the $\sin(\frac{1}{y})$ term will continue to oscillate forever from 1 to -1 and so the limit will never converge. Similarly the limit will not converge in the x direction either. Hence none of the requested limits exists.

Question 5.

Consider a point $x \in [0,1] \cap \mathbb{Q}$ then let $\epsilon = 1/2$ and note that for all δ there exists $\xi \in (x - \delta, x + \delta)$ irrational. Hence $|f(x) - f(\xi)| = 1 > 1/2 = \epsilon$. So f is not continuous at any rational point x. To show f is not continuous at irrational points one can just flip the proof and do it the opposiate way.

Now we show that g is continuous at 0, Let $\epsilon > 0$ and let $\delta < \epsilon$, Then if $x \in \mathbb{Q}$ we have

$$|0 - x| = |x| < \delta \implies |g(0) - g(x)| = |0 - x| = x < \epsilon.$$

and if $x \notin \mathbb{Q}$ then $|g(0) - g(x)| = 0 - 0 = 0 < \epsilon$.. Hence g is continuous at 0.

Now at any rational point $p \neq 0$, Let $\epsilon < p$, for all δ we have that there exists x_0 irrational in the delta nbhd around p so that $|p - x_0| < \delta \implies |g(p) - 0| = p > \epsilon$. Conversely given an irrational point and a ϵ less than it, for all delta nbhds there exists rationals just a little larger so that the image under g is greater than epsilon. Thus g is not continuous at any other point than 0.

To see that h is continuous at every irrational number ξ, remember that as rational approximations to irrational numbers get better, the numerator and denominator grow, hence for any epsilon greater than zero, fix N st $n > N$ implies $\frac{1}{n} < \epsilon$, Then the appropriate delta nbhd, is the one where all rational numbers around ξ are better approximations than m/n for any m. Then they will have larger denominators and their map under h will be less than ϵ. To prove that it is discontinuous at rationals, note that since $h(m/n) = 1/n$, in any nbhd on m/n I can find a irrational that will map to 0. So then we will have $|\xi - m/n| < \delta$ but $|0 - 1/n|$, then we see the function is not continuous when we select $\epsilon < 1/n$.

Question 6.

Proof. Restrict f to $[x_1, x_2]$ since this is closed subset of a compact space, it is compact. Then since f is continuous we may apply the extreme value thm to obtain that f must achieve a minimum on its image. This cannot be x_1 or x_2 since if it were we would contradict the existence of a nbhd where $f(x_1)$ or $f(x_2)$ is a local max. Then since this is a minimum for all $x \in [x_1, x_2]$ there will exist a nbhd making this x_3 a local minimum. □

Question 7.

Consider the sequence $(1/2^n)$ in the metric space $(0, \infty)$. This is cauchy, but its image under $f(x) = 1/x$ is not cauchy. And it is clear that f is continuous on $(0, \infty)$ because it it the quotient of two continuous functions.

Question 8.

Proof. note that $(0, 1)$ is connected and the function given is continuous restricted to this domain so the image is connected. the interval $[-1, 0]$ is also connected. hence we at least have two connected components. So our only hope is that the two sets them self form a separation, but $[-1, 0]$ is not open in the subspace topology induced on our set so there can be no separation

□

Question 9.

Proof. Let (x_n) be cauchy, Let $\epsilon > 0$ and then there exits δ such that

$$|a - b| < \delta \implies |f(a) - f(b)| < \epsilon$$

using the fact that (x_n) is cauchy, fix $N \in \mathbb{N}$ such that $n, m > N$ implies $|x_m - x_m| < \delta$, Then we will have $|f(x_n) - f(x_m)| < \epsilon$. Hence it is cauchy. □

Question 10.

Proof. Let m be large enough such that $\alpha_m < 1$, $\alpha_{m+1} < 1$, and $\alpha_{m^2+m} < 1$. Then f^m and f^{m+1} are contraction mappings and by the fixed point thm we have a fixed point p of f^m and q of f^{m+1}. We show $p = q$. Since $f^m(p) = p$ we have

$$f^{m^2+m}(p) = f^m(f^m(\dots f^m(p))) = p$$

and

$$f^{m^2+m}(q) = f^{m+1}(f^{m+1}(\dots f^{m+1}(q))) = q$$

but since f^{m^2+m} is also a contraction mapping, there is a unique fixed point, so $p = q$. Then

$$p = f^m(p)$$

$$f(p) = f(f^m(p)) = p$$

So p is a fixed point of f. □