Homework 2

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Disclosure: Morgan Prior and I worked together on most of the problems, speficially on number 2.

Problem 1. *The base case for $n = 2$ is clear($0 = 0$), now suppose that the theorem is true for all $n \geq 2$ and let T be a tree on $n+1$ vertices. Fix v as a leaf in T and let u be adjacent to v. Now consider the tree $T' \stackrel{\text{def}}{=} T \setminus v$. Then we may apply our induction hypothesis to T' and say*

$$t'_3 + \dots (n-3)t'_{n-1} = t'_1 - 2 \tag{1}$$

We proced with two cases on the degree of u in T'. First suppose that $deg_{T'}(u) = 1$. Then when adding v back we remove a vertex of degree 1 (u) and we add back a vertex of degree 1(v); this is the same as adding and subtracting 1 from the RHS of equation (1) so that equality still holds for T. Now on the other hand suppose that $deg_{T'}(u) = k \geq 2$. Then adding v back to T' gives T and we have that the RHS of (1) increases by one (since v has degree 1). however, we also have that $k+1 = deg_T(u) \geq 3$. Effectivly $t'_k - 1 = t_k$ and $t'_{k+1} + 1 = t_{k+1}$ where t_i denotes the same thing as t'_i but for T rather than T'. Since the coefficents are increasing by 1 for each consecutive term, the LHS of the equation will increase by 1 when we add v back, we can calculate the LHS of ! for T in the following way,

$$t_3 + 2t_4 + \dots + (n-3)t_{n-1} =$$

$$t_3 + 2t_4 + \dots + (k-2)t_k + (k-1)t_{k+1} + \dots + (n-3)t_{n-1} =$$

$$t'_3 + 2t'4 + \dots + (k-2)(t'_k - 1) + (k-1)(t'_{k+1} + 1) + \dots + (n-3)t'_{n-1} =$$

$$t'_3 + 2t'4 + \dots + (k-2)(t'_k) + (k-1)(t'_{k+1}) - (k-2) + (k-1) + \dots + (n-3)t'_{n-1} =$$

$$t'_3 + 2t'4 + \dots + (k-2)(t'_k) + (k-1)(t'_{k+1}) + \dots + (n-3)t'_{n-1} + 1$$

hence adding v back to T' increases the LHS of equation (1) by 1, since the RHS increses by the same quanity the equation holds in T.

Now part (b) will be clear since adding 2 to both sides gives

$$t_3 + 2t_4 + \dots + (n-3)t_{n-1} + 2 = t_1$$

we just verify that the lhs is always at least the maximall degree of T. if the max degree is $3 \leq k \leq n-1$, then we have

$$t_1 > (k-2)t_k + 2 \geq k$$

and this will be a term on the LHS of the above. All other terms only serve to increase the value of t_1 more. If $k = 1, 2$, then the lhs side is 0 so the proposistion is true.

Problem 2. *Since n^{n-2} is the number of labeled trees on n vertices, it is clear that we need to show that the number of labeled trees that use a specfic edge e is $2n^{n-3}$. Then subtracting this quanity from the total number of spaning trees will give the number of labeled spanning trees in $K_n \setminus e$. To do this we use the same bipartite technique that we learned in class. Let G_1 be the set of all labeled trees on n vertices, and G_2 be the set of all edges in K_n Then $|G_1| = n^{n-1}$ by cayleys formula and $|G_2| = \binom{n}{2}$. Then we add an edge to our bipartite graph from a fixed labeled tree to a vertice in G_2 (which is an edge in K_n) if the given tree contains the edge. This is the correct setup since we want to find the number of labeled trees that use a spefic edge, which will be the degree of the vertice in G_2. Since we know that each tree contains exactly $n-1$ edges, each vertice in G_1 must have that many edges, so the degree of each vertice in G_1 is $n-1$. We also know that the sum of the degrees of the vertices of both partite sets are equal. The sum of the degrees in G_1 is then $n - 1(n^{n-2})$ and this must be equal to the sum of the deg rees in G_2 which is $\sum_{k=1}^{\binom{n}{2}} d(k)$, but each degree will be the same since for each edge the number of labeled spanning trees in K_n which contain said edge is constant, that is, it doesnt matter which edge we removed in the statement of the question. So the sum above becomes $t\binom{n}{2}$ where t is the number of (labeled spanning) trees that use a given edge in K_n. Now solving $n - 1(n^{n-2}) = t\binom{n}{2}$ for t gives us*

$$(n-1)(n^{n-2}) = t \cdot \frac{n(n-1)}{2}$$
$$2n^{n-2} = nt$$
$$2n^{n-3} = t$$

. So $t = 2n^{n-3}$ this is the number of trees that use the edge e. Now subtracting this from cayleys formula n^{n-2} must give the total number of labeld trees in $K_n \setminus e$.

Problem 3. *Suppose that G is bipartite and let B_i for $i = 1, 2$ be the partite sets for G. For each H subgraph of G. Consider $A_i = V(G) \cap B_i$ for $i = 1, 2$, clearly, A_i is an independant set and $|A_1| + |A_2| = |H|$ so that at least one of them must contain at least half the vertices of H.*

On the other hand, suppose that G is not bipartite, then there exits an odd cycle C_m subgraph of G. Let $|C_m| = 2k + 1$. We want to show that any set $X \subset V(C_m)$ such that $|X| \geq k + 1/2$ contains two neighbors in C_m. To do this, fix a vertex $v \in X$, now since X is supposed to be independant it cannot contain *neighbors of v, so it can only contain points of even distance from v. In fact we would have to contain all points of even distance from v so that X contains at least half the elements of X (there are k vertices of even distance and including v gives $k+1 > k+\frac{1}{2}$). However, it will be the case that there will exists a pair of adjcent vertices with even distence to v, so that X is not independant. Namley the furthest two vertices of even distance, will be adjacent; otherwise we contradict C_m being an odd cycle. Given the two furthest vertices of even distance, there could at most be one vertice between them (otherwise they are not furthest); if this vertice y exists, then it is the unique vertice of distance l from v, but this contradicts C_m being odd, since it implies $m = 2l - 2$ which is even.*

Problem 4. *Base case is clear for $n = 2$ and $n = 3$, since one can very quickly draw the bipartite graph using the method discussed in class and the bipartite graph obtained is unique.*

Suppose that for $n \geq 3$ that T_n has a leaf in its larger partite set or both if they are equal. Let $v, u \in T_{n+1}$ with v a leaf and $vu \in E(T_{n+1})$ now set $T = T_{n+1} \setminus v$. Clearly $|T| = n$ so that we may apply the induction hypothesis. Let A_1, A_2 denote the partite sets of T and suppose $|A_1| \geq |A_2|$. Then we may fix a leaf in A_1. If this leaf is any vertex other than u, we are done since adding v back could at worst (if $u \in A_1$) make $|A_1| = |A_2|$ in which case there would be a leaf in both and at best (if $u \in A_2$) we just add another leaf to A_1. Now we need to pay attention to the case where u is the only leaf in A_1 since adding v back could concivably make it so that there are no leaves in A_1, in this case we prove that $|A_1| = |A_2|$. Indeed, assume that u is the only leaf in A_1. Then we can consider $T \setminus u$ and apply the induction hypothesis, since u was the only leaf in A_1, $A_1' \stackrel{\text{def}}{=\!=} A_1 \setminus u$ now has no leaves, so that it cannot be larger than $A_2' \stackrel{\text{def}}{=\!=} A_2$ or we would obtain a contradiction; so $|A_1'| \leq |A_2'|$ but this just says $|A_1| - 1 \leq |A_2|$, Hence adding u back and recalling we assumed $|A_2| \leq |A_1|$, must make $|A_1| = |A_2|$. and then adding v back makes $|A_1| < |A_2|$, with $v \in A_2$ (since u was in A_1) so that T_{n+1} contains a leaf in its larger parite set.