MTH 316 Homework 3

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Question 1.

Let G be a group of permutations on a set A. For $a \in A$ define $\text{stab}(a) = \{\sigma \in G | \sigma(a) = a\}$. Prove $\text{stab}(a) \leq G$.

Proof. It is clear that $\epsilon(a) = a$ for all $a \in A$; thus $\epsilon \in \text{stab}(a)$. Now we use the two step subgroup test so let $\sigma, \tau \in \text{stab}(a)$. Then

$$(\sigma \circ \tau)(a) = \sigma(\tau(a)) = \sigma(a) = a$$

since both σ and τ fix a. Now we need to show that if σ fixes a then σ^{-1} also fixes a. since

$$\sigma(a) = a$$

we can take σ^{-1} of both sides

$$\sigma^{-1}(\sigma(a)) = \sigma^{-1}(a)$$
$$a = \sigma^{-1}(a)$$

and this completes the proof.

$\square$

Question 2.

Let σ, τ be permutations. Prove $\sigma\tau$ is even if and only if σ and τ are both even or both odd.

Proof. ($\Leftarrow$) Assume that σ and τ are both even or both odd. We have

$$\sigma = \alpha_1\alpha_2...\alpha_s$$

and

$$\tau = \gamma_1\gamma_2...\gamma_t$$

Where α_i and γ_i are two cycles and s, t have the same parity. We can write $\sigma\tau$ as

$$\sigma\tau = \alpha_1...\alpha_s\gamma_1...\gamma_t$$

then it is clear that $\sigma\tau$ can be written as a product of $t+s$ two cycles. Since an even number plus an even number is even and an odd number plus an odd number is also even, $\sigma\tau$ can be written as an even number of two cylces.

($\Rightarrow$) To prove the opposite direction we use the contrapositive, so assume without loss of generality that σ is even and τ is odd; we prove that $\sigma\tau$ is odd. Just like before we can decompose σ and τ into two cycles of the form

$$\sigma = \alpha_1\alpha_2...\alpha_s$$

and

$$\tau = \gamma_1\gamma_2...\gamma_t$$

where s is even and t is odd. We may then write $\sigma\tau$

$$\sigma\tau = \alpha_1...\alpha_s\gamma_1...\gamma_t$$

So $\sigma\tau$ can be written as a product of $s + t$ two cycles where s is even and t is odd. Then since an odd number plus an even number is odd the result follows.

$\square$