

# MTH 525: Topology

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# Contents

<b>1</b>	<b>Topology and Basis</b>	<b>2</b>
1.1	Topology . . . . .	2
1.2	Basis for a Toplogy . . . . .	3
<b>2</b>	<b>Order and Product Topology</b>	<b>7</b>
2.1	Order Topology . . . . .	7
2.2	Product Topology . . . . .	8

# Chapter 1

## Topology and Basis

### 1.1 Topology

#### Definition 1.1.1

*Let  $X$  be a set. A topology on  $X$  is a collection of subsets  $T$  such that*

- 1.  $X, \emptyset \in \mathfrak{T}$ .*
- 2. Closed under arbitrary unions.*
- 3. Closed under finite intersections.*

On a given set there may be many different topologies. That can be defined on that set. Let  $X = \{1, 2, 3, 4, 5\}$ , we will write down several examples of topologies on  $X$ .

1.  $\mathfrak{T} = \{\emptyset, X\}$
2.  $\mathfrak{T} = \mathfrak{p}(X)$
3.  $\mathfrak{T} = \{\emptyset, X, \{1\}\}$
4.  $\mathfrak{T} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$

are all topologies on  $X$ . The first is called the trivial topology or the indiscrete topology and the second is the discrete topology. We will now give an example of another topology, the finite complement topology.

#### Definition 1.1.2

*Finite complement Topology Let  $X$  be a set and let  $\mathfrak{T}$  consist of all subsets  $U \subset X$  such that the complement of  $U$  in  $X$  is finite or  $X$ .*

**Proposition 1.1.3**

*The finite complement topology is a topology*

*Proof.* We must show all three conditions are true, first we show that  $X$  and  $\emptyset$  are in  $\mathfrak{T}$ . Note,  $X - \emptyset = X$  and  $X - X = \emptyset$  and both of these satisfy are conditions on  $\mathfrak{T}$ . Now we show  $\mathfrak{T}$  is closed under arbitrary unions. Let  $\{U_\alpha\}_{\alpha \in J} \subset \mathfrak{T}$  and we want to show that  $(\bigcup_\alpha U_\alpha)^c \in \mathfrak{T}$ . We have by DeMorgans laws

$$(\bigcup_{\alpha \in J} U_\alpha)^c = \bigcap_{\alpha \in J} U_\alpha^c$$

and since each  $U_i \in \mathfrak{T}$  we now that each  $U_i^c$  is finite, thus since the intersection of finite sets are finite we are done. Now we must show closure under finite intersections, so let  $\{U_1, \dots, U_n\}$  be a subset of  $\mathfrak{T}$ . Then we have

$$(\bigcap U_i)^c = \bigcup U_i^c$$

and since each  $U_i^c$  is finite, and we have a finite number of sets to union, the result is finite. Hence we have showed the finite complement topology is indeed a topology.  $\square$

We could replace the finite condition with countable and we would still have a topology since the union of countable sets is again countable.

Given two topologies on a set we can also compare them.

**Definition 1.1.4**

*Let  $X$  be a set and let  $\mathfrak{T}$  and  $\mathfrak{T}'$  be topologies on  $X$ . We say  $\mathfrak{T}$  is finer than  $\mathfrak{T}'$  if  $\mathfrak{T}' \subset \mathfrak{T}$ . We say  $\mathfrak{T}$  is corser in the reverse situation.*

## 1.2 Basis for a Topology

**Definition 1.2.1**

*Basis Let  $X$  be a set, we say  $\mathfrak{B}$  is a Basis if*

1. *For all  $x \in X$  there exists  $B \in \mathfrak{B}$  such that  $x \in B$ .*
2. *If  $B_1, B_2 \in \mathfrak{B}$  and  $x \in B_1 \cap B_2$  then there exists  $B_3 \in \mathfrak{B}$  such that  $x \in B_3 \subset B_1 \cap B_2$ .*

*We define the topology generated by  $\mathfrak{B}$  as the collection  $\mathfrak{T}$  such that for any  $U \subset X$ , if for all  $x \in U$  there exists  $B \in \mathfrak{B}$  such that  $x \in B \subset U$  then  $U \in \mathfrak{T}$ . For any  $U \in \mathfrak{T}$  we say that  $U$  is open.*

**Proposition 1.2.2**

*The topology generated by a basis is a topology.*

*Proof.* The first condition for a basis gives us  $X$  as an open set and the empty set satisfies our condition vacuously. Now we must prove closure under arbitrary unions. Let  $\{U_\alpha\}_{\alpha \in J} \subset \mathfrak{T}$  be a collection of open sets and consider their union. Then for  $x \in \bigcup U_\alpha$  we must have  $x$  appearing in some  $U_\alpha$  since it is in the union, but  $U_\alpha$  is by assumption open so there exists  $B \in \mathfrak{B}$  such that

$$x \in B \subset U_\alpha \subset \bigcup_{\alpha \in J} U_\alpha$$

as desired. Now we must show that the finite intersection of open sets is again open, for that let  $\{U_1, \dots, U_n\}$  be a collection of open sets. Then if  $x$  lies in their intersection it must lie in each  $U_i$ . Thus there exists a family of basis elements  $\{B_1, \dots, B_n\}$  such that  $x \in B_i \subset U_i$ . It follows then that  $x \in \bigcap_{i=1}^n B_i$ . Now to see that this intersection must be a basis element, we use induction on the second part of the definition of a basis.  $\square$

We now look at another way to define the topology generated by a basis.

**Lemma 1.2.3**

*Let  $X$  be a topological space and let  $\mathfrak{B}$  be a basis for the topology on  $X$ . Then  $\mathfrak{T}$  is equal to set containing all unions of elements of  $\mathfrak{B}$*

*Proof.* Let  $U \in \mathfrak{T}$ , we want to write  $U$  as a union of basis elements. By definition we know that for each  $x \in U$  there exists  $B_x \in \mathfrak{B}$  satisfying  $x \in B_x \subset U$ . Taking the union over all  $B_x$  gives us the desired result. Now since Basis elements are open, any union of them must be contained in  $\mathfrak{T}$ , by definition.  $\square$

It may be helpful to be able to check whether or not a given set of subsets forms a basis for the topology.

**Lemma 1.2.4**

*Let  $X$  be a topological space. Let  $\mathfrak{C}$  be a collection of open sets such that for all  $U \in \mathfrak{T}$  and all  $x \in U$  there exists  $C \in \mathfrak{C}$  such that*

$$x \in C \subset U$$

*Then  $\mathfrak{C}$  is a basis for the topology on  $X$*

*Proof.* The first condition of a basis is satisfied by assumption. Now suppose  $x \in C_1 \cap C_2$  we must show there exists  $C_3 \in \mathfrak{C}$  such that  $x \in C_3 \subset C_1 \cap C_2$ . We may use the fact that  $\mathfrak{C}$  is a collection of opensets together with our assumption to produce such an element.

Now we must show that  $\mathfrak{C}$  generates the correct topology. Let  $\mathfrak{C}$  generate  $\mathfrak{T}'$ . If  $U$  is open in  $\mathfrak{T}$  then by assumption it is open in  $\mathfrak{T}'$ . If  $U$  is open in  $\mathfrak{T}'$  then it is a union of elements of  $\mathfrak{C}$ , since  $\mathfrak{C}$  is a collection of open sets of  $\mathfrak{T}$ ,  $U$  must be open in  $\mathfrak{T}$ .  $\square$

Now we may wish to tell whether one topology is finer than another, we can use the following lemma.

**Lemma 1.2.5**

*Let  $X$  be a set and let  $\mathfrak{T}, \mathfrak{T}'$  be topologies on  $X$  with bases  $\mathfrak{B}$  and  $\mathfrak{B}'$  respectively. Then the following are equivalent.*

1.  $\mathfrak{T}'$  is finer than  $\mathfrak{T}$
2. For all  $B \in \mathfrak{B}$  and  $x \in B$  there exists  $B' \in \mathfrak{B}'$  such that  $x \in B' \subset B$ .

*Proof.* (2)  $\implies$  (1): Let  $U \in \mathfrak{T}$ , then for every  $x \in U$  there exists  $B \in \mathfrak{B}$  with  $x \in B \subset U$ . Then by assumption we have there exists  $B' \in \mathfrak{B}'$  with  $x \in B' \subset B \subset U$ . Hence  $U$  is open in  $\mathfrak{T}'$ .

(1)  $\implies$  (2): Assume that  $\mathfrak{T}'$  is finer than  $\mathfrak{T}$ . Then since  $\mathfrak{T}$  is a topology and  $\mathfrak{T} \subset \mathfrak{T}'$ , for all  $x \in B$  there must exist  $B' \in \mathfrak{B}'$  such that  $x \in B' \subset B$ .  $\square$

To end this section we discuss subbases.

**Definition 1.2.6**

*Let  $X$  be a set. A subbasis is a collection of subsets  $A$  such that the union over  $A$  is  $X$ . We define the topology generated by the subbasis as the collection of all unions of all intersections of elements of  $A$*

Of course we must prove that this is indeed a topology, but first notice that the definition of a subbasis is just the first axiom of a basis. Thus, every basis is a subbasis, and if  $\mathfrak{B}$  is a basis considering it as a subbasis will generate the same topology. So Subbases are a generalization of a basis. Can you give an example of a subbasis which is not also a basis?

**Proposition 1.2.7**

*The topology generated by a subbasis is a topology*

*Proof.* It is sufficient to show that the collection of all finite intersections of elements of  $A$  is a basis. Since the union of all elements of  $A$  is  $X$ , the first condition of a basis is clearly satisfied. Now suppose  $x \in a_1 \cap a_2$ , where  $a_1, a_2$  are finite intersections of elements of  $A$ . Then  $a_1 \cap a_2$  is a finite intersection with length equal to the sum of the lengths of  $a_1$  and  $a_2$ .  $\square$

## Chapter 2

# Order and Product Topology

### 2.1 Order Topology

#### Definition 2.1.1

Let  $X$  be a set, we define the order topology on  $X$  as the topology generated by the basis  $\mathfrak{B}$  such that  $\mathfrak{B}$  contains all elements of the form

1.  $(a, b)$  for  $a, b \in X$ .
2.  $[a_0, b)$  where  $a_0$  is the minimal element of  $X$ .
3.  $(a, b_0]$  where  $b_0$  is the maximal element of  $X$ .

If  $X$  has no maximal or minimal elements, then  $\mathfrak{B}$  consists only of elements of the first type.

Now we must of course prove that this choice of  $\mathfrak{B}$  does indeed form the basis of a topology, but it is clear since the intersection of any of these sets yeilds another set of the same type. There are several cases to check.

#### Example 2.1.1

Consider the order topology on  $\mathbb{R}$ , since  $\mathbb{R}$  has no maximal or minimal elements the order topology is generated by  $\mathfrak{B} = \{(a, b) | a, b \in \mathbb{R}\}$ . Then it is clear that this coencides with the standard topology on  $\mathbb{R}$

#### Example 2.1.2

Consider the order topology on  $\mathbb{Z}_+$  given the usual order. Then the order topology is equivalent to the discrete topology.

#### Example 2.1.3

Consider  $\mathbb{R} \times \mathbb{R}$  equiped with the dictonary order, then the basis elements are of the form  $(a, b) \times (c, d)$  where  $a < c$  or  $(a = c) \wedge (b < d)$ .



## 2.2 Product Topology

Given two sets  $A$  and  $B$  we may want to define a topology on the cartesian product  $A \times B$ . The cartesian product comes equipped with two functions  $\pi_A(x) : A \times B \rightarrow A$  and  $\pi_B(x) : A \times B \rightarrow B$  called the projections of  $A \times B$ .