Online Supplementary Materials for

Cooperative Coevolution for Non-Separable Large-Scale Black-Box Optimization: Convergence Analyses and Distributed Accelerations

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https://github.com/Evolutionary-Intelligence/DCC/blob/main/SupplementaryMaterials.pdf

1. Convergence to a Pure Nash Equilibrium for Cooperative Coevolution (CC):

***Theorem* 1:** Given *any* partition $p = \{g_1, \dots, g_m\}$ of the objective function $f(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, where $1 < m \leq n$, we say that the convergence point of CC under this partition is also one *pure Nash equilibrium (PNE)*.

Proof: If $\boldsymbol{x}^* = (x_1^*, x_2^*, \cdots, x_n^*) = (\boldsymbol{x}_{g_1}^*, \cdots, \boldsymbol{x}_{g_m}^*)$ is a convergence point of CC under the partition $p = \{g_1, \dots, g_m\}$, then $\boldsymbol{x}_{g_i}^*$ is one of the global optima of the function $f\left(\boldsymbol{x}_{g_1}^*, \cdots, \boldsymbol{x}_{g_i}, \cdots, \boldsymbol{x}_{g_m}^*\right), i = 1, \cdots, m$.

If $x^* = (\boldsymbol{x}_{g_1}^*, \cdots, \boldsymbol{x}_{g_m}^*)$ is not one *pure Nash equilibrium* (w.r.t. p), according to the definition of **Pure Nash Equilibrium (PNE),** we have that $\exists i \in \{1, \cdots, m\}$, $\exists \breve{\boldsymbol{x}}_{g_i} \in \mathbb{R}^{|g_i|} \backslash \{\boldsymbol{x}_{g_i}\}$, such that $f\left(\boldsymbol{x}_{g_1}^*, \cdots, \breve{\boldsymbol{x}}_{g_i}, \cdots, \boldsymbol{x}_{g_m}^*\right) < f\left(\boldsymbol{x}_{g_1}^*, \cdots, \boldsymbol{x}_{g_i}^*, \cdots, \boldsymbol{x}_{g_m}^*\right)$, namely, then $\boldsymbol{x}_{g_i}^*$ is not the global optimum of $f\left(\boldsymbol{x}_{g_1}^*, \cdots, \boldsymbol{x}_{g_i}, \cdots, \boldsymbol{x}_{g_m}^*\right)$. It is a contradiction.

2. Convergence Analyses on Four Representative Test Functions:

$$\boldsymbol{f_1(x,y) = 7x^2 + 6xy + 8y^2}$$

Since its Hessian matrix $\begin{bmatrix} 7 & 6 \\ 6 & 8 \end{bmatrix}$ is positive definite, f_1 is differentiable strictly convex, then it has a unique global optimum $(0,0)$.

Given (x_0, y_0) is a PNE, by the definition of PNE, (x_0, y_0) is the unique global optimum of differentiable strictly convex $f_1(x, y_0)$ and $f_1(x_0, y)$.

Then $\frac{\partial f_1(x,y_0)}{\partial x} = 14x + 6y_0 = 0$, we have $x = -\frac{6y_0}{14} = x_0$, $\frac{\partial f_1(x_0,y)}{\partial y} = 16y + 6x_0 = 0$, we have $y = -\frac{6x_0}{16} = y_0$. So, $(x_0, y_0) = (0,0)$.

$$\boldsymbol{f_2(x,y) = x^2 + 10^6 y^2}$$

Since its Hessian matrix $\begin{bmatrix} 1 & 0 \\ 0 & 10^6 \end{bmatrix}$ is positive definite, f_1 and its rotation variant are all strictly convex and have the unique global optimum $(0,0)$.

Like the above proof, they have only one PNE, namely global optimum $(0,0)$.

$$\boldsymbol{f_3(x,y) = 100(x^2 - y)^2 + (x-1)^2}$$

Obviously, $f_3 \geq 0$ and

$$\begin{cases} \dfrac{\partial f_3(x,y)}{\partial x} = 400x(x^2 - y) + 2(x-1) = 0 \\ \dfrac{\partial f_3(x,y)}{\partial y} = 200(y - x^2) = 0 \end{cases},$$

So, it has a unique global optimum $(1,1)$.

Suppose (x_0, y_0) is a PNE, then x_0 and y_0 are the global minima of differentiable $f_3(x, y_0)$ and $f_3(x_0, y)$, respectively. Then

$$\begin{cases} \dfrac{df_3(x,y_0)}{dx} = 400x^3 - 400xy_0 + 2x - 2 = 0 \\ \dfrac{df_3(x_0,y)}{dy} = 200y - 200x_0^2 = 0 \end{cases},$$

we have $200x^3 - 200xx_0^2 + x - 1 = 0, y = x_0^2 = y_0$.

Since x_0 is one of solutions of equation $200x^3 - 200xx_0^2 + x - 1 = 0$, we conclude that $x_0 = 1,\ y_0 = 1$.

$$\boldsymbol{f_4(x,y) = |x-y| - \min(x,y) = \begin{cases} x - 2y, x > y \\ -x, x = y \\ y - 2x, x < y \end{cases}}$$

For any (x_0, y_0), the function $f_4(x, y_0)$ and $f_4(x_0, y)$ obtain their global minima at (y_0, y_0) and (x_0, x_0), respectively. So, the set $\{(x,y) | x = y\}$ is the set of PNEs.

3. Convergence Analysis on A Function (with Loss of Gradients):

Corollary 3: For the *Schwefel's Problem 2.21*, $min\,(f(\boldsymbol{x})) = min\,(\max_{i=1,\dots,n}(|\boldsymbol{x_i}|))$, defined on an open set[1] $\Omega \subseteq \mathbb{R}^n$, the set of global pure Nash equilibria w.r.t. any partition $p = \{g_1, \dots, g_m\}$ is $\{\boldsymbol{x}, \max_{j\in g_1}|\boldsymbol{x}_j| = \cdots = \max_{j\in g_m}|\boldsymbol{x}_j|\}$. There is a unique strict global Nash equilibrium $\boldsymbol{x} = (0, \dots, 0)$ w.r.t. any partition set, which equals the global optimum, and vice versa.

Proof: Given $\boldsymbol{x} = (\boldsymbol{x}_{g_1}, \cdots, \boldsymbol{x}_{g_m})$ is a PNE w.r.t. any partition $p = \{g_1, \dots, g_m\}$ of $f(\boldsymbol{x}) = \max_{i=1,\dots,n}(|\boldsymbol{x_i}|)$, owing to the definition of PNE, we have $\boldsymbol{x}_{g_i} \in \mathbb{R}^{|g_i|}$ for each $g_i \in p, \forall i \in \{1, \dots, m\}$, satisfies

$$f(\boldsymbol{x}_{g_i}, \boldsymbol{x}_{\neq g_i}) \le f(\boldsymbol{x}_{\tilde{g}_i}, \boldsymbol{x}_{\neq g_i}),\ \forall \boldsymbol{x}_{\tilde{g}_i} \in \mathbb{R}^{|g_i|} \backslash \{\boldsymbol{x}_{g_i}\},$$

namely, for any $i \in \{1, \cdots, m\}$, $\boldsymbol{x}_{\tilde{g}_i} \in \mathbb{R}^{|g_i|} \backslash \{\boldsymbol{x}_{g_i}\}$, $max\{\max_{j\in g_i}|\boldsymbol{x}_j|, \max_{j\in\neq g_i}|\boldsymbol{x}_j|\} \le max\{\max_{j\in g_i}|\boldsymbol{x}_{\tilde{g}_i}|, \max_{j\in\neq g_i}|\boldsymbol{x}_j|\}$, only if, $max\{\max_{j\in g_i}|\boldsymbol{x}_j|, \max_{j\in\neq g_i}|\boldsymbol{x}_j|\} \le min\{\max_{j\in g_i}|\boldsymbol{x}_{\tilde{g}_i}|, \max_{j\in\neq g_i}|\boldsymbol{x}_j|\}$, only if,

$$\begin{cases} \max_{j\in g_i}|\boldsymbol{x}_j| \le \min\left\{\max_{j\in g_i}|\boldsymbol{x}_{\tilde{g}_i}|\right\} \\ \max_{j\in g_i}|\boldsymbol{x}_j| \le \max_{j\in\neq g_i}|\boldsymbol{x}_j| \end{cases},$$

owing to the definition of PNE,

$$\max_{j\in g_i}|\boldsymbol{x}_j| \le min\left\{\max_{j\in g_i}|\boldsymbol{x}_{\tilde{g}_i}|\right\},$$

so, we only need for any $i \in \{1, \cdots, m\}$,

$$\max_{j\in g_i}|\boldsymbol{x}_j| \le \max_{j\in\neq g_i}|\boldsymbol{x}_j|,$$

we have $\max_{j\in g_1}|\boldsymbol{x}_j| = \cdots = \max_{j\in g_m}|\boldsymbol{x}_j|$.

Since $f(\boldsymbol{x}) \ge 0$, $\boldsymbol{x_0} = (0, \cdots, 0)$ is a PNE, and $f(x_0) = f(0, \cdots, 0) = 0$, $\boldsymbol{x_0}$ is the unique global optimum, thus it is a unique strict global Nash equilibrium, and vice versa.

[1]Here it is *implicitly* assumed that there is (at least) one global optimum in this open set.