# GÖDEL'S INCOMPLETENESS THEOREMS

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I trust in Nature for the stable laws
Of beauty and utility. Spring shall plant
And Autumn garner to the end of time.
I trust in God,—the right shall be the right
And other than the wrong, while he endures.
I trust in my own soul, that can perceive
The outward and the inward

*A Soul's Tragedy* - Robert Browning

## 1. A History and a Result

It has been assumed, by mathematicians and laymen alike, that if something were true it could be proven. In 1931 Kurt Gödel showed this assumption to be mistaken. Gödel first published this result in a short paper titled "*On Formally Undecidable Propositions of Principia Mathematica and Related Systems*" as a response to questions that had arose about uncertainties at the foundations of mathematics. Gödel showed that such uncertainties in mathematics are inexorable and in so doing forever changed perceptions of mathematics, reasoning, and philosophy.

1.1. **The Axiomatic System is Developed.** To understand what it was exactly that Gödel proved we will first discuss the general development of mathematics until that time when Gödel published his result. As with most histories of math, we will start with Euclid, as it was he who made math what it is.

Euclid wrote *The Elements* in around 300 B.C. and with it established math as a *deductive*, as opposed to *empirical*, discipline. This meant hypotheses were not evaluated by their consistency with observations, as is the case with science, but rather by a demonstration of their inevitability when certain basic assumptions are made, these basic assumptions being called axioms. Euclid gave five axioms to describe geometry, and with them proved results of incredible complexity. It is difficult to overstate the impact this book has had on thinkers of all kinds throughout history; soon after it was first written it was copied and recopied, and occasionally experienced additions. In the sixth century Isidore of Miletus, the architect of the Hagia Sophia and compiler of the works of Archimedes, for example, wrote a Book XV[1]. In 1482 it became one of the earliest books to be put into print, and has since enjoyed over 1000 editions, second only to the Bible. Abraham Lincoln even cited it as the means by which he learned to be a lawyer, saying

[1] *The Elements* is a collection of 13 books

> At last I said, 'Lincoln, you never can make a lawyer if you do not understand what demonstrate means'; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means, and went back to my law studies.

In a private, unpublished note from 1854, Lincoln even used the principles of deductive reasoning to offer a challenge to slavery.

> If A. can prove, however conclusively, that he may, of right, enslave B. — why may not B. snatch the same argument, and prove equally, that he may enslave A?– You say A. is white, and B. is black. It is color, then; the lighter, having the right to enslave the darker? Take care. By this rule, you are to be slave to the first man you meet, with a fairer skin than your own. You do not mean color exactly?–You mean the whites are intellectually the superiors of the blacks, and, therefore have the right to enslave them? Take care again. By this rule, you are to be slave to the first man you meet, with an intellect superior to your own. But, say you, it is a question of interest; and, if you can make it your interest, you have the right to enslave another. Very well. And if he can make it his interest, he has the right to enslave you.

*The Elements* has never gone out of print, but starting in the $20^{\text{th}}$ century the popularity it had enjoyed for millenia began to wane, ceasing to be a standard text for all educated people. To understand why, we need to look at the last axiom Euclid employed.

1.2. **The Odd One Out.** There are few sentences that have inspired as much intellectual activity as this:

> If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

At first glance it seems rather inaesthetic and unappealing. It tends to stay that way for subsequent glances as well. Yet this is Euclid's fifth axiom nonetheless, often called the Parallel Postulate. Other axioms Euclid employs include such things as the idea that all right angles are equal, and that two points may be connected by a line. This axiom by comparison is lengthy and unwieldy and so it became the wish of countless mathematicians to prove it as a consequence of the other axioms, relegating it to the status of a mere theorem.

Some suggested simply throwing the postulate out. In his commentary on *The Elements* Proclus remarks

> This [fifth postulate] ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties which Ptolemy, in a certain book, set himself to solve, and it requires for the demonstration of it a number of definitions as well as theorems.

Proclus mentions here the work of Ptolemy, who, in the second century, sought to prove the parallel postulate from the others. And he did. The only problem was that he unwittingly assumed the Parallel Postulate in his proof of the Parallel Postulate, falling into circular reasoning. In his "proof" Proclus assumed that two lines that are parallel always maintain the the same distance between them, but this assumption is in fact equivalent to the Parallel Postulate. This assumption implies that given a line and a point not on that line there exists exactly one line through the given point parallel to the given line, and this formulation of the Parallel Axiom is known as Playfair's Axiom.

A more novel approach was given over a thousand years later in 1663 by John Wallis, who proved the Parallel Postulate using the assumption that for every figure there exists a similar one at every size. This assumption, as natural as it is, is also, in fact, equivalent to the parallel postulate.

In 1733 Giovanni Saccheri made what has become the most famous attempt in his book *Euclid Freed of Every Flaw*, which purported to prove the Parallel Postulate from the other axioms. He attempted to do so by contradiction, assuming the negation of the Parallel Postulate as an axiom and trying to find an inconsistency. To do this he considered the two alternatives to Playfair's Axiom; that given a point and line there exist A, no parallels, or B, multiple. If no parallels exist then it can be shown that lines can only ever be finite, which contradicts Euclid's second axiom, which states that lines can be extended indefinitely, so Saccheri had success in this direction. If, however, it was assumed that there exist multiple lines through a given point parallel to a given line, then no contradiction can in fact be found. However, there are many counterintuitive results, one of which is the fact that a quadrilateral may have three right angles and one acute angle, to which Saccheri reacted by saying

> The hypothesis of the acute angle is absolutely false because it is repugnant to the nature of straight lines.

And thusly concluded his proof[2].

The reason Saccheri's treatment has enjoyed its popularity (some may say infamy) is that the two alternatives to Playfair's Axiom actually give rise to two different entirely consistent types of geometry, and many of Saccheri's results are now theorems in these alternatives.

1.3. **An Alternative Geometry.** The first person to publish on an alternative to Euclidean Geometry, while acknowledging the fact that there were indeed alternatives, was Nikolai Lobachevsky. In 1840, in *Geometrical Investigations on the Theory of Parallels*, he wrote

> In geometry I find certain imperfections which I hold to be the reason why this science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid. As belonging to these imperfections, I consider the obscurity in the fundamental concepts of the geometrical magnitudes and in the manner and method of representing the measuring of

[2]It has been theorized that Saccheri, a highly skilled logician, did not actually hold this view but rather wanted to avoid criticism.

> these magnitudes, and finally the momentous gap in the theory of parallels, to fill which all efforts of mathematicians have been so far in vain. ... Yet I am of the opinion that the Theory of Parallels should not lose its claim to the attention of geometers, and therefore I aim to give here the substance of my investigations...

He goes on to strictly define the properties of a straight line, which, with its many connotations, lead to many unnamed assumptions being used by other mathematicians. In this paper and others Lobachevsky adopted the assumption that for each line and point there exist multiple parallels, and with it gave rise to what is now called hyperbolic geometry, or in Russia, Lobachevskian geometry.

1.4. **A Bigger Problem.** What Lobachevsky's treatment of geometry showed was that Euclid's fifth axiom was independent of the rest: that the first four axioms were not sufficient to prove or disprove certain statements in geometry. This quality is called *incompleteness*. We call a set of axioms *incomplete* if there exists a well-formed statement without free variables $p$ such that neither $p$ nor *not* $p$ is provable from the axioms.

By well-formed what is meant is simply that the statement makes sense within the system. So within geometry a well-formed statement might be "the points $a$ and $b$ are connected by the line $x$", or "the lines $x$ and $y$ intersect at a right angle". Statements that are not well-formed are statements like "right angle the line $x$ with the point $a$", because even though all the words the statement comprises belong to geometry, this combination is nonsensical, or "1 plus 1 equals 2", as this, despite being true, is not a statement within geometry, or "bacon", because bacon, for better or worse, is not within the scope of geometry.

But of course, even the examples of well-formed statements above are not decidable, as their truth of falsity depends on what values the variables take. So while "the points $a$ and $b$ are connected by the line $x$" is well-formed, it cannot be proven or disproven, because the truth value of this statement depends upon what exactly $a$, $b$, and $x$ are. More will be said on this matter later.

What came to the attention of many mathematicians in the late 1800s was the fact that, even with the fifth axiom, Euclid's axioms were incomplete. To see why, let us consider all of Euclid's axioms and his first proposition, the first thing that he proved with said axioms.

Euclid's axioms are as follows.

(1) To draw a line from any point to any point
(2) To produce a finite line continuously
(3) To describe a circle with any center and distance
(4) That all right angles are equal
(5) If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

(We can see why the fifth was not favored.)

With these we will now go through Euclid's first proposition, where he proposed the ability to construct an equilateral triangle from a finite straight line.

Given a finite straight line, which we will call AB (points A and B being the endpoints of the line), we can, by axiom (3), form a circle with center A and distance AB, and likewise form a circle with center B and distance BA. The point at which these circles intersect (it may be either point) we will call C. By axiom (1) we can form a line between A and C and between B and C. We have thus formed an equilateral triangle ABC, completing the proof.

But there's a problem here. We don't know whether or not the circles we constructed intersect. In fact, the only axiom that deals with intersection is (5), which was never invoked. We, as Ptolemy did, unwittingly made an assumption critical to proving what we already believed to be the case. Euclid's passive assumptions are not obvious, and they are very natural, but they are assumptions nonetheless and in a rigorous treatment of geometry must be stated as such.

Rigor is what defines the modern mathematical approach. The reason Euclid fell out of fashion was because he was simply no longer rigorous enough for mathematicians in the late 1800s. Mathematicians needed to flesh out each and every assumption being made. A rigorous treatment of geometry was given in the book *Foundations of Geometry*, published by David Hilbert in 1899, which contained 21 axioms, filling in the gaps that Euclid had simply assumed. One of these axioms was shown to be redundant and was removed in later editions.

1.5. **A New Foundation for Mathematics.** In the midst of all this change in the field of geometry there was a rather modest 2-page paper in number theory that was published in 1874 under the title "*On a Property of the Collection of All Real Algebraic Numbers*". It was written by 29 year-old German mathematician Georg Cantor and it changed mathematics forever.

In this paper Cantor was merely re-proving a result first proved by Louisville, that there exist numbers that are not algebraic, what we now call transcendental numbers, in a more efficient way. The manner in which he did this was to identify that there were more numbers in an interval of real numbers than there were algebraic numbers. As Cantor said in his paper, "I have uncovered the essential difference between a so-called continuum and a set such as the set of all real algebraic numbers".

For millenia paradoxes and problems with infinity perplexed thinkers. In his *Dialogue Concerning the Two Chief World Systems* Galileo writes two characters, Salviati and Simplicio, who are having a discussion of the infinite. Salviati is meant to represent Galileo, the correct view of the world, and Simplicio his rivals, the backward view of the world. In this discussion Salviati claims that "infinity and indivisibilty are in their very nature incomprehensible to us", and declares that "we cannot speak of infinite quantities as being the one greater or less than or equal to another. To prove this I have in mind an argument". His argument is that for every square number there exists a pair in the natural numbers (namely its square root), and that both are infinite, yet clearly there are fewer squares than naturals, so this is a contradiction. Thus we cannot compare infinities.

In *The Life and Opinions of Tristram Shandy, Gentleman*, the titular main character wonders if he would ever be able to complete his autobiography if he lived forever; it takes two days to write about one day, so surely he could never finish, yet any given point of his life would eventually be committed to paper.

Even Gauss "protest[ed] against the use of infinite magnitude... which is never permissible in mathematics".

The problems surrounding infinity seemed intractable.

Yet they weren't. In his short paper Cantor arrived at the heart of infinity, and discovered the "essential difference" between different infinities. The problem with previous approaches to distinguishing sizes of infinities, which rarely even considered the notion that there may be different sizes to begin with, was that they relied on methods applicable only to finite objects. To determine whether one finite quantity is larger than another we establish the size of each and then compare the sizes. What Cantor did was to reverse the order in which this procedure was carried out by comparing sets and subsequently evaluating their size.

Given two infinite sets we can say they are of equal size if there exists a function that maps each member of one to exactly one member of the other, and does so uniquely. Such a function is called a *bijection.* Cantor showed that there cannot exist a bijection between the set of algebraic numbers and the set of real numbers in a given interval. This was the first demonstration of the rather astounding fact that there are different sizes of infinity. Salviati, the correct view of the world, was incorrect.

This result invited vehement debate among mathematicians about the fundamental assumptions being made about their subject. Some mathematicians, like Leopold Kronecker and Henri Poincaré, were fierce opponents of this theory of sets, or set theory. Poincaré believed that "later generations will regard [Cantor's] *Mengenlehre* as a disease", while other mathematicians, like Richard Dedekind and David Hilbert, saw it as the means to tie together all of mathematics and contributed a great deal to the theory. It was ultimately the latter group that won out.

1.5.1. *Arithmetic, Set Theory, and Logic.* To understand how and why set theory became a foundation for math, it is important to understand a parallel development in arithmetic. Logician Bertrand Russell writes in his 1919 book *An Introduction to Mathematical Philosophy*,

> All traditional pure mathematics, including analytical pure geometry, may be regarded as consisting wholly of propositions about the natural numbers. That is to say, the terms which occur can be defined by means of the natural numbers, and the propositions can be deduced from the propositions of the natural numbers - with the addition, in each case, of the ideas and propositions of pure logic.

Adding that this discovery was recent, but that "it had long been expected". The discovery of this fact was recent because ancient mathematicians thought it to be foreclosed. This was due to the discovery of incommensurables, what we now call irrational numbers, two centuries prior to Euclid, in Pythagoras' time. These numbers could not be defined as a ratio between natural numbers and thus it was thought they could not be defined at all by them, yet work by Dedekind showed that in fact all real numbers could be defined by the naturals.

Once the natural numbers were shown to be the basis of all of math, there was an aim to simplify them into as few primitive concepts as possible. A primitive concept is a concept that is only defined by its relation with other things, and not by explicit definition in itself. Because all definitions require other concepts, eventually there must always exist primitive concepts to start the actual development of theory. As Russell goes on to say, the work of defining the natural numbers in terms of very few primitive concepts was done by Peano, who "showed that the entire theory of the natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of pure logic." The three primitive notions here identified were *0*, *number*, and *successor*. The five primitive propositions are

(1) 0 is a number.
(2) The successor of a number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.
(5) Any property that belongs to 0 and also the successor of any number that has the property belongs to all numbers. (induction)

However, *number* can be further defined, a fact realized and explicated by German logician Gottlob Frege in his 1884 book *The Foundations of Arithmetic*. In this work, Frege shows that comparing sizes of sets is a more primitive notion than actually enumerating the sizes of those sets. Thusly, he defines the *number* of a set as the set of all sets of equal size to it. So number in the general sense is anything that is the number for some set.

So *number* can be defined within the theory of sets. But what is a set? In *Contributions in Support of Transfinite Set Theory* Cantor describes sets as "...a gathering together into a whole of definite, distinct objects of our perception or of our thought".

But Frege gave a more rigorous definition. He showed that any set can be described using a defining property, such as the set of natural numbers, or the set of points on a circle, or the set of people living in London, and so a set can be thought of as a collection of things united by a common property. But there was a problem lurking in this unassuming assumption.

1.6. **A Crisis in the Foundation.** The problem was the natural assumption that any property may be used to define a set. This assumption was mistaken.

In 1901 Russell sent a letter to Frege to demonstrate this fact. He writes

> There is just one point where I have encountered a difficulty. You state that a function too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let $w$ be the predicate: to be a predicate that cannot be predicated of itself. Can $w$ be predicated of itself? From each answer its opposite follows.

What Russell is asking of Frege is to consider a set whose defining property is "to have a property that excludes oneself", or in other words, consider the set of sets that do not not contain themselves. There are many sets that do not contain themselves, but if we call the above set $\Lambda$, does $\Lambda$ contain itself? If it does not, then it fulfills its defining property, and so must be contained within itself. If it does, then it fails its defining property, so must not be contained within itself. This is Russell's Paradox, and it showed that a reformulation of set theory was needed, as allowing any property to define sets led to contradictions. The question was, what sort of restrictions could be placed on the construction of sets to avoid such contradictions? The difficulty was that, unlike the reformulation of geometry, which only required filling in some gaps and exploring new territory, the reformulation of set theory necessitated an entirely new conception of the most basic notion of the study.

Fortunately, the man who destroyed the foundation was there to set it right.

1.7. **A New Theory of Sets.** Russell was not among the mathematicians who opposed set theory. Quite the opposite, he viewed it as the only possible way to put mathematics on a firm footing without any ambiguity. The inconsistency he discovered in set theory showed to him what he thought of as the bane of any system attempting to be consistent. If a system is to function, he thought, it must be rid entirely of self-reference.

Self-reference in normal communication is almost impossible to avoid. Self reference is present any time the word "I", the eleventh most common word in English, is spoken. This essay self-references, not only in this sentence, but in the previous sentence, when it uses the word "word". Despite the obvious utility of self-reference, Russell sought to do away with it in his system of set theory, and it took him quite a bit to do so.

The means by which he accomplished, at least ostensibly, the banishment of self-reference was through an idea called *Ramified Type Theory.* In this theory, there exist terms of first type, which are not sets, terms of second type, which are sets that contain only terms of first type, terms of third type, which contain only elements of first and second type, and so on. In this scheme no set may contain itself, so concern for self-reference is, ideally, obviated.

Between 1910 and 1913 Russell and Alfred North Whitehead published, in three volumes, *Principia Mathematica*, which explicated their theory. It served as a vehicle for a particular philosophical view of Russell as well. As Russell writes in his 1907 essay *The Study of Mathematics*,

> Mathematics takes us still further from what is human, into the region of absolute necessity, to which not only the world, but every possible world, must conform.

Russell's view of math is that it is equivalent to logic. This position, called Logicism, formed part the motivation behind *Principia.* Another part of the motivation was the reevaluation of basic concepts. As Russell writes in *An Introduction to Mathematical Philosophy*,

> We shall find that by analyzing our ordinary mathematical notions we acquire fresh insight, new powers, and the means of reaching whole new mathematical subjects by adopting fresh lines of advance after our backward journey.

To demonstrate that going forward once again was possible after his significant backward journey Russell gave, in a mere 379 pages, a proof of the arithmetic statement "$1+1=2$", accompanied by a caption that reads "the above proposition is occasionally useful". And useful it is indeed. This was the first step onto the bridge between Russell's logical foundation of math and arithmetic, the system capable of describing all of mathematics.

Motivated by this and other successes in reformulations of set theory, mathematicians became optimistic and emboldened, and sought, once and for all, to rid themselves of all doubt.

1.8. **The Culmination of Mathematical History.** In 1921 David Hilbert issued a challenge to the mathematicians of his day. His goal was two-fold: to formalize all of mathematics, and to prove the completeness and consistency of this formalization. A completion of this task would mark the end of all ambiguity in the subject and essentially complete the project started by Euclid thousands of years prior.

Not only this, but just as Russell used *Principia* to advocate for his logicist view of math, this project reflected Hilbert's disposition toward math as a *formalist.* A formalist sees math as the declaration of axioms and the manipulation of these axioms to reach conclusions that can only be considered true within that axiomatic system. Any coincidence with the real world is just that, and reflects more our arbitrary choice to look at things that comport with our preconceptions of the world.

To fully understand this project, it is important to understand what a formalization is. A formalization of an area of math is the reduction of the concepts of that area to a finite number of symbols representing the basic concepts necessary to discuss that area, using these symbols to construct what are called strings, which represent statements, and then providing rules of transformation so that we may derive new statements from old. Let's go through an example.

If we consider arithmetic, the addition and multiplication of natural numbers, some basic concepts are addition, multiplication, and equality. In addition to these concepts, in order to talk about these things we need logical connectors to represent ideas like "and" and "not" and "there exists", etc. So a formalization of arithmetic might consist in part of the following symbols:

$$\begin{aligned} \text{Addition} &: \quad + \\ \text{Multiplication} &: \quad \cdot \\ \text{Equality} &: \quad = \\ \text{And} &: \quad \wedge \\ \text{Not} &: \quad \neg \\ \text{There Exists} &: \quad \exists \\ \text{For All} &: \quad \forall \end{aligned}$$

With these symbols we can make strings that represent statements:

$$\begin{aligned} x \textit{ divides } y &: \quad \exists a(x \cdot a = y) \\ x \textit{ is prime} &: \quad a\forall b\forall(\neg(a=1) \wedge \neg(b=1))\neg(a \cdot b = x) \end{aligned}$$

In both these examples we use symbols that are not part of our formalization, like "(" and ")" and "1", but we will deal with these later. For now we will content ourselves with a general demonstration of the idea behind a formalization. Also, note that $x\forall p$ denotes "for all $x$, $p$ is true". This quirk is due to linguistic differences in English and German[3].

Once the symbols have been set in place, it is important to provide rules of transformation. These rules mirror logical rules. For example, one such rule might be “From $A$ and $A$ *implies* $B$ produce $B$”.

In principle all proofs can be reduced to some formalization. The concepts can be matched up with basic symbols and logical steps can be identified as certain rules of transformation. Of course, in practice, especially with complicated proofs, this is not practical, but that has never deterred mathematicians.

As important as the formalization was, its purpose is only to serve as a vehicle for the second step, which is a proof of the formalization’s completeness and consistency. In 1929 Polish mathematician Mojżesz Presburger proved the completeness and consistency for a simplified version of arithmetic that contained only addition on the integers. However, two years later, in 1931, 25 year-old Austrian logician Kurt Gödel showed a similar proof for when multiplication is adopted into this system to be impossible.

1.9. **The Result.** In the opening to *On Formally Undecidable Propositions of Principia Mathematica and Related Systems* Gödel writes

> The development of mathematics towards greater exactness has, as is well-known, lead to large tracts of it being formalized such that all methods of proof may be carried out according to a few mechanical rules. The most comprehensive formal systems yet devised are the system of Principia Mathematica (PM) on the one hand, and the Zermelo-Fraenkel set theory on the other. These two systems are so extensive that all methods of proof used in mathematics today have been formalized in them, that is, reduced to a few axioms and rules of transformation. It may therefore be surmised that these deduction rules are sufficient to decide all mathematical questions that can in any way at all be expressed in those systems. It is shown below that this is not the case.

In the pages that follow Gödel demonstrates this fact, and in so doing divorces in the minds of mathematicians, for the first time, truth and provability, and in so doing illustrates the folly of strict formalism and logicism. In this essay we will go through how exactly this was done.

## 2. The Proof

What Gödel showed was that any system of axioms capable of describing arithmetic, such as Peano’s axioms or Principia Mathematica, will always contain a well-formed statement that cannot be decided, making the system incomplete. This means that within these systems there exists a well-formed statement $p$ such that neither $p$ nor *not* $p$ can be proved from the axioms. Not only this, but he gave an example of such a statement, and showed that this fact implies that these systems

[3]We can think of $x\forall p$ as “$x$ generalizes for $p$”.

are incapable of proving their own consistency, and thus any proof of their consistency must rely on systems more complex than arithmetic, whose own consistency may be in doubt.

2.1. **Necessary Considerations.** Gödel's proof was written to a small group of specialists in a niche field of mathematics using completely novel methods of argument. It contains some difficult language. Here, before going into the details of the proof, we will go through some requisite definitions and concepts.

2.1.1. *Logic.* There are two types of logic necessary for our discussion, propositional logic and predicate logic. These are also known as zero-order and first-order logic, respectively. We will go through a formalization of them to familiarize ourselves with the symbols.

Propositional logic deals with statements, or propositions, and their combinations in arguments. Its utility is in the generalization of the forms of valid arguments. The relevant ideas and the symbols for them are as follows.

$$\begin{aligned} \text{And} &: \wedge \\ \text{Or} &: \vee \\ \text{Not} &: \neg \\ \text{Implies} &: \implies \\ \text{Ordering} &: \text{“(” and “)”} \end{aligned}$$

Some of these, however, are redundant. In the stead of $p \wedge q$, we can write $\neg(\neg p \vee \neg q)$ (DeMorgan's Laws). And for $p \implies q$ we can write $\neg p \vee q$ (Material Implication). Thus we see that we can write any statement in propositional logic using only $\neg$, $\vee$, and the parentheses. This simplification will become important when we go through Gödel's proof.

Predicate logic extends propositional logic and enables us to talk not only about statements, but variables. It consists of what are called *quantifiers*, which allow for the quantification of variables. The relevant concepts and symbols for them are

$$\begin{aligned} \text{There Exists} &: \exists \\ \text{For All} &: \forall \end{aligned}$$

We call $\exists$ the existential quantifier and $\forall$ the universal quantifier.

And again we can do away with a certain term. Instead of writing $\exists x(p)$ (there exists an $x$ such that $p$ is true), we can write $\neg(x\forall\neg p)$. Thus we can make any statement in predicate logic using only the universal quantifier and the symbols from propositional logic.

An important fact about predicate logic is that every variable must be paired with a quantifier in order for the statement to be considered true or false. Consider the statement "$x$ is prime". Is that statement true or false? The correct answer is no; it is not true or false. We can, however, assign a truth value to the statements "$\exists x$ such that $x$ is prime" and "$x\forall x$ is prime" (these statements being true and false, respectively). When a variable is quantified it is called *bound*. When it is not bound it is called *free*. And when there exists a free variable in a statement we know the statement cannot be assigned a truth value, and thus cannot be proved nor disproved.

Thus, the elimination of the existential quantifier allows us to test whether a statement can be assigned a truth value with greater ease, as we need only test whether every variable is paired with the universal quantifier. This will be useful in Gödel's proof, as we will need to make many very specific statements about the exact structure of strings (which we recall are the formal representations of statements).

An interesting note; both these systems of logic are complete, a fact established in 1929 by none other than Gödel in his doctoral thesis.

2.1.2. *Primitive Recursive.* Primitive recursive is a fancy term for a fairly basic idea. If a formula is primitive recursive all that is meant is that we can assign a finite bound to how long it will take to evaluate it. By formula we mean either what is commonly called a function, like $x^2$ or $x!$, something that takes an input and transforms it in some way, or a statement, in which case an evaluation would yield *true* or *false*.

Examples of primitive recursive formulae include $x!$, as we know this will take at most $x$ steps to evaluate, and $x < y$, as we know this will take at most $y$ steps to very its truth or falsity[4]. In reality these will take far longer, as evaluating $x!$, for example, requires $x$ steps of multiplication, which will in turn take many steps of addition, which will in turn take many steps of the successor function, but all of these are primitive recursive operations, and successive iterations of finite operations will always be finite. All common arithmetic formulae, such as addition, multiplication, exponentiation, factorials, equality, etc. are primitive recursive.

For an example of a formula that is not primitive recursive think of just about any unsolved problem in number theory. The Collatz Conjecture, for example, considers the function

$$f(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even} \end{cases}$$

and claims that continuously applying the function to any positive integer will always result, eventually, in getting a 1. The problem is that there doesn't seem to be any way to assign a bound to the "eventually" part. Even though every number we've tried goes 1, we have no idea how to calculate how long it might take as a function of the number with which we start.

The importance of primitive recursive formulae is that if an arithmetic formula is primitive recursive it can be expressed within an axiomatic system for arithmetic of equivalent or greater power than Peano arithmetic, such as Principia Mathematica or Zermelo-Fraenkel. This means that if a formula can be shown to be primitive recursive we can be sure that there exists a formalization of that formula (that is, a string of symbols representing the formula) within our system. This is important to Gödel's work, as a large portion of his proof is dedicated to showing that certain arithmetic relations can be expressed in the system of Principia Mathematica.

[4]simply add every number up to $y$ to $x$ and see if the result is $y$.

2.2. **The Proof.** We will now prove the incompleteness of arithmetic.

2.2.1. *Defining the System P.* We will start by going through the details of the formal system in which we are working, which we will call P. P is Principia Mathematica augmented by the Peano axioms. Including the Peano axioms is actually redundant, but they are useful in simplifying things. The basic symbols we will use are "$\neg$" (not), "$\vee$" (or), "$\forall$" (for all), "0", "$s$" (successor), "(", and ")".

In addition to these, we will need an infinite amount of symbols to represent variables. Recall that Principia Mathematica advanced *Ramified Type Theory*, in which there are terms of first type, which are numbers[5], terms of second type, which are sets of numbers, terms of third type, which are sets of sets of numbers, and so on. Thus, we will have variables of different types, which we will indicate with a subscript. So $x_1, y_1, z_1 \ldots$ will be variables that stand for numbers, $x_2, y_2, z_2 \ldots$ will be variables that stand for sets, and so on.

Thus, our basic symbols are

$$\neg, \vee, \forall, 0, s, (, )$$

$$x_1, y_1, z_1, \ldots$$

$$x_2, y_2, z_2, \ldots$$

$$\vdots$$

We now define our axioms.

**I - Axioms from Peano arithmetic**.

1. $\neg(sx_1 = 0)^6$
2. $sx_1 = sy_1 \implies x_1 = y_1$
3. $[x_2(0) \wedge x_1\forall(x_2(x_1) \implies x_2(sx_1))] \implies x_1\forall(x_2(x_1))^7$

**II - Axioms from Propositional Logic**

1. $p \vee p \implies p$
2. $p \implies p \vee q$
3. $p \vee q \implies q \vee p$
4. $(p \implies q) \implies (r \vee p \implies r \vee q)$

Where any formulae may be substituted for $p$, $q$, and $r$.

**III - Axioms from Predicate Logic**

1. $v\forall a \implies a\ \mathrm{Sub}(v, b)^8$
2. $v\forall(c \vee a) \implies c \vee v\forall a$

[5]They are numbers in our case because our system is arithmetic, but in general can be anything.

[6]$=$, $\wedge$, and $\implies$ are for abbreviation only; the actual axioms do not contain them. $x_1 = y_1$ can be expressed in our system as $x_2\forall(\neg(x_2(x_1)) \vee x_2(y_1))$. $x_2(x_1)$ denotes $x_1 \in x_2$.

[7]This axiom establishes induction.

[8]$a\ \mathrm{Sub}(v, b)$ denotes substituting $v$ with $b$ in $a$. This axiom can be thought of as saying "if a is true for any value it is true for a given value".

Where $a$ may be any formula, $v$ any variable, $b$ any variable of the same type as $v$ which does not contain a variable bound in $a$ where $v$ is free (more will be said on this later), and $c$ any formula in which $v$ is bound.

**IV - Axiom of Comprehension**

1. $\exists u(v\forall(u(v) \iff a))$

Where $v$ and $u$ may be substituted for a variable of type $n$ and $n+1$, respectively, and $a$ may be any formula in which $u$ is bound. This axiom establishes that sets are defined by rules, sometimes called intensions.

**V - Axiom of Extensionality**

1. $x_1\forall(x_2(x_1) \iff y_2(x_1)) \implies x_2 = y_2$

Or any version of this formula where the type of all variables are increased by the same amount (called a *type-lift*). This axiom defines set equality.

We now define the relation *immediate consequence of*. A formula $c$ is an immediate consequence of $a$ if $a$ is the formula $v\forall c$, where $v$ is any given variable. It is the immediate consequence of $a$ and $b$ if $a$ is the formula $b \implies c$. It is a fact of great import that all statements within any proof are produced from previous statements using only these logical steps. Thus, the set of provable formulae is the set that contains the axioms and is closed with respect to *immediate consequence of*. This fact will prove of great use, as we will later need to describe relations that define provability.

2.2.2. *Gödel Numbering.* This next step is the key to the proof. What we will do is assign each formula and proof (sequence of formulae) a number, which we will call the Gödel number[9]. We will do this in such a way that if the proof $a$ is a proof of the formula $b$, then the Gödel number for $a$ will have a certain arithmetic relation with the Gödel number for $b$. And because this relation is arithmetic, it can be described within our system P. This will enable P to, in a sense, talk about itself. As things turn out, self-reference can never be banished from math.

We will begin by defining a function $\Gamma : \{\text{Strings in P}\} \to \mathbb{N}$ which will take strings in P and yield their Gödel number. We will define this function in terms of the basic symbols to start, and use that definition to allow statements and proofs into the domain of $\Gamma$.

$$\Gamma(0) = 1 \qquad \Gamma(\neg) = 5 \qquad \Gamma(\forall) = 9 \qquad \Gamma(() = 11$$
$$\Gamma(s) = 3 \qquad \Gamma(\vee) = 7 \qquad \Gamma()) = 13$$

Further, $\Gamma$ will assign to each variable a number. Variables of first type, $x_1, y_1, z_1$, and so on, it will assign primes greater than 13. So

$$\Gamma(x_1) = 17 \qquad \Gamma(y_1) = 19 \qquad \Gamma(z_1) = 23 \ \ldots$$

For variables of greater type we will do the same, but raise the primes to powers equal to the type. So in general

$$\Gamma(x_n) = 17^n \qquad \Gamma(y_n) = 19^n \qquad \Gamma(z_n) = 23^n \ \ldots$$

[9]Gödel was not so bold as to name these numbers after himself, but these numbers now universally bear his name.

Now we will use this numbering to assign numbers to formulae. Every formula can be thought of as a sequence of symbols, and as such may be represented by a sequence of Gödel numbers. For example, the statement

$$e = \neg x_2 \forall\ \neg(x_1 \forall \neg(x_2(x_1))),$$

which states the existence of the empty set, would be given the sequence

$$5,\ 17^2,\ 9,\ 5,\ 11,\ 17,\ 9,\ 5,\ 11,\ 17^2,\ 11,\ 17,\ 13,\ 13,\ 13.$$

We can use such sequences to produce a unique number for every formula by taking the first prime to the power of the first number in the sequence, the second prime to power of the second number in the sequence, and so on. Thus, the Gödel number for the above formulae would be

$$\Gamma(e) = 2^5 \cdot 3^{17^2} \cdot 5^9 \cdot 7^5 \cdot 11^{11} \cdot 13^{17} \cdot 17^9 \cdot 19^5 \cdot 23^{11} \cdot 29^{17^2} \cdot 31^{11} \cdot 37^{17} \cdot 41^{13} \cdot 43^{13} \cdot 47^{13}$$

Which is approximately $2.4 \cdot 10^{743}$. These numbers get very large very quickly, and as such we will not calculate any of them explicitly. This assignment is used because it gives every formula a unique number[10] that has convenient arithmetic properties.

Now that we have a unique number for every formula we can get a unique number for every proof by simply repeating the process described above. A proof can be thought of as a sequence of formulae, so if a proof $a$ consisted of formulae that gave the sequence of Gödel numbers $a_1, \ldots a_n$ the Gödel number for the proof would be

$$\Gamma(a) = \prod_{k=1}^{n} p_k^{a_k}$$

Where $p_k$ is the $k^{\text{th}}$ prime. This assignment makes $\Gamma$ injective. If we restrict the codomain of $\Gamma$ to Gödel numbers, we get a bijective function $\Gamma : \{\text{Strings in P}\} \to \{\text{Gödel Numbers}\}$, which allows to formulate $\Gamma^{-1}$, which gives the statement for a given Gödel number.

We can now use the $\Gamma$ function to allow P to talk about itself. For example, if we consider our statement $e = \neg x_2 \forall\ \neg(x_1 \forall \neg(x_2(x_1)))$ and wish to say of it that its first symbol is "$\neg$", we could say that $2^5$ divides $\Gamma(e)$, but $2^6$ does not. This gives an arithmetic interpretation of a statement about a statement of P.

To show that something about P may be said within P it is sufficient to show that the arithmetic relation corresponding to that something is primitive recursive. In our example, this would mean that in order to be sure that P can say that "$\neg$" is the first symbol in $e$ we would show that "$2^5$ divides $x$, while $2^6$ does not" is a primitive recursive relation. Fortunately it is, and this is easily shown, but more complicated and esoteric relations are not so readily shown. Our goal will be to show that P is capable of saying "$x$ is a proof of $y$", and to do so we will show that the arithmetic relation corresponding to this statement is primitive recursive.

2.2.3. *A Primitive Recursive Series.* It can easily be shown that addition, multiplication, exponentiation, less than, and equality are primitive recursive, so we will start from these and build up a series of 45 formulae, each employing previous ones, of which the final will be an arithmetic relation between two numbers $x$ and $y$ such that, if satisfied, $\Gamma^{-1}(x)$ will be a proof of $\Gamma^{-1}(y)$.

[10] By the Fundamental Theorem of Arithmetic.

For each formula we will indicate a bound on the number of steps necessary to evaluate the formula in order to ensure it is primitive recursive, unless it is an explicit function, as in [4][11], [9], [10], etc.

**1** $x/y$ - $x$ is divisible by $y$.

$\exists z \leq x$ such that

$x = y \cdot z$

**2** Prim($x$) - $x$ is prime.

$\neg\exists z < x$ such that

$z \neq 1$

$x/z$ [1]

**3** $n$Pr $x$ - $n^{th}$ prime in $x$.[12]

0Pr $x = 0$[13]

$(n+1)$Pr $x =$ smallest $y$ such that

$n$Pr $x < y < x$

Prim($y$) [2]

$x/y$ [1]

**4** $x!$

$0! = 1$

$(n+1)! = (n+1) \cdot n!$

**5** Pr($n$) - $n^{th}$ prime.

Pr$(0) = 0$

Pr$(n+1) =$ smallest $y$ such that

Prim($y$) [2]

Pr$(n) <$ Pr$(n+1) \leq$ Pr$(n)!$ [4]

Thus far the formulae have been purely arithmetic with no relation to describing Gödel numbers. From here on out the arithmetic formulae will be designed to operate on Gödel numbers and will correspond to some description of the statements those Gödel numbers represent. As such, the corresponding description the arithmetic formulae represent will be given in place of arithmetic descriptions. The arithmetic descriptions will however, as above, be given explicitly. The reader is encouraged to go through the arithmetic definition to verify its correspondence with the stated interpretation. The [n]'s are useful references for this.

[11] We will use a [n] to designate the reference of the $\mathrm{n}^{\mathrm{th}}$ formula on this list.

[12] e.g. $168 = 2^3 \cdot 3 \cdot 7$; 1 Pr 168 = 2, 2 Pr 168 = 3, 3 Pr 168 = 7 etc.

[13] Another way to show a formula to be primitive recursive is to provide a recursive definition.

**6** $n$Trm $x$ - Gives $n^{th}$ term in $x$.[14]
smallest $y \leq x$ such that
$x/[(n\text{Pr } x)^y]$ [1, 3]
$\neg x/[(n\text{Pr } x)^{y+1}]$

**7** $l(x)$ - Length of $x$.[15]
smallest $y \leq x$ such that
$y\text{Pr } x > 0$ [3]
$(y+1)\text{Pr } x = 0$

**8** $x * y$ - Gives the concatenation of $x$ and $y$.
smallest $z \leq [\text{Pr}(l(x) + l(y))]^{x+y}$ such that [5, 7]
for all $n \leq l(x)$ [7]
$n\text{Trm } z = n\text{Trm } x$ [6]
for all $n \leq l(y)$
$[n + l(x)]\text{Trm } z = n\text{Trm } y$

**9** R($x$) - String consisting of only $x$.
$2^x$

**10** P($x$) - Puts $x$ in parentheses.
$\text{R}(11) * x * \text{R}(13)$ [8, 9]

**11** $n$Var $x$ - $x$ is a variable of $n^{th}$ type.
$\exists z \leq x$ such that
$z > 13$
Prim($z$) [2]
$x = z^n$

**12** Var($x$) - $x$ is a variable.
$\exists n \leq x$ such that
$n$Var $x$ [11]

**13** Neg($x$) - Negation of $x$.
$\text{R}(5) * \text{P}(x)$ [8, 9, 10]

**14** $x$Dis $y$ - $x$ or $y$ (disjunction of $x$ and $y$).
$\text{P}(x) * \text{R}(7) * \text{P}(y)$ [8, 9, 10]

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[14] $n^{th}$ symbol in a lone formula and $n^{th}$ formula in a proof
[15] In terms of symbols for a formula or formulae for proof

**15** $x$Gen $y$ - $x$ generalizes for $y$ (for all $x$, $y$ is true).

$\mathrm{R}(x) * \mathrm{R}(9) * \mathrm{P}(y)$ [8, 9, 10]

**16** $n\mathrm{S}x$ - $n^{th}$ successor of $x$.

$0\mathrm{S}x = x$

$(n+1)\mathrm{S}x = \mathrm{R}(3) * n\mathrm{S}x$ [8, 9]

**17** $\mathrm{N}(n)$ - symbol for the number $n$.[16]

$\mathrm{N}(n) = n\mathrm{S}[\mathrm{R}(1)]$ [9, 16]

**18** $\mathrm{Typ_1}'(x)$ - $x$ is a term of first type.[17]

$\exists m, n \leq x$ such that

$m = 1$ or 1Var $m$ [11]

$x = n\mathrm{S}[\mathrm{R}(m)]$ [9, 16]

**19** $\mathrm{Typ}_n(x)$ - $x$ is a term of $n^{th}$ type.

$n = 1$ and $\mathrm{Typ_1}'(x)$ [18]

\- or -

$n > 1$ and $\exists v \leq x$ such that

$n$Var $v$ [11]

$x = \mathrm{R}(v)$ [9]

We will pause for a moment and summarize some of the things we have found. We now have arithmetic formulae for finding any term within a string [6], for joining two strings together [8], for assessing whether a string is a variable and of what type ([12] and [11] respectively), and for placing various logical operations on strings [13, 14, 15], among other things.

While these formulae have been a sort of hodge-podge of concepts, our next four formulae will be dedicated to producing an arithmetic formula for the concept "$x$ is a formula". To do this we recognize that all formulae can be built with only four concepts - set membership, negation, disjunction, and generalization. So we will define in [20] an arithmetic formulae that tests whether a string is of the form $x_{n+1}(x_n)$, which we recall denotes $x_n \in x_{n+1}$. Then in [21] we will give a formula that tests whether the string is a negation, disjuntion, or generalization. In [22] we will test whether the string is a series of formulae[18], which we will use in [23] to isolate the last formula, which we will know must be a formula.

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[16] e.g. N(4) corresponds to "ssss0".

[17] e.g. 0, ss...s0, $x_1$, ss...s$x_1$.

[18] It may seem circular that we are testing for being a string of formulae before we test for being a formula. What is happening is that we know all formulae can be produced from the basic symbols using certain operations, so we test for whether a sequence consists of strings such that each is produced from the previous by such operations, and then if there exists such a sequence whose last term is $x$, we can know with certainty that $x$ is a formula.

**20** Set($x$) - $x$ expresses set inclusion.

$\exists y, z, n \leq x$ such that

Typ$_{\mathrm{n}}$(y) [19]

Typ$_{\mathrm{n+1}}$(z)

$x = z * \mathrm{P}(y)$ [8, 10]

**21** Log($x, y, z$) - $x$ is the negation, disjunction with $z$, or generalization of $y$.

$x = \mathrm{Neg}(y)$ [13]

-or-

$x = y\mathrm{Dis}\ z$ [14]

-or-

$\exists v \leq x$ such that

Var($v$)

$x = v\mathrm{Gen}\ y$ [15]

**22** Fs($x$) - $x$ is a formula series.

$l(x) > 0$

For all $n \leq l(x)$, $n \neq 0$

Set($n$Trm $x$) [6, 20]

-or-

$\exists p, q < n$ where $p, q \neq 0$ such that

Log($n$Trm $x$, $p$Trm $x$, $q$Trm $x$) [21]

**23** Form($x$) - $x$ is a formula.

$\exists n \leq (\mathrm{Pr}[l(x)^2])^{xl(x)^2}$ such that[19] [5, 7]

Fs($n$) [22]

x = $[l(n)]$Trm $n$ [6]

We now have a way to take the Gödel number for a string and determine whether that string is a formula using the arithmetic formula defined above.

In the next eight formulae we will develop the ability to talk about free and bound variables in formulae. Recall that a variable is bound if it is quantified, which in our case means that it is quantified with the universal quantifier. If it is not bound it is free. The importance of this is that the only formulae that can be assigned a truth value are those in which every variable is bound. In [31] we will have an arithmetic formula that corresponds to the notion of replacing all free variables with something else.

---

[19] $\mathrm{Pr}[l(x)^2])^{xl(x)^2}$ is the bound here for the following reason: the length of the shortest series of formulae that produces $x$ can at most be equal to the number of formulae that make up the formula $x$. There are at most $l(x)$ formulae of length 1, $l(x) - 1$ formulae of length 2, and so on, so at most $\frac{1}{2}l(x)[l(x)+1] \leq l(x)^2$. The prime numbers in $n$ can thus be assumed to all be less than $\mathrm{Pr}[l(x)^2]$ and the quantity of them less than $l(x)^2$ and their exponents less than $x$.

**24** $v$Bnd $n, x$ - $v$ is bound at the $n^{th}$ place in $x$.

Var($v$) [12]

Form($x$) [23]

$\exists a, b, c \leq x$ such that

$x = a * (v\text{Gen } b) * c$ [8, 15]

Form($b$)

$l(a) + 1 \leq n \leq l(a) + l(v\text{Gen } b)$ [7]

**25** $v$Fr $n, x$ - $v$ is free at the $n^{th}$ place in $x$.

Var($v$) [12]

Form($x$) [23]

$v = n\text{Trm } x$ [6]

$n \leq l(x)$ [7]

$\neg(v\text{Bnd } n, x)$ [24]

**26** $v$Free $x$ - $v$ occurs in $x$ as a free variable.

$\exists n \leq l(x)$ such that [7]

$v$Fr $n, x$ [25]

**27** $x$Su $n, y$ - Substitutes the $n^{th}$ term in $x$ with $y$.

smallest $z \leq (\text{Pr}[l(x) + l(y)])^{x+y}$ such that [5, 7]

$\exists u, v \leq x$ such that

$x = u * \text{R}(n\text{Trm } x) * v$ [6, 8, 9]

$z = u * y * v$

$n = l(u) + 1$

**28** $k$Pl $v, x$ - The $(k+1)^{th}$ place from the end of $x$ where $v$ is free.[20]

0Pl $v, x$ = smallest $n \leq l(x)$ such that [7]

$v$Fr $n, x$ [25]

$\neg\exists p \leq l(x)$ such that

$p > n$

$v$Fr $p, x$

$(k+1)$Pl $v, x = n < k$Pl $v, x$ such that

$v$Fr $n, x$

$\exists p < k$Pl $v, x$ such that

$p > n$

$v$Fr $p, x$

[20]This means $k = 0$ gives the last place $v$ is free, $k = 1$ gives the penultimate place, etc.

**29** $\mathrm{T}(v, x)$ - Total number of places in $x$ where $v$ is free.
smallest $n \leq l(x)$ such that [7]
$n\mathrm{Pl}\ v, x = 0$ [28]

**30** $x\mathrm{Sub}_k(v, y)$ - Substitutes last $k$ free $v$'s with $y$.
$x\mathrm{Sub}_0(v, y) = x$
$x\mathrm{Sub}_{\mathrm{k+1}}(v, y) = [x\mathrm{Sub}_{\mathrm{k}}(v, y)]\mathrm{Su}\ ([k\mathrm{Pl}\ v, x],\ y)$ [27, 28]

**31** $x\mathrm{Sub}(v, y)$ - Substitute all free $v$'s with $y$.
$x\mathrm{Sub}(v, y) = x\mathrm{Sub}_{\mathrm{T}(v,\, x)}(v, y)$ [29, 30]

We now have an arithmetic formula that allows us to take the Gödel numbers for a formula, a variable, and a string and get a new number, which will be the Gödel number for that same formula, except that everywhere where the variable was free in the formula it is replaced with the string. This will become very important as it will allow us to make every variable in a formula quantified, which we recall is necessary for assigning a truth value to a statement.

We will now define two formulae that will help us define the axioms arithmetically. In [32] we will actually define four formulae in one go, which will use our previously defined formulae for negation, disjunction, and generalization [13, 14, 15] to make formulae representing implication, conjunction, logical equivalence, and the existential qualifier. These will help in defining the axioms. In [33] we will define a formula that represents type-lifting, where the type of every variable in a formula is raised by the same amount. The only use of this will be to define the Axiom of Extensionality in [41].

**32** - Implication, conjunction, logical equivalence, and existence, respectively.
$x\mathrm{Imp}\ y = [\mathrm{Neg}(x)]\mathrm{Dis}\ y$ [13, 14]
$x\mathrm{Con}\ y = \mathrm{Neg}([\mathrm{Neg}(x)]\mathrm{Dis}\ [\mathrm{Neg}(y)])$
$x\mathrm{Equ}\ y = (x\mathrm{Imp}\ y)\mathrm{Con}\ (y\mathrm{Imp}\ x)$ [32]
$v\mathrm{Ex}\ y = \mathrm{Neg}(v\mathrm{Gen}\ [\mathrm{Neg}(y)])$

**33** $n\mathrm{Tl}\ x$ - $n^{th}$ type-life of $x$.
smallest $y \leq x^{(x^n)}$ such that
For all $k \leq l(x)$ [7]
$k\mathrm{Trm}\ x \leq 13$ and $k\mathrm{Trm}\ y = k\mathrm{Trm}\ x$ [6]
-or-
$k\mathrm{Trm}\ x > 13$ and $k\mathrm{Trm}\ y = (k\mathrm{Trm}\ x)\cdot(1\mathrm{Pr}\ [x\mathrm{Trm}\ x])^n$ [3]

Now that we have these formulae we will be able to define arithmetic formulae that assess whether a number is the Gödel number for an axiom. This will be accomplished in [42].

The axioms from Peano arithmetic are statements that only make use of our basic symbols[21] and as such have Gödel numbers of their own. So we will define $\alpha_1$, $\alpha_2$, and $\alpha_3$ to be the Gödel numbers for axioms I - 1, 2, and 3 respectively.

**34** $\text{Ax}_{\text{I}}(x)$ - $x$ is an axiom of Peano Arithmetic.

$x = \alpha_1$

-or-

$x = \alpha_2$

-or-

$x = \alpha_3$

The axioms of propositional logic and predicate logic and the axiom of comprehension are not written using our basic symbols, but rather are general forms of statements involving formulae and variables. So instead of seeing, as in [34], whether or not the Gödel number for a formula equals the Gödel number of an axiom, we will have to determine whether the formula is of a certain form. In [35] we will give an example of such a form for axiom II - 1, but won't do so for axioms II - 2, 3, and 4, as it will be easy to see how they are made given the formula in [35]. In [36] we will test whether a formula comes from the axioms for propositional logic.

**35** $\text{Ax}_{\text{II-1}}(x)$ - $x$ is derived from the first axiom of propositional logic.

$\exists y \leq x$ such that

Form$(y)$ [23]

$x = (y\text{Dis } y)\text{Imp } y$ [32]

**36** $\text{Ax}_{\text{II}}(x)$ - $x$ is derived from an axiom of propositional logic.

$\text{Ax}_{\text{II-1}}(x)$ [35]

-or-

$\text{Ax}_{\text{II-2}}(x)$

-or-

$\text{Ax}_{\text{II-3}}(x)$

-or-

$\text{Ax}_{\text{II-4}}(x)$

For the axioms of propositional logic we only needed to check that the things we were substituting were formulae. Predicate logic allows us to talk about variables, and as such we will need to develop formulae that allow us to test for certain constraints on variables.

The first axiom for predicate logic is $v\forall a \implies a\ \text{Sub}(v, b)$. In this $a$ can be any formula, $v$ any variable, and $b$ any variable of the same type as $v$ that does not contain a variable bound in $a$ where $v$ is free. It makes sense why $b$ must be

[21] We gave an abbreviation of them using other symbols, but they can easily be rewritten.

the same type as $v$, but what does the second qualification mean? Recall that a term of first type is something of the form 0, ss...s0, $x_1$, or ss...s$x_1$. So consider the case when $v$ is a variable of first type, $a$ is the statement "$\exists y_1(v = y_1)$" and $b$ is the variable of first type "s$y_1$". Upon substituting $v$ for $b$, as this axiom suggests we can do, we get "$\exists y_1(\mathrm{s}y_1 = y_1)$", which is plainly false. The reason for this is that $b$ contains a variable, $y_1$, that is bound in $a$, and upon substitution with $v$ the quantification of $y_1$ in $a$ applies to the thing being substituted, which it shouldn't. So in [37] we will develop a formula that tests for this condition.

**37** $\mathrm{Q}(z, y, v)$ - $z$ has no variables bound in $y$ at a position where $v$ is free.

$\exists n \leq l(y)$, $\exists m \leq l(z)$, $\exists w \leq z$ such that [7]

$w = m\mathrm{Trm}\ z$ [6]

$w\mathrm{Bnd}\ n, y$ [24]

$v\mathrm{Fr}\ n, y$ [25]

**38** $\mathrm{Ax}_{\text{III-1}}(x)$ - $x$ is derived from the first axiom of predicate logic.

$\exists v, y, z, n \leq x$ such that

$n\mathrm{Var}\ v$ [11]

$\mathrm{Typ_n(z)}$ [19]

$\mathrm{Form}(y)$ [23]

$\mathrm{Q}(z, y, v)$

$x = (v\mathrm{Gen}\ y)\mathrm{Imp}\ [y\mathrm{Sub}(v, z)]$ [15, 31, 32]

**39** $\mathrm{Ax}_{\text{III-2}}(x)$ - $x$ is derived from the second axiom of predicate logic.

$\exists v, q, p \leq x$ such that

$\mathrm{Var}(v)$ [12]

$\mathrm{Form}(p)$ [23]

$v\mathrm{Free}\ p$ [26]

$\mathrm{Form}(q)$

$x = [v\mathrm{Gen}\ (p\mathrm{Dis}\ q)]\mathrm{Imp}\ [p\mathrm{Dis}\ (v\mathrm{Gen}\ q)]$ [32]

We now have means to test whether a formula is derived from the axioms of predicate logic. We will now develop the same for the axiom of comprehension.

**40** $\mathrm{Ax_{IV}}(x)$ - $x$ is derived from the axiom of comprehension.

$\exists u, v, y, n \leq x$ such that

$n\mathrm{Var}\ v$ [11]

$n + 1\ \mathrm{Var}\ u$

$u\mathrm{Free}\ y$ [26]

$\mathrm{Form}(y)$ [23]

$x = u\mathrm{Ex}\ [v\mathrm{Gen}\ ([\mathrm{R}(u) * \mathrm{P}(\mathrm{R}(v))]\mathrm{Aeq}y)]$

The axiom of extensionality is, like the axioms for Peano Arithmetic, written in terms of our basic symbols, and so has a Gödel number, but this axiom applies to any type-lift of itself, which is when the type of every variable is raised by the same amount. Thus, we define $\alpha_4$ to be the Gödel number for the axiom of extensionality at the lowest type (see its definition above), and will test whether a formula is a type-lift of this formula (or this formula itself), in [41].

**41** $\mathrm{Ax}_{\mathrm{V}}(x)$ - $x$ is a type lift of the axiom of extensionality.

$\exists n \leq x$ such that

$x = n\mathrm{Tl}\ \alpha_4$ [33]

**42** $\mathrm{Ax}(x)$ - $x$ is derived from an axiom.

$\mathrm{Ax}_{\mathrm{I}}(x)$ [34]

-or-

$\mathrm{Ax}_{\mathrm{II}}(x)$ [36]

-or-

$\mathrm{Ax}_{\text{III-1}}(x)$ [38]

-or-

$\mathrm{Ax}_{\text{III-2}}(x)$ [39]

-or-

$\mathrm{Ax}_{\mathrm{IV}}(x)$ [40]

-or-

$\mathrm{Ax}_{\mathrm{V}}(x)$ [41]

We now have a means to test whether a formula is derived from an axiom. Recall that valid proofs are those sequences of statements that include the axioms and are closed with respect to the relation *immediate consequence of*. In [43] we will develop a means to test whether a formula is an immediate consequence of two other formulae, then in [44] we will test whether a string is closed with respect to this relation (making it a proof). In a proof of this nature what is proved is the last formula it comprises, so in [45] we will finally be able to test if $x$ is a proof of $y$ by testing whether $y$ is the last formula that constitutes the proof $x$.

**43** $x\mathrm{IC}(y, z)$ - $x$ is an immediate consequence of $y$ and $z$.

$y = z\mathrm{Imp}\ x$ [32]

-or-

$\exists v \leq x$ such that

$\mathrm{Var}(v)$ [12]

$x = v\mathrm{Gen}\ y$ [15]

**44** Prf($x$) - $x$ is a valid proof.

For all $n \leq x$,

Ax($n$Trm $x$) [6, 42]

-or-

$\exists p, q \leq n$ such that

($n$Trm $x$)IC($p$Trm $x$, $q$Trm $x$) [43]

**45** $x$Dem $y$ - $x$ is a proof of $y$ ($x$ demonstrates $y$).

Prf($x$) [44]

$[l(x)]$Trm $x = y$ [6, 7]

In $x$Dem $y$ we have a primitive recursive arithmetic formula between $x$ and $y$ that, if satisfied, means that $\Gamma^{-1}(x)$ is a proof of $\Gamma^{-1}(y)$. What this means is that there is a statement within our system P that can say whether or not two other strings constitute a valid proof within P. In the next section we will use this to create an undecidable statement.

2.2.4. *Constructing the Undecidable.* We will now construct a statement that is undecidable within P.

Consider the statement

$$\neg\exists x_1(x_1\text{Dem } y_1).$$

This is a statement claiming that there does not exist an $x_1$ such that $\Gamma^{-1}(x_1)$ is a proof of $\Gamma^{-1}(y_1)$ (see [45]), or in other words, that $\Gamma^{-1}(y_1)$ is not provable.

We now consider the statement

$$p = \neg\exists x_1(x_1\text{Dem } [y_1\text{Sub}(19, y_1)]).$$

Which states that $y_1\text{Sub}(19, y_1)$ is not provable[22]. We recall (see [31]) that $y_1\text{Sub}(19, y_1)$ represents going into $\Gamma^{-1}(y_1)$ and wherever $\Gamma^{-1}(19)$ is free substituting $y_1$. It is important to note that $y_1$ is being treated as a Gödel number in its first appearance and a "normal" number in the second, that is, we are interested in the statement $y_1$ represents in one instance, but only its numerical value in the other. Why we would want to do this may not be clear as of yet, but already we see the interplay between the arithmetic and the interpretation of the arithmetic. As of now $p$ cannot be assigned a truth value, as it has a free variable, $y_1$, but $p$ is nonetheless well-formed, and as such has a Gödel number, which we will call $n$, so that $\Gamma(p) = n$.

We now consider the statement

$$r = \neg\exists x_1(x_1\text{Dem } [n\text{Sub}(19, n)])$$

Which states that $n\text{Sub}(19, n)$ is not provable. In this case there are no free variables, and as such $p$ should have a truth value. To determine what this truth value is, we must examine $n\text{Sub}(19, n)$ more closely.

The statement $n\text{Sub}(19, n)$ represents going into $\Gamma^{-1}(n)$, and wherever $\Gamma^{-1}(19)$ is free, substituting $n$. So what is $\Gamma^{-1}(n)$? We defined $n$ such that $\Gamma(p) = n$,

[22]Strictly speaking it states that $\Gamma^{-1}(y_1\text{Sub}(19, y_1))$ is not provable, but we will do away with this notation when it is unambiguous as to whether we are talking about a statement or the Gödel number for that statement.

so $\Gamma^{-1}(n) = p$. Thus, we will go into $p = \neg\exists x_1(x_1\text{Dem }[y_1\text{Sub}(19, y_1)])$ and wherever $\Gamma^{-1}(19)$ is free replace it with $n$. We know $\Gamma^{-1}(19) = y_1$, so going into $\neg\exists x_1(x_1\text{Dem }[y_1\text{Sub}(19, y_1)])$ and replacing free instances of $y_1$ with $n$ we get $\neg\exists x_1(x_1\text{Dem }[n\text{Sub}(19, n)])$ as the meaning of $n\text{Sub}(19, n)$. But recall that $r = \neg\exists x_1(x_1\text{Dem }[n\text{Sub}(19, n)])$, so we can say that

$$r = \neg\exists x_1(x_1\text{Dem }[n\text{Sub}(19, n)]) = \neg\exists x_1(x_1\text{Dem } r)$$

Or in other words, $r$ says that $r$ is not provable.

This is a truly remarkable statement. We have uncovered a way to form a statement in P that says *of itself* that it is not provable in P. We now look to see if this is indeed the case. Is it provable? And if not, is its negation provable?

If $r$ is provable, then there exists a proof of it, so we can prove that $\exists x_1(x_1\text{Dem } r)$. But that is in fact the negation of $r$, so if $r$ is provable, then so is $\neg r$. If P is consistent then this cannot be the case.

If $\neg r$ is provable, then $\exists x_1(x_1\text{Dem } r)$ is provable (this being the negation of $r$), so there exists a proof of $r$. So if $\neg r$ is provable then $r$ is provable. Again, this cannot be the case if P is consistent.

We can thus conclude that if P is consistent then neither $r$ nor $\neg r$ is provable, meaning P is incomplete. This concludes the proof of Gödel's first incompleteness theorem.

2.2.5. *A Problem with Consistency.* We have shown that if P is consistent then it is incomplete. More specifically, we have shown that consistency implies $r$, as $r$ is the statement that $r$ cannot be proven, which is indeed the case if P is consistent. We can represent this relation succinctly as

$$\text{C} \Rightarrow r$$

Where C is the statement claiming the consistency of P.

We then come to another remarkable conclusion. Because $r$ follows from the consistency of P, if there exists a proof of the consistency of P within P then $r$ would be provable. But we have shown above that there cannot exist a proof of $r$, so likewise there cannot exist a proof of the consistency of P within P. So not only is P incomplete, but it is incapable of proving its own consistency. This is Gödel's Second Incompleteness Theorem. This fact contrasts arithmetic systems such as propositional and predicate logic, both of which are capable of an internal proof of consistency. The consistency of P can, however, be proven by auxiliary means, and indeed Gentzen proved its consistency using transfinite induction in 1936, but any such proof will necessarily rely on assumptions that go beyond those of arithmetic, so if the consistency of arithmetic is in doubt, a system capable of proving its consistency must share similar doubts.

## 3. Closing Thoughts

Since Gödel's proof there have been more "natural" statements shown to be unprovable in Peano Arithmetic. The first we just mentioned. Gentzen proved the the consistency of arithmetic using transfinite induction, and so by Gödel's Second Incompleteness Theorem arithmetic is incapable of proving transfinite induction.

The second was the Strengthened Finite Ramsey Theorem (SFRT), which states the following:

> For all $n, k, m \in \mathbb{Z}^+$ there exists an $N$ such that if we color each of the $n$-element subsets of $\mathrm{S} = \{1, 2, 3, \ldots, N\}$ with one of $k$ colors, then we can find a $\mathrm{Y} \subseteq \mathrm{S}$ with at least $m$ elements, such that all $n$-element subsets of Y have the same color, and the number of elements of Y is at least the smallest element of Y.

The fact that this is unprovable is known as the Paris-Harrington Theorem, published in 1972. The Paris-Harrington Theorem showed that the SFRT implies the consistency of arithmetic, and so by Gödel's Second Incompleteness Theorem cannot be provable within arithmetic.

The third is Goodstein's Theorem, which states that all Goodstein sequences terminate at 0. Its unprovability was also established in part by Paris. This was done by showing Goodstein's Theorem to be equivalent to transfinite induction, already shown to be unprovable by Gentzen (albeit indirectly).

What all these statements have in common is that they rely on Gödel's results to establish unprovability. It has been conjectured by Douglas Hofstadter (a cognitive scientist, not a mathematician) that all unprovable statements are unprovable due to such a connection with Gödel. Such a conjecture is undoubtedly too vague to be provable, but thus far it seems to be, at the very least, not even wrong.

3.1. **Human Reasoning.** Chesterton (certainly not a mathematician) once wrote, "Thoroughly worldly people never understand even the world". As the world goes, so goes mathematics (or perhaps it's the other way 'round); any engagement with a system of mathematics strictly within its formal limits necessarily restrains the understanding of that system. Gödel's result applies not only to mathematics, but to philosophy, computer science, and the nature of reasoning itself.

Gödel's result placed limits on our ability to ascertain the truth, yet at the same time demonstrated our ability to abstract and look beyond a given system. Since the Incompleteness Theorems were published people have wondered as to their relevance to human reasoning. A certain hypothesis is that it has no relevance, as the way people reason is not entirely consistent. But this is not a very satisfactory answer given that consistent reasoning is the only kind of reasoning about which anyone cares, and Gödel's result still very much applies there.

What Gödel's result hinges upon is the ability to "step outside" the system in which we would normally be working to gain a higher perspective. In his theorems, Gödel steps out of arithmetic so as to garner the ability to assume its consistency, an assumption to which we would normally not have access. Philosopher J.R. Lucas has posited that this option is always available to us, that we can always "step out" and view things from an outside perspective. He writes, "Gödel's theorem seems to me to prove that mechanism is false, that minds cannot be explained by machines". His reasoning is that machines always operate upon what is essentially a formal system, and if that system is capable of arithmetic (which it had better be if it wants to be useful in any capacity), then there must exist certain truths which will perpetually escape the machine's "intelligence". This, however, needn't be so for human beings (or so Lucas claims). His thought is that we, humans, are always capable of performing the same sort of reasoning Gödel employed, of stepping out of the system.

This sort of argument is not capable of mathematical proof or disproof, as it is too ill-defined to speak of in a rigorous way, but it can certainly be questioned as to whether or not we are always capable of removing ourselves from a given system. It took a great deal of ingenuity for Gödel to do what he did, and it doesn't take much curiosity to wonder how far our intellect may carry us before the complexity of a system is simply too overwhelming. Gödel's proof is still only completely understood by a handful of experts, and yet Gödel's result is for arithmetic, perhaps the most basic mathematical system possible, and certainly the most widely known. If it takes someone like Gödel to step out of a system of arithmetic, it is certainly questionable as to whether or not men are capable of stepping outside of every conceivable system.

3.2. **Philosphical Import.** Gödel's result is easily the most consequential mathematical result to mathematical philosophy, and perhaps the only time in history a philosophical position has been thoroughly disproven.

We have mentioned logicism before. This position, champoined by Russell, presumed that mathematics was, for lack of a better word, perfect. The view of logicists was that only the most basic assumptions of logic were necessary for a complete understanding of mathematics. In *Principles of Mathematics* Russel writes that "The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself". Russell went through great lengths to show this. In *Principia Mathematica* he makes very few assumptions, and those that he does make are almost entirely unobjectionable. Despite the paucity of assumptions, however, he was able to show that any primitive recursive truth was capable of being shown within his system, which grants it an enormous amount of power. Nevertheless, however, Gödel showed that no amount of assumptions will ever be sufficient to capture the truths of arithmetic.

While Russell's philosophy was essentially foreclosed, the relationship between Gödel's result and Hilbert's philosophy, formalism, is more complicated. Most of this complication results, however, from the ambiguity of certain philosophical positions, as is often the case in philosophy. If we consider formalism to be logicism but amended with extra axioms, then certainly it is false, as no amount of axioms will ever be sufficient to make arithmetic complete. If, however, formalism is made as general as possible to say simply that mathematics as a discipline is the practice of declaring assumptions and deriving conclusions, then Gödel's result has almost no relevance. At that point it comes down to what mathematics is.

3.3. **What is Mathematics?** If we are not satisfied with the quip that "mathematics is what mathematicians do", then we would be in want of a more precise definition, or at least a more restrictive conception, of what mathematics is. There are two aspects to any discipline: the object of its study and the means by which that study occurs. Certainly the means by which mathematical objects are studied is through deduction, but what are mathematical objects themselves?

The main divide among mathematical philosophers is whether or not mathematical objects are created are discovered. Intuitionists believe that math is a result of human cognition, that objects in math are discovered only insofar as they are unearthed from their source in the mind of man. John Fraleigh and Raymond Beauregard write that "Numbers exist only in our minds. There is no physical entity that is number 1. If there were, 1 would be in a place of honor in some great museum of science, and past it would file a steady stream of mathematicians gazing at 1 in wonder and awe". Formalists see mathematical objects as arising only from the constructs of certain sets of axioms. Logicists would see the objects of study in math as being necessary consequences of the rules of logic.

But what of those that see mathematical objects as discovered entities? A certain philosophy that embraces this notion is Platonism, so called as it originated with Plato. This was Gödel's philosophy, and it holds that mathematical objects are real things, just like cars and bricks and chairs. As Gödel put it,

> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions.

The only difference between the reality of these sorts of objects with the reality of objects our senses perceive is the domain in which existence obtains and the means by which truths relating objects come about. Plato viewed mathematical objects as existing in a world "higher" than our own; roughly speaking, that there exists the world in which we live where there are objects that are perceptible but unintelligible, and there exists the world in which mathematical objects exist, the world of forms, where objects are imperceptible but intelligible. Hungarian mathematician John von Neumann gave a speech at a computing conference in which he claimed that "If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is". Platonism claims this is precisely because those objects with which we interact in life are unintelligible, and in the world of forms, "the higher beings are connected to the others by analogy, not by composition", as Gödel would say, and thus these higher beings are much more well-behaved.

This conception of mathematics is certainly the most natural, evidenced by that fact that it was held almost universally by mathematicians for thousands of years. Certainly for children learning arithmetic the conception of numbers is that they are real in a sense, and this conception persists for triangles, curves, sets, and all sorts of other things learned within early mathematical education.

The instantiation of higher forms in our world was the motivation of mathematics from its start. While formalism holds that mathematical truths follow from axioms, Platonists would say the opposite is the case; axioms are merely means to describe the realities of what mathematical objects are. Mathematician Richard Hamming sums up this view when he writes

> The idea that theorems follow from the postulates does not correspond to simple observation. If the Pythagorean theorem were found to not follow from the postulates, we would again search for a way to alter the postulates until it was true. Euclid's postulates came from the Pythagorean theorem, not the other way around.

While in a strict formalist sense primitive terms are considered undefined, in the construction and application of axiomatic systems this is plainly not the case. In geometry, the notions of a point, a line, and a plane are “undefined”, yet our conception of those things serves as the means by which we define the axioms which purport to govern them. There is a sense in which a point *really is* that which has no parts.

Consider the means by which Gödel proved his results. He took for granted facts that any system of arithmetic must have (e.g. the Fundamental Theorem of Arithmetic, the infinitude of primes), assumed the consistency of the system, and proved that the system was insufficient to capture everything the system in fact implied. This, at the very least, supports the proposition that arithmetic really is something out there with an objective reality, that we can only attempt to capture with a set of axioms. Gödel himself viewed his conception of mathematical reality as indispensable to the creation of his proof.

Ultimately what Gödel did raises more questions than it answers, and raises questions that will probably never be answered. But as Plato said, “I am the wisest man alive, for I know one thing, and that is that I know nothing”. And thanks to Gödel, we all at least know one thing.

## 4. Bibliography


Carl Boyer, *A History of Mathematics*, Princeton University Press, 1985.

Henry Ketcham, *The Life of Abraham Lincoln*, AL Burt, 1901.

Abraham Lincoln, *Collected Works of Abraham Lincoln Vol. 2*, Wildside Press LLC, Springfield, Ill, 2008.

Proclus, *A Commentary on the First Book of Euclid's Elements*, Princeton University Press, 1992.

Alexander Bogomolny, *Attempts to Prove Euclid's Fifth Postulate*, Interactive Mathematics Miscellany and Puzzles, 2018.

Peter Wolff, *Breakthroughs in Mathematics*, New American Library, New York, NY, 1970.

Galileo Galilei, *Dialogue Concerning the Two Chief World Systems*, University of California Press, 1997.

Laurence Sterne, *The Life and Opinions of Tristram Shandy, Gentleman*, Oxford University Press, New York, NY, 1983.

Morris Kline, *Mathematics and Western Culture*, Oxford University Press, 1953.

Georg Cantor, *Contributions in Support of a Transfinite Set Theory*, Dover Publications, 1915.

Georg Cantor, *On a Property of the Collection of all Real Algebraic Numbers*, 1874.

Gottlob Frege, Michael Beaney, *The Frege Reader*, Wiley, 1997.

Bertrand Russell, Alfred North Whitehead, *Principia Mathematica*, Cambridge University Press, 1910.

Kurt Gödel, *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, Basic Books, 1962.

G.K. Chesterton, *Orthodoxy*, Word on Fire Catholic Ministries, 2017.

Douglas Hofstadter, *Gödel, Escher, Bach*, Basic Books, 1979.