On integration and asymptotic domination

January 4, 2017

1 Original problem statement

> *Supposons une fonction f :R+->R continue telle que f(x)=a+o(1) avec a réel.*
>
> *Peut-on affirmer que F(x)/x, avec F primitive de f, tend vers a quand x tend vers +inf ?*
>
> *(je n'ai pas la réponse, mais mon intuition est que oui)*
> — (@LHomme_Qui_Rit)

In other words

Proposition 1.1
Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function such that $f \underset{+\infty}{\longrightarrow} a \in \mathbb{R}$ and F a primitive of f.
Then

$$\frac{F(x)}{x} \xrightarrow[x \to +\infty]{} a$$

2 Proof of the original conjecture

Let us begin by proving the following corollary, which is the particular case for $a = 0$.

Corollary 2.1
Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function, such that $f \underset{+\infty}{\longrightarrow} 0$ and F a primitive of f.
Then

$$\frac{F(x)}{x} \xrightarrow[x \to +\infty]{} 0$$

Proof of corollary 2.1
Let $x \in \mathbb{R}_*^+$ (since we only care about the limit in $+\infty$ anyway), then

$$\frac{F(x)}{x} = \frac{1}{x}\int_{\alpha}^{x} f(t)\,dt$$

for some $\alpha \in \mathbb{R}$.

Let $\varepsilon > 0$, since $f \xrightarrow[+\infty]{} 0$ there exists $A \in \mathbb{R}$ (for convenience, let us take it positive, too) such that for all $x \geqslant A$, $|f(x)| \leqslant \varepsilon$. Thus

$$\begin{aligned}
\left|\frac{F(x)}{x}\right| &\leqslant \frac{1}{x}\int_{\alpha}^{x} |f(t)|\,dt \\
&\leqslant \frac{1}{x}\underbrace{\int_{\alpha}^{A} |f(t)|\,dt}_{\leqslant |A-\alpha| \sup_{[\alpha,A]}|f|} + \frac{1}{x}\int_{A}^{x} \underbrace{|f(t)|}_{\leqslant\varepsilon}\,dt \\
&\leqslant \underbrace{\frac{|A-\alpha| \sup_{[\alpha,A]}|f|}{x}}_{\xrightarrow[x\to+\infty]{} 0} + \underbrace{\frac{|x-A|}{x}}_{\xrightarrow[x\to+\infty]{} 1}\varepsilon \xrightarrow[x\to+\infty]{} \varepsilon
\end{aligned}$$

Thus, for all $\varepsilon > 0$, $\lim_{x\to+\infty}\left|\frac{F(x)}{x}\right| \leqslant \varepsilon$, and so

$$\frac{F(x)}{x} \xrightarrow[x\to+\infty]{} 0$$

□

We can then easily prove proposition 1.1

Proof of proposition 1.1

Let $f = a + w$, with $w : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\lim_{+\infty} w = 0$. If F is a primitive of f, we then have for all x

$$F(x) = ax + W(x) + C$$

with $W(x) = \int_0^x w(t)\,dt$ and $C \in \mathbb{R}$.

So, by corollary 2.1

$$\begin{aligned}
\frac{F(x)}{x} &= a + \underbrace{\frac{W(x)}{x}}_{\xrightarrow[x\to\infty]{} 0} + \frac{C}{x} \\
&\xrightarrow[x\to+\infty]{} a
\end{aligned}$$

□

3 Further results

The previous section proved the original conjecture in its restated form. But let us recall the original statement (emphasis mine) "*Supposons une fonction f :R⁺->R continue telle que **f(x)=a+o(1)** avec a réel*". Indeed, if we keep this formulation, corollary 2.1 becomes

Corollary 3.1
Let $f = \underset{+\infty}{o}(1)$ and F a primitive of f. Then $\mathrm{F} = \underset{+\infty}{o}(x)$.

Which begs the question : "What of $f = o(x)$?".