

On integration and asymptotic domination

January 4, 2017

1 Problem statement

“Supposons une fonction $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ continue telle que $f(x) = a + o(1)$ avec a réel.

Peut-on affirmer que $F(x)/x$, avec F primitive de f , tend vers a quand x tend vers $+\infty$?

(je n'ai pas la réponse, mais mon intuition est que oui)

— (@LHomme_Qui_Rit)

In other words

Proposition 1.1

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function, such that $f \xrightarrow{+\infty} a \in \mathbb{R}$ and F a primitive of f .

Then

$$\frac{F(x)}{x} \xrightarrow{x \rightarrow +\infty} a$$

2 Proof

Let us begin by proving the following corollary, which is the particular case for $a = 0$ and $F(0) = 0$.

Corollary 2.1

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function, such that $f \xrightarrow{+\infty} 0$. Let

$$F : x \mapsto \int_0^x f(t) dt$$

Then

$$\frac{F(x)}{x} \xrightarrow{x \rightarrow +\infty} 0$$

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Proof. Let $x \in \mathbb{R}$, then

$$\frac{F(x)}{x} = \frac{1}{x} \int_0^x f(t) dt$$

Let $\varepsilon > 0$, since $f \xrightarrow{+\infty} 0$ there exists $A \in \mathbb{R}$ (for convenience, let us take it positive, too) such that for all $x \geq A$, $|f(x)| \leq \varepsilon$. Thus

$$\begin{aligned} \frac{F(x)}{x} &= \frac{1}{x} \underbrace{\int_0^A f(t) dt}_{\leq A \sup_{[0,A]} f} + \frac{1}{x} \int_A^x \underbrace{f(t)}_{\leq \varepsilon} dt \\ &\leq \underbrace{\frac{A \sup_{[0,A]} f}{x}}_{\xrightarrow{x \rightarrow +\infty} 0} + \underbrace{\frac{|x-A|}{x}}_{\xrightarrow{x \rightarrow +\infty} 1} \varepsilon \xrightarrow{x \rightarrow +\infty} \varepsilon \end{aligned}$$

Thus, for all $\varepsilon > 0$, $\lim_{x \rightarrow +\infty} \frac{F(x)}{x} \leq \varepsilon$, and so

$$\frac{F(x)}{x} \xrightarrow{x \rightarrow +\infty} 0$$

□

We can then easily prove proposition 1.1

proposition 1.1. Let $f = a + w$, with $w : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\lim_{+\infty} w = 0$. If F is a primitive of f , we then have for all x

$$F(x) = ax + W(x) + C$$

with $W(x) = \int_0^x w(t) dt$ and $C \in \mathbb{R}$.

So, by corollary 2.1

$$\begin{aligned} \frac{F(x)}{x} &= a + \underbrace{\frac{W(x)}{x}}_{\xrightarrow{x \rightarrow \infty} 0} + \frac{C}{x} \\ &\xrightarrow{x \rightarrow +\infty} a \end{aligned}$$

□