On integration and asymptotic domination

January 4, 2017

1 Problem statement

> *Supposons une fonction f:R⁺->R continue telle que f(x)=a+o(1) avec a réel.*
>
> *Peut-on affirmer que F(x)/x, avec F primitive de f, tend vers a quand x tend vers +inf ?*
>
> *(je n'ai pas la réponse, mais mon intuition est que oui)*
> — (@LHomme_Qui_Rit)

In other words

Proposition 1.1
Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function, such that $f \xrightarrow[+\infty]{} a \in \mathbb{R}$ and F a primitive of f.

Then

$$\frac{\mathrm{F}(x)}{x} \xrightarrow[x\to+\infty]{} a$$

2 Proof

Let us begin by proving the following corollary, which is the particular case for $a = 0$ and $\mathrm{F}(0) = 0$.

Corollary 2.1
Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function, such that $f \xrightarrow[+\infty]{} 0$. Let

$$\mathrm{F} : x \longmapsto \int_0^x f(t)\,\mathrm{d}t$$

Then

$$\frac{\mathrm{F}(x)}{x} \xrightarrow[x\to+\infty]{} 0$$

Proof of corollary 2.1
Let $x \in \mathbb{R}_*^+$ (since we only care about the limit in $+\infty$ anyway), then

$$\frac{F(x)}{x} = \frac{1}{x} \int_0^x f(t)\,dt$$

Let $\varepsilon > 0$, since $f \xrightarrow[+\infty]{} 0$ there exists $A \in \mathbb{R}$ (for convenience, let us take it positive, too) such that for all $x \geqslant A$, $|f(x)| \leqslant \varepsilon$. Thus

$$\begin{aligned}
\left|\frac{F(x)}{x}\right| &\leqslant \frac{1}{x}\int_0^x |f(t)|\,dt \\
&\leqslant \frac{1}{x}\underbrace{\int_0^A |f(t)|\,dt}_{\leqslant A\sup_{[0,A]}|f|} + \frac{1}{x}\int_A^x \underbrace{|f(t)|}_{\leqslant \varepsilon}\,dt \\
&\leqslant \underbrace{\frac{A\sup_{[0,A]}|f|}{x}}_{\xrightarrow[x\to+\infty]{}0} + \underbrace{\frac{|x-A|}{x}}_{\xrightarrow[x\to+\infty]{}1}\varepsilon \xrightarrow[x\to+\infty]{} \varepsilon
\end{aligned}$$

Thus, for all $\varepsilon > 0$, $\lim_{x\to+\infty}\left|\frac{F(x)}{x}\right| \leqslant \varepsilon$, and so

$$\frac{F(x)}{x} \xrightarrow[x\to+\infty]{} 0$$

□

We can then easily prove proposition 1.1

Proof of proposition 1.1
Let $f = a + w$, with $w : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\lim_{+\infty} w = 0$. If F is a primitive of f, we then have for all x

$$F(x) = ax + W(x) + C$$

with $W(x) = \int_0^x w(t)\,dt$ and $C \in \mathbb{R}$.

So, by corollary 2.1

$$\begin{aligned}
\frac{F(x)}{x} &= a + \underbrace{\frac{W(x)}{x}}_{\xrightarrow[x\to\infty]{}0} + \frac{C}{x} \\
&\xrightarrow[x\to+\infty]{} a
\end{aligned}$$

□