Final Proofs for MATH450 Real Analysis

Quinn Gates, Evan Walker

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Problem 1.

Theorem 1 (3.24 Bolzano-Weierstrass)**.** *If a bounded set $S \subseteq \mathbb{R}^n$ contains infinitely many points, then there is at least one point in $\mathbb{R}^n$ which is an accumulation point of S.*

Proof. (Case $n = 1$, $\mathbb{R}^1$) Let $S \subset \mathbb{R}^1$ and S bounded, $\Rightarrow$ since S is bounded, S lies completely within an interval $[a, b]$. Since S contains infinitely many points,

$$[a, \frac{a+b}{2}] \quad \text{and} \quad [\frac{a+b}{2}, b]$$

may either both contain infinitely many points or only one contains infinitely many points. Now define $[a_1, b_1]$ to be one of the previously stated intervals with infinitely many points. Then

$$[a_1, \frac{a_1+b_1}{2}] \quad \text{and} \quad [\frac{a_1+b_1}{2}, b_1]$$

again may either both contain infinitely many points or only one contains infinitely many points. Repeating this process k times, we arrive at an interval $[a_k, b_k]$ such that

$$sup(a_k) = inf(b_k) = x$$

where $x \in S$. We aim to show that x is an accumulation point of S. Consider k to be large enough such that $b_k - a_k < \frac{r}{2}$, $\Rightarrow [a_k, b_k] \subseteq B(x, r)$. Since, by construction, $B(x, r)$ contains infinitely many points, $(B(x, r) - \{x\}) \cap S \neq \emptyset$ and hence x is an accumulation point of S.

(Case $n = 2$, $\mathbb{R}^2$) Let $S \subset \mathbb{R}^2$ and S bounded, $\Rightarrow$ since S is bounded, S lies completely within a 2-dimensional interval J defined by the cartesian product of two 1-dimensional intervals.

$$J_1 = [a, b] \times [a, b]$$

similar to the case for $n = 1$ above, we continually divide the interval into halfs, choosing the interval with infinitely many points. Let $I_1^{(1)} = [a, b]$, and $I_2^{(1)} = [a, b]$, Then let the next division of the interval be $I_1^{(2)} = [a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b] = [a_1, b_1]$, and $I_2^{(2)} = [a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b] =$

$[a_2, b_2]$, Again choosing whichever interval has infinitely many points. If they both contain infinitely many points, our choice is arbitrary. Hence, define $J_n = I_1^n \times I_2^n$. For a large enough n, we have that J_n has the property that

$$sup(a_1^{(n)}) = inf(b_1^{(n)}) = x_1 \quad \text{and} \quad sup(a_2^{(n)}) = inf(b_2^{(n)}) = x_2$$

We assert that $x = (x_1, x_2)$ is an accumulation point of S. Note that $b_k^{(n)} - a_k^{(n)} = \frac{a}{2^{n-2}}$ for $k = 1, 2$. take a ball in $\mathbb{R}^2$, $B(x, r)$ and let n be large enough such that $\frac{a}{2^{n-2}} < \frac{r}{2}$. Hence $J_n \subseteq B(x, r)$, $\Rightarrow$ $B(x, r)$ also contains infinitely many points and we have that,

$$(B(x, r) - \{x\}) \cap S \neq \emptyset$$

This proves that x is an accumulation point of S.

$\square$

Applications of 3.24

Theorem 2 (3.25 The Cantor Intersection Theorem)**.** *Let $\{Q_1, Q_2, ...\}$ be a countable collection of nonempty sets in $\mathbb{R}^2$ such that.*

i) $Q_{k+1} \subseteq Q_k \qquad (k = 1, 2, 3, ...)$

ii) *Each set Q_k is closed and Q_1 is bounded*

$\Longrightarrow \quad \cap_{k=1}^{\infty} Q_k \neq \emptyset$ *and is closed.*

Proof. (Case 1: $Q_m = Q_k \quad \forall m \geq k$)
$\Rightarrow \quad Q_{k+2} \subseteq Q_{k+1} \subseteq Q_k \subseteq ...Q_2 \subseteq Q_1 \Rightarrow \cap_{n=1}^{\infty} Q_n = Q_k \neq \emptyset$
(Case 2: $Q_{k+1} \nsubseteq Q_k$)
$\Rightarrow$ we can build $A = \{x_1, x_2, x_3, ..., x_k, ...\}$, an infinite set. where $x_k \in Q_k \setminus Q_{k+1} \quad x_k \in Q_k$.
$\Rightarrow$ (by the Bolzano-Weierstrass Theorem) $\exists x \in \mathbb{R}^2$ such that x is an accumulation point of A. We want to show $x \in \cap_{k=1}^{\infty} Q_k$. since x is an accumulation point of A, x is also an accumulation point for $Q_k \quad \forall k$. Since Q_k is closed, it contains all of its accumulation points $\Rightarrow x \in Q_k$ and therefore $x \in \cap_{k=1}^{\infty} Q_k$.

$\square$

Problem 2.

Theorem 3 (3.27)**.** *Let $G = \{A_1, A_2, ...\}$ denote the countable collection of all n-balls having rational radii and centers at points with rational coordinates. Assume $x \in \mathbb{R}^n$ and let S be an open set in $\mathbb{R}^n$ and $x \in S$. Then $x \in A_k \subseteq S$ for some k. That is, x is contained by some n-ball in G which is contained by S.*

Proof. Theorem 2.27 states that if G is a countable collection of countable sets, then

$$\cup_{k=1}^{n} A_k \qquad A_k \in G$$

is also countable. Since $x \in S$, and S is open, we know that there exists an n-ball such that

$$B(x, r) \subseteq S$$

We want to find a rational point $y \in S$ such that $x \in B(y, R) \in G$, or equally stated by $B(y, R) \subseteq B(x, r) \subseteq S$. Let $x = (x_1, x_2, ..., x_n)$ and let y_k be a rational number such that

$$|y_k - x_k| < \frac{r}{4n} \qquad \text{for each } k = 1, 2, 3, ..., n$$

Then

$$\|y - x\| \leq \mid y_1 - x_1 \mid + ... + \mid y_n - x_n \mid < \frac{r}{4}$$

Let us define R as a rational number such that $\frac{r}{4} < R < \frac{r}{2}$. Then $x \in B(y, R)$ since we found the distance between y and x to be less than $r/4$, so surely x must be contained in $B(y, R)$. By the same reasoning, we have that $B(y, R) \subseteq B(x, r) \subseteq S$. since both y and R are rational by construction, we have that $B(y, R) \in G$. Hence, we've proved the assertion. □

Applications of 3.27

Theorem 4 (3.28 Lindeloff Covering Theorem)**.** *Assume $A \subseteq \mathbb{R}^n$ and let f be an open covering of A. Then: $\exists$ a countable subcollection of f that also covers A.*

$$A \subseteq \cup_{S \in f'} S$$

Proof. Let $G = \{A_1, A_2, \ldots, A_k \ldots\}$ countable subset of $B_n(y, q)$ where y and q are both rational numbers. Then take $x \in A \quad \Rightarrow \quad \exists S \in f$ such that $x \in S \subset \mathbb{R}^n$ Where S is an open subset. Therefore, we can use Theorem 3.27 to say:

$$\exists A_k \in G \text{ such that } x \in A_k \subseteq S$$

Define $m = m(x) = min\{k|x \in A_k, A_k \in G\}$

$$G' = \{A_{m(x)}|x \in A\} \iff f' = \{S|A_{m(x)} \in S\}$$

Both G' and f' are countable so:

$$\Rightarrow \quad A \subseteq \cup_{S \in f'} S$$

□

Problem 3

Theorem 5 (3.29 Heine-Borel)**.** *Let F be an open covering of a closed and bounded set $S \subseteq \mathbb{R}^n$. Then a finite subcollection of F also covers S.*

Proof. Assume that F is an open covering of the set S. A countable subcollection of F, say $\{I_1, I_2, \ldots\}$ covers S. Now we consider the finite union:

$$S_m = \bigcup_{k=1}^{m} I_k$$

This set is open since it is the union of open sets. We shall show that for some value of m the union S_m covers S.

Consider the complement $R^n \backslash S_m$, which is closed. Now define a countable collection of sets $\{Q_1, Q_2, \ldots\}$ as follows: $Q_1 = S$ and for $m > 1$,

$$Q_m = S \cap (R^n \backslash S_m)$$

Then Q_m consists of all those points of S that are outside of S_m. Now we just need to show that there is some m for which Q_m is empty which will show that for this m no point of S lies outside of S_m. Consider the following:

Each set Q_m is closed since it is the intersection of closed S and closed $R^n \backslash S_m$

The sets Q_m are decreasing since the S_m are increasing. That is: $Q_{m+1} \subseteq Q_m$.

The sets Q_m being subsets of S are all bounded.

Therefore, if no set Q_m is empty we can use the $Cantor Intersection Theorem$ to conclude that the intersection:

$$\bigcap_{k=1}^{\infty} Q_k \neq \emptyset$$

That means that there is some point in S which is in all the sets Q_m or outside all of the sets S_m. But this is a contradiction since $S \subseteq \bigcup_{k=1}^{\infty} S_k$. Therefore some Q_m must be empty. □

Applications of 3.29

The Heine-Borel Theorem always applies when in a Euclidean space R^n, but if an extra condition is added then it will apply in any metric space. In a metric space $M = (S, d)$ M must be both complete and totally bounded. This means that M must have the property that every Cauchy sequence in M converges in M, and it must be possible to cover M with a finite number of balls with of fixed size. So having your space be closed and bounded takes care of half but for the theorem to hold in a metric space the hypothesis must also require that the metric space be complete.

Problem 4

Theorem 6 (3.31)**.** *Let $S \subseteq \mathbb{R}^n$. Then the following statements are equivalent*

a) *S is compact.*

b) *S is closed and bounded.*

c) *Every infinite subset of S has an accumulation point in S.*

Proof. To prove equivalence, we will prove that a implies b, that b implies c, that c implies b, b implies a by $Thm3.29$, the $Heine-Borel$ theorem. Once these parts are proven, we will have equivalence of the statements.

$a \implies b$

Assume that S is compact. First we will prove that S is bounded. For some $p \in S \; \exists \; T :=$ a collection of n-balls: $B(p, k), k = 1, 2, 3 \ldots$ such that T is an open covering of the space S. Since S is compact, there is a finite subset of that collection that is still an open covering of S. Therefore, S is bounded. :)

Next we must prove that S is closed. This will be done through contradiction so first assume that S is not closed. Then $\exists y \notin S$ where y is an accumulation point of S. Now lets say $\exists x \in S$ and let $r_x = \frac{\|x-y\|}{2}$ (half of the distance between x and y) We know that each radius r_x is positive since $y \notin S$, and the collection fo balls $B(x, r_x)$ is an open covering of S. Since we know that S is compact, there is a finite number, n of these balls that still covers S which is basically the union of all of the balls.

$$S \subset \bigcup_{k=1}^{n} B(x_k, r_k)$$

We have a finite number of radii since we have a finite number of balls so lets say r is the smallest radius. Now we want to prove that $B(y,r)$ has no points in common with any of the other balls that cover $S\left(B\left(x_k, r_k\right)\right)$ So if we assume $x \in B(y,r)$ then $\|x-y\| < r < r_k$. So by the triangle inequality, we have:

$$\|y - x_k\| \leq \|y - x\| + \|x - x_k\|$$

$$\Rightarrow \|x - x_k\| \geq \|y - x_k\| - \|x - y\| = 2r_k - \|x - y\| > r_k$$

$$\Rightarrow x \notin B(x_k, r_k) \Rightarrow B(y,r) \cap S = \emptyset$$

$\Rightarrow\Leftarrow$ Contradicts the fact that y is an accumulation point This shows that S is therefore closed and $a \implies b$ ✓

$b \implies a$ by the $Heine - BorelTheorem$

Proved in number 3

$b \implies c$

Assume that b holds. This will be used to prove c: Every infinite subset of S has an accumulation point in S. If T is an infinite subset of S, since S is bounded, T must also be bounded because T is a subset of S. By the $BolzanoWeierstrassTheorem$, if a bounded set in R^n contains infinitely many points, then there is at least one point in R^n which is an accumulation point of S. Lets name that accumulation in T, x. Since S is closed, we know that the accumulation point of T is also the accumulation point of S. $\Rightarrow b \implies c$ ✓

$c \implies b$

Assume that c holds. $\rightarrow$ every infinite subset of S has an accumulation point in S. We want to prove b which says that S is closed and bounded. First, we are going to say that S is unbounded, and prove by contradiction. So assume that S is unbounded $\Rightarrow \forall m > 0 \exists x_m \in S$ such that $\|x_n\| > m$ There is a collection of points $T := \{x_1, x_2, \ldots\}$ that is an infinite subset of S, and since we are assuming (c), T has an accumulation point $y \in S$. For $m > 1 + \|y\|$ we have:

$$\|x_m - y\| \geq \|x_m\| - \|y\| > m - \|y\| > 1$$

This contradicts the fact that y is an accumulation point of T. Therefore, S is bounded. :)

Next we have to prove that S is closed. Let x be an accumulation point of S. Since every set containing a ball with center x contains infinitely many points of S, we can consider $B(x, \frac{1}{k})$, where $k = 1, 2, 3, \ldots$, and get a countable set of distinct points, $T := \{x_1, x_2, \ldots\} \subset$

S, such that $x_k \in B(x, \frac{1}{k})$ The point x is also an accumulation point of T since $T \subset S$ Since T is an infinite subset of S, (c) tells us that T must have an accumulation point in S Next, we have to prove that x is the only accumulation point of T. Suppose that $y \neq x$. By the triangle inequality we have:

$$\|y - x\| \leq \|y - x_k\| + \|x_k - x\| < \|y - x_k\| + \frac{1}{k} \mid x_k \in T$$

If k_0 is taken to be $k_0 = \lfloor \frac{2}{\|y-x\|} \rfloor \Rightarrow \frac{1}{k} < \frac{1}{2}\|y - x\|$ whenever $k \geq k_0$, the last inequality leads to $\frac{1}{2}\|y - x\| < \|y - x_k\| \implies x_k \notin B(y, r)$ when $k \geq k_0$ if $r = \frac{1}{2}\|y - x\| \implies y$ is not an accumulation point of T. ✓ *QED*

Theorem 3.31 does not hold in a general metric space. This is because the requirements for a metric space to be compact are more rigorous than the requirements for a euclidean space. In a metric space $M = (S, d)$ M must be both complete and totally bounded. This means that M must have the property that every Cauchy sequence in M converges in M, and it must be possible to cover M with a finite number of balls with of fixed size. For this reason, we still have that $a \implies b$ but the theorem will fail when we try to prove that $b \implies a$.

Consider the metric space $R^2 \backslash (0, 0)$ with the usual metric from R^2. The set

$$D = \{(x, y) \mid 0 < x^2 + y^2 \leq 1\}$$

is closed and bounded since it is the interior of the unit circle. But D is not compact since it will "want" to converge to $(0, 0)$ but $(0, 0)$ is not an element of the metric so it is not compact hence b does not imply a. Therefore, Theorem 3.31 does not apply to the general metric space. □

The next 3 theorems are with respect to problem 5.

Theorem 7 (theorem 4.16)**.** *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T), and assume $p \in S$. Then f is continuous at p iff for every sequence $\{x_n\}$ in S convergent to p, the sequence $\{f(x_n)\}$ in T converges to $f(p)$. Symbolically,*

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

Proof. $(\implies)$ Assume that f is continuous at p. The definition of continuity at a point states that for every $\epsilon > 0$ there is a $\delta > 0$ such that,

$$d_T(f(x), f(p)) < \epsilon \qquad \text{whenever } d_S(x, p) < \delta$$

By Definition 4.1(def convergence) if the sequence $\{x_n\} \to p$ in S, then $\forall \delta > 0 \quad \exists N$ s.t.

$$d_S(x_n, p) < \delta, \qquad \text{whenever } n \geq N$$

Hence, we have that

$$d_T(f(x_n), f(p)) < \epsilon \quad \text{(By Definition 4.15(continuity))}$$

and therefore, by using Definition 4.1 twice

$$\lim_{n\to\infty} f(x_n) = f(p)$$

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

($\Longleftarrow$) Assume $\{f(x_n)\} \to f(p)$ in T and Then by Definition 4.1, we have

$$d_T(f(x_n), f(p)) < \epsilon \qquad \text{for } \epsilon > 0$$

By Definition 4.11, we have that when $\lim_{n\to\infty} f(x_n) = f(p), \forall \epsilon > 0 \quad \exists \delta > 0$ such that $d_T(f(x_n), f(p)) < \epsilon \quad$ whenever $d_S(x_n, p) < \delta$, and hence δ is defined by ϵ. Therefore we have that $\forall \epsilon > 0, \quad \exists \delta > 0$ such that

$$d_T(f(x), f(p)) < \epsilon \qquad \text{whenever } d_S(x, p) < \delta$$

and hence, f is continuous at p.

□

Theorem 8 (theorem 4.23)**.** *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Then f is continuous on S iff for every open set Y in T, the inverse image $f^{-1}(Y)$ is open on S.*

Proof. ($\Longrightarrow$) Assume Y is open and f is continuous, Since Y is open we have that $B_T(y, \epsilon) \subseteq Y$ for some $\epsilon > 0$. Let $p \in f^{-1}(Y)$ and $y = f(p)$, And Since f is continuous on S we have that f is continuous at p and hence $\exists \delta > 0$ such that $f(B_S(p, \delta)) \subseteq B_T(y, \epsilon)$. Hence,

$$B_S(p, \delta) \subseteq f^{-1}[f(B_S(p, \delta))] \subseteq f^{-1}[B_T(y, \epsilon)] \subseteq f^{-1}(Y)$$

Then p is an interior point of $f^{-1}(Y)$ and by Definition 3.6, if a set contains all its interior points then it is open, and since p can be any point in $f^{-1}(Y)$ we have that $f^{-1}(Y)$ contains

all its interior points and hence is open.
($\Longleftarrow$) Conversely, assume $f^{-1}(Y)$ is open in S for every open subset set Y in T. Take $p \in S$ and again let $y = f(p)$, then we want to show f is continuous at p. $\forall \epsilon > 0$, $B_T(y, \epsilon)$ is open in T, hence $f^{-1}(B_T(y, \epsilon))$ is open in S. Since $f^{-1}(y) = p$, $p \in f^{-1}(B_T(y, \epsilon))$, then $\exists \delta > 0$ such that,

$$B_S(p, \delta) \subseteq f^{-1}(B_T(y, \epsilon)) \Rightarrow f(B_S(p, \delta)) \subseteq B_T(y, \epsilon)$$

By definition 4.15, a function f is continuous at p iff $\forall \epsilon > 0, \quad \exists \delta > 0$ such that $f(B_S(p, \delta)) \subseteq B_T(f(p), \epsilon)$, and hence we have that f is continuous at p.

$\square$

Theorem 9 (theorem 4.25)**.** *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). If f is continuous on compact subset X of S, then the image $f(X)$ is a compact subset of T; in particular, $f(X)$ is closed and bounded in T.*

Proof. By Definition 3.30, a set S is compact iff every open covering of S contains a finite subcovering, that is, a finite subcovering that also covers S. And by Theorem 3.38, if S is compact then it is also closed and bounded.
Hence our goal is to find a finite open subcovering of $f(X)$ and the assertion will be proved. Define F to be an open covering of $f(X)$, so that $f(X) = \cup_{A \in F} A$. Since f is continuous on S and hence every subset of S, we can apply Theorem 4.23 to see that $\forall A \in F$, $f^{-1}(A)$ is also open in (X, d_S). Hence, $X \subseteq \cup_{A \in F} f^{-1}(A)$. Since X is compact, we know by the definition of a compact set that there exists an open subcovering of X, hence $X \subseteq \cup_{k=1}^{n} f^{-1}(A_k)$ for $A_k \in F$. And hence,

$$f(X) \subseteq f(\cup_{k=1}^{n} f^{-1}(A_k)) = \cup_{k=1}^{n} f(f^{-1}(A_k)) = \cup_{k=1}^{n} A_k$$

$\Rightarrow f(X) \subseteq \cup_{k=1}^{n} A_k$ and we have found our open subcovering, and hence $f(X)$ is closed and compact in T.

$\square$

Problem 6

Theorem 10 (Uniform Continuity Thm. 4.47 (Heine)). *Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Let A be a compact subset of S and assume that f is continuous on A. Then f is uniformly continuous on A.*

Proof. Let $\epsilon > 0$ be given. Then each point $a \in A$ has associated with it a ball $B_S(a, r)$, with r depending on a such that,

$$d_T(f(x), f(a)) < \frac{\epsilon}{2} \qquad \text{whenever } x \in B_S(a, r) \cap A$$

Consider the collection of balls $B_S(a, r/2)$. Since A is compact, a finite number of them also cover A, say

$$A \subseteq \cup_{k=1}^{n} B_S(a_k, \frac{r_k}{2})$$

in any ball with twice the radius, $B_S(a_k, r_k)$, we have

$$d_T(f(x), f(a_k)) < \frac{\epsilon}{2} \qquad \text{whenever } x \in B_S(a_k, r_k) \cap A$$

Let $\delta = min(r_n/2)$. We want to show that δ satisfies the condition of uniform continuity. Take $x \in A$ and $p \in A$ with $d_S(x, p) < \delta$. it is clear that $x \in B_S(a_k, r_k/2)$, hence

$$d_T(f(x), f(a_k)) < \frac{\epsilon}{2}$$

By the Triangle inequality and our defined δ, we have

$$d_S(p, a_k) \leq d_S(p, x) + d_S(x, a_k) < \delta + \frac{r_k}{2} \leq \frac{r_k}{2} + \frac{r_k}{2} = r_k$$

Hence, $d_S(p, a_k) \leq r_k$ and thus $p \in B_S(a_k, r_k) \cap S$. Using the Triangle inequality and the fact that $d_T(f(p), f(a_k)) < \epsilon/2$, we have

$$d_T(f(x), f(p)) \leq d_T(f(x), f(a_k)) + d_T(f(a_k), f(p)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

hence we have that $d_T(f(x), f(p)) < \epsilon$ and $d_S(x, p) < \delta$, thus fulfilling the condition for uniform continuity.

$\square$

The following 2 theorems are with respect to problem 7

Theorem 11 (Thm 5.10 (Rolle))**.** *Let f be a functions, having a derivative (finite or infinite) at each point of an open interval (a, b) and continuous at the endpoints a and b. if $f(a) = f(b)$, then there is at least one interior point c at which*

$$f'(c) = 0$$

Proof. Assume $f'(x) \neq 0 \quad \forall x \in (a, b)$, we want to arrive at a contradiction. Since f is continuous on a compact interval (a, b), $\exists c_1, c_2 \in (a, b)$ such that

$$f(c_1) = min(f(x)) \quad \text{and} \quad f(c_2) = max(f(x))$$

by Theorem 5.9, c_1, c_2 cannot be interior points since this implies $f'(c_1) = 0$ and $f'(c_2) = 0$. This implies that $f(c_1)$ and $f(c_2)$ are endpoints. Since $f(a) = f(b) \Rightarrow f(c_1) = f(c_2)$ and hence f is constant on $[a, b]$. However this contradicts our assumption that $f'(x) \neq 0 \quad \forall x \in (a, b)$. Therefore $f'(c) = 0$ for some $c \in (a, b)$.

□

Theorem 12 (Thm 5.12 (Generalized Mean Value))**.** *Let f and g be two functions, each having a derivative (finite or infinite) at each point of an open interval (a, b) and each continuous at the endpoints a and b. Assume also that there is no interior point x at which both $f'(x)$ and $g'(x)$ are infinite. Then for some interior point c we have*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

Proof. Let $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$. Since $h(x)$ is a linear combination of $f(x)$ and $g(x)$, $h(x)$ inherits many of the assumptions about $f(x)$ and $g(x)$, which is exactly what we want in order to use Rolle's theorem. Note that,

$$\begin{aligned}
\text{h(a)} &= \text{f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)]} \\
&= \text{f(a)g(b) - g(a)f(b)} \\
&= \text{- f(b)g(a) + g(b)f(a)} \\
&= \text{f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)]} \\
&= \text{h(b)}
\end{aligned}$$

So $h(a) = h(b)$, $\Rightarrow$ by Rolle's theorem we have that $\exists\ c$ such that $h'(c) = 0$. And Hence,

$$h'(c) = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0$$

$\Rightarrow$

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

This proves the assertion.

$\square$

Applications of 5.12

Theorem 13 (Mean-Value Theorem(5.11))**.** *Assume that f has a derivative (finite or infinite) at each point of (a, b), and assume that f is continuous at both endpoints a and b. Then $\exists c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. Consider Theorem 5.12, The General Mean-Value Theorem. We have that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

Letting $g(x) = x$, then if we can find $g'(x)$ we could substitute and simplify to get

$$f(b) - f(a) = f'(c)(b - a)$$

Let's first find $g'(x)$

$$\begin{aligned} \text{g'(x)} &= \lim_{x\to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x\to c} \frac{x - c}{x - c} \\ &= 1 \end{aligned}$$

Now substituting $g(x) = x$ and $g'(x) = 1$ into the result of Theorem 5.12 we get,

$$f'(c)[b - a] = 1[f(b) - f(a)]$$

After little simplification, we get exactly what we wanted,

$$f(b) - f(a) = f'(c)(b - a)$$

$\square$

<u>The next 3 theorems are with respect to problem 8.</u>

Theorem 14 (7.8)**.** *Assume $f \in R(\alpha)$ on $[a,b]$ and assume that α has a continuous derivative α' on $[a,b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and we have*

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

Proof. Let $g(x) = f(x)\alpha'(x)$ and consider a Riemanm sum

$$S(P,g) = \sum_{k=1}^{n} g(t_k)\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k$$

Using the same P and t_k we can form the following Riemann-Stieljes integral

$$S(P,f,\alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k$$

From the M.V.T. we can write,

$$\Delta\alpha_k = \alpha'(v_k)\Delta x_k \qquad \text{whenever } v_k \in (x_{k-1}, x_k)$$

$\Longrightarrow$

$$\begin{aligned} S(P,f,\alpha) - S(P,g) &= \sum_{k=1}^{n} f(t_k)\alpha'(v_k)\Delta x_k - \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\Delta x_k \\ &= \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\Delta x_k \end{aligned}$$

Since f is bounded, $|f(x)| \leq M \quad \forall x \in [a,b]$, where $M > 0$. Continuity of α' on $[a,b]$ implies uniform continuity on $[a,b]$. $\Rightarrow$ for $\epsilon > 0$, $\quad \exists \delta > 0$ such that

$$0 \leq |x-y| < \delta \quad \text{implies} \quad |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}$$

If we take a partition P'_ϵ with norm $\|P'_\epsilon\| < \delta$, then for any finer partition P we will have

$$|\alpha'(v_k) - \alpha'(t_k)| < \frac{\epsilon}{2M(b-a)} \Rightarrow |S(P,f,\alpha) - S(P,g)| < \frac{\epsilon}{2}$$

Since $f \in R(\alpha)$ on $[a,b]$, $\exists P''_\epsilon$ such that

$$\left| S(P,f,\alpha) - \int_a^b f d\alpha \right| < \frac{\epsilon}{2}$$

Hence, combining the last two inequalities we have that

$$\left| S(P,g) - \int_a^b f d\alpha \right| < \epsilon$$

This proves the assertion.

$\square$

Theorem 15 (7.9)**.** *Given $a < c < b$. Define α on $[a, b]$ as follows: The values $\alpha(a)$, $\alpha(c)$, $\alpha(b)$ are arbitrary;*

$$\alpha(x) = \alpha(a) \text{ if } a \leq x < c$$

and

$$\alpha(x) = \alpha(b) \text{ if } c \leq x < b$$

Let f be defined on $[a, b]$ in such a way that at least one of the functions f or α is continuous from the left at c and at least one is continuous from the right at c. Then $f \in R(\alpha)$ on $[a, b]$ and we have

$$\int_a^b f d\alpha = f(c)[\alpha(c+) - \alpha(c-)]$$

Proof. if $c \in P$, every term in the sum $S(P, f, \alpha)$ is zero except the two terms arising from the subinterval separated by c. To account the nonzero terms, write

$$S(P, f, \alpha) = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)]$$

where $t_{k-1} \leq c \leq t_k$. Consider subtracting by $f(c)[\alpha(c+) - \alpha(c-)]$ on both sides of the above equation,

$$S(P, f, \alpha) - f(c)[\alpha(c+) - \alpha(c-)] = \Delta = (f(t_{k-1}) - f(c))[\alpha(c) - \alpha(c-)] + (f(t_k) - f(c))[\alpha(c+) - \alpha(c)]$$

Hence we have the following about Δ

$$|\Delta| \leq |f(t_{k-1}) - f(c)|\, |\alpha(c) - \alpha(c-)| + |f(t_k) - f(c)|\, |\alpha(c+) - \alpha(c)|$$

With respect to $|\Delta|$, if f is continuous at $c, \Rightarrow \quad \forall \epsilon > 0, \quad \exists \delta > 0$ such that $\|P\| > \delta$ implies

$$|f(t_{k-1}) - f(c)| < \epsilon \quad \text{and} \quad |f(t_k) - f(c)| < \epsilon$$

Substituting our inequalities back into our equation for $|\Delta|$ we get,

$$|\Delta| \leq \epsilon\, |\alpha(c) - \alpha(c-)| + \epsilon\, |\alpha(c+) - \alpha(c)|$$

This inequality holds true whether or not f is continuous at c. By changing f's continuity from the left or right, we change the above inequality. If discontinuous from both sides at c, then $\Delta = 0$. If discontinuous from the right at c, then $\alpha(c) = \alpha(c+)$ and if discontinuous from the left at c, then $\alpha(c) = \alpha(c-)$. Either way, the assertion is proved.

$\square$

Theorem 16 (7.12)**.** *Every finite sum can be written as a Riemann-Stieltjes integral. In fact, given a sum* $\sum_{k=1}^{n} a_k$*, define* f *on* $[0, n]$ *as follows:*

$$f(x) = a_k \text{ if } k-1 < x \leq k$$

for $k = 1, 2, ..., n$*, and* $f(0) = 0$*. Then*

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)d[x]$$

where $[x]$ *is the greatest integer* $\leq x$*.*

Proof. The greatest integer function is a type of step-function, hence $d[x]$ is continuous from the right and jumps 1 at each integer by definition of $[x]$. The function f is continuous from the left at $1, 2, ..., n$ by construction of f. Hence we meet the conditions to apply Theorem 7.11, and the sum can therefore be written as a Riemann-Stieljes integral.

□

Ch.7 Examples:

Let $\{a_n\}$ be a sequence of real numbers. For $x \geq 0$, define

$$A(x) = \sum_{n \leq x} a_n = \sum_{n=1}^{[x]} a_n$$

Where $[x]$ is the greatest integer in x and empty sums are interpreted as zero. Let f have a continuous derivative in $[1, a]$. Use Stieltjes integrals to derive:

$$\sum_{n \leq a} a_n f(n) = -\int_1^a A(x)f'(x)dx + A(a)f(a)$$

Proof. Since

$$\int_1^a A(x)f'(x)dx = \int_1^a A(x)df(x) \text{ since f has a continuous derivative on } [1, a]$$

$$= -\int_1^a f(x)dA(x) + A(a)f(a) - A(1)f(1) \text{ From integration by parts}$$

$$= -\sum_{n \leq a} a_n f(n) + A(a)f(a) \text{ by } \int_1^a f(x)dA(x) = \sum_{n=2}^{[a]} a_n f(n) \text{ and } A(1) = a_1$$

Therefore we know that:

$$\sum_{n \leq a} a_n f(n) = -\int_1^a A(x)f'(x)dx + A(a)f(a)$$

□

The following 3 theorems are with respect to problem 9

Theorem 17 (7.30 First Mean-Value Theorem for Riemann-Stieltjes integrals)**.** *Assume that α is increasing and let $f \in R(\alpha)$ on $[a,b]$. Let M and m denote, respectively, the sup and inf of the set $\{f(x) : x \in [a,b]\}$. Then $\exists c \in \mathbb{R}$ satisfying $m \leq c \leq M$ such that*

$$\int_a^b f(x)d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$$

In particular, if f is continuous on $[a,b]$, then $c = f(x_0)$ for some $x_0 \in [a,b]$.

Proof. if $\alpha(a) = \alpha(b) \Rightarrow \int_a^b f(x)d\alpha(x) = c[\alpha(b) - \alpha(a)] = 0$ and the condition holds trivially. Hence, assume $\alpha(a) < \alpha(b)$. Consider the following inequality that holds true for all upper and lower sums

$$m[\alpha(b) - \alpha(a)] \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

$\int_a^b f(x)d\alpha$ must be lie in this interval. This is clear since if m is the smallest $f(x)$, then clearly, the smallest "area" possible is m multiplied by the length of the integrand and similarly M multiplied by the length of the integrand will be the largest possible area of the integral. Therefore,

$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f(x)d\alpha \leq M[\alpha(b) - \alpha(a)]$$

$$m \leq \frac{\int_a^b f(x)d\alpha}{[\alpha(b) - \alpha(a)]} \leq M$$

$$m \leq \frac{\int_a^b f(x)d\alpha}{\int_a^b d\alpha} \leq M$$

Hence, define

$$c = \frac{\int_a^b f(x)d\alpha}{\int_a^b d\alpha}$$

Clearly, if f is continuous on $[a,b]$, then we can apply the intermediate value to find that $\exists x_0 \in [a,b]$ such that $f(x_0) = c$

$\square$

Theorem 18 (7.32)**.** *Let α be of bounded variation on $[a,b]$ and assume that $f \in R(\alpha)$ on $[a,b]$. Define F by*

$$F(x) = \int_a^x f d\alpha \qquad \text{if } x \in [a,b]$$

a) *F is of bounded variation $[a,b]$.*

b) *Every point of continuity of α is also a point of continuity of F.*

c) *if α is increasing on $[a,b]$, then $\exists F'(x)$ at each point $x \in [a,b]$ where $\alpha'(x)$ exists and where f is continuous. For such x, we have*

$$F'(x) = f(x)\alpha'(x)$$

Proof. a) if $x \neq y$ and α increasing on $[a,b]$, then consider the following

$$\begin{aligned}
F(y) - F(x) &= \int_a^x f d\alpha - \int_a^y f d\alpha \\
&= \int_x^y f d\alpha \qquad \text{(by Theorem 7.4, } \int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha\text{)} \\
&= c[\alpha(y) - \alpha(x)] \qquad \text{(by Theorem 7.30)} \\
F(y) - F(x) &= c[\alpha(y) - \alpha(x)] \qquad (1)
\end{aligned}$$

where $m \leq c \leq M$. if α is of bounded variation on $[a,b]$, then

$$\sum_{k=1}^{n} |\Delta\alpha_k| \leq T$$

where T is a positive number. Then consider

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta F_k| &= \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \\
&= \sum_{k=1}^{n} |c_k[\alpha(x_k) - \alpha(x_{k-1})]| \quad \text{by equation (1)} \\
&= \sum_{k=1}^{n} |c_k \Delta\alpha_k| \\
&\leq T \sum_{k=1}^{n} |c_k|
\end{aligned}$$

Hence $\sum_{k=1}^{n} |\Delta F_k|$ is bounded by a positive number $T\sum_{k=1}^{n} |c_k|$, therefore F is of bounded variation on $[a,b]$.

b) By Theorem 7.30, we have that $F(x) = \int_a^x f d\alpha = c[\alpha(x) - \alpha(a)]$. Hence, if α is continuous at a point x, then F is also continuous at x.

c) Consider $F(y) - F(x) = c[\alpha(y) - \alpha(x)]$, then by dividing $y - x$ on both sides, we have

$$\frac{F(y) - F(x)}{y - x} = c\frac{\alpha(y) - \alpha(x)}{y - x}$$

taking the limit as $y \to x$ we have,

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x} = \lim_{y \to x} c\frac{\alpha(y) - \alpha(x)}{y - x}$$
$$F'(x) = c\alpha'(x)$$

Since $c = f(x_0)$ for some $x_0 \in [x, y]$, we have that when we take the limit as $y \to x$, then $x_0 \in [x, x]$. Hence $x_0 = x$, and $c = f(x)$. Therefore

$$F'(x) = f(x)\alpha'(x)$$

□

Theorem 19 (7.34 Second fundamental theorem of integral calculus)**.** *Assume that $f \in R$ on $[a, b]$. Let g be a function defined on $[a, b]$ such that $\exists g' \in (a, b)$ and has the value*

$$g'(x) = f(x) \textit{ for every } x \in (a, b)$$

At the endpoints assume that $g(a+)$ and $g(b-)$ exist and satisfy

$$g(a) - g(a+) = g(b) - g(b-)$$

Then we have

$$\int_a^b f(x)dx = \int_a^b g'(x)dx = g(b) - g(a)$$

Proof. For every partition on $[a, b]$, we can rewrite $g(b) - g(a)$ by

$$g(b) - g(a) = \sum_{k=1}^{n} [g(x_k) - g(x_{k-1})] = \sum_{k=1}^{n} g'(t_k)\Delta x_k = \sum_{k=1}^{n} f(t_k)\Delta x_k$$

where $t_k \in (x_{k-1}, x_k)$ determined by the Mean Value Theorem by $g'(t_k) = \frac{g(x_k) - g(x_{k-1})}{x_k - x_{k-1}}$. Taking $\epsilon > 0$, the partition can be made finer such that

$$\left| g(b) - g(a) - \int_a^b f(x)dx \right| = \left| \sum_{k=1}^{n} f(t_k)\Delta x_k - \int_a^b f(x)dx \right| < \epsilon$$

Hence,

$$\int_a^b f(x)dx = g(b) - g(a)$$

This proves the assertion.

□

Ch.7 Examples

Example 1: The Second Mean-Value Theorem for Riemann Integrals(Thm.7.37) states that, if f and g be function defined on $[a, b]$ with f increasing and g continuous, then

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{x_0} g(x)dx + f(b)\int_{x_0}^b g(x)dx$$

for some $x_0 \in [a, b]$.

Proof. If $G(x) = \int_a^x g(x)dx$ and

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x) = f(a)(G(x_0) - G(a)) + f(b)(G(b) - G(x_0))$$

for some $x_0 \in [a, b]$ by the Theorem 7.30, the First Mean-Value Theorem. □

Example 2: Let α be a continuous function of bounded variation on $[a, b]$. Assume $g \in R(\alpha)$ on $[a, b]$ and define $\beta(x) = \int_a^x g(t)d\alpha(t)$ if $x \in [a, b]$. Show that:
If f only increases on $[a, b]$, then $\exists x_0 \in [a, b]$ such that:

$$\int_a^b f d\beta = f(a)\int_a^{x_0} g d\alpha + f(b)\int_{x_0}^b g d\alpha$$

Proof. Since α is a continuous function of bounded variation on $[a, b]$, and $g \in R(\alpha)$ on $[a, b]$, we know that $\beta(x)$ is a continuous function of bounded variation on $[a, b]$, by Theorem 7.32. Hence, by Second Mean-Value Theorem for Riemann-Stieltjes integrals, we know that:

$$\int_a^b f d\beta = f(a)\int_a^{x_0} \beta(x) + f(b)\int_{x_0}^b d\beta(x)$$

which implies that, By Theorem 7.26,

$$\int_a^b f d\beta = f(a)\int_a^{x_0} g d\alpha + f(b)\int_{x_0}^b g d\alpha$$

□

The next 4 theorems are with respect to problem 10.

Theorem 20 (8.22)**.** *If* $|x| < 1$*, the series* $1 + x + x^2 + ...$ *converges and has the sum* $\frac{1}{1-x}$*. If* $|x| \geq 1$*, the series diverges.*

Proof. the series can be written in summation form by $\sum_{k=0}^{n} x^k$, and by multiplying it by $1 - x$, we can simplify our sum to be in terms of only x and n,

$$(1-x)\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} x^k - x\sum_{k=0}^{n} x^k$$

$$= \sum_{k=0}^{n}(x^k - x^{k+1})$$

$$= 1 - x^{n+1}$$

dividing $(1 - x)$ on both sides, we get

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{(1-x)}$$

if $|x| < 1$ and as $n \to \infty$ we have that

$$lim_{n\to\infty} x^{n+1} = 0$$

and hence

$$lim_{n\to\infty}\frac{1 - x^{n+1}}{(1-x)} = \frac{1}{(1-x)}$$

if $|x| \geq 1 \quad \Rightarrow$ as $n \to \infty$, we have that $lim_{n\to\infty} x^{n+1}$ diverges and hence $\frac{1-x^{n+1}}{(1-x)}$ diverges. (NOTE: This Theorem can be proved briefly using Theorem 8.10 on the convergence of telescopic series, letting $a_n = x^k - x^{k+1}$) □

Theorem 21 (8.23 integral test)**.** *Let* f *be a positive decreasing function defined on* $[1, \infty)$ *such that* $\lim_{x\to+\infty} f(x) = 0$*. For* $n = 1, 2, ...,$ *define*

$$s_n = \sum_{k=1}^{n} f(k), \qquad t_n = \int_1^n f(x)dx, \qquad d_n = s_n - t_n$$

Then we have:

a) $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1), \qquad$ *for* $n = 1, 2, ...$

b) $\lim_{n\to\infty} d_n$ *exists*

c) $\sum_{n=1}^{\infty} f(n)$ *converges iff the sequence* t_n *converges*

d) $0 \le d_k - \lim_{n\to\infty} d_n \le f(k), \qquad$ *for* $k = 1, 2, ...$

Proof. a)

$$t_{n+1} = \int_1^{n+1} f(x)dx = \sum_{k=1}^{n} \int_k^{k+1} f(x)dx \le \sum_{k=1}^{n} \int_k^{k+1} f(k)dx = \sum_{k=1}^{n} f(k) = s_n$$

Which implies that $f(n+1) = s_{n+1} - s_n \le s_{n+1} - t_{n+1} = d_{n+1} \Rightarrow 0 < f(n+1) \le d_{n+1}$. But we also have:

$$d_n - d_{n+1} = t_{n+1} - t_n - (s_{n+1} - s_n) = \int_n^{n+1} f(x)dx - f(n+1) \ge \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Therefore $d_{n+1} \le d_n \le d_1 = f(1)$. ✓

b) In a), we proved that $0 < f(n+1) \le d_{n+1} \le d_n \le f(1)$ so $\lim_{n\to\infty} d_n$ must exist since the sequence is always decreasing but has a lower bound. ✓

c) Since b is true, we know that since d_n converges then s_n and t_n converge as well, so $\sum_{n=1}^{\infty} f(n)$ converges whenever $\int_1^{\infty} f(x)dx$ converges. ✓

d) From above we can write:

$$0 \le d_n - d_{n+1} \le \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

$$\text{and summing on n: } 0 \le \sum_{n=k}^{\infty}(d_n - d_{n+1}) \le \sum_{n=k}^{\infty}(f(n) - f(n+1)) \quad \text{if } k \ge 1$$

Each part of that inequality telescopes and breaks down into: $0 \le d_k - \lim_{n\to\infty} d_n \le f(k)$ ✓

□

Theorem 22 (8.25 ratio test)**.** *Given a series* $\sum a_n$ *of nonzero complex terms, let*

$$r = \lim_{n\to\infty} inf \left|\frac{a_{n+1}}{a_n}\right|, \qquad R = \lim_{n\to\infty} sup \left|\frac{a_{n+1}}{a_n}\right|$$

a) *The series* $\sum a_n$ *converges absolutely if* $R < 1$.

b) *The series* $\sum a_n$ *diverges if* $r > 1$.

c) *The test is inconclusive if* $r \le 1 \le R$.

Proof. Assume that $R < 1$ and take x such that $R < x < 1$. Since $R < 1$, we know that $a_{n+1} < a_n$. This implies $\exists N$ such that $|\frac{a_{n+1}}{a_n}| < x$ whenever $n \geq N$. Since $x = \frac{x^{n+1}}{x^n}$:

$$\frac{|a_{n+1}|}{x^{n+1}} < \frac{|a_n|}{x^n} \leq \frac{|a_N|}{x^N} \qquad \text{if } n \geq N$$

$\Rightarrow \quad |a_n| \leq cx^n$ if $n \leq N$, where $c = |a_N|x^{-N}$. We now have the requirements fulfilled to apply the comparison test $\Rightarrow \quad \sum a_n$ converges and a) is confirmed. Now we assume that $r > 1$. $\Rightarrow \quad |a_{n+1}| > a_n \quad \forall n \geq N$ for some N and therefore $\lim_{n\to\infty} \neq 0$ so $\sum a_n$ diverges and b) is confirmed. Finally, to confirm c) consider the two summations:

$$\sum n^{-1} \quad \sum n^{-2}$$

In both summations, $r = R = 1$ but we know that $\sum n^{-1}$ diverges but that $\sum n^{-2}$ converges so the test is inconclusive confirming c). □

Theorem 23 (8.26 root test)**.** *Given a series $\sum a_n$ of complex terms, let*

$$p = \lim_{n\to\infty} sup \sqrt[n]{\mid a_n \mid}$$

a) *The series $\sum a_n$ converges absolutely if $p < 1$.*

b) *The series $\sum a_n$ diverges if $p > 1$.*

c) *The test is inconclusive if $p = 1$.*

Proof. Assume that $\rho < 1$ and take x such that $\rho < x < 1$. Since $\rho < 1$ we know that $\exists N$ such that $|a_n| < x^n$ whenever $n \geq N$. We now have the elements required to run the comparison test and since x^n converges to 0 and $|a_n| < x^n$, we know that $\sum a_n$ also converges confirming a).
Now assume that $\rho > 1$. This implies that $|a_n| > 1 \quad \forall a_n$ so $\lim_{n\to\infty} a_n \neq 0$ and therefore $\sum a_n$ diverges confirming b).
Consider the two summations:

$$\sum n^{-1} \quad \sum n^{-2}$$

In both summations, $\rho = 1$ but we know that $\sum n^{-1}$ diverges but that $\sum n^{-2}$ converges so the test is inconclusive confirming c). □

Ch.8 Examples:

Test for convergence: $\sum_{n=1}^{\infty} n^3 e^{-n}$

By Root Test, $\lim_{n\to\infty} \sup(\frac{n^3}{e^n})^{1/n} = \frac{1}{e} < 1 \quad \Rightarrow$ The sum converges.

Test for convergence: $\sum_{n=3}^{\infty} \frac{1}{nlogn(loglogn)^p}$

By Integral Test. The function $f(x) = \frac{1}{nlogn(loglogn)^p}$ is positive, decreasing, and continuous on $[a, \infty)$ so the test is usable. Consider:

$$\int_a^{\infty} \frac{dx}{xlogx(loglogx)^p} = \int_{logloga}^{\infty} \frac{dy}{y^p}$$

Which implies that the series converges if $p > 1$ and it diverges if $p \leq 1$