Midterm Project 4

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I. PROOFS FOR THEOREMS 5.12 AND 5.13

Theorem 1 (Generalized Mean-Value Theorem(5.12))**.** *(also known as Cauchys Mean-Value Theorem)Let f and g be two functions, each having a derivative (finite or infinite) at each point of an open interval (a, b) and each continuous at the endpoints a and b. Assume also that there is no interior point x at which both $f'(x)$ and $g'(x)$ are infinite. Then for some interior point c we have*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

Note: easier to visualize why this is when in the form $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Proof. Let $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$. Since $h(x)$ is a linear combination of $f(x)$ and $g(x)$, $h(x)$ inherits many of the assumptions about $f(x)$ and $g(x)$, which is exactly what we want in order to use Rolle's theorem. Note that,

$$\begin{aligned} \text{h(a)} &= \text{f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)]} \\ &= \text{f(a)g(b) - g(a)f(b)} \\ &= \text{- f(b)g(a) + g(b)f(a)} \\ &= \text{f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)]} \\ &= \text{h(b)} \end{aligned}$$

So $h(a) = h(b)$, $\Rightarrow$ by Rolle's theorem we have that $\exists$ c such that $h'(c) = 0$. And Hence,

$$h'(c) = f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0$$

$\Rightarrow$

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

This proves the assertion. □

Theorem 2 (5.13 in our book)**.** *Let f and g be two functions, each having a derivative (finite or infinite) at each point of (a, b). At the endpoints assume that the limits $f(a+)$,$g(a+)$,$f(b-)$, and $g(b-)$ exist as finite values. Assume further that there is no interior point x at which both $f'(x)$ and $g'(x)$ are infinite. Then for some interior point c we have*

$$f'(c)[g(b-) - g(a+)] = g'(c)[f(b-) - f(a+)]$$

Proof. Define $F(x)$ and $G(x)$ as,

$$F(x) = f(x) \text{ and } G(x) = g(x) \text{ for } x \in (a, b)$$

and define the end points of F and G at a and b as,

$$F(a) = f(a+) \text{ and } F(b) = f(b-)$$

$$G(a) = g(a+) \text{ and } G(b) = g(b-)$$

So G and F are continuous on $[a, b]$ since we've defined F and G on the interval (a, b) and at the endpoints a and b. Now, by the General Mean Value Theorem we have that $\exists$ c such that,

$$F'(c)[G(b) - G(a)] = G'(c)[F(b) - F(a)]$$

since c is an interior point, we have that $G'(c) = g'(c)$ and $F'(c) = f'(c)$. and plugging in our values for a and b we get,

$$f'(c)[g(b-) - g(a+)] = g'(c)[f(b-) - f(a+)]$$

□

II. PROOFS FOR THEOREMS 5.11, 5.14, AND 5.16 AS EXAMPLES

Theorem 3 (Mean-Value Theorem(5.11))**.** *Assume that f has a derivative (finite or infinite) at each point of (a, b), and assume that f is continuous at both endpoints a and b. Then $\exists c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a)$$

Note: easier to visualize when in the form $f'(c) = \frac{f(b)-f(a)}{b-a}$, consider the graph above

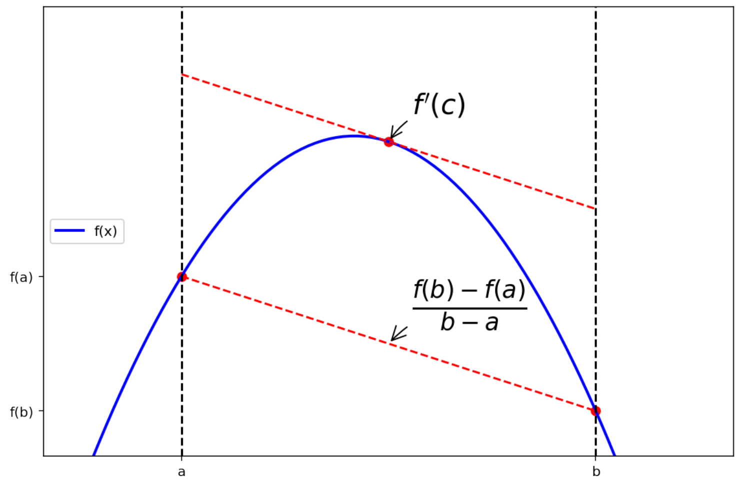

FIG. 1. We can see that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Consider Theorem 5.12, The General Mean-Value Theorem. We have that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

Letting $g(x) = x$, then if we can find $g'(x)$ we could substitute and simplify to get

$$f(b) - f(a) = f'(c)(b - a)$$

Let's first find $g'(x)$

$$\begin{aligned} \text{g'(x)} &= \text{lim}_{x\to c} \frac{g(x) - g(c)}{x - c} \\ &= \text{lim}_{x\to c} \frac{x - c}{x - c} \\ &= 1 \end{aligned}$$

Now substituting $g(x) = x$ and $g'(x) = 1$ into the result of Theorem 5.12 we get,

$$f'(c)[b - a] = 1[f(b) - f(a)]$$

After little simplification, we get exactly what we wanted,

$$f(b) - f(a) = f'(c)(b - a)$$

$\square$

Theorem 4 (5.14 in our book)**.** *Assume that f has a derivative (finite or infinite) at each point of an open interval (a,b), and that f is continuous at both endpoints a and b.*

(a) *if $f'(x) > 0$, $\forall x \in (a,b) \Rightarrow f$ is strictly increasing on $[a,b]$.*

(b) *if $f'(x) < 0$, $\forall x \in (a,b) \Rightarrow f$ is strictly increasing on $[a,b]$.*

(c) *if $f'(x) = 0$, $\forall x \in (a,b) \Rightarrow f$ is constant on $[a,b]$.*

Proof. (a) by definition 4.50, strictly increasing implies that $\forall(n,m)$ where $n < m$ then $f(n) < f(m)$, which is equivalent to $m-n > 0$ implies $f(m) - f(n) > 0$. Let $(n,m) \in (a,b)$ and $n < m$. Using the General Mean-Value theorem where $g(x) = x$,

$$\text{f'(c)[g(m) - g(n)]} = \text{g'(c)[f(m) - f(n)]} \tag{1}$$

$$\text{f'(c)(m - n)} = \text{f(m) - f(n)} \qquad \text{(since g(x) = x)}$$

$$\text{f'(c)} = \frac{f(m) - f(n)}{m-n} \tag{2}$$

We want to show that if $f'(c) > 0 \Rightarrow f(m) - f(n) \geq 0$. Since we assumed $m - n > 0$, then by equation (2) above and the order axioms, we have that $f(m) - f(n) > 0$, and thus fulfilling the definition of strictly increasing.

(b) by definition 4.50, strictly decreasing implies that $\forall(n,m)$ where $n < m$ then $f(n) > f(m)$, which is equivalent to $m-n > 0$ implies $f(m) - f(n) < 0$. Let $(n,m) \in (a,b)$ and $n < m$. Using the General Mean-Value theorem where $g(x) = x$,

$$\text{f'(c)[g(m) - g(n)]} = \text{g'(c)[f(m) - f(n)]} \tag{3}$$

$$\text{f'(c)(m - n)} = \text{f(m) - f(n)} \qquad \text{(since g(x) = x)}$$

$$\text{f'(c)} = \frac{f(m) - f(n)}{m-n} \tag{4}$$

We want to show that if $f'(c) < 0 \Rightarrow f(m) - f(n) < 0$. Since we assumed $m - n > 0$, then by equation (4) above and the order axioms, we have that $f(m) - f(n) < 0$, and thus fulfilling the definition of strictly decreasing.

(c) Let $f'(x) = 0, \forall x \in (a,b)$. We want to show that $f(x) = c$ on $\forall x \in [a,b]$ and for some constant c. Again using the General Mean-Value theorem where $g(x) = x$,

$$\text{f'(c)[g(m) - g(n)]} = \text{g'(c)[f(m) - f(n)]} \tag{5}$$

$$\text{f'(c)(m - n)} = \text{f(m) - f(n)} \qquad \text{(since g(x) = x)}$$

$$\text{f'(c)} = \frac{f(m) - f(n)}{m - n} \tag{6}$$

Since $f'(x) = 0, \forall x \in (a, b)$, then $f'(c) = 0$. By equation (6) above, $f(m) - f(n) = 0$ which implies $f(m) = f(n)$, $\forall n, m \in [a, b]$. And hence, f is constant on the interval $[a, b]$

□

Theorem 5 (Intermediate-value theorem for derivatives(5.16))**.** *Assume that f is defined on a compact interval $[a, b]$ and that f has a derivative(finite or infinite) at each interior point. Assume also that f has finite one-sided derivatives $f'_+(a)$ and $f'_-(b)$ at the endpoints, with $f'_+(a) \neq f'_-(b)$. Then, if c is a real number between $f'_+(a)$ and $f'_-(b)$, there exists at least one interior point n such that $f'(n) = c$*

Proof. Define $r(x)$ and $l(x)$ as,

$$l(x) = \begin{cases} \dfrac{f(a) - f(x)}{a - x} & \text{for } x \neq a \\ f'_+(a) \text{ for } x = a \end{cases}$$

and

$$r(x) = \begin{cases} \dfrac{f(x) - f(b)}{x - b} & \text{for } x \neq b \\ f'_-(b) \text{ for } x = b \end{cases}$$

Note that $r(a) = f'_+(a)$ and $l(b) = f'_-(b)$. If $f'_+(a) < c < f'_-(b) \Rightarrow r(a) < c < l(b)$. We want to show that $\exists\, n \in (a, b)$ s.t. $f'(n) = c$.

Also note that by the General Mean-Value theorem $l(x) = f'(k)$ for some $k \in (a, x)$. likewise for r, $r(x) = f'(k)$ for some $k \in (x, b)$. Hence f' ranges over $r(x) \cup l(x)$ $\forall x \in [a, b]$. Note both r and l are continuous on the closed interval $[a, b]$. Then we have that $\forall x \in [a, b]$,

$$l(a) \leq l(x) \leq l(b)$$

$\Rightarrow$

$$f'_+(a) \leq l(x) \leq \frac{f(a) - f(b)}{a - b} \tag{1}$$

and

$$r(a) \leq r(x) \leq r(b)$$

$\Rightarrow$

$$\frac{f(a) - f(b)}{a - b} \leq r(x) \leq f'_-(b) \tag{2}$$

Then by (1) and (2) we have that,

$$f'_+(a) \leq l(x) \leq r(x) \leq f'_-(b)$$

Then by the Intermediate Value Theorem, c must be in either $l(x)$ or $r(x)$. If $c \in l(x)$ then,

$$c = \frac{f(a) - f(s)}{a - s} \text{ for some s} \in (a, b)$$

and by the Mean-Value Theorem, $\exists n$ s.t.

$$c = \frac{f(a) - f(s)}{a - s} = f'(n)$$

If $c \in r(x)$ then,

$$c = \frac{f(s) - f(b)}{s - b} \text{ for some s} \in (a, b)$$

and by the Mean-Value Theorem, $\exists n$ s.t.

$$c = \frac{f(s) - f(b)}{s - b} = f'(n)$$

In either case, n exists and the assertion is proved. $\square$