

Results chapter 2: Observability and Information Content **DRAFT**

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1 Introduction (to be included in literature review chapter)

1.1 Observability

Observability is a mathematical concept from control theory. A system is said to be observable if it is possible to determine the state by measuring only the output. The following definition is taken from Barnett and Cameron [1985], For the linear time varying system defined as,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (1)$$

$$\mathbf{y} = \mathbf{C}(t)\mathbf{x}(t) \quad (2)$$

where \mathbf{A} is $n \times n$, \mathbf{B} is $n \times m$ and \mathbf{C} is $r \times n$ is *completely observable* if for any t_0 and any initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ there exists a finite time $t_i > t_0$ such that knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ for $t_0 \leq t \leq t_i$ suffices to uniquely determine \mathbf{x}_0 . There is no loss of generality in assuming $\mathbf{u}(t)$ is identically zero throughout the whole interval; this is the case for data assimilation.

Theorem 1.1. *When \mathbf{A} , \mathbf{B} and \mathbf{C} are time-invariant the system is completely observable if and only if the $nr \times n$ observability matrix*

$$\mathbf{V} = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{pmatrix} \quad (3)$$

has rank n .

This result can be applied to the data assimilation problem [Johnson et al., 2005], where for 4D-Var the observability matrix corresponds to

$$\hat{\mathbf{H}} = \begin{pmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \mathbf{M}_0 \\ \vdots \\ \mathbf{H}_N \mathbf{M}_{N,0} \end{pmatrix} \quad (4)$$

as defined in section **REF**. In Appendix B of Zou et al. [1992] it is shown that for the linear data assimilation problem it is possible to obtain a unique analysis state over a specific assimilation window with no background term if the rank of $\hat{\mathbf{H}}$ is equal to n , the size of \mathbf{x}_0 . For the non-linear data assimilation problem the rank of $\hat{\mathbf{H}}$ being equal to n ensures a locally unique analysis can be found without including a background term. In practice a background term is typically included in the cost function for 4D-Var data assimilation which regularises the problem and means that we always have a unique solution.

2 Information content measures

In data assimilation we combine prior estimates with observations to improve our knowledge of the state of a system. In this process some observations will have a greater impact on our results than others. Many measures exist for understanding observation impact on the analysis.

Information content measures have been used to quantify the different levels of information provided by observations in the development of satellite instruments [Engelen and Stephens, 2004, Stewart et al., 2008] and in operational data assimilation schemes [Fisher, 2003, Singh et al., 2013]. In Fowler and Van Leeuwen [2013] it is discussed that in these operational schemes information content measures have been used for

- Removing observations with a lesser impact in order to improve the efficiency of the assimilation process [Rabier et al., 2002, Rodgers, 1998, Singh et al., 2013].
- Diagnosing erroneous observations and assumed statistics [Desroziers et al., 2009].
- Improving data assimilation results by adding observations which theoretically have a high impact. Defining target observations [Palmer et al., 1998]. Designing new observing systems [Eyre, 1990, Wahba, 1985].

For the following measures the data assimilation problem is assumed to be Gaussian with a linear function mapping the state to observation space (\mathbf{H}), following the derivation from Kalnay [2003] we have,

$$\mathbf{x}_a = \mathbf{x}_b + \mathbf{K}(\mathbf{y} - \mathbf{H}\mathbf{x}_b), \quad (5)$$

where \mathbf{K} is the Kalman gain matrix,

$$\mathbf{K} = (\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H} + \mathbf{B}^{-1})^{-1} \mathbf{H}^T \mathbf{R}^{-1}. \quad (6)$$

In order to consider observations over a 4D-Var time window we rewrite equation (5) as,

$$\mathbf{x}_a = \mathbf{x}_b + \hat{\mathbf{K}}(\hat{\mathbf{y}} - \hat{\mathbf{H}}\mathbf{x}_b), \quad (7)$$

using the defined matrices in section REF, with $\hat{\mathbf{K}} = (\hat{\mathbf{H}}^T \hat{\mathbf{R}}^{-1} \hat{\mathbf{H}} + \mathbf{B}^{-1})^{-1} \hat{\mathbf{H}}^T \hat{\mathbf{R}}^{-1}$.

Making the assumption of a linear and Gaussian data assimilation problem is clearly a limitation. These measures are therefore limited to a period where the forecast model remains reasonably linear. The implications of assuming Gaussian error statistics are discussed in Fowler and Van Leeuwen [2013].

2.1 Sensitivity of analysis to observations

The influence matrix measures the sensitivity of the analysis in observation space to the observations [Cardinali et al., 2004] and is defined by,

$$\mathbf{S} = \frac{\partial \mathbf{H}\mathbf{x}_a}{\partial \mathbf{y}}. \quad (8)$$

From equation (5) we see that,

$$\mathbf{S} = \mathbf{K}^T \mathbf{H}^T, \quad (9)$$

here \mathbf{S} will be a $p \times p$ matrix, where p is the number of observations. The diagonal elements of \mathbf{S} are $\mathbf{S}_{i,i} = \frac{\partial (\mathbf{H}\mathbf{x}_a)_i}{\partial \mathbf{y}_i}$ and represent the ‘self-sensitivity’ of the i^{th} modelled observation to the i^{th} observation. The off-diagonal elements of \mathbf{S} represent the ‘cross-sensitivity’ and are given by $\mathbf{S}_{i,j} = \frac{\partial (\mathbf{H}\mathbf{x}_a)_i}{\partial \mathbf{y}_j}$. If we wish to consider the influence matrix for observations over a 4D-Var time window we can re-write equation (8) as,

$$\mathbf{S} = \frac{\partial \hat{\mathbf{H}}\mathbf{x}_a}{\partial \hat{\mathbf{y}}} = \hat{\mathbf{K}}^T \hat{\mathbf{H}}^T. \quad (10)$$

The Kalman gain matrix $\hat{\mathbf{K}}$ can be re-written as,

$$\hat{\mathbf{K}} = \mathbf{A}\hat{\mathbf{H}}^T\hat{\mathbf{R}}^{-1}, \quad (11)$$

where \mathbf{A} is the analysis error covariance,

$$\mathbf{A} = (\hat{\mathbf{H}}^T\hat{\mathbf{R}}^{-1}\hat{\mathbf{H}} + \mathbf{B}^{-1})^{-1}. \quad (12)$$

Inserting equation (11) into (10) we find,

$$\mathbf{S} = \hat{\mathbf{R}}^{-1}\hat{\mathbf{H}}\mathbf{A}\hat{\mathbf{H}}^T. \quad (13)$$

We can therefore see the sensitivity of the analysis to observations is inversely proportional to the observation error and proportional to the analysis error. This means that the most influential observations are those with the smallest error variance providing information about regions of state space with the largest prior error [Cardinali et al., 2004].

2.2 Degrees of freedom for signal

The degrees of freedom for signal (*dfs*) indicates the number of elements of the state that have been measured by the observations. If we consider a state vector \mathbf{x} with n elements (or n degrees of freedom) then the maximum value the *dfs* could obtain would be n , in this case all elements of the state would have been measured. Conversely if *dfs* = 0 then no elements of the state would have been measured by our observations [Fowler and Van Leeuwen, 2013].

For symmetric positive definite prior and posterior error covariance matrices \mathbf{B} and \mathbf{A} , we can define the degrees of freedom for signal by means of a transform \mathbf{L} that reduces the prior error covariance matrix, \mathbf{B} to the $n \times n$ identity [Fisher, 2003]. Each diagonal element of the transformed matrix \mathbf{B} then corresponds to a single degree of freedom with the trace being equal to n , the total degrees of freedom.

The transform \mathbf{L} can also be represented by $\mathbf{Q}^T\mathbf{L}$, where \mathbf{Q} is an orthogonal matrix. So that $\mathbf{Q}^T\mathbf{L}\mathbf{B}\mathbf{L}^T\mathbf{Q} = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}_{n \times n}$. By defining \mathbf{Q} to be the matrix of the eigenvectors of $\mathbf{L}\mathbf{A}\mathbf{L}^T$, we reduce \mathbf{B} to the identity and $\mathbf{L}\mathbf{A}\mathbf{L}^T$ to the diagonal matrix of its eigenvalues, $\mathbf{\Lambda}$. The eigenvalues λ_i of $\mathbf{L}\mathbf{A}\mathbf{L}^T$ can be interpreted as the fractional reduction in uncertainty for the n state members. If an eigenvalue is close to zero the corresponding state member has been well observed, if it is close to one the corresponding state member has not been constrained by the assimilated observations [Stewart et al., 2008]. We then define the degrees of freedom for signal as,

$$\begin{aligned} dfs &= \text{trace}(\mathbf{Q}^T\mathbf{L}\mathbf{B}\mathbf{L}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{L}\mathbf{A}\mathbf{L}^T\mathbf{Q}) \\ &= \text{trace}(\mathbf{I}_{n \times n} - \mathbf{\Lambda}) \\ &= n - \text{trace}(\mathbf{\Lambda}) \\ &= n - \text{trace}(\mathbf{L}\mathbf{A}\mathbf{L}^T) \\ &= n - \text{trace}(\mathbf{B}^{-1}\mathbf{A}). \end{aligned} \quad (14)$$

In Rodgers et al. [2000] it is shown that the *dfs* can also be calculated as the trace of the influence matrix \mathbf{S} (defined in section 2.1) with,

$$dfs = \text{trace}(\mathbf{S}) = \sum_i \lambda_i, \quad (15)$$

where λ_i is the i^{th} eigenvalue of \mathbf{S} .

2.3 Shannon information content

Shannon Information Content (SIC) is a measure of the reduction in entropy (uncertainty) given a set of observations. When a measurement is made, the entropy or uncertainty in our state decreases. The *SIC* of an observation is a measure of the factor by which the uncertainty decreases [Cover and Thomas, 1991]. We can define this using the prior, $p(\mathbf{x})$, and posterior, $p(\mathbf{x}|\mathbf{y})$, distributions as,

$$SIC = \int p(\mathbf{x}) \ln p(\mathbf{x}) d\mathbf{x} - \int p(\mathbf{y}) \int p(\mathbf{x}|\mathbf{y}) \ln p(\mathbf{x}|\mathbf{y}) d\mathbf{x} d\mathbf{y}. \quad (16)$$

From Rodgers et al. [2000], for the Gaussian case SIC becomes a function of the prior and posterior error covariance matrices with,

$$SIC = \frac{1}{2} \ln \frac{|\mathbf{B}|}{|\mathbf{A}|}. \quad (17)$$

The SIC can also be defined in terms of the eigenvalues of the influence matrix \mathbf{S} with,

$$SIC = -\frac{1}{2} \sum_i \ln |1 - \lambda_i| \quad (18)$$

where λ_i is the i^{th} eigenvalue of \mathbf{S} . In Eyre [1990] using SIC is shown to be beneficial over solely measuring the change in error variances before and after assimilation as the SIC also uses information about the change in error covariances. This is also true for the *dfs*.

3 Results

3.1 Chapter overview

Draft, redo once chapter complete: In this chapter we aim to analyse the information content in the observations used for assimilation with the DALEC1 and DALEC2 model. We begin by considering the observability of our system to see if we have enough information from the observations available to us in order to construct a unique solution to our data assimilation problem. In practice we include a background term in our assimilation (**REF DA section**) to ensure we can always find a locally unique solution. However, it is informative to show that observations alone provide us with enough information to find a unique solution.

We then consider different information content measures applied to our system in order to show how the information content varies for the different observation types available to us for both DALEC1 and DALEC2. Using these measure also allows us to consider the effect of including error correlations in our data assimilation algorithm (see previous results chapter **REF**) on the information content in the observations.

3.2 DALEC1 observability

DALEC1 is the original version of the DALEC2 model introduced in section **REF**. At the start of the PhD project work was undertaken with DALEC1 before the DALEC2 model was released. The version of DALEC1 used was an evergreen only model; further details of the model can be found in section **REF** and Williams et al. [2005].

We initially consider observability of the DALEC1 state estimation system. DALEC1 is a smaller model and allows us to understand the concept of observability before moving onto work with the more complicated DALEC2 joint state and parameter estimation system. DALEC1 was implemented in a 4D-Var data assimilation scheme for state estimation, with the tangent linear model being computed analytically by

hand. Using this analytic implementation of the tangent linear model we can compute the observability of the model for differing sets of observations. We have the tangent linear model,

$$\mathbf{M}_i = \frac{\partial \mathbf{m}_{i-1 \rightarrow i}(\mathbf{x}_i)}{\partial \mathbf{x}_i} \quad (19)$$

$$= \begin{pmatrix} (1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^i & 0 & 0 & 0 & 0 \\ f_{roo}(1 - f_{fol})(1 - f_{auto})\zeta^i & (1 - \theta_{roo}) & 0 & 0 & 0 \\ (1 - f_{roo})(1 - f_{fol})(1 - f_{auto})\zeta^i & 0 & (1 - \theta_{woo}) & 0 & 0 \\ \theta_{fol} & \theta_{roo} & 0 & (1 - (\theta_{min} + \theta_{lit})\chi^{i-1}) & 0 \\ 0 & 0 & \theta_{lit} & \theta_{min}\chi^{i-1} & (1 - \theta_{som}\chi^{i-1}) \end{pmatrix},$$

where $\mathbf{x}_i = (C_{fol}^i, C_{roo}^i, C_{woo}^i, C_{lit}^i, C_{som}^i)^T$, $\zeta^i = \partial GPP^i(C_{fol}^{i-1}, \Psi) / \partial C_{fol}^{i-1}$ and $\chi^{i-1} = e^{\Theta T^{i-1}}$ with the parameters and symbols having the same meaning as in section [REF](#).

We can use the linearised model with the linearised observation operator \mathbf{H}_i to form the matrix in equation (4) and compute the observability. We will need at least 5 observations of any type for the system to be observable as the state \mathbf{x}_0 is of size 5 in the DALEC1 state estimation case. We first consider the observability for 5 observations of LAI. For DALEC1 LAI takes the form

$$LAI^i = \frac{C_{fol}^i}{c_{lma}}. \quad (20)$$

We then have the linearised observation operator

$$\mathbf{H}_i = \frac{\partial LAI^i}{\partial \mathbf{x}_i} = \begin{pmatrix} \frac{1}{c_{lma}} & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (21)$$

Using the linearised observation operator and the linear model from equation 19 we can compute $\hat{\mathbf{H}}$ for 5 observations of LAI with

$$\hat{\mathbf{H}} = \begin{pmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \mathbf{M}_0 \\ \vdots \\ \mathbf{H}_4 \mathbf{M}_{3,0} \end{pmatrix} \quad (22)$$

$$= \begin{pmatrix} \frac{1}{c_{lma}} & 0 & 0 & 0 & 0 \\ \frac{1}{c_{lma}} \prod_{i=0}^1 ((1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^i) & 0 & 0 & 0 & 0 \\ \frac{1}{c_{lma}} \prod_{i=0}^2 ((1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^i) & 0 & 0 & 0 & 0 \\ \frac{1}{c_{lma}} \prod_{i=0}^3 ((1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^i) & 0 & 0 & 0 & 0 \\ \frac{1}{c_{lma}} \prod_{i=0}^4 ((1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^i) & 0 & 0 & 0 & 0 \end{pmatrix},$$

so that no matter how many observations of LAI we add, our system will not be observable as the rows of $\hat{\mathbf{H}}$ are all linearly dependant, so that $\hat{\mathbf{H}}$ in this case has rank 1. We can repeat this for different observations to see for which observation types our system is observable.

From figure 1 we can see that our system is observable for 5 observations of the soil and organic matter carbon pool C_{som} . In figure 1 we have shown results for the rank of $\hat{\mathbf{H}}$ when we have 5 observations in each case; this has also been tested with increasing numbers of observations being added to the system with the results from figure 1 remaining unchanged.

The system being observable for observations of C_{som} physically makes sense as all the carbon in the system that is not respired to the atmosphere eventually ends up in C_{som} , so that by taking observations of

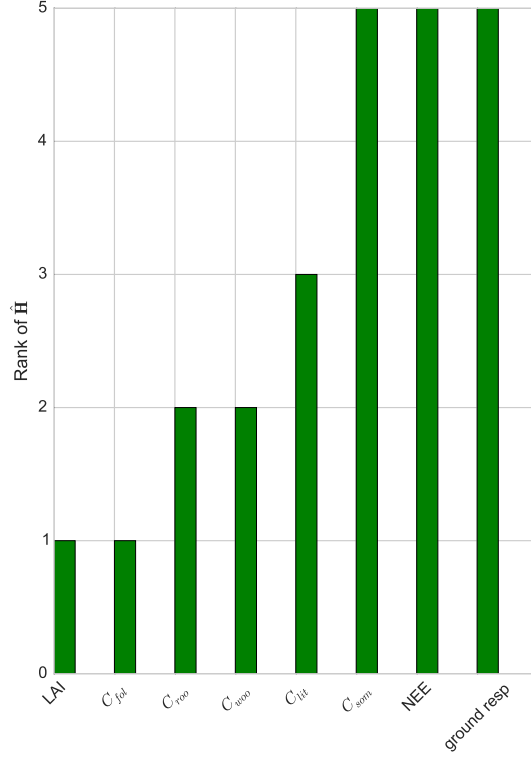


Figure 1: Rank of the observability matrix $\hat{\mathbf{H}}$ for 5 observations of different types. The ranks shown here are computed analytically using SymPy [Joyner et al., 2012].

this pool we observe all the others. In a similar way $\hat{\mathbf{H}}$ is also full rank for observations of NEE and ground respiration. We can see from the form of these observations in DALEC1 that they both contain indirect observations of C_{som} with NEE taking the form

$$NEE^i = -(1 - f_{auto})GPP^i(C_{fol}^{i-1}, \Psi) + \theta_{lit}C_{lit}e^{\Theta T^i} + \theta_{som}C_{som}e^{\Theta T^i} \quad (23)$$

with a corresponding linearised observation operator

$$\mathbf{H}_i = \frac{\partial NEE^i}{\partial \mathbf{x}_i} = \begin{pmatrix} -(1 - f_{auto})\zeta^i & 0 & 0 & \theta_{lit}e^{\Theta T^i} & \theta_{som}e^{\Theta T^i} \end{pmatrix}, \quad (24)$$

and for ground respiration

$$G_{resp}^i = \frac{1}{3}f_{auto}GPP^i(C_{fol}^{i-1}, \Psi) + \theta_{lit}C_{lit}e^{\Theta T^i} + \theta_{som}C_{som}e^{\Theta T^i} \quad (25)$$

(here we have assumed the fraction of total autotrophic respiration from below ground to be $\frac{1}{3}$) with a corresponding linearised observation operator

$$\mathbf{H}_i = \frac{\partial G_{resp}^i}{\partial \mathbf{x}_i} = \begin{pmatrix} \frac{1}{3}f_{auto}\zeta^i & 0 & 0 & \theta_{lit}e^{\Theta T^i} & \theta_{som}e^{\Theta T^i} \end{pmatrix}. \quad (26)$$

At flux tower sites NEE is the most observed quantity, these results give us confidence that we can construct a unique solution when working with flux tower data. We will further explore the concept of observability for the joint parameter and state estimation case with DALEC2 in section 3.3.

For the state and parameter estimation case we will not be able to compute the observability of the system analytically, it is therefore important to check that the numerical calculation of the rank of $\hat{\mathbf{H}}$ for DALEC1 is equal to the rank when calculated analytically. This will give us confidence that our implementation of the numeric rank is correct for DALEC2 when applied to a well-conditioned problem as the implementation is the same in both cases. In table 1 we show that for both numeric and analytic implementations we have the same results for the rank of $\hat{\mathbf{H}}$.

Observation	Rank of $\hat{\mathbf{H}}$ (numeric)	Rank of $\hat{\mathbf{H}}$ (analytic)
LAI	1	1
C_{fol}	1	1
C_{roo}	2	2
C_{woo}	2	2
C_{lit}	3	3
C_{som}	5	5
NEE	5	5
G_{resp}	5	5

Table 1: Rank of $\hat{\mathbf{H}}$ for 5 observations of different types for both numeric and analytic implementations with DALEC1.

3.3 DALEC2 observability

For DALEC2 we perform joint parameter and state estimation and have an augmented state of size $n = 23$. The augmented state is made up of the 6 carbon pool state members and 17 model parameters as described in section REF. As we are also estimating the parameters of DALEC2 the concept of observability for our system is closely linked to the concept of identifiability [Navon, 1998]. A system is identifiable if given observations of the state variables and knowledge of the model dynamics it is possible to obtain a unique deterministic set of model parameter values [Ljung, 1998]. If a model parameter is not observable it will not be identifiable [Jacquez and Greif, 1985]. It is therefore useful to compute the observability of the DALEC2 joint parameter and state estimation system.

We compute observability in the same way as in section 3.2 by finding the rank of $\hat{\mathbf{H}}$ for a given set of observations. As discussed in section 3.2 we cannot compute the observability analytically for the DALEC2 joint parameter and state estimation scheme. We have tested our numeric implementation for the state estimation case with DALEC1 and find the same results for the rank of $\hat{\mathbf{H}}$ as for the analytic case. We calculate the rank of the $\hat{\mathbf{H}}$ matrix using a singular value decomposition (SVD) which can have issues if the condition number of $\hat{\mathbf{H}}$ is large [Paige, 1981]. This is a problem we encounter in the DALEC2 case when trying to calculate the rank of $\hat{\mathbf{H}}$ directly.

In figure 2a we see that for 23 observations of NEE our system is unobservable as we have a rank deficient $\hat{\mathbf{H}}$. However, we cannot trust the rank calculation of $\hat{\mathbf{H}}$ in this case. Figure 2b shows that for 23 observations of NEE $\hat{\mathbf{H}}$ has a condition number in the order of 10^{19} . The condition number of a matrix corresponds to the ratio of the largest to the smallest singular values. A condition number of this size means that we have very small singular values. In the calculation of the rank of a matrix using an SVD we define the rank to be the number of singular values greater than the threshold $\text{tol} = \max(S) * \max(n, m) * \text{eps}$ [Press et al., 2007], where S is the vector of singular values, n and m are the rows and columns of the matrix whose rank we wish to calculate and eps is the machine accuracy for the datatype of S (In this case a double-precision float with $\text{eps} = 2.22\text{e-}16$). For 23 observations of NEE $\hat{\mathbf{H}}$ is classed as being rank

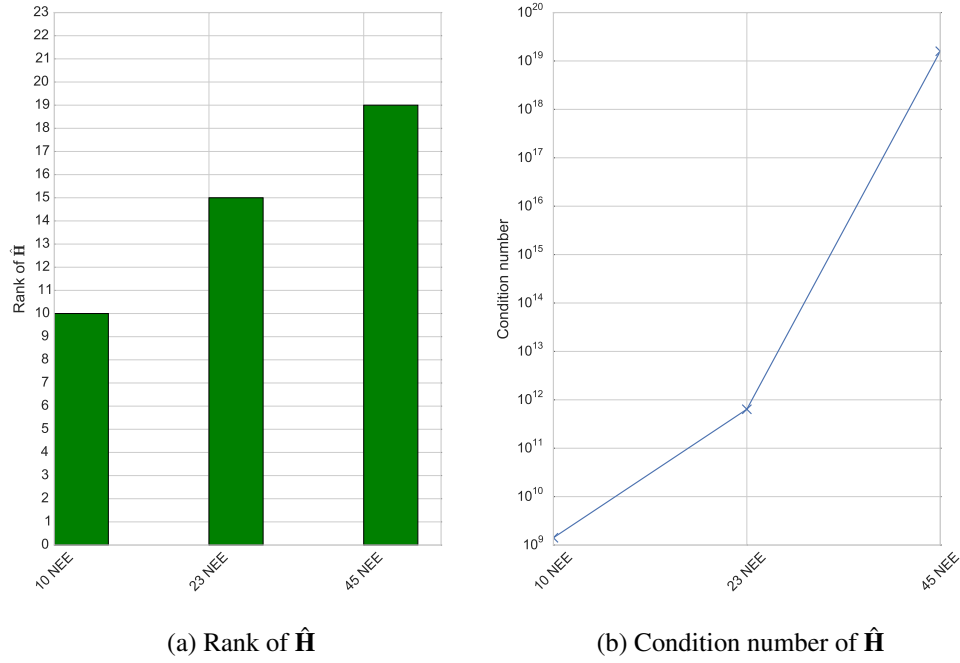


Figure 2: Observability of DALEC2 for $\hat{\mathbf{H}}$ with an increasing number of NEE observations displayed alongside the condition number for the $\hat{\mathbf{H}}$ matrices.

deficient as $\text{tol} = 1.02\text{e-}10$ and the three smallest singular values of $\hat{\mathbf{H}}$ are $[1.39\text{e-}11, 7.84\text{e-}15, 1.46\text{e-}15]$ but here we are working past the accuracy of the computer and so cannot have confidence that $\hat{\mathbf{H}}$ is rank deficient in this case.

In order to address the problem of ill-conditioning of the $\hat{\mathbf{H}}$ matrix we can instead calculate the rank of the control variable transform observability matrix, $\hat{\mathbf{R}}^{-1/2}\hat{\mathbf{H}}\mathbf{D}^{1/2}$, where the symbols have the same meaning as in section REF section where CVT is described, $\mathbf{D} = \text{diag}\{\mathbf{B}\}$. The rank of $\hat{\mathbf{R}}^{-1/2}\hat{\mathbf{H}}\mathbf{D}^{1/2}$ and $\hat{\mathbf{H}}$ are the same since $\hat{\mathbf{R}}$ and \mathbf{D} are both full rank matrices. From figure 3b we can see this matrix is much better conditioned than $\hat{\mathbf{H}}$ and for 23 observations of NEE we now have an observable system. Although the condition numbers here are still large we can have more confidence in these results as we are working within the precision of the computer.

In the previous experiments we have considered increasing numbers of NEE observations taken on adjacent days. It is also useful to consider the observability of the system when we have a number of observations randomly distributed throughout a time window. This is more consistent with what we expect from the real data we have to work with.

In figure 4 we see that having the observations randomly distributed throughout a 1 year assimilation window has improved the conditioning of $\hat{\mathbf{H}}$ in comparison to figure 2. This is due to the observations being randomly distributed rather than adjacent. The rows of $\hat{\mathbf{H}}$ are more distinct when being evolved to different times in the year by the tangent linear model rather than evolved to adjacent days only. However, we still have a rank deficient $\hat{\mathbf{H}}$ for the 23 NEE observation case. From figure 4b we see that this is the case where the condition number peaks. As we add more randomly distributed observations the condition number of $\hat{\mathbf{H}}$ is reduced by an order of 10^2 and we have a full rank $\hat{\mathbf{H}}$.

In figure 5 we again see that using the CVT observability matrix has much improved the conditioning of the problem in comparison to figure 4. We now have that the DALEC2 system is observable when we have

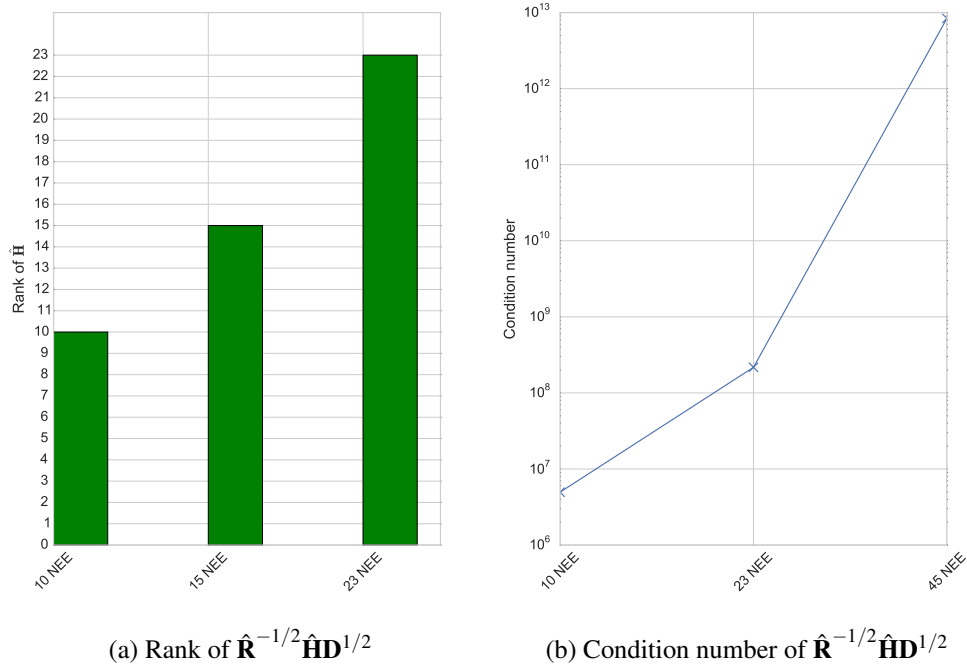


Figure 3: Observability of the CVT DALEC2 for $\hat{\mathbf{R}}^{-1/2} \hat{\mathbf{H}} \mathbf{D}^{1/2}$ with an increasing number of NEE observations displayed alongside the condition number for the $\hat{\mathbf{R}}^{-1/2} \hat{\mathbf{H}} \mathbf{D}^{1/2}$ matrices.

23 observations of NEE randomly distributed throughout the 1 year assimilation window. We have more confidence that this is the case as the condition numbers for the CVT observability matrix are almost half the values of those for $\hat{\mathbf{H}}$. We again see a similar pattern in figure 5 for the condition numbers with a peak for 23 NEE observations and then a reduction of order 10^2 when more observations are added.

We have tested the observability of the system for observations of NEE when we have different driving data, linearising around different states and with different distributions of observations throughout our assimilation window and in every case we have an observable system given an adequate number of NEE observations (at least 23). We can therefore have confidence that for the available data, typically 60-80 observations of daily NEE for any years window, we can construct a unique solution with the observations alone.

3.4 DALEC1 information content

3.4.1 Information content for observations at a single time

For the DALEC1 state estimation we can calculate the analytic representation of the information content measures discussed in section 2. This will allow us to understand how the information content changes for differing numbers of observations, different observation types and the effect of including observation error correlations in the assimilation scheme, before moving onto work with the larger DALEC2 joint parameter and state estimation case. For these experiments the elements of the state vector have corresponding

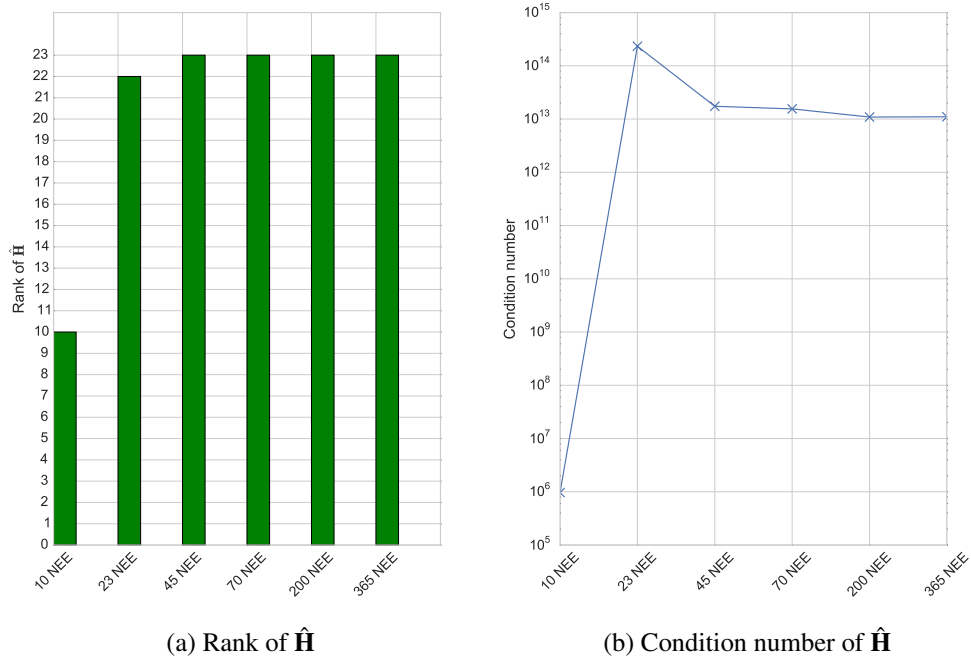


Figure 4: Observability of DALEC2 for a $\hat{\mathbf{H}}$ with an increasing number of NEE observations randomly distributed through a 1 year assimilation window (left). Condition number for the $\hat{\mathbf{H}}$ matrices (right).

background standard deviations $\sigma_{cfol,b}$, $\sigma_{croo,b}$, $\sigma_{cwoo,b}$, $\sigma_{clit,b}$, $\sigma_{csom,b}$. We then have

$$\mathbf{B} = \begin{pmatrix} \sigma_{cfol,b}^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{croo,b}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{cwoo,b}^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{clit,b}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{csom,b}^2 \end{pmatrix}. \quad (27)$$

We begin by considering the Shannon Information Content (SIC) and degrees of freedom for signal (dfs) for a single observation of LAI. We have the linearised observation operator

$$\mathbf{H}_i = \frac{\partial LAI^i}{\partial \mathbf{x}_i} = \frac{\partial}{\partial \mathbf{x}_i} \left(\frac{C_{fol}^i}{c_{lma}} \right) = \begin{pmatrix} \frac{1}{c_{lma}} & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

As we have a single observation at one time, our observation error covariance matrix, \mathbf{R} , is just the variance of our observation of LAI at time t_0 ($\sigma_{LAI,o}^2$). Therefore,

$$\mathbf{R}_i = \sigma_{LAI,o}^2. \quad (29)$$

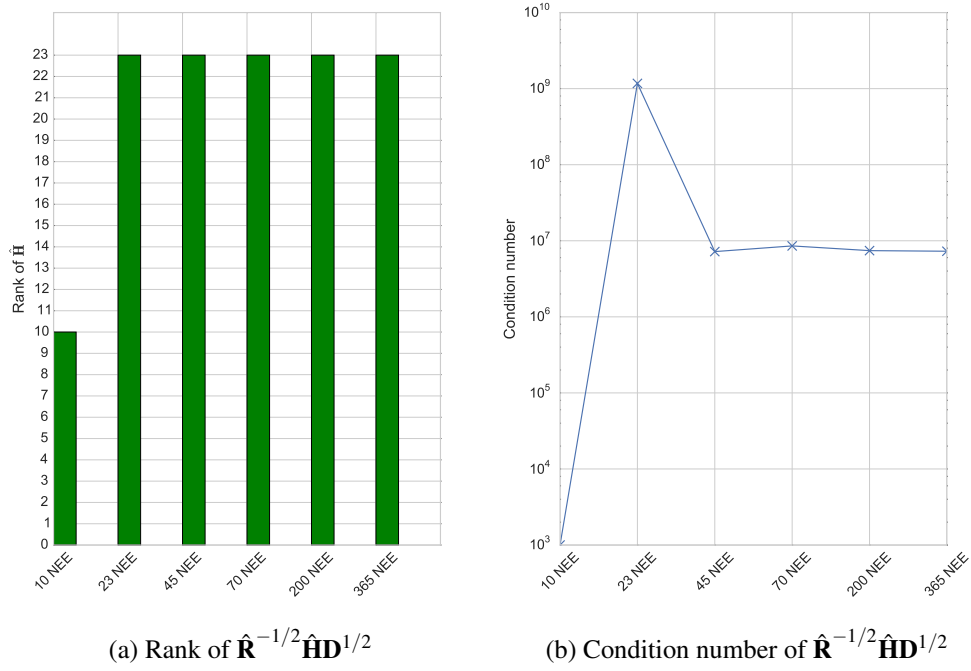


Figure 5: Observability of the CVT DALEC2 system for $\hat{\mathbf{R}}^{-1/2} \hat{\mathbf{H}} \mathbf{D}^{1/2}$ with an increasing number of NEE observations randomly distributed through a 1 year assimilation window (left). Condition number for the $\hat{\mathbf{R}}^{-1/2} \hat{\mathbf{H}} \mathbf{D}^{1/2}$ matrices (right).

We then have from equation [REF A mat eqn from DA section](#),

$$\begin{aligned}
 \mathbf{A} &= (\mathbf{J}'')^{-1} \\
 &= (\mathbf{B}^{-1} + \hat{\mathbf{H}}^T \hat{\mathbf{R}}^{-1} \hat{\mathbf{H}})^{-1} \\
 &= (\mathbf{B}^{-1} + \mathbf{H}_0^T \mathbf{R}_0^{-1} \mathbf{H}_0)^{-1} \\
 &= \begin{pmatrix} \frac{c_{lma}^2 \sigma_{LAI,o}^2 \sigma_{cfol,b}^2}{\sigma_{cfol,b}^2 + c_{lma}^2 \sigma_{LAI,o}^2} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{croo,b}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{cwo,b}^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{clit,b}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{csom,b}^2 \end{pmatrix}.
 \end{aligned} \tag{30}$$

We can now derive the SIC and dfs using equation (17) and (14) as,

$$\text{SIC} = \frac{1}{2} \ln \frac{|\mathbf{B}|}{|\mathbf{A}|} = \frac{1}{2} \ln \frac{(c_{lma}^2 \sigma_{LAI,o}^2 + \sigma_{cfol,b}^2)}{c_{lma}^2 \sigma_{LAI,o}^2} = \frac{1}{2} \ln \left(1 + \frac{\sigma_{cfol,b}^2}{c_{lma}^2 \sigma_{LAI,o}^2} \right) \tag{31}$$

and

$$dfs = n - \text{tr}(\mathbf{B}^{-1} \mathbf{A}) = 5 - \left(\frac{c_{lma}^2 \sigma_{LAI,o}^2}{(c_{lma}^2 \sigma_{LAI,o}^2 + \sigma_{cfol,b}^2)} + 4 \right) = 1 - \frac{c_{lma}^2 \sigma_{LAI,o}^2}{(c_{lma}^2 \sigma_{LAI,o}^2 + \sigma_{cfol,b}^2)}. \tag{32}$$

We see that in general for a direct observation of any of the carbon pools C we have

$$\text{SIC} = \frac{1}{2} \ln \left(1 + \frac{\sigma_{c,b}^2}{\sigma_{c,o}^2} \right) \tag{33}$$

and

$$dfs = 1 - \frac{\sigma_{c,o}^2}{(\sigma_{c,o}^2 + \sigma_{c,b}^2)}, \quad (34)$$

where $\sigma_{c,o}$ and $\sigma_{c,b}$ are the observation and background standard deviations respectively, corresponding to any of the 5 carbon pools. We see the SIC for a single observation of one of the carbon pools is dependent on the ratio between the observation and background variances. The carbon pool observation which will give us the highest SIC is the observation with the largest ratio $\frac{\sigma_{c,b}^2}{\sigma_{c,o}^2}$. This is also the case for dfs . Assuming a fixed background standard deviation, the carbon pool observation which will give us the highest information content is the pool which we can measure most accurately, as expected. From equations (31) and (32) for an observation of LAI the information content is also dependent on c_{lma} the parameter describing leaf mass area.

Next we consider the information content in a single observation of NEE. We have

$$\mathbf{H}_i = \frac{\partial NEE^i}{\partial \mathbf{x}_i} = \begin{pmatrix} -(1 - f_{auto})\zeta^i & 0 & 0 & \theta_{lit}e^{\Theta T^i} & \theta_{som}e^{\Theta T^i} \end{pmatrix} \quad (35)$$

and

$$\mathbf{R}_i = \sigma_{NEE,o}^2. \quad (36)$$

We then find

$$SIC = \frac{1}{2} \ln \left(1 + \frac{(f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{cfol,b}^2 + (e^{\Theta T^i})^2 (\theta_{som}^2 \sigma_{csom,b}^2 + \theta_{lit}^2 \sigma_{clit,b}^2)}{\sigma_{NEE,o}^2} \right) \quad (37)$$

and

$$dfs = 1 - \frac{\sigma_{NEE,o}^2}{(f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{cfol,b}^2 + (e^{\Theta T^i})^2 (\theta_{som}^2 \sigma_{csom,b}^2 + \theta_{lit}^2 \sigma_{clit,b}^2) + \sigma_{NEE,o}^2}. \quad (38)$$

We see that Equations (37) and (38) have a similar form to Equations (33) and (34). The information content is again dependent on the ratio between the observation and background variances. The information content for the observations of NEE is also dependent on the magnitude of the first derivative of GPP with respect to C_{fol} and the magnitude of the exponential function of temperature controlling the rate of heterotrophic respiration, $e^{\Theta T^i}$. Both the first derivative of GPP and $e^{\Theta T^i}$ will be of greater magnitude when we have higher mean daily temperatures. This means that observations of NEE made at times with higher temperatures will have higher information content and more of an impact on data assimilation results. In Figure REF we show how closely SIC is related to mean daily temperature for observations throughout a years using daily driving data from a pine stand in Oregon.

In Figure 6 we show how closely SIC is related to mean daily temperature for NEE observations throughout a years using daily driving data from a pine stand in Oregon. Higher information content in summer observations of NEE makes physical sense. In summertime fluxes of carbon through the forest ecosystem are of greater magnitude than in winter, with more photosynthesis and respiration occurring. This gives us more information about the fluxes of carbon through our system in summertime observations of NEE. It is important to consider this result when planning for down time or routine maintenance at flux tower sites measuring NEE. The temperature dependence of information content will also hold true for other observations whose observation operators include the nonlinear temperature term controlling heterotrophic respiration. These observations include ground respiration, measured using soil respiration chambers and total ecosystem respiration, estimated from nighttime NEE measurements.

In Figure 6a we have assumed constant prior and observation standard deviations. This is an accurate assumption for our prior errors. However, it has been shown that NEE errors are heteroskedastic [Richardson

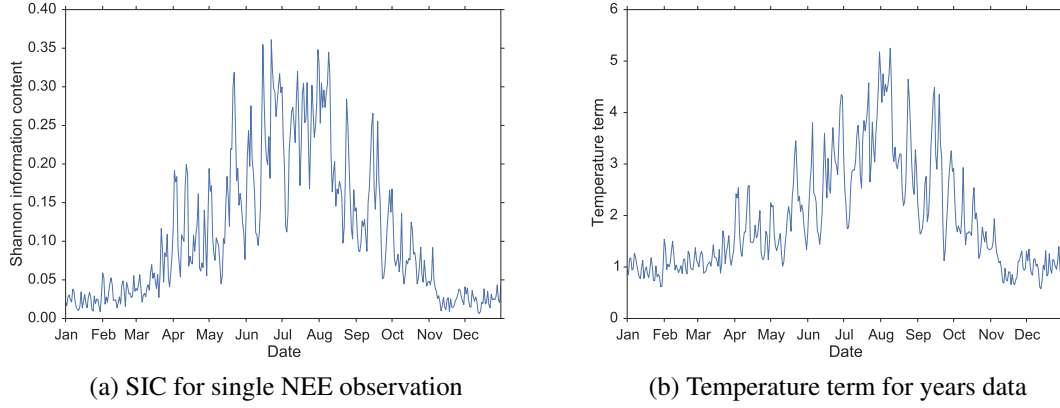


Figure 6: SIC for a single NEE observation changing throughout a years window using driving data from a pine stand in Oregon taken in 2007 (left). Temperature term $e^{\Theta T^i}$ (where T^i is the mean daily temperature) using the same driving data (right).

et al., 2008] and therefore scale with the magnitude of the flux. This would reduce the magnitude of the results shown in Figure 6a, as our standard deviation in observations of summer NEE would be larger, reducing the information content. Although the true NEE error is heteroskedastic for data assimilation the results in Figure 6a suggest assuming a constant standard deviation is a good approximation, if our aim is to best fit the data. A larger standard deviation in summer observations of NEE will reduce their weight in the assimilation algorithm. This will mean our assimilation will attempt to fit the winter observations more closely than the summer observations. If in fact the summer observations are providing the most information about the fluxes of carbon through our system they will help constrain our state better than winter observations. Hence assuming a constant standard deviation for NEE errors may help improve data assimilation forecasts. **TEST THIS??? (maybe with DALEC2? show improvement in forecast with constant σ vs. σ scaled by flux magnitude)**

For Figure 6a we have used a numerical implementation in Python to calculate the SIC varying for 365 days of driving data. It is important to test our numerical implementation for correctness. In table 2 we show the SIC and dfs calculated both analytically and numerically. From this table we can see that both analytic and numerical implementations give us the same result to machine precision. This gives us a degree of confidence that our implementation is also correct for DALEC2. In this table we have assumed constant prior and observation standard deviations for the carbon pools.

Obs.	SIC analytic value	SIC numeric value	dfs analytic value	dfs numeric value
NEE	0.0209343224569909	0.0209343224569913	0.0410042587324008	0.0410042587324008
C_{fol}	0.8047189562170501	0.8047189562170515	0.7999999999999998	0.7999999999999998
C_{roo}	0.1838623900626585	0.1838623900626572	0.3076923076923075	0.3076923076923083
C_{woo}	0.8047189562170501	0.8047189562170515	0.7999999999999998	0.7999999999999998
C_{lit}	0.1838623900626585	0.1838623900626572	0.3076923076923075	0.3076923076923074
C_{som}	0.1838623900626585	0.1838623900626572	0.3076923076923075	0.3076923076923074

Table 2: Correctness tests showing numeric and analytic values of information content calculated using 2007 driving data and parameter values from an Oregon pine forest.

We next consider the SIC when we have more than one observation at a single time. Here we will investigate the representation of information content when assimilating an observation of NEE with an

observation of a carbon pool state member. We begin with a single observation of NEE and an observation of C_{fol} . We have the linearised observation operator,

$$\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{fol}^i) = \begin{pmatrix} -(1-f_{auto})\zeta^i & 0 & 0 & \theta_{lit}e^{\Theta T^i} & \theta_{som}e^{\Theta T^i} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (39)$$

and observation error covariance matrix

$$\mathbf{R}_i = \begin{pmatrix} \sigma_{NEE,o}^2 & 0 \\ 0 & \sigma_{C_{fol},o}^2 \end{pmatrix}. \quad (40)$$

We then find,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{C_{fol},b}^2}{\sigma_{C_{fol},o}^2} + \frac{\xi^i}{\sigma_{NEE,o}^2} + \frac{\sigma_{C_{fol},b}^2 (e^{\Theta T^i})^2 (\theta_{som}^2 \sigma_{csom,b}^2 + \theta_{lit}^2 \sigma_{clit,b}^2)}{\sigma_{NEE,o}^2 \sigma_{C_{fol},o}^2} \right) \quad (41)$$

where, $\xi^i = (f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{C_{fol},b}^2 + (e^{\Theta T^i})^2 (\theta_{som}^2 \sigma_{csom,b}^2 + \theta_{lit}^2 \sigma_{clit,b}^2)$. From equation (41) we can see that we have the first order terms for both NEE and C_{fol} as in equations (33) and (37). We also have a second order term for the combination of these observations. We can repeat this for the other carbon pools and find for $\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{roo}^i)$,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{C_{roo},b}^2}{\sigma_{C_{roo},o}^2} + \frac{\xi^i}{\sigma_{NEE,o}^2} + \frac{\sigma_{C_{roo},b}^2 ((f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{C_{fol},b}^2 + (e^{\Theta T^i})^2 (\theta_{som}^2 \sigma_{csom,b}^2 + \theta_{lit}^2 \sigma_{clit,b}^2))}{\sigma_{NEE,o}^2 \sigma_{C_{roo},o}^2} \right), \quad (42)$$

for $\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{woo}^i)$,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{C_{woo},b}^2}{\sigma_{C_{woo},o}^2} + \frac{\xi^i}{\sigma_{NEE,o}^2} + \frac{\sigma_{C_{woo},b}^2 ((f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{C_{fol},b}^2 + (e^{\Theta T^i})^2 (\theta_{som}^2 \sigma_{csom,b}^2 + \theta_{lit}^2 \sigma_{clit,b}^2))}{\sigma_{NEE,o}^2 \sigma_{C_{woo},o}^2} \right), \quad (43)$$

for $\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{lit}^i)$,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{C_{lit},b}^2}{\sigma_{C_{lit},o}^2} + \frac{\xi^i}{\sigma_{NEE,o}^2} + \frac{\sigma_{C_{lit},b}^2 ((f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{C_{fol},b}^2 + (e^{\Theta T^i})^2 \theta_{som}^2 \sigma_{csom,b}^2)}{\sigma_{NEE,o}^2 \sigma_{C_{lit},o}^2} \right) \quad (44)$$

and for $\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{som}^i)$,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{C_{som},b}^2}{\sigma_{C_{som},o}^2} + \frac{\xi^i}{\sigma_{NEE,o}^2} + \frac{\sigma_{C_{som},b}^2 ((f_{auto} - 1)^2 (\zeta^i)^2 \sigma_{C_{fol},b}^2 + (e^{\Theta T^i})^2 \theta_{lit}^2 \sigma_{clit,b}^2)}{\sigma_{NEE,o}^2 \sigma_{C_{som},o}^2} \right). \quad (45)$$

Assuming constant prior and observation standard deviations across our carbon pool observations we see that the information content will be largest in equations (42) and (43). For both $\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{roo}^i)$ and $\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{woo}^i)$ we have an extra term in the numerator for our second order term corresponding to the combination of the two observations. If we consider the linearised observation operator for both these cases,

$$\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{roo}^i) = \begin{pmatrix} -(1-f_{auto})\zeta^i & 0 & 0 & \theta_{lit}e^{\Theta T^i} & \theta_{som}e^{\Theta T^i} \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (46)$$

and

$$\mathbf{H}_i = \frac{\partial}{\partial \mathbf{x}_i}(NEE^i, C_{woo}^i) = \begin{pmatrix} -(1-f_{auto})\zeta^i & 0 & 0 & \theta_{lit}e^{\Theta T^i} & \theta_{som}e^{\Theta T^i} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad (47)$$

we can see that these observations provide an orthogonal constraint to the observation of NEE. Neither or these pools are observed with a single observation of NEE. The information content being greater when assimilating C_{roo} or C_{woo} alongside NEE is therefore expected.

In practice we cannot assume constant prior and observation errors across the different carbon pools. Root carbon is hard to measure accurately [Brown, 2002]. However, woody biomass (C_{woo}) is regularly measured using mensuration [Husch et al., 2002] or point-centred quarter methods [Dahdouh-Guebas and Koedam, 2006] at forest sites. Advancements in Light Detection And Ranging (LiDAR) scanning [Lefsky et al., 1999] mean that we have increasingly more accurate observations of woody biomass. The European Space Agency BIOMASS mission [Le Toan et al., 2011] will also provide a much more abundant source of woody biomass measurements in the future. If we consider NEE to be the main observation currently used in ecosystem data assimilation, then the increasing number of available woody biomass measurements will benefit assimilation schemes greatly.

3.4.2 Information content in successive observations

In section 3.4.1 we investigate the information in observation for DALEC1 at a single time. In this section we will consider successive observations in time. It has been shown that the SIC in observations is additive with successive observations in time. The proof for this can be found in appendix A.1 of Fowler and Jan Van Leeuwen [2012]. We can see this if we calculate the SIC for successive observations of foliar carbon, C_{fol} . We have the linearised observation operator and observation error covariance matrix at time t_i ,

$$\mathbf{H}_i = \frac{\partial C_{fol}^i}{\partial \mathbf{x}_i} = (1 \quad 0 \quad 0 \quad 0 \quad 0) \quad \text{and} \quad \mathbf{R}_i = \sigma_{cfol,o}^2. \quad (48)$$

For two successive observations of C_{fol} we have,

$$\hat{\mathbf{H}} = \begin{pmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \mathbf{M}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ (1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (49)$$

and

$$\hat{\mathbf{R}} = \begin{pmatrix} \mathbf{R}_0 & 0 \\ 0 & \mathbf{R}_1 \end{pmatrix} = \begin{pmatrix} \sigma_{cfol,o}^2 & 0 \\ 0 & \sigma_{cfol,o}^2 \end{pmatrix}. \quad (50)$$

We then have,

$$SIC = \frac{1}{2} \ln \frac{|\mathbf{B}|}{|\mathbf{A}|} = \frac{1}{2} \ln \left(1 + \frac{\sigma_{cfol,b}^2}{\sigma_{cfol,o}^2} + \frac{\sigma_{cfol,b}^2 \eta_0^2}{\sigma_{cfol,o}^2} \right), \quad (51)$$

where $\eta_i = (1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^i$. We see this is similar to equation (33) for the SIC of a single carbon pool observation but with an added term evolved by the linearised model. Here the second term is multiplied by η_0^2 which is the square of the first element of the linearised model \mathbf{M}_0 . We can continue adding more observations at successive times. For three observations at successive times we have,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{cfol,b}^2}{\sigma_{cfol,o}^2} + \frac{\sigma_{cfol,b}^2 \eta_0^2}{\sigma_{cfol,o}^2} + \frac{\sigma_{cfol,b}^2 \eta_0^2 \eta_1^2}{\sigma_{cfol,o}^2} \right), \quad (52)$$

for four,

$$SIC = \frac{1}{2} \ln \left(1 + \frac{\sigma_{cfol,b}^2}{\sigma_{cfol,o}^2} + \frac{\sigma_{cfol,b}^2 \eta_0^2}{\sigma_{cfol,o}^2} + \frac{\sigma_{cfol,b}^2 \eta_0^2 \eta_1^2}{\sigma_{cfol,o}^2} + \frac{\sigma_{cfol,b}^2 \eta_0^2 \eta_1^2 \eta_2^2}{\sigma_{cfol,o}^2} \right). \quad (53)$$

Using a simple proof by induction we find that for n successive observations we have,

$$SIC \text{ for } n \text{ successive observations of } C_{fol} = \frac{1}{2} \ln \left(1 + \frac{\sigma_{cfol,b}^2}{\sigma_{cfol,o}^2} \left(1 + \sum_{k=0}^{n-2} \prod_{i=0}^k \eta_i^2 \right) \right) \quad (54)$$

This demonstrates that SIC is additive for successive observations in time. In Figure 7 we have plotted the SIC and dfs for increasing numbers of observations of C_{fol} , using a year of meteorological driving data from a pine stand in Oregon. We see that as successive observations are added the information content tends to a limit where we are adding no new information with extra observations of C_{fol} . For dfs this limit is one as we are only observing a single degree of freedom so cannot constrain more than a single element of the state. For SIC we add a decreasing amount of information as observations are made further away from the initial state. We find similar results for all other carbon pools. This suggests making observations of any individual carbon pool for a forest site too often is not cost effective as after just a few observations the information you are adding to your system begins to decrease.

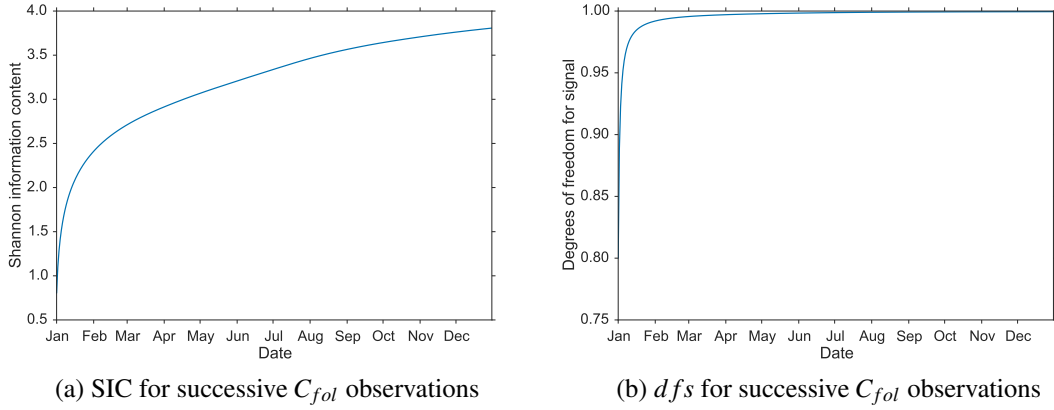


Figure 7: SIC and dfs for as successive C_{fol} observations are added throughout a years window using driving data from a pine stand in Oregon taken in 2007.

In section 3.4.1 it was shown that observations of NEE made during the summer had significantly higher information content than those made during winter for an evergreen forest site. In figure 8 we show that 27 days of successive winter NEE observations (made from January 1st 2007) are required to give the same information content as a single summer observation of NEE (taken on 22nd June 2007).

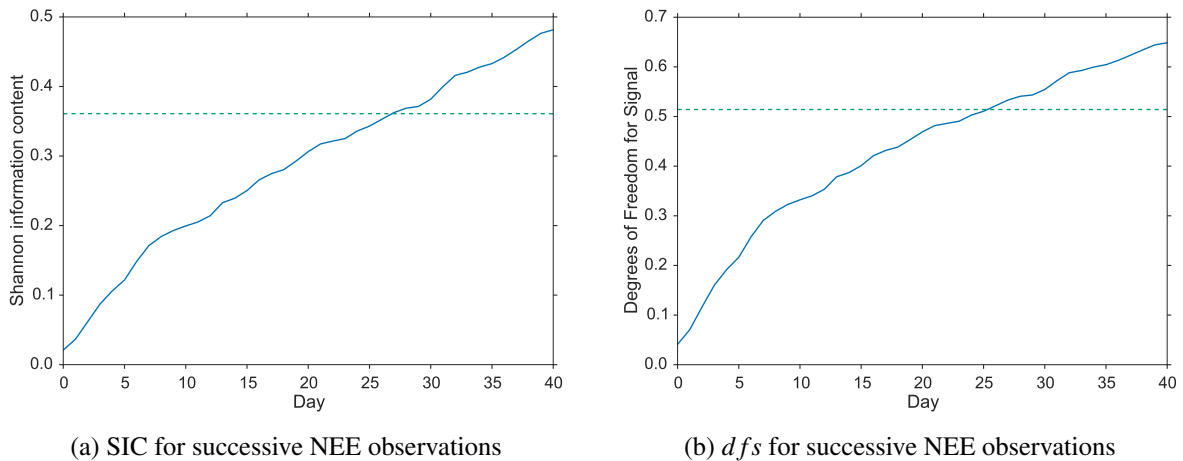


Figure 8: Blue line: SIC and dfs for as successive NEE observations are added for 40 days from the 1st January 2007 using driving data from a pine stand in Oregon, green dotted line: SIC and dfs for a single summer observation of NEE made on 22nd June 2007.

In chapter REF we have investigated the effect of including correlations in time between observation errors on the results from data assimilation with DALEC2. We can see the effect on the analytic representation of information content for two successive observations of C_{fol} when including an off-diagonal correlation term in the matrix $\hat{\mathbf{R}}$. We have

$$\mathbf{H}_i = \frac{\partial C_{fol}^i}{\partial \mathbf{x}_i} = (1 \ 0 \ 0 \ 0 \ 0) \quad \text{and} \quad \mathbf{R}_i = \sigma_{cfol,o}^2. \quad (55)$$

For two successive observations of C_{fol} we have,

$$\hat{\mathbf{H}} = \begin{pmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \mathbf{M}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ (1 - \theta_{fol}) + f_{fol}(1 - f_{auto})\zeta^0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (56)$$

and

$$\hat{\mathbf{R}} = \begin{pmatrix} \mathbf{R}_0 & \sigma_{1,2}^2 \\ \sigma_{2,1}^2 & \mathbf{R}_1 \end{pmatrix} = \begin{pmatrix} \sigma_{cfol,o}^2 & \sigma_{1,2}^2 \\ \sigma_{2,1}^2 & \sigma_{cfol,o}^2 \end{pmatrix}, \quad (57)$$

here $\hat{\mathbf{R}}$ is symmetric and we have $\sigma_{1,2}^2 = \sigma_{2,1}^2 = \sigma_{corr}^2$. We then find,

$$SIC = \frac{1}{2} \ln \frac{|\mathbf{B}|}{|\mathbf{A}|} = \frac{1}{2} \ln \left(1 + \frac{\sigma_{cfol,b}^2}{\sigma_{cfol,o}^2 - \frac{\sigma_{corr}^2}{\sigma_{cfol,o}^2}} + \frac{\sigma_{cfol,b}^2 \eta_0^2}{\sigma_{cfol,o}^2 - \frac{\sigma_{corr}^2}{\sigma_{cfol,o}^2}} - \frac{2\sigma_{cfol,b}^2 \sigma_{corr} \eta_0}{\sigma_{cfol,o}^4 - \sigma_{corr}^2} \right), \quad (58)$$

We have a similar expression as in equation (51) qith no correlations. Terms 2 and 3 in equation (58) will be slightly increased as $\frac{\sigma_{corr}^2}{\sigma_{cfol,o}^2}$ has been subtracted from the denominator of each term. These terms will only be slightly increased as $\frac{\sigma_{corr}^2}{\sigma_{cfol,o}^2}$ is bound above by 1 and will tend to be smaller as the correlation between the two successive observation errors will tend to be closer to zero than 1. As we have an additional 4th term in equation (58) subtracted from the other 3 terms the included time error correlation has the overall effect of reducing the information content in the two successive C_{fol} observations. The reduction in information content in successive observations makes sense as we are assimilating less independent information. This backs up the results seen in the previous chapter REF where including correlations in time between observations of NEE reduces overfitting to the observations in the assimilation window and gives an improvement to the model forecast.

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