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Topics on scattering amplitudes

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Abstract: This note covers various topics of scattering amplitudes, including the new conformal symmetry, off-shell extension of CHY, amplitudes in four dimensions, string theory and so on.

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Contents

1 Introduction

I write various new process on scattering amplitudes and problems deserve further study in this note.

2 Preparations

2.1 Decompositions of tree-level GR and color-ordered YM amplitudes

Gravitons as two copies of gluons A tree-level scattering amplitude $M^{\text{GR}}(1,2,\dots,n)$ for n gravitons is a rational function of Lorentz contractions of external polarization tensors $\epsilon_i^{\mu\nu}$ $(i=1,\dots,n)$ and momenta k_i^{μ} $(i=1,\dots,n)$, where $\epsilon_i^{\mu\nu}$ is a *traceless symmetric tensor* which satisfies

$$k_{i\,\mu}\epsilon_i^{\mu\nu}=k_{i\,\nu}\epsilon_i^{\mu\nu}=0. \tag{2.1}$$

In Yang-Mills (YM) theory, an external polarization (of a gluon) ϵ_i^{μ} carries only one Lorentz index and satisfies the condition

$$k_{i\,\mu}\cdot\epsilon_i^{\mu}=0. \tag{2.2}$$

Thus it seems that a graviton polarization tensor may be written as two copies of gluon polarizations:

$$\epsilon_i^{\mu\nu}\to\frac{1}{2}(\epsilon_i^{\mu}\widetilde{\epsilon}_i^{\nu}+\epsilon_i^{\mu}\widetilde{\epsilon}_i^{\nu})-\epsilon_i^{\mu}\epsilon_{i\,\mu}.^{\text{[Eq:Polaization1]}} \tag{2.3}$$

A polarization ϵ_i^μ depends on the choice of gauge and the corresponding momentum k_i^μ. Once we choose the same gauge for ϵ_i^μ and $\widetilde{\epsilon}_i^\nu$, $\epsilon_i^\mu \widetilde{\epsilon}_i^\nu$ is naturally a symmetric tensor. Furthermore, in four dimensions, one can always construct YM polarizations such that $\epsilon_i^\mu \epsilon_{i\,\mu} = 0$. In this sense, we will use

$$\epsilon_i^{\mu\nu} \to \epsilon_i^\mu \widetilde{\epsilon}_i^\nu \text{[Eq:Polaization2]} \tag{2.4}$$

instead of eq. (2.3). In the following, all gravitons and gluons are treated as massless particles, thus we always have $k_i^2 = 0$ if we are considering the *on-shell amplitudes*. Another condition for momenta is momentum conservation $\sum_{i=1}^{n} k_i^\mu = 0$.

Decompositions of GR and YM amplitudes Tree-level scattering amplitudes for n gravitons can be written as

$$M^{\mathrm{GR}}(1,2,\ldots,n) = \sum_{\boldsymbol{\sigma}\in S_{n-2}} N_{1|\boldsymbol{\sigma}|n} A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n), \text{[Eq:ExpansionGR]} \tag{2.5}$$

where $N_{1,\boldsymbol{\sigma},n}$ are called BCJ numerators in DDM basis. In the above equation, $A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n)$ denote color-ordered YM amplitudes, while $\boldsymbol{\sigma}$ denote the $(n-2)!$ permutations (i.e. an element in S_{n-2}) of the external particles $2,3,\ldots,n-1$. In general, the numerator $N_{1|\boldsymbol{\sigma}|n}$ is a rational function of $k_i \cdot k_j$, $\epsilon_i \cdot k_j$ and $\epsilon_i \cdot \epsilon_j$, while the other copies of polarizations $\widetilde{\epsilon}_i$ are included in the color ordered YM amplitudes $A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n)$. A BCJ numerator $N_{1|\boldsymbol{\sigma}|n}$ does not have unique form because it is gauge dependent.

A color-ordered YM amplitude $A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n)$ can be decomposed similarly:

$$A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n) = \sum_{\boldsymbol{\rho}\in S_{n-2}} \widetilde{N}_{1|\boldsymbol{\rho}|n} A^{\mathrm{BS}}(1,\boldsymbol{\sigma},n|1,\boldsymbol{\rho},n). \text{[Eq:ExpansionYM]} \tag{2.6}$$

Here $\widetilde{N}_{1|\boldsymbol{\rho}|n}$ (the other copy of BCJ numerator) is a function of Lorentz contractions $\widetilde{\epsilon}_i \cdot \widetilde{\epsilon}_j$, $\widetilde{\epsilon}_i \cdot k_j$ and $k_i \cdot k_j$. $\widetilde{N}_{1|\boldsymbol{\rho}|n}$ can be constructed by a same rule with $N_{1|\boldsymbol{\rho}|n}$. The amplitude $A^{\mathrm{BS}}(1,\boldsymbol{\rho},n)$ is called bi-scalar amplitude which depends on both permutations $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ and does not contain any polarization.

Inserting the eq. (2.6) into the gravity amplitude eq. (2.10), we get

$$M^{\mathrm{GR}}(1,2,\ldots,n) = \sum_{\boldsymbol{\sigma}\in S_{n-2}} \sum_{\boldsymbol{\rho}\in S_{n-2}} N_{1|\boldsymbol{\sigma}|n} A^{\mathrm{BS}}(1,\boldsymbol{\sigma},n|1,\boldsymbol{\rho},n) \widetilde{N}_{1|\boldsymbol{\rho}|n}, \text{[Eq:ExpansionGR1]} \tag{2.7}$$

which expresses a gravity amplitude $M^{\mathrm{GR}}(1,2,\ldots,n)$ by bi-scalar ones associated with two copies of BCJ numerators. Apparently, information of all polarizations are included in the BCJ numerators.

2.2 The rule for bi-scalar amplitudes

Now we present the construction rule for bi-scalar amplitudes $A^{\mathrm{BS}}(1,\boldsymbol{\sigma},n|1,\boldsymbol{\rho},n)$. A bi-scalar amplitude is given by summing over all possible *trivalent Feynman diagrams* (i.e. diagrams constructed by only *cubic vertices*) that are allowed by both permutations $1,\boldsymbol{\sigma},n$ and $1,\boldsymbol{\rho},n$. Each diagram contributes the product

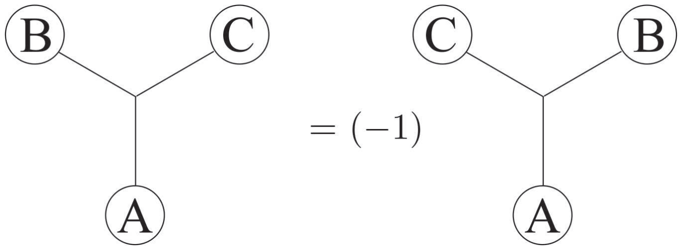

Figure 1. Two graphs which are related by exchanging two branches attached to a same cubic vertex must have opposite signs.

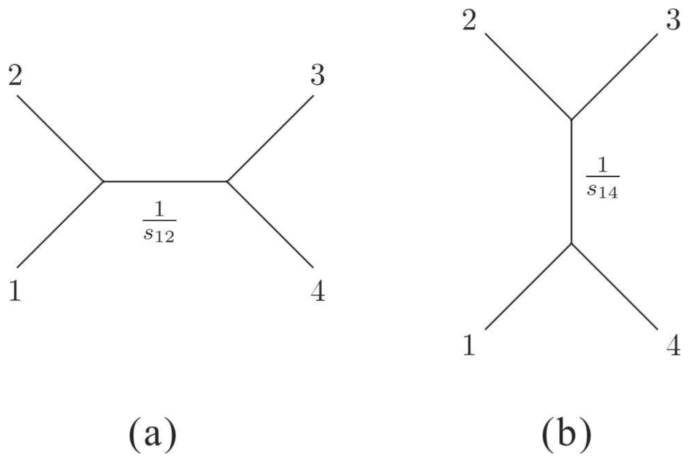

Figure 2. Both (a) and (b) are Feynman diagrams for $A^{\mathrm{BS}}(1,2,3,4|1,2,3,4)$. Only (b) is allowed by $A^{\mathrm{BS}}(1,2,3,4|1,3,2,4)$.

of propagators therein, with a proper sign. The sign is determined as follows: if two diagrams contribute the same propagators, they must be related by exchanging branches attached to cubic vertices. Such an exchanging introduces a minus sign (see Fig. 2). Consequently, if the number of such exchangings is even, the sign should be (+1), otherwise, (-1).

Let us take the four-point BS amplitudes as examples. For the amplitude $A^{\mathrm{BS}}(1,2,3,4|1,2,3,4)$, $\boldsymbol{\sigma} = \boldsymbol{\rho} = \{2,3\}$. The Feynman diagrams allowed by both permutations are shown by Fig. 2 (a) and (b). Therefore, the amplitude is

$$A^{\mathrm{BS}}(1,2,3,4|1,2,3,4) = \frac{1}{s_{12}} + \frac{1}{s_{14}}. \tag{2.8}$$

Here, $s_{ij} \equiv (k_i + k_j)^2 = 2k_i \cdot k_j$.

Another four-point amplitude $A^{\mathrm{BS}}(1,2,3,4|1,3,2,4)$ is only given by Fig. 2 (b). This is because *(i)* both diagrams (a) and (b) in Fig. 2 are allowed by the permutation $1,2,3,4$ *(ii)* the diagrams Fig. 2 (a), (b) when exchanging $2 \leftrightarrow 3$ are those allowed by the permutation $1,3,2,4$. The common diagram of *(i)* and *(ii)* is only Fig. 2 (b) (although for (ii), we should exchange 2 and 3 in Fig. 2 (b), it contributes the

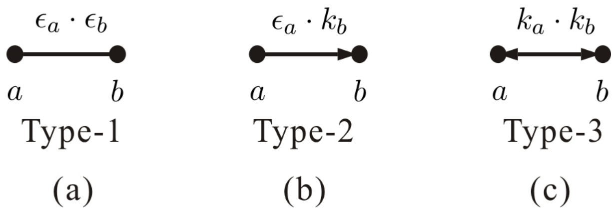

Figure 3. Line styles

same propagator $\frac{1}{s_{14}}$). The exchanging number is 1 for Fig. 2 (b). Thus

$$A^{\rm BS}(1,2,3,4|1,3,2,4)=-\frac{1}{s_{14}}. \tag{2.9}$$

Exercise-1 What are the Feynman diagrams for the following five-point bi-scalar amplitudes: $A^{\rm BS}(1,2,3,4,5|1,2,3,4,5)$, $A^{\rm BS}(1,4,3,2,5|1,3,2,4,5)$ and $A^{\rm BS}(1,2,3,4,5|1,4,3,2,5)$? Calculate these amplitudes.

2.3 Refined graphic rule for GR and color-ordered YM amplitudes

Instead of presenting a construction rule for the BCJ numerators with a permutation $\boldsymbol{\sigma}$ explicitly (although we are able to do this), we provide another expression of GR amplitude here:

$$M^{\rm GR}(1,2,\dots,n)=\sum_{\mathcal{F}}C^{[\mathcal{F}]}\left[\sum_{\boldsymbol{\sigma}^{\mathcal{F}}}A^{\rm YM}(1,\boldsymbol{\sigma}^{\mathcal{F}},n)\right],^{\text{[Eq:ExpansionGR]}} \tag{2.10}$$

where all possible refined graphs $\mathcal{F}$ have been summed over. For each graph, we have *(i)* a coefficient $C^{[\mathcal{F}]}$ which is a polynomial of factors $k_i{\cdot}k_j$, $\epsilon_i{\cdot}\epsilon_j$ and $\epsilon_i{\cdot}\epsilon_j$, *(ii)* a sum of amplitudes with permutations $\boldsymbol{\sigma}^{\mathcal{F}}$ that are allowed by this graph. When $M^{\rm GR}(1,2,\dots,n)$ on the LHS is replaced by the color-ordered YM amplitude $A^{\rm YM}(1,\boldsymbol{\sigma},n)$ and $A^{\rm YM}(1,\boldsymbol{\sigma}^{\mathcal{F}},n)$ on the RHS is replaced by the bi-scalar amplitude $A^{\rm BS}(1,\boldsymbol{\sigma}^{\mathcal{F}},n|1,\boldsymbol{\rho},n)$, the expression eq. (2.10) becomes an expansion of YM amplitudes. We will explain the coefficients $C^{[\mathcal{F}]}$ and permutations $\boldsymbol{\sigma}^{\mathcal{F}}$ in detail.

Refined graphic rule

Reference order Consider each graviton as a node, define a reference order

$$\mathsf{R}=\{1,\rho(1),...,\rho(n-1),n\},^{\text{[Eq:ReferenceOrder]}} \tag{2.11}$$

where the two gravitons 1 and 2 are considered as the first and the last elements in R (of cause, you can chose other two elements instead). The permutation $\boldsymbol{\rho}\equiv\{\rho(2),\rho(3),\dots,\rho(n-1)\}$ is an arbitrary permutation of 2, 3, ..., $n-1$. The position of each element in R is called its *weight*. Apparently, n is the highest-weight element and 1 is the lowest-weight one in $\mathsf{R}=\{1,\rho(1),...,\rho(n-1),n\}$.

Nodes and line styles Every external particle is considered as a node. To construct a graph, we

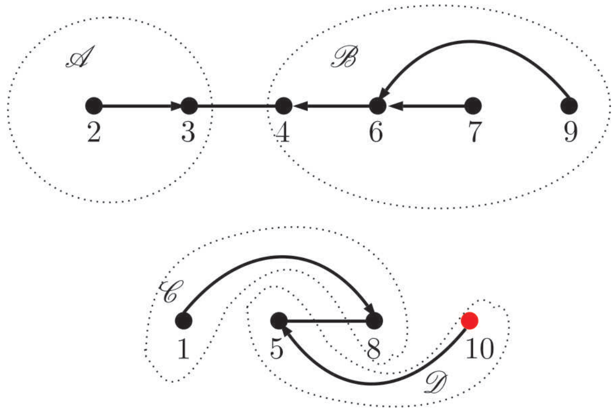

Figure 4.

need to connect lines between nodes. There are three types of lines Fig. 3 (a), (b) and (c) which correspond to factors $\epsilon_a \cdot \epsilon_b$, $\epsilon_a \cdot k_b$ and $k_a \cdot k_b$.

The construction of graphs Graphs are constructed as follows:

(i) **Grouping nodes** We divide the gravitons $1, \ldots, n$ into several sets such that *(a)* the highest- and the lowest-weight elements in the reference order eq. (2.11) are always in a same group, *(b)* every group must contains at least two elements. **Exercise-2: Supposing that the reference order for 4-point amplitude is $\mathsf{R} = \{1, 2, 3, 4\}$, what are the possible groupings?**

(ii) **Constructing components for each group of elements** Given a grouping, **for the set containing the highest-weight and the lowest-weight elements** 1, n (for example the set $\{1, 5, 8, 10\}$ in Fig. 4), find out a path (which may passes through other nodes in this group) from n to 1 (for example the path in Fig. 4 is $10 \to 5 \to 8 \to 1$) and connect arbitrary two adjacent nodes on this path via a type-1 line (for example the nodes 5 and 8 are connected by a type-1 line), which is called the *kernel*. Any other pair of adjacent nodes on this path are connected by a type-2 line (Fig. 3 (b)) whose arrow points to the kernel (for example, the 1,8 nodes and the 10, 5 nodes are connected by type-2 lines (with arrows pointing to the kernel) respectively). We also connect other nodes in this set arbitrarily via type-2 lines pointing towards those nodes (**except the node** n) on the path from n and 1. Then all nodes in this set form a *fully connected tree graph, which is called a component. For a set which does not contain* 1 *and* n (for example the set $\{2, 3, 4, 6, 7, 9\}$ in Fig. 4), pick out arbitrary two elements (in Fig. 4 the elements 3 and 4) and connect them by a type-1 line (the kernel). Other nodes in this set point towards the kernel via type-2 lines in an arbitrary way. Then nodes in this set also form a fully connected tree graph (a *component*). A graph with a given grouping and a given configuration of components for each set in this grouping is called a *skeleton* (for example, Fig. 4 is a skeleton for 10-point amplitude). Apparently, each skeleton consists of at least one component. **Exercise-3: Find out all possible skeletons for four-point amplitudes.**

(iii) **Weights of components, the top and the bottom sides of components** Each component is separated into two parts by its kernel. We define the *top side* by the part which containing the highest-weight node in this component. The opposite side is called the *bottom side*. For example, if the reference order is defined by the regular order $\mathsf{R} = \{1, 2, 3, \dots, 9, 10\}$, the top side of components containing $2, 3, 4, 6, 7, 9$ in Fig. 4 is the part $\mathscr{B}$, while the bottom side is the part $\mathscr{A}$. The top side of the component containing $1, 5, 8, 10$ is $\mathscr{D}$, while the bottom side is $\mathscr{C}$. We define the *weight of each component* by the weight of its highest-weight node.

(vi) **From skeletons to fully connected graphs** If a skeleton has only one component, it is already a connected graph. If a skeleton consists of more than one mutually disjoint components, we should connect these components, via type-3 lines, in a proper way to form a fully connected tree graph.

- ***Step-1***, we arrange those components, which does not contain 1 and n, into an ordered set $\mathsf{R}^{\mathscr{C}} = \{\mathscr{C}_1, \mathscr{C}_1, \dots, \mathscr{C}_N\}$ (the reference order for components) according to their weights. We define the component containing the node 1 as the root part $\mathcal{R}$.
- ***Step-2*** Pick out the highest weight component $\mathscr{C}_N$ and other components $\mathscr{C}_{i_1}, \dots, \mathscr{C}_{i_j}$ arbitrarily and construct a chain of components towards the root part $\mathsf{R}^{\mathscr{C}}$ as follows:

$$\mathbb{CH} = \Big[(\mathscr{C}_N)_t - (\mathscr{C}_N)_b \leftrightarrow (\mathscr{C}_{i_j})_{t(\,\mathrm{or}\,b)} - (\mathscr{C}_{i_j})_{b(\,\mathrm{or}\,t)} \leftrightarrow \cdots \leftrightarrow (\mathscr{C}_{i_1})_{t(\,\mathrm{or}\,b)} - (\mathscr{C}_{i_1})_{b(\,\mathrm{or}\,t)} \leftrightarrow \mathcal{R} \setminus \{n\}\Big].^{\text{[Eq:Chain]}} \tag{2.12}$$

  Here, a subscript t or b denotes the top or bottom side of the corresponding component. The notation '$-$' between two sides of a component denotes the kernel. A '$\leftrightarrow$' denotes the type-3 line between two nodes in the corresponding regions. For example the '$\leftrightarrow$' between $(\mathscr{C}_N)_b$ and $(\mathscr{C}_{i_j})_{t(\,\mathrm{or}\,b)}$ means that we connect arbitrary nodes x, y ($x \in (\mathscr{C}_N)_b, y \in (\mathscr{C}_{i_j})_{t(\,\mathrm{or}\,b)}$) by a type-3 line. $\mathcal{R}\setminus\{n\}$ is used to recall that n cannot be considered as a root. **Note that all the kernels of the components $\mathscr{C}_N, \mathscr{C}_{i_j}, \dots, \mathscr{C}_{i_1}$ in eq. (2.12) on this chain are passed through by the path that starts from the highest-weight element in $\mathscr{C}_N$ and ends at the node 1!** After this construction, we redefine the reference order $\mathsf{R}^{\mathscr{C}}$ and the root part $\mathcal{R}$:

$$\mathsf{R}^{\mathscr{C}} \to \mathsf{R}^{\mathscr{C}} \setminus \{\mathscr{C}_1, \mathscr{C}_1, \dots, \mathscr{C}_N\}, \mathcal{R} \to \mathcal{R} \cup \{\mathscr{C}_1, \mathscr{C}_1, \dots, \mathscr{C}_N\}. \tag{2.13}$$

- ***Step-3*** Repeating Step-2 until the ordered set $\mathsf{R}^{\mathscr{C}}$ becomes empty. Then a fully connected graph is produced.

All possible graphs (i.e. all possible groupings, all possible skeletons for a given grouping, all possible graphs constructed from a given skeleton) produced along the above line together are all graphs $\mathcal{F}$ in eq. (2.10)! Exercise-4: Construct all possible graphs for 4-point amplitudes!

(Hint: classify the graphs according to the number of components. Although the above construction rule seems very complicated, it is not hard to treat the 4-point case.)

The factor $C^{[\mathcal{F}]}$ and permutations $\boldsymbol{\sigma}^{\mathcal{F}}$ The factor $C^{[\mathcal{F}]}$ is given by the product of all lines (defined by Fig. 3) inside a graph with a proper sign $(-1)^{\mathcal{N}(\mathcal{F})}$ where $\mathcal{N}(\mathcal{F})$ is the total number of arrows pointing away from the node 1. The permutations $1, \boldsymbol{\sigma}^{\mathcal{F}}, n$ are determined by: (i) 1 and n are always considered as the first and the last elements, (ii) if x and y are two adjacent elements and x is nearer to the node 1 than y, the relative orders of them must satisfy $(\boldsymbol{\sigma}^{\mathcal{F}})^{-1}(x) < (\boldsymbol{\sigma}^{\mathcal{F}})^{-1}(y)$ [1]. (iii) If two substructures are attached to a same node, we should *shuffle* the permutations corresponding to these two substructures together. As an example, if we connect the nodes 3 and 8 in Fig. 4 via a type-3 line, the factor $C^{[\mathcal{F}]}$ is given by

$$\Big[(-\epsilon_1 \cdot k_8)(\epsilon_5 \cdot \epsilon_8)(\epsilon_{10} \cdot k_5)\Big](k_3 \cdot k_8)\Big[(\epsilon_2 \cdot k_3)(\epsilon_3 \cdot \epsilon_4)(\epsilon_6 \cdot k_4)(\epsilon_9 \cdot k_6)(\epsilon_7 \cdot k_6)\Big] \tag{2.14}$$

and the permutations $\boldsymbol{\sigma}^{\mathcal{F}}$ are given by

$$\boldsymbol{\sigma}^{\mathcal{F}} \in \Big\{\{8,5\} \shuffle \{3,\{2\} \shuffle \{4,6,\{9\} \shuffle \{7\}\}\}\Big\}. \tag{2.15}$$

Exercise-5: Write down the coefficients $C^{[\mathcal{F}]}$ and all amplitudes $A(1, \boldsymbol{\sigma}^{\mathcal{F}}, n)$ for all the graphs $\mathcal{F}$ constructed in Exercise-4.

3 New conformal symmetry

4 On off-shell extension

5 One-loop amplitudes

6 String amplitudes

References

[1] Here we use $\sigma^{-1}(x)$ to denote the position of x in the permutation $\boldsymbol{\sigma}$