Visualizing viscoelasticity with virtual experiments

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1 Purpose

This document provides a virtual experiment platform to help users build intuition for rheologically-complex materials and aid in integrating these function-valued properties into their design. The aim is to build intuition for designing with linear viscoelastic materials by seeing the effects of viscoelastic properties and on the performance of a simulated system.

This is intended to be a standalone document to complement the vibration isolation virtual experiment

2 Designing with rheological complexity

Rheologically complex materials cover a wide range of behaviors. These materials can show responses to forcing or deformation that are nonlinear in nature and time/history dependent. Engineers commonly design with more traditional materials (Newtonian fluids, elastic solids) but incorporating complex materials can add expanded functionality to designed systems.

When designing with rheological complexity, rather than attempting to match existing or discovered materials to possible end use, the design process should begin with intended performance targets (such as vibration isolation performance). The process to design with complex materials should follow the top-down design hierarchy outlined in previously published work[?]. The next step is to identify the function-valued material properties as well as the material geometry to optimize this performance. Unlike simple materials, the properties of viscoelastic material properties are function-valued not single constants. Defining and identifying the target material properties represents the first step in the design process, followed later by material selection or design to achieve these targets. Therefore, it is essential to understand the relationship between material properties and performance of the designed system.

3 Viscoelasticity

This virtual experiment focuses on a specific class of rheological complexity: viscoelasticity. In particular, the (virtual) materials will be linear viscoelastic.

If a purely viscous liquid is subjected to a deformation, any internal stress relaxes instantly to zero as soon as the strain becomes constant. Alternately, an elastic solid would show no stress relaxation. Some materials, such as polymeric materials, many biological materials, and others, exhibit gradual stress relaxation with time when they are subject to deformation or forcing. We call this time-dependent response viscoelasticity. The materials that exhibit viscous or elastic characteristics at different timescale of stress or strain input are called viscoelastic materials. Linear viscoelastic materials show behavior that depends only on the timescale of the input and are linear with respect to forcing amplitude.

Figure 2 below can best demonstrate the time-dependent properties of viscoelastic materials. The top left corner of Fig.2. shows a high speed camera capturing a ball made of a viscoelastic material hitting a solid surface. Within the time span of 4 ms, the ball bounced off the solid surface with only slight deformation. This shows that the ball is behaving as an elastic solid at small time scale of observation. On the other hand, The bottom of the Fig.2 shows

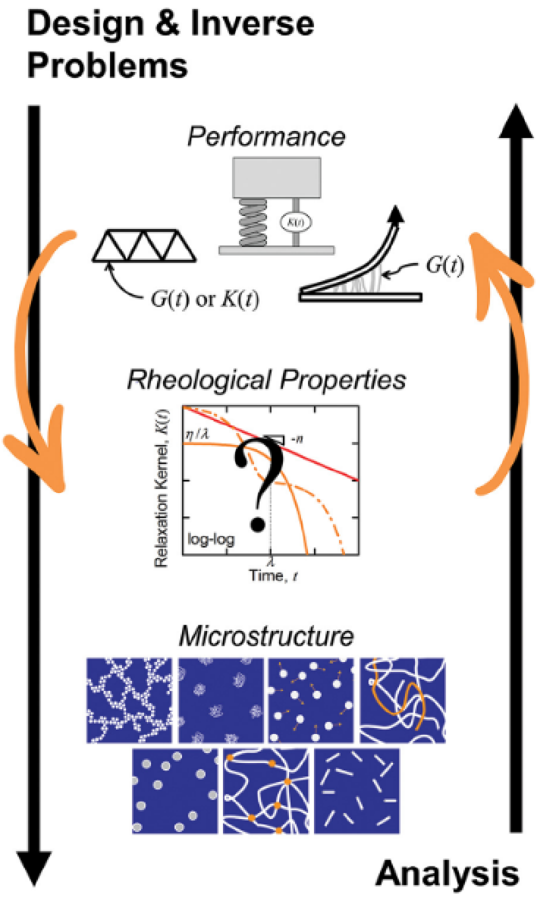

Figure 1: Traditional materials science represents a bottom-up design strategy wherein a specific material or material class is characterized and catalogued for potential exploit in some potential outcome. More effectively, we can use a top-down design methodology which begins by thinking about end-use performance. From here, the necessary rheological property targets (i.e. the viscoelastic relaxation modulus) are determined to meet these needs.

the ball will deform gradually and flow like a viscous liquid when it sits on a surface with the force of gravity applied on it over a long period of time.

The time-dependent properties of a viscoelastic material can be best described in terms of a dimensionless number called Deborah number (De). It is defined as the ratio of a characteristic material relaxation time (characterizing the time it takes for a material to relax after applied stresses or deformations) (λ) and a timescale of observation ($t_{observation}$).

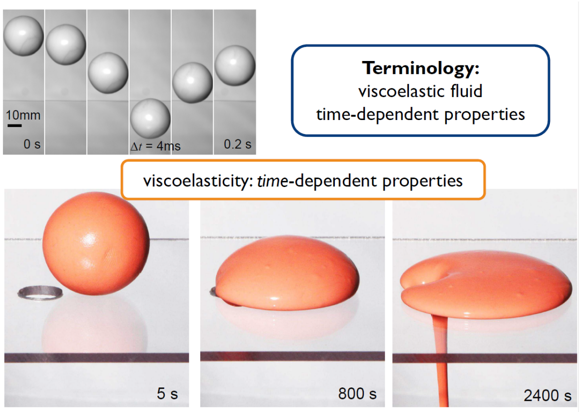

Figure 2: Viscoelasitic materials, such as Silly Putty or physical therapy putty (Theraputty), show time-dependent material properties. Top: when stimuli (such as force) are applied over a short time, such as during the contact with a surface during a bounce, the material responds like an elastic solid. Bottom: when a stimuli is applied over long timescales, such as the force due to gravity when the material sits on a surface, the material can flow similar to a viscous liquid. (For video, see Ewoldt Research Group Youtube, https://www.youtube.com/watch?v=UsE6x2NYec4)

$$De = \frac{\lambda}{t_{observation}} \tag{1}$$

We expect the material to exhibit liquid-like behavior at Deborah number less than one and solid-like behavior at Deborah number greater than one.

The stress response of the material is often normalized by the input strain. This gives the viscoelastic material property, the relaxation modulus:

$$G(t) = \frac{\tau(t)}{\gamma} \tag{2}$$

When there is no dependence on the strain-input amplitude for any given time (or equivalently, Deborah number), the response is said to be linear viscoelastic. Shown in the "Pipkin Map" of rheological complexity below, linear viscoelastic lies on the horizontal axis while purely non-linear response (i.e. shear thinning) lies on the vertical axis. Non-linear viscoelasticity (amplitude AND time dependence) lie within the interior region of the x-y plane.

For linear viscoelastic materials, the relaxation modulus can be used to form

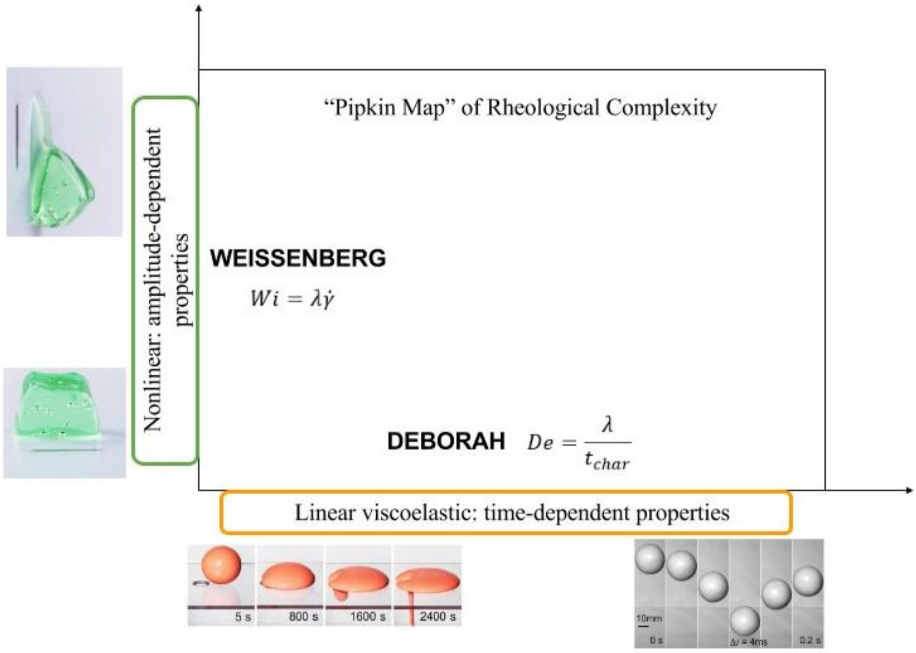

Figure 3: The "Pipkin Map" of Rheological Complexity represents a plane of rheologically complex material behavior and descriptions. On the horizontal axis is the purely time-depenedent material response, or linear viscoelasiticity. On the vertical axis is the purely non-linear with respect to input-amplitude. The interior of the plane is non-linear viscoelastic material behavior.

a constitutive equation relating stress and strain. The one-dimensional constitutive model for linear viscoelastic behavior is given by:

$$\tau = \int_{-\infty}^{0} G\left(t - t'\right) \dot{\gamma}\left(t - t'\right) dt' \tag{3}$$

Where t' is the past time variable running from the infinite past to the present time t. The constitutive equation reflects that viscoelastic behavior is a function of the entire history of the material.

A simplified model for viscoelastic relaxation is given by the Maxwell model:

$$G(t) = G_0 e^{-\frac{t}{\lambda}} \tag{4}$$

The model predicts that stress decays exponentially with time following a single dominant relaxation time, λ. For viscoelastic relaxation behavior, strain has two components: elastic and viscous.

A Maxwell material can be alternatively represented through a mechanical model by a purely viscous damper (damping coefficient, η_0) and a purely elastic spring (spring constant, G_0) connected in series, shown in Fig.4, below.

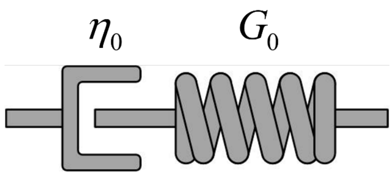

Figure 4: The mechanical representation of a Maxwell element is a spring and dashpot connected in series.

Based on the single Maxwell model, we can write the concept of linear viscoelasticity in a differential from :

$$\tau + \lambda \frac{d\tau}{dt} = \eta\dot{\gamma} \tag{5}$$

Where we define $\lambda = \frac{\eta_0}{G_0}$ as the characteristic the timescale of the simple Maxwell model.

4 Problem Details

In many applications, engineers seek to reduce mechanical vibration arising from rotating equipment, vortex shedding, or other sources of cyclic force input . The standard design problem[?] might assume a simple combination of springs and dashpots in parallel to isolate the vibration, as shown in Fig. 5. To incorporate rheological-complexity into the design, we can generalize the linear dashpot to be a parallel viscoelastic connection with relaxation modulus an arbitrary G(t). Here, we will demonstrate the added performance from a viscoelastic connection and modify G(t) to change the performance of the system.

For the particular system in Fig.5 with a generalized viscoelastic element, the equation of motion of the mass is given by:

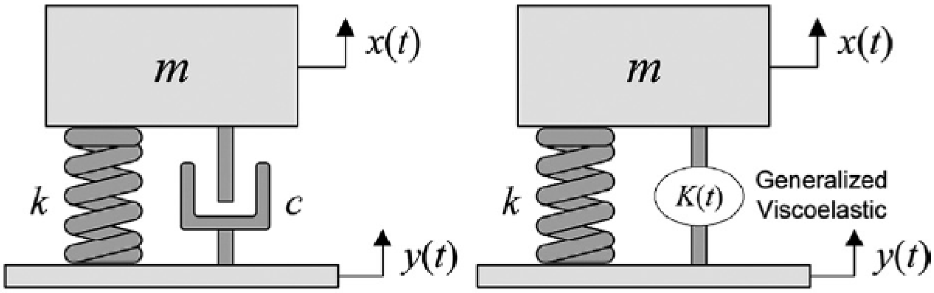

Figure 5: Traditional vibration isolation systems rely upon a combination of mechanical springs and linear dampers to reduce the transmission of vibrations (left). We replace the traditional linear damper with a generalized viscoelastic element whose material response can lie in between that of a solid and that of a fluid (right).

$$-k\left(x-y\right)-\int_{0}^{t}K(s)\left[\dot{x}\left(t-s\right)-\dot{y}\left(t-s\right)\right]ds=m\ddot{x}(t) \tag{6}$$

Where $s=t-t'$ and $\dot{x}-\dot{y}=\dot{X}$, the rate of deformation of the viscoelastic element. The first term on the left-hand side is the force by the spring element and the second term is the force by the viscoelastic element, modified from the constitutive model mentioned in Sec. 3.

For a sinusoidal input, the input displacement, $y(t)$, and displacement of the mass, $x(t)$, has the form:

$$y(t)=Y\sin\left(\omega t\right) \tag{7}$$

$$x(t)=X_R\sin\left(\omega t\right)+X_i\cos\left(\omega t\right) \tag{8}$$

Using the equations above and the relaxation modulus for the single Maxwell model, we can obtain the analytical form of the displacement and acceleration of the mass, $x_{ss}(t)$ and $\ddot{x}_{ss}(t)$, at steady state.

In the applet, the nondimensionalized results are shown for simplicity in understanding. The nondimensional variables are $\tilde{G}=\frac{G}{kY}$, $\tilde{\lambda}=\frac{\lambda}{\sqrt{\frac{m}{k}}}$, $\tilde{\omega}=\frac{\omega}{\sqrt{\frac{k}{m}}}$, $\tilde{\eta}=\frac{\eta}{\sqrt{km}}$, and the nondimensional outputs are $|\tilde{x}|=\frac{|x|}{Y}$ and $|\tilde{\ddot{x}}|=\frac{|\ddot{x}|}{\frac{Yk}{m}}$

5 The applet

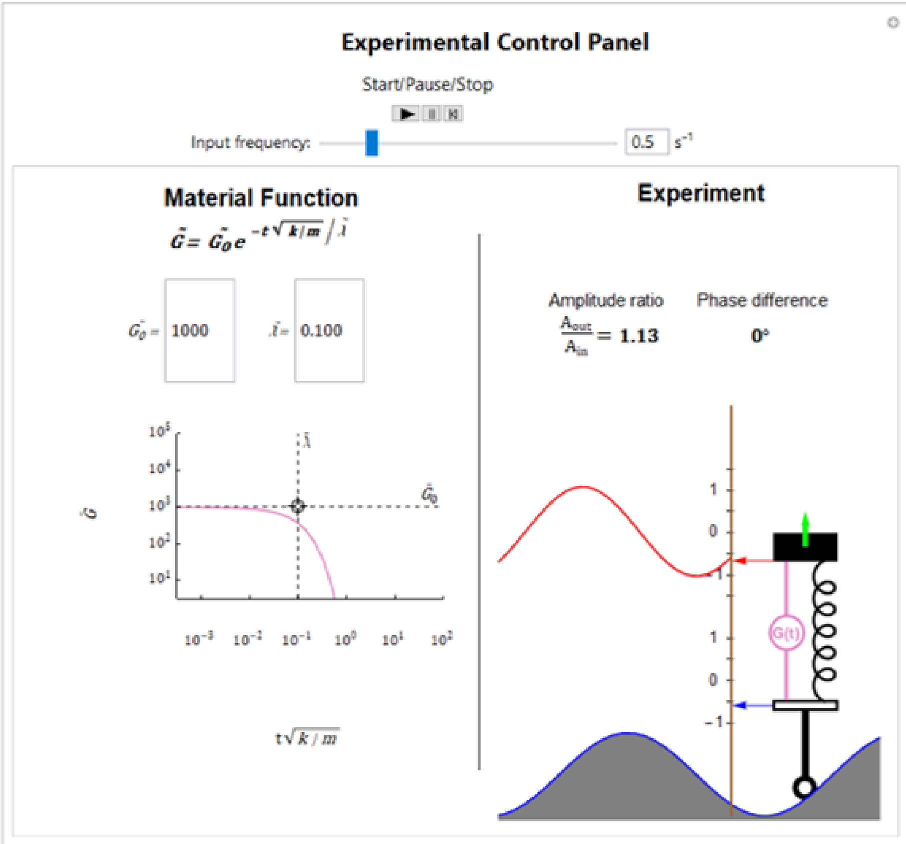

Figure 6: The applet, designed using Mathematica Demonstration Toolbox is divided into two panels. The Material Function panel (left) allows for user input of the material properties, through the relaxation modulus. The Experiment panel (right) shows the behavior of the vibration isolation system with a generalized viscoelastic component with the specified material property

Figure 6 shows the main screen for the applet. The **Material Function** panel (left) is where the user can adjust the material in the generalized viscoelastic connection through its relaxation modulus. There are two modes of input: (1) Input text boxes (top) where the user can manually input values through key board input for the material stiffness ($\tilde{G}_0$) and the material relaxation time ($\tilde{\lambda}$) and (2) graphical input where the user can click and drag the graphical representation of the material spectrum to adjust both parameters simultaneously.

The **Experiment** panel (right) shows the response of the vibrating system with the material designed by the user as defined in the equations of motions, above. In addition to numerical read-outs of the amplitude attenuation or amplification ("Amplitude ratio") and the phase difference between the input signal and the response signal, the **Experiment** panel also gives a visual vibration response of the system (in red) when it is subjected to the input signal (represented by the blue curve).

The user also has the ability to change the frequency of the input signal (between 0.3 and 1.5 s^{-1}) through either a slider bar or text input at the top of the panel.

6 Key Results

6.1 Improved performance

One key metric for "better" performance is an attenuation of the input signal (as indicated by the ratio of $\frac{A_{out}}{A_{in}}$. This can be achieved by tuning the relaxation time of the material in relation to the characteristic frequency of the input signal (or vice versa). This is clearly demonstrated in Fig. 7 where there is a demonstrably better result of the same material relaxation time at different input frequencies (0.3 s^{-1} on the left and 1.5 s^{-1} on the right.

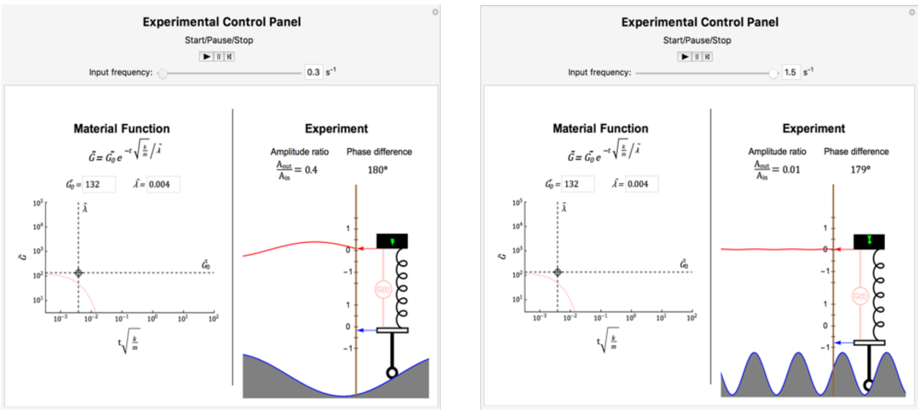

Figure 7: Performance improvement with increasing input frequency beyond resonance

6.2 System resonance

Another interesting things to explore are the location of the resonance in the system, as demonstrated by extreme amplification of the input signal (or a high ratio of $\frac{A_{out}}{A_{in}}$). At different locations in the design space (i.e.different material properties), resonance occurs at different frequencies, as shown in Fig. 8. Users can use the input fields at the top of the "Material Function" panel to make the subtle change in ($\tilde{\lambda}$,$\tilde{G}$) in order to find the resonance at any frequency.

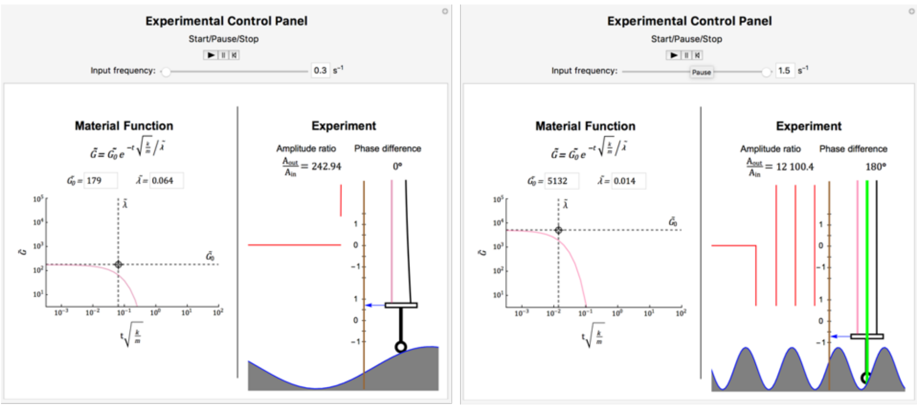

Figure 8: The location of the resonance in the design space is different at different input frequencies.

7 Additional Notes

The appelet was designed using the Mathematica demonstration toolbox with the help of Lianghao Cao in 2016.