DensityFluctuations

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1 A generator of a random scalar field with given correlation

The problem is generating a possibly large number of samples of a scalar random field of which we know the spectrum of two-point correlation function only. The field is meant to represent turbulent fluctuations of the particle density in a magnetized plasma. As such, we can consider the scalar field to be defined on a two-dimensional domain, as turbulence in magnetized plasmas is highly anisotropic with a very small wave number in the direction of the equilibrium magnetic field.

1.1 General set-up

We consider a two-dimensional box $\Omega = [x_1, x_2] \times [z_1, z_2]$ on the (x, z)-plane, together with a probability space $(A, \mathcal{F}, \mu)$ with probability measure $d\mu(\alpha)$ where $\alpha \in A$. A real-valued random field over Ω is a measurable function

$$\delta n : \Omega \times A \to \mathbb{R}.$$

We choose to represent such a field in terms of normalized coordinates

$$\xi = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_1}{a_x},$$
$$\zeta = \frac{z - z_1}{z_2 - z_1} = \frac{z - z_1}{a_z},$$

and to make use of a truncated Fourier-like series representation

$$\delta n(x, z) = \sum_{k=-M_x}^{M_x} \sum_{\ell=-M_z}^{M_z} \widehat{\delta n}_{k,\ell}(x, z, \alpha) e^{2\pi i (k\xi + \ell\zeta)},$$

where the dependence on $\alpha \in A$ is implied in $\delta n(x, z)$. Here, M_x, M_z are positive integers representing the maximum frequency present in the spectrum of the field. One can also notice that the spectral components $\widehat{\delta n}_{k,\ell}(x, z, \alpha)$, in addition to their natural dependence on $\alpha \in A$, are allowed to depend on the point location $(x, z) \in \Omega$, thus this is not strictly speaking a Fourier series. The choice of the probability space A is crucial to the statistics of the field. The only criteria for the choice of A and the amplitudes $\widehat{\delta n}_{k,\ell}(x, z, \alpha)$ are *(i)* the fact that the field $\delta n(x, z)$ is real valued, hence,

$$\overline{\widehat{\delta n}_{k,\ell}(x, z, \alpha)} = \widehat{\delta n}_{-k,-\ell}(x, z, \alpha),$$

with over-lines denoting complex conjugation; *(ii)* the fact that the fluctuations should not change the background and thus average out to zero; *(iii)* the Wigner funciton of the two-point correlation, which is known from either turbulence simulation or experimental observations.

By definition, the two-point correlation function of a random field $\delta n(x, y)$ is a deterministic function $C : \Omega \times \Omega \to \mathbb{R}$ defined by

$$C(x, z, x', z') = \mathbb{E}\big(\delta n(x, z)\delta n(x', z')\big),$$

where $\mathbb{E}$ is the expectation value operator on $(A, \mathcal{F}, \mu)$. For the proposed model we have

$$C(x, z, x', z') = \sum_{k=-M_x}^{M_x} \sum_{\ell=-M_z}^{M_z} \sum_{k'=-M_x}^{M_x} \sum_{\ell'=-M_z}^{M_z} \mathbb{E}\big(\widehat{\delta n}_{k,\ell}(x, z, \cdot)\widehat{\delta n}_{k',\ell'}(x', z', \cdot)\big) e^{2\pi i(k\xi+\ell\zeta)} e^{2\pi i(k'\xi'+\ell\zeta')},$$

where ξ', ζ' are the normalized coordinates corresponding to the point (x', z').

Upon assuming that the fluctuations and the corresponding correlation function are extended to the whole space by periodicity, the Wigner function is defined by

$$\Gamma(k_0, x, z, N_x, N_z) = \int e^{-ik_0(N_x s_x + N_z s_z)} C(x + \tfrac{1}{2}s_x, z + \tfrac{1}{2}s_z, x - \tfrac{1}{2}s_x, z - \tfrac{1}{2}s_z) ds_x ds_z,$$

where $k_0 = \omega/c$ is the wave vector in free space of a monochromatic electromagnetic waves of frequency ω and c is the speed of light. The variable (N_x, N_z) is the wave refractive index. The integral is not absolutely convergent and must be treated in sense of distributions.

Imposing the condition of real-valued fields

In one dimension, the fact that the field is real valued means that we only need to specify the amplitude of, e.g., non-negative frequencies. In two or more dimensions, there is a certain degree of freedom in choosing the independent harmonics. One can isolate the "zero frequencies" by rewriting the Fourier series in the form

$$\begin{aligned}
\delta n(x, z) = {} & \widehat{\delta n}_{0,0}(x, z, \alpha) \\
& + \sum_{k=1}^{M_x} \Big[\widehat{\delta n}_{k,0}(x, z, \alpha) e^{2\pi i k\xi} + \widehat{\delta n}_{-k,0}(x, z, \alpha) e^{-2\pi i k\xi}\Big] \\
& + \sum_{\ell=1}^{M_z} \Big[\widehat{\delta n}_{0,\ell}(x, z, \alpha) e^{2\pi i \ell\zeta} + \widehat{\delta n}_{0,-\ell}(x, z, \alpha) e^{-2\pi i \ell\zeta}\Big] \\
& + \sum_{k=1}^{M_x} \sum_{\ell=1}^{M_z} \Big[\widehat{\delta n}_{k,\ell}(x, z, \alpha) e^{2\pi i(k\xi+\ell\zeta)} + \widehat{\delta n}_{-k,-\ell}(x, z, \alpha) e^{-2\pi i(k\xi+\ell\zeta)}\Big] \\
& + \sum_{k=1}^{M_x} \sum_{\ell=1}^{M_z} \Big[\widehat{\delta n}_{-k,\ell}(x, z, \alpha) e^{2\pi i(k\xi+\ell\zeta)} + \widehat{\delta n}_{k,-\ell}(x, z, \alpha) e^{-2\pi i(k\xi+\ell\zeta)}\Big].
\end{aligned}$$

The reality condition determines the second term in each sum given the first. We can single out the $(0, 0)$ harmonic which does not have a corresponding "negative harmonic" and identify the set of the remaining statistically independent harmonics $\mathcal{M}$ as the union of four index sets, namely,

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4,$$

where

$$\begin{aligned}
\mathcal{M}_1 &= \{(k,0) \mid 1 \le k \le M_x\},\\
\mathcal{M}_2 &= \{(0,\ell) \mid 1 \le \ell \le M_z\},\\
\mathcal{M}_3 &= \{(k,\ell) \mid 1 \le k \le M_x, 1 \le \ell \le M_z\},\\
\mathcal{M}_4 &= \{(-k,\ell) \mid 1 \le k \le M_x, 1 \le \ell \le M_z\}.
\end{aligned}$$

The field $\delta n(x,z)$ of random fluctuations is then represented as

$$\delta n(x,z) = \widehat{\delta n}_{0,0}(x,z,\alpha) + \sum_{(k,\ell)\in\mathcal{M}} \Big[\widehat{\delta n}_{k,\ell}(x,z,\alpha) e^{2\pi i(k\xi+\ell\zeta)} + \text{c.c}\Big],$$

where "c.c." stands for complex conjugate. This expression remove redundant degree of freedom and make it explicit that the field is real-valued.

For real-valued fields, we can write the correlation in terms of the statistically independent harmonics only, namely,

$$\begin{aligned}
C(x,z,x',z') =& \mathbb{E}\big(\widehat{\delta n}_{0,0}(x,z,\cdot)\widehat{\delta n}_{0,0}(x,z,\cdot)\big)\\
&+ \sum_{(k,\ell)\in\mathcal{M}} \Big[\mathbb{E}\big(\widehat{\delta n}_{0,0}(x,z,\cdot)\widehat{\delta n}_{k,\ell}(x',z',\cdot)\big) e^{2\pi i(k\xi+\ell\zeta)}\\
&\qquad + \mathbb{E}\big(\widehat{\delta n}_{0,0}(x,z,\cdot)\overline{\widehat{\delta n}_{k,\ell}(x',z',\cdot)}\big) e^{-2\pi i(k\xi+\ell\zeta)}\\
&\qquad + \mathbb{E}\big(\widehat{\delta n}_{k,\ell}(x,z,\cdot)\widehat{\delta n}_{0,0}(x',z',\cdot)\big) e^{2\pi i(k\xi+\ell\zeta)}\\
&\qquad + \mathbb{E}\big(\overline{\widehat{\delta n}_{k,\ell}(x,z,\cdot)}\widehat{\delta n}_{0,0}(x',z',\cdot)\big) e^{-2\pi i(k\xi+\ell\zeta)}\Big]\\
&+ \sum_{(k,\ell)\in\mathcal{M}} \sum_{(k',\ell')\in\mathcal{M}} \Big[\mathbb{E}\big(\widehat{\delta n}_{k,\ell}(x,z,\cdot)\,\widehat{\delta n}_{k',\ell'}(x',z',\cdot)\big) e^{2\pi i(k\xi+\ell\zeta+k'\xi'+\ell'\zeta')}\\
&\qquad + \mathbb{E}\big(\widehat{\delta n}_{k,\ell}(x,z,\cdot)\,\overline{\widehat{\delta n}_{k',\ell'}(x',z',\cdot)}\big) e^{2\pi i(k\xi+\ell\zeta-k'\xi'-\ell'\zeta')}\\
&\qquad + \mathbb{E}\big(\overline{\widehat{\delta n}_{k,\ell}(x,z,\cdot)}\,\widehat{\delta n}_{k',\ell'}(x',z',\cdot)\big) e^{-2\pi i(k\xi+\ell\zeta-k'\xi'-\ell'\zeta')}\\
&\qquad + \mathbb{E}\big(\overline{\widehat{\delta n}_{k,\ell}(x,z,\cdot)}\,\overline{\widehat{\delta n}_{k',\ell'}(x',z',\cdot)}\big) e^{-2\pi i(k\xi+\ell\zeta+k'\xi'+\ell'\zeta')}\Big].
\end{aligned}$$

The correlation function $C(x,z,x',z')$ is also manifestly real-valued. Again the actual values of the expectation-value operators involved in the foregoing expression depend on the choice of the probability space $(A,\mathcal{F},\mu)$. A convenient way to generate the statistics of the field makes use of random phases.

1.2 Method of random phases

Probably the simplest way to generate the harmonics, the random phase method consists in the Ansatz

$$\begin{aligned}
\widehat{\delta n}_{0,0}(x,z,\alpha) &= \Phi_{0,0}(x,z)\cos(2\pi\alpha_{0,0}),\\
\widehat{\delta n}_{k,\ell}(x,z,\alpha) &= \Phi_{k,\ell}(x,z)e^{2\pi i\alpha_{k,\ell}}, \qquad\qquad \text{for } (k,\ell)\in\mathcal{M}.
\end{aligned}$$

where $\Phi_{k,\ell}(x,z)$ are deterministic functions over Ω, and $\alpha_{k,\ell}$ are independent random numbers uniformly distributed in the interval $[0,1]$. Since a random phase shift is added, we can take the functions $\Phi_{k,\ell}(x,z)$ to be real-valued, and

we only need to define the independent harmonics $(k,\ell) \in \{(0,0)\} \cup \mathcal{M}$. Therefore, the configuration space for $\alpha = (\alpha_{k,\ell})_{(k,\ell)\in\{0,0)\}\cup\mathcal{M}}$ is the unit hypercube $A = [0,1]^N$, where $N = 1 + |\mathcal{M}|$, $|\mathcal{M}|$ being the number of elements of $\mathcal{M}$; then, $\mathcal{F}$ is the usual Borel σ-algebra of A, and μ is the uniform probability measure $d\mu(\alpha) = \prod_{(k,\ell)\in\mathcal{M}} d\alpha_{k,\ell}$. The $(0,0)$-harmonic is real-valued by definition.

The zero-average condition $\mathbb{E}(\delta n) = 0$ is automatically satisfied as

$$\begin{aligned}\mathbb{E}\big(\cos(2\pi\alpha_{0,0})\big) &= \int_A \cos(2\pi\alpha_{0,0}) d\mu(\alpha) = 0,\\ \mathbb{E}\big(\exp(2\pi i\alpha_{k,\ell})\big) &= \int_A \exp(2\pi i\alpha_{k,\ell}) d\mu(\alpha) = 0.\end{aligned}$$

It remains to compute the two-point correlation function in order to find the deterministic amplitudes $\Phi_{k,\ell}(x,z)$.

The two-point correlation function

In order to compute the two-point correlation function, we need the following integrals:

$$\begin{aligned}\int_A \cos^2(2\pi\alpha_{0,0}) d\mu(\alpha) &= \int_0^1 \cos^2(2\pi\alpha_{0,0}) d\alpha_{0,0} = 1/2,\\ \int_A \cos(2\pi\alpha_{0,0}) e^{\pm 2\pi i\alpha_{k,l}} d\mu(\alpha) &= \int_0^1 \cos(2\pi\alpha_{0,0}) d\alpha_{0,0} \int_0^1 e^{\pm 2\pi i\alpha_{k,l}} d\alpha_{k,\ell} = 0,\\ \int_A e^{2\pi i(\alpha_{k,\ell}+\alpha_{k',\ell'})} d\mu(\alpha) &= \int_A e^{-2\pi i(\alpha_{k,\ell}+\alpha_{k',\ell'})} d\mu(\alpha) = 0,\\ \int_A e^{2\pi i(\alpha_{k,\ell}-\alpha_{k',\ell'})} d\mu(\alpha) &= \int_A e^{-2\pi i(\alpha_{k,\ell}-\alpha_{k',\ell'})} d\mu(\alpha) = \delta_{kk'}\delta_{\ell\ell'}.\end{aligned}$$

Correspondingly, the correlation function reads

$$\begin{aligned}C(x,z,x',z') = \frac{1}{2}\Phi_{0,0}(x,z)\Phi_{0,0}(x',z') + \sum_{(k,\ell)\in\mathcal{M}} \Big[&\Phi_{k,l}(x,z)\Phi_{k,\ell}(x',z') e^{2\pi i\left[k(\xi-\xi')+\ell(\zeta-\zeta')\right]}\\ &+ \Phi_{k,l}(x,z)\Phi_{k,\ell}(x',z') e^{-2\pi i\left[k(\xi-\xi')+\ell(\zeta-\zeta')\right]}\Big].\end{aligned}$$

Again we observe that the correlation function is manifestly real-valued and depends on the difference $(\xi-\xi')$, $(\zeta-\zeta')$ rather than on (ξ,ζ,ξ',ζ') separately. This simplification is the main motivation for working with the Fourier basis (although, in principle other basis could be used). If we define the "negative harmonics" by

$$\Phi_{k,\ell}(x,z) = \Phi_{-k,-\ell}(x,z), \quad \text{for } (-k,-\ell) \in \mathcal{M},$$

where $\Phi_{k,\ell}(x,z)$ are assumed to be real valued are given for $(k,\ell) \in \mathcal{M}$, we can write the correlation function in the compact form

$$C(x,z,x',z') = \frac{1}{2}\Phi_{0,0}(x,z)\Phi_{0,0}(x',z') + \sum_{0<|k|\le M_x} \sum_{0<|\ell|\le M_z} \Phi_{k,\ell}(x,z)\Phi_{k,\ell}(x',z') e^{2\pi i[k(\xi-\xi')+\ell(\zeta-\zeta')]}.$$

The amplitudes $\Phi_{k,\ell}(x,z)$ must be selected so that the corresponding Wigner transform of the correlation matches or at least approximates the desired model.

The Wigner function

Explicitly, we have

$$\Gamma(k_0,x,z,N_x,N_z)=\frac{1}{2}\int e^{-ik_0(N_xs_x+N_zs_z)}\Phi_{0,0}(x+\tfrac{1}{2}s_x,z+\tfrac{1}{2}s_z)\Phi_{0,0}(x-\tfrac{1}{2}s_x,z-\tfrac{1}{2}s_z)ds_xds_z$$
$$+\sum_{0<|k|\leq M_x}\sum_{0<|\ell|\leq M_z}\int e^{-ik_0(N_xs_x+N_zs_z)}\Phi_{k,\ell}(x+\tfrac{1}{2}s_x,z+\tfrac{1}{2}s_z)\Phi_{k,\ell}(x-\tfrac{1}{2}s_x,z-\tfrac{1}{2}s_z)$$
$$\times e^{2\pi i[ks_x/a_x+\ell s_z/a_z]}ds_xds_z.$$

As usual we are interested in the principal part of the Wigner function, i.e., in the leading order in the formal high-frequency limit, namely, $k_0\to+\infty$ (more precisely, the limit parameter is $k_0L\to+\infty$ where L is the length scale of the problem). Taylor's formula with integral remainder gives

$$\Phi_{k,\ell}(x+\tfrac{1}{2}s_x,z+\tfrac{1}{2}s_z)=\Phi_{k,\ell}(x,z)+s_xR_x(x,z,s_x,s_z)+s_zR_z(x,z,s_x,s_z),$$

where R_x and R_z are the remainders. By using the identity

$$s_xe^{-ik_0N_xs_x}=\frac{i}{k_0}\frac{\partial}{\partial N_x}e^{-ik_0N_xs_x},$$

and the analogous for s_z together with formal integration by parts, we see that remainders corresponds to integrals of the same type but multiplied by $1/k_0$. The Wigner function therefore amounts to

$$\Gamma(k_0,x,z,N_x,N_z)=\frac{1}{2}[\Phi_{0,0}(x,z)]^2\int e^{-ik_0(N_xs_x+N_zs_z)}ds$$
$$+\sum_{0<|k|\leq M_x}\sum_{0<|\ell|\leq M_z}[\Phi_{k,\ell}(x,z)]^2\int e^{-ik_0(N_xs_x+N_zs_z)}e^{2\pi i[ks_x/a_x+\ell s_z/a_z]}ds_xds_z$$
$$+k_0^{-1}\Gamma_R(k_0,x,z,N_x,N_z),$$

where Γ_R is the contribution due to the remainders. The integral is now readily evaluated by

$$\int e^{-ik_0(N_x-\frac{2\pi k}{k_0a_x})s_x}ds_x=\frac{2\pi}{k_0}\delta(N_x-\tfrac{2\pi k}{k_0a_x}),$$

and analogously for the integral in ds_z. At last, the Wigner function reads,

$$\Gamma(k_0,x,z,N_x,N_z)=\frac{4\pi^2}{k_0^2}\Big[\frac{1}{2}[\Phi_{0,0}(x,z)]^2\delta(N_x)\delta(N_z)$$
$$+\sum_{0<|k|\leq M_x}\sum_{0<|\ell|\leq M_z}[\Phi_{k,\ell}(x,z)]^2\delta(N_x-\tfrac{2\pi k}{k_0a_x})\delta(N_z-\tfrac{2\pi\ell}{k_0a_z})\Big]$$
$$+k_0^{-1}\Gamma_R(k_0,x,z,N_x,N_z).$$

This is the final form of the Wigner function for the considered model of fluctuations. One can observe that the leading order term amounts to a Dirac's comb, which is a measure. This is expected due to the fact that the considered fluctuations are periodic with spatial periods a_x and a_z in the two directions.

We recall that the derivation of the wave kinetic equation assumes high regularity for Γ as a function of the refractive index. Therefore, The Dirac's comb cannot be used to build a scattering operators in the wave kinetic equation. It is however possible to construct a regular Wigner function which approaches the Dirac's comb in the sense of distributions in the high-frequency limit.

Approximation with a regular measure and definition of $\Phi_{k,\ell}(x,z)$

Let $\Phi(k_0,x,z,N_x,N_z)$ be a smooth real-valued function with compact support in the box

$$\hat{\Omega}=[-N_x^*,N_x^*]\times[-N_z^*,N_z^*],$$

satisfying the duality relation

$$\frac{k_0a_x}{2\pi}N_x^*\in\mathbb{N},\quad \frac{k_0a_z}{2\pi}N_z^*\in\mathbb{N}.$$

Here, the box is fixed, i.e., both N_x^* and N_z^* do not depend on the frequency. In practice one picks a rapidly decreasing function $\Phi(k_0,x,z,N_x,N_z)$ and select N_x^*,N_z^* sufficiently large to include the approximate support (cutting out small values) of $\Phi(k_0,x,z,.)$ for $(x,z)\in\Omega$. Then, we define the maximum number of harmonics by setting M_x and M_z to the foregoing integer numbers, namely,

$$M_x=\frac{k_0a_x}{2\pi}N_x^*\in\mathbb{N},\quad M_z=\frac{k_0a_z}{2\pi}N_z^*\in\mathbb{N}.$$

As a result, the number of harmonics grows linearly with the frequency, which enters through the parameter k_0, while the size of the box $\hat{\Omega}$ is independent of k_0.

Let us consider the integral

$$\langle\chi,[\Phi(k_0,x,z,\cdot)]^2\rangle=\int\chi(N_x,N_z)[\Phi(x,z,N_x,N_z)]^2dN_xdN_z,$$

where $\chi(N_x,N_z)$ is a test function of class C^∞, i.e., we consider $|\Phi(k_0,x,z,\cdot)|^2$ as a compactly supported distribution.

This integral can be approximated by Riemann quadrature formula on the uniform mesh of points

$$N_{x,k}=\frac{2\pi k}{k_0a_x},\quad N_{z,\ell}=\frac{2\pi\ell}{k_0a_z},$$

for $k\in\{-M_x,\dots,M_x\}$ and $\ell\in\{-M_z,\dots,M_z\}$. The step sizes are

$$\Delta N_x=\frac{2\pi}{k_0a_x},\quad \Delta N_z=\frac{2\pi}{k_0a_z}.$$

The integral over the total domain $\hat{\Omega}$ is written as the sum of integrals over the grid cells

$$\hat{\Omega}_{k,\ell}=\Big[N_{x,k}-\frac{\Delta N_x}{2},N_{x,k}+\frac{\Delta N_x}{2}\Big]\times\Big[N_{z,\ell}-\frac{\Delta N_z}{2},N_{z,\ell}+\frac{\Delta N_z}{2}\Big],$$

centered around the (k,ℓ) point, and we make use of the Taylor formula with integral remainder for the product $\chi(\cdot)[\Phi(k_0,x,z,\cdot)]^2$, namely,

$$\chi(N_x,N_z)[\Phi(k_0,x,z,N_x,N_z)]^2 = \chi(N_{x,k},N_{z,\ell})[\Phi(k_0,x,z,N_{x,k},N_{z,\ell})]^2 + (N_x-N_{x,k})R_{x,k\ell}(k_0,x,z,N_x,N_z) + (N_z-N_{z,\ell})R_{z,k\ell}(k_0,x,z,N_x,N_z),$$

with remainders

$$R_{x,k\ell}(k_0,x,z,N_x,N_z) = \int_0^1 \partial_{N_x}(\chi(N_x,N_z)[\Phi(k_0,x,z,N_x,N_z)]^2)\Big|_{\substack{N_x=N_{x,k}+t(N_x-N_{x,k})\\ N_z=N_{z,\ell}+t(N_z-N_{z,\ell})}} dt,$$
$$R_{z,k\ell}(k_0,x,z,N_x,N_z) = \int_0^1 \partial_{N_z}(\chi(N_x,N_z)[\Phi(x,z,N_x,N_z)]^2)\Big|_{\substack{N_x=N_{x,k}+t(N_x-N_{x,k})\\ N_z=N_{z,\ell}+t(N_z-N_{z,\ell})}} dt.$$

We obtain the exact identity

$$\langle\chi,[\Phi(k_0,x,z,\cdot)]^2\rangle = \sum_{k,\ell}\Big[\chi(N_{x,k},N_{z,\ell})[\Phi(k_0,x,z,N_{x,k},N_{z,\ell})]^2\Delta N_x\Delta N_z + I_{x,k\ell}(\chi)\Delta N_x^2\Delta N_z + I_{z,k\ell}(\chi)\Delta N_x\Delta N_z^2\Big],$$

where (k,ℓ) run over the considered mesh so that the sum is finite, and

$$I_{x,k\ell}(\chi) = \frac{1}{\Delta N_x\Delta N_z}\int_{\hat\Omega_{k\ell}}\frac{N_x-N_{x,k}}{\Delta N_x}R_{x,k\ell}(k_0,x,z,N_x,N_y)dN_xdN_z,$$
$$I_{z,k\ell}(\chi) = \frac{1}{\Delta N_x\Delta N_z}\int_{\hat\Omega_{k\ell}}\frac{N_z-N_{z,\ell}}{\Delta N_z}R_{x,k\ell}(k_0,x,z,N_x,N_y)dN_xdN_z,$$

are two linear continuous functionals depending of the mesh sizes $\Delta N_x, \Delta N_z$. At last, we have the exact expression

$$\langle\chi,[\Phi(k_0,x,z,\cdot)]^2\rangle = \frac{4\pi^2}{k_0^2}\sum_{k,\ell}\Big[\chi(N_{x,k},N_{z,\ell})\frac{[\Phi(k_0,x,z,N_{x,k},N_{z,\ell})]^2}{a_xa_z} + \frac{1}{k_0}\Big[\frac{2\pi}{a_x}I_{x,k\ell}(\chi)+\frac{2\pi}{a_z}I_{z,k\ell}(\chi)\Big]\Big].$$

With those definitions both $I_{x,k\ell}$ and $I_{z,k\ell}$ are bounded in k_0, so that for large k_0 their contribution is small with respect to the leading term. On the other hand, upon testing Γ against χ, we have

$$\langle\chi,\Gamma(k_0,x,z,\cdot)\rangle = \frac{4\pi^2}{k_0^2}\Big[\frac{1}{2}\chi(0,0)[\Phi_{0,0}(x,z)]^2 + \sum_{0<|k|\le M_x}\sum_{0<|\ell|\le M_z}\chi(N_{x,k},N_{z,\ell})[\Phi_{k,\ell}(x,z)]^2\Big] + \frac{1}{k_0}\langle\chi,\Gamma_R(k_0,x,z,\cdot)\rangle,$$

since the mesh points $(N_{x,k},N_{z,\ell})$ coincide with the support of the principal part in $\Gamma(k,0,x,z,\cdot)$.

We want to choose the smooth function $\Phi(k_0,x,z,N_x,N_z)$ so that the two foregoing expressions agree to lowest order in $1/k_0$, that is,

$$\Phi_{0,0}(x,z) = \sqrt{\frac{2}{a_x a_z}}\Phi(k_0,x,z,0,0),$$
$$\Phi_{k,\ell}(x,z) = \frac{1}{\sqrt{a_x a_z}}\Phi(k_0,x,z,N_{x,k},N_{z,\ell}).$$

We notice that Φ must be such that *(a)* the reality condition $\Phi_{k,\ell} = \Phi_{-k,-\ell}$ is satisfied and *(b)* Φ evaluated on grid points should not depend on k_0 since the amplitudes of the density fluctuations have no relation with the considered frequency; this means that Φ must be a function of the wave numbers $k_0 N_x$ and $k_0 N_z$.

Provided that we can find such a function Φ, we have the exact identity,

$$\Gamma(k_0,x,z,N_x,N_z) = [\Phi(k_0,x,z,N_x,N_z)]^2 + \frac{1}{k_0}\Gamma_1,$$

in the weak sense, testing with a smooth function χ. Here, the first order term is defined as the functional

$$\langle \chi, \Gamma_1 \rangle = \langle \chi, \Gamma_R(k_0,x,z,\cdot)\rangle - \frac{4\pi^2}{k_0^2}\Big[\frac{2\pi}{a_x} I_{x,k\ell}(\chi) + \frac{2\pi}{a_z} I_{z,k\ell}(\chi)\Big].$$

This calculation shows that, in the high frequency regime, the exact Wigner function Γ for the considered fluctuation model can be written as a smooth compactly supported function plus a lowest order remainder of $O(1/k_0)$, this identity being valid in the weak sense only (with smooth test functions, i.e., in the space of compactly supported distributions).

This result allows us to represent the density fluctuations with a Dirac's comb spectral distribution in the framework of the wave kinetic equation and of the WKBeam code in particular.

A standard assumption for the lowest order term in Γ is the Gaussian

$$\Gamma_{\text{WKBeam}}(k_0,x,z,N_x,N_z) = 2\pi\big[n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z)\big]^2 L_\perp^2 e^{-\frac{L_\perp^2}{2}(k_0^2 N_x^2 + k_0^2 N_z^2)},$$

which is a function of the wave numbers $k_0 N_x, k_0 N_z$ only, is real-valued, and symmetric in N_x, N_z, and thus it satisfies the requirements *(a)* and *(b)* above. Here, $n_0(x,z)$ is the background field, and $F(x,z)$ is the envelope of the fluctuations. This is assumed to take values in [0,1] and it allows us to describe the spatial localization of turbulent fluctuations. The strength of the fluctuations is determined by the root mean square $\langle \delta n/n_0\rangle$ relative to the background. The scale-length $L_\perp$ is the perpendicular correlation length of the turbulence (and it can in general depend on the point (x,z), but this level of generality is not exploited in the following).

If we set $[\Phi(k_0,x,z,N_x,N_z)]^2 = \Gamma_{\text{WKBeam}}(k_0,x,z,N_x,N_z)$, the amplitudes of the Fourier series of the fluctuation model amount to

$$\Phi_{0,0}(x,z) = \sqrt{\frac{4\pi}{a_x a_z}} n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z) L_\perp,$$
$$\Phi_{k,\ell}(x,z) = \sqrt{\frac{2\pi}{a_x a_z}} n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z) L_\perp e^{-\frac{L_\perp^2}{a_x^2}\pi^2 k^2 - \frac{L_\perp^2}{a_z^2}\pi^2 \ell^2}.$$

This result, together with the random phases, fully characterizes the turbulence model. In the following, the foregoing procedure is implemented in a Python class for the generator of samples of the random field $\delta n(x,z)$ with the Gaussian amplitudes given above.

Summary on the results of the random phase method

The fluctuating field is written in the form

$$\delta n(x,z) = n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z)\Big[\tilde{\Phi}_{0,0}\cos(2\pi\alpha_{0,0}) + \sum_{(k,\ell)\in\mathcal{M}} \tilde{\Phi}_{k,\ell}\, 2\cos\big(2\pi(k\xi+\ell\zeta+\alpha_{k,\ell})\big)\Big],$$

where the independent harmonics have been singled out (thus avoiding unnecessarily long loops) and the amplitudes have been re-written in terms of

$$\begin{aligned}\tilde{\Phi}_{0,0} &= \sqrt{\frac{4}{\pi}\sigma_x\sigma_z},\\ \tilde{\Phi}_{k,\ell} &= \sqrt{\frac{2}{\pi}\sigma_x\sigma_z}\; e^{-\sigma_x^2k^2-\sigma_z^2\ell^2},\end{aligned}$$

where

$$\sigma_x = \pi L_\perp/a_x, \quad \sigma_z = \pi L_\perp/a_z,$$

are dimension-less parameters accounting for the ratio of the correlation length to the box sizes. One can note that each realization of this random field is smooth by definition.

By construction the mean field is zero, namely,

$$\mathbb{E}(\delta n) = 0.$$

Since the field is smooth, we can compute its square and thus the variance,

$$\mathbb{V}(\delta n) = \mathbb{E}(\delta n^2) - \mathbb{E}(\delta n)^2 = \mathbb{E}(\delta n^2),$$

or equivalently

$$\mathbb{V}\big(\delta n(x,z)\big) = C(x,z,x,z),$$

where $C(x,z,x',z')$ is the correlation function: The variance is then the restriction of the correlation to the diagonal $x=x'$ $z=z'$. Then, one readily finds

$$\begin{aligned}\mathbb{V}\big(\delta n(x,z)\big) &= \frac{1}{2}[\Phi_{0,0}(x,z)]^2 + \sum_{0<|k|\le M_x}\sum_{0<|\ell|\le M_z}[\Phi_{k,\ell}(x,z)]^2,\\ &= [n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z)]^2\Big[\frac{1}{2}[\tilde{\Phi}_{0,0}]^2 + \sum_{0<|k|\le M_x}\sum_{0<|\ell|\le M_z}[\tilde{\Phi}_{k,\ell}(x,z)]^2\Big]\\ &= [n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z)]^2\frac{2}{\pi}\sigma_x\sigma_z\sum_{k,\ell} e^{-2\sigma_x^2k^2-2\sigma_z^2\ell^2}.\end{aligned}$$

The last sum can be approximated by an integral with the result that

$$\frac{2}{\pi}\sigma_x\sigma_z\sum_{k,\ell} e^{-2\sigma_x^2k^2-2\sigma_z^2\ell^2} \approx \frac{2}{\pi}\int e^{-2(u^2+v^2)}dudv = \frac{1}{\pi}\int e^{-(u^2+v^2)}dudv = 1.$$

We can therefore write an estimator for the variance, namely,

$$\mathbb{V}\big(\delta n(x,z)\big) \approx [n_0(x,z)\langle\frac{\delta n}{n_0}\rangle F(x,z)]^2.$$

Usually, the envelope F attains values in $[0,1]$ so that we can interpret the parameter $\langle\delta n/n_0\rangle$ as the maximum root mean square of fluctuations.

The maximum value of the fluctuations relative to n_0 can be significantly larger than the root mean square. In fact, with $F(x,z)\in[0,1]$ we have the bound,

$$\begin{aligned}\Big|\frac{\delta n(x,z)}{n_0(x,z)}\Big| \le \langle\frac{\delta n}{n_0}\rangle\Big[\big|\tilde\Phi_{0,0}\big| + 2\sum_{(k,\ell)\in\mathcal{M}}\big|\tilde\Phi_{k,\ell}\big|\Big] &= \langle\frac{\delta n}{n_0}\rangle\Big[\sqrt{\frac{4}{\pi}\sigma_x\sigma_z} + \sqrt{\frac{2}{\pi}\sigma_x\sigma_z}\sum_{k\neq0}\sum_{\ell\neq0} e^{-\sigma_x^2k^2-\sigma_z^2\ell^2}\Big]\\ &= \langle\frac{\delta n}{n_0}\rangle\sqrt{\frac{2\pi}{\sigma_x\sigma_z}}\Big[\frac{1}{\pi}\sigma_x\sigma_z\sum_{k,\ell} e^{-\sigma_x^2k^2-\sigma_z^2\ell^2} + \frac{\sqrt{2}-1}{\pi}\sigma_x\sigma_z\Big].\end{aligned}$$

The remaining sum is deterministic and can be evaluated once for all samples, thus providing a sharp bound for the maximum oscillation. Alternatively, we can estimate the sum by an integral as above,

$$\frac{1}{\pi}\sigma_x\sigma_z\sum_{k,\ell} e^{-\sigma_x^2k^2-\sigma_z^2\ell^2} \approx \frac{1}{\pi}\int e^{-(u^2+v^2)}dudv = 1,$$

thus obtaining the approximate bound

$$\Big|\frac{\delta n(x,z)}{n_0(x,z)}\Big| \lessapprox \langle\frac{\delta n}{n_0}\rangle\sqrt{\frac{2\pi}{\sigma_x\sigma_z}},$$

where "$\lessapprox$" means that the bound is given within the error associated to the approximation of the sum with the corresponding integral. The last term in the square bracket has been dropped as it is of the same order as the error on the sum. Since σ_x and σ_z are typically small as they amount to essentially the ratios of the correlation length to the box sizes, fluctuations can be significantly larger than $\langle\delta n/n_0\rangle$.

This also implies that, unless $\langle\delta n/n_0\rangle$ is sufficiently small, there is a significant probability of negative fluctuations of the total field $n_0+\delta n$, even when n_0 is strictly positive. When the total field $n_0+\delta n$ represents a density, this is clearly unphysical and should be used with care.

1.3 Sample generator with the random phase method

In the following, a python class for the implementation of the fluctuation model based on random phases is proposed and tested. The use of this class is exemplified by generating fluctuations in a two dimensional box with both uniform and non uniform envelope and over different background.

Sample generator

The following code import the class Fluctuations_RandomPhases which defines generators of samples of a random field as described in the theory above. The class is imported from the module DensityFlucutations.py, cf. also its documentation string.

In [1]:
```
# Import standard packages and plotting tools
%matplotlib inline
from pylab import *

# Import the random field generator
from DensityFluctuations import Fluctuations_RandomPhases
```

Testing the sample generator

In [2]:
```
# Parameters
x1 = -1.
x2 = +1.
z1 = -0.5
z2 = +0.5
Lperp = 0.2
n0 = lambda x,z : ones_like(x)
dn_n0 = 1.
F = lambda x,z : ones_like(x)

# Setting up the test case
parameters = {
    'x1': x1,
    'x2': x2,
    'z1': z1,
    'z2': z2,
    'Lperp': Lperp,
    'n0' : n0,
    'dn_n0': dn_n0,
    'F' : F,
    'tol' : 1.e-10,
    }

# Construction of the object
deltaN_simple = Fluctuations_RandomPhases(parameters)

# Approximation of fluctuations upper bound
approx_bound = dn_n0 * sqrt((2./pi)*(x2-x1)*(z2-z1)/Lperp**2)

# Some data
Mx = deltaN_simple.Mx
Mz = deltaN_simple.Mz
Nh = deltaN_simple.Nharm
MaxF = deltaN_simple.MaxRelativeFluctuation
print( 'Required max. sample frequency Mx = {}'.format(Mx) )
print( 'Required max. sample frequency Mz = {}'.format(Mz) )
print( 'Number of independent harmonics N = {}'.format(Nh) )
print( 'Max. relative fluctuation = {}'.format(MaxF) )
print( 'Estimated max. relative fluctuation = {}'.format(approx_bound) )
```

```
Required max. sample frequency Mx = 16
Required max. sample frequency Mz = 8
Number of independent harmonics N = 281
Max. relative fluctuation = 5.788730720378465
Estimated max. relative fluctuation = 5.641895835477563
```

In [3]:
```
# Test correlated sampling
x0 = z0 = array([0])
dN1 = deltaN_simple(x0, z0)
```

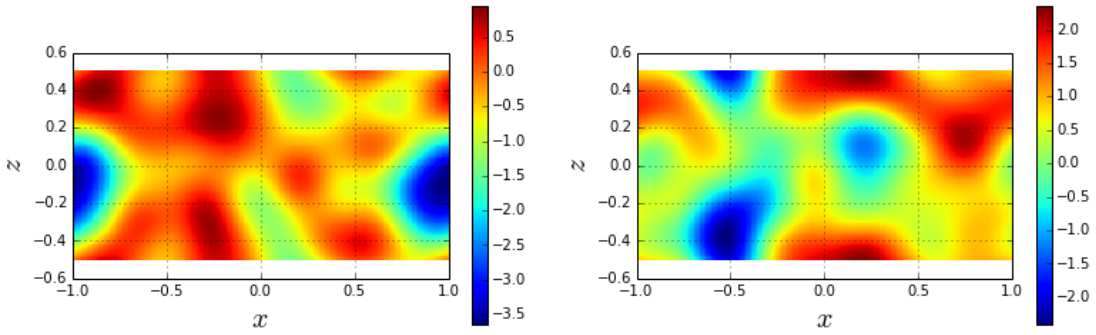
```
dN2 = deltaN_simple(x0, z0)
dN3 = deltaN_simple(x0, z0, correlated=True)

# Print the comparison of the samples
print( 'dN1 is NOT equal to dN2: {}'.format(dN1!=dN2) )
print( 'dN2 is indeed equal to dN3: {}'.format(dN2==dN3) )
```

```
dN1 is NOT equal to dN2: [ True]
dN2 is indeed equal to dN3: [ True]
```

In [4]:
```
# Define the sample grid
nptx = 200
nptz = 150
X, Z = deltaN_simple.grid(nptx, nptz)
```

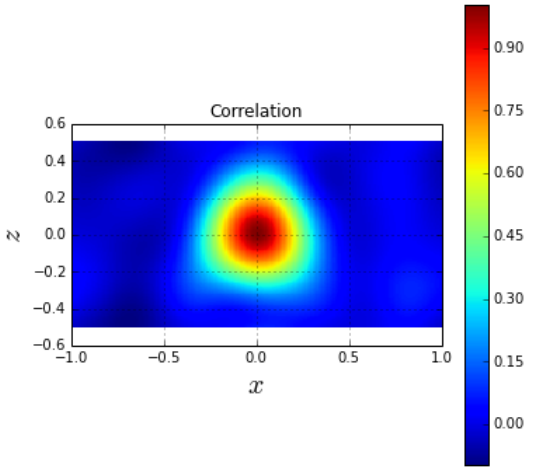
In [5]:
```
# Generate and plot two samples
sample1 = deltaN_simple(X, Z)
sample2 = deltaN_simple(X, Z)

# Maximum fluctuation
max1 = np.max(abs(sample1))
max2 = np.max(abs(sample2))
print( 'Maximum fluctuation of sample1 = {}'.format(max1) )
print( 'Maximum fluctuation of sample2 = {}'.format(max2) )

# Plotting
figure(1, figsize=(13,4))

subplot(121, aspect='equal')
pcolormesh(X, Z, sample1)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
colorbar()
grid('on')

subplot(122, aspect='equal')
pcolormesh(X, Z, sample2)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
colorbar()
grid('on')
```

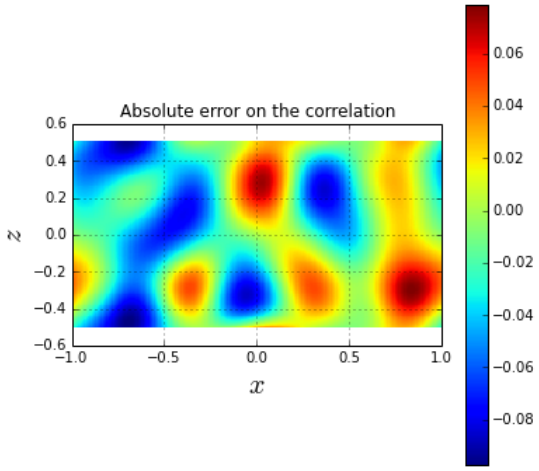
```
Maximum fluctuation of sample1 = 3.6500431209060764
Maximum fluctuation of sample2 = 2.4145260752843223
```

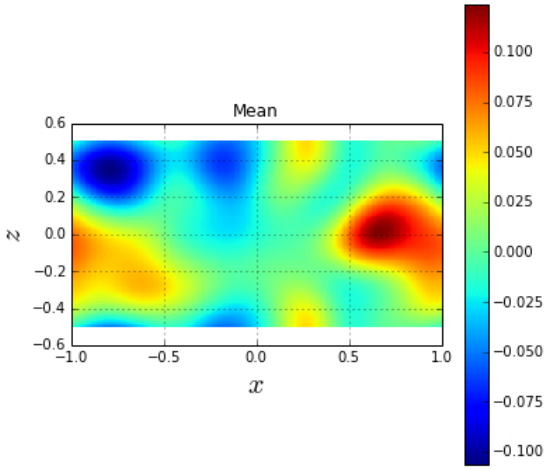


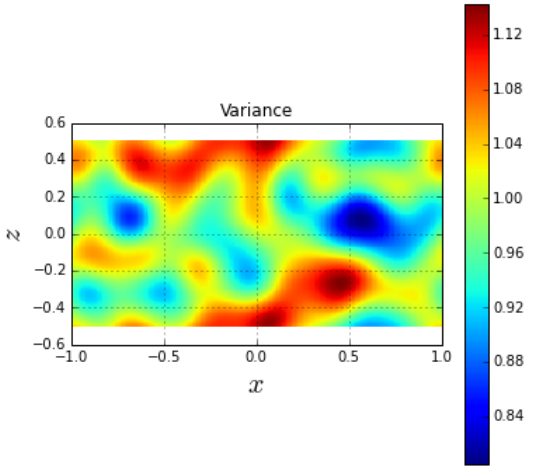
In [6]:
```
# Checking the correlation function (This will take a while ...)
nsamples = 500 # (Reasonable convergence requires about 1000 samples)
x0 = z0 = 0.
C, M, V = deltaN_simple.statistics(nsamples, x0, z0, X, Z)

# The theoretical value is a Gaussian centered in (0, 0)
Vtheory = n0(X, Z)**2 * dn_n0 * F(X, Z)**2
```

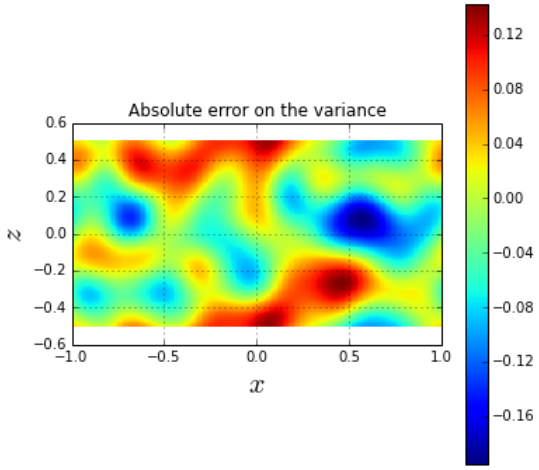
```
Ctheory = Vtheory * np.exp(-0.5 * (X**2 + Z**2) / Lperp**2)

# Plotting
figure(2, figsize=(13,20))

subplot(321, aspect='equal')
pcolormesh(X, Z, C)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
title('Correlation')
colorbar()
grid('on')

subplot(322, aspect='equal')
pcolormesh(X, Z, C - Ctheory)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
title('Absolute error on the correlation')
colorbar()
grid('on')

subplot(323, aspect='equal')
pcolormesh(X, Z, M)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
title('Mean')
colorbar()
grid('on')

subplot(325, aspect='equal')
pcolormesh(X, Z, V)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
title('Variance')
colorbar()
grid('on')

subplot(326, aspect='equal')
pcolormesh(X, Z, V - Vtheory)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
title('Absolute error on the variance')
colorbar()
grid('on')
```


Correlation
z
x
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
0.90
0.75
0.60
0.45
0.30
0.15
0.00


Absolute error on the correlation
z
x
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08


Mean
z
x
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100


Variance
z
x
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
1.12
1.08
1.04
1.00
0.96
0.92
0.88
0.84


Absolute error on the variance
z
x
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−1.0
−0.5
0.0
0.5
1.0
0.12
0.08
0.04
0.00
−0.04
−0.08
−0.12
−0.16

ASDEX - like slab

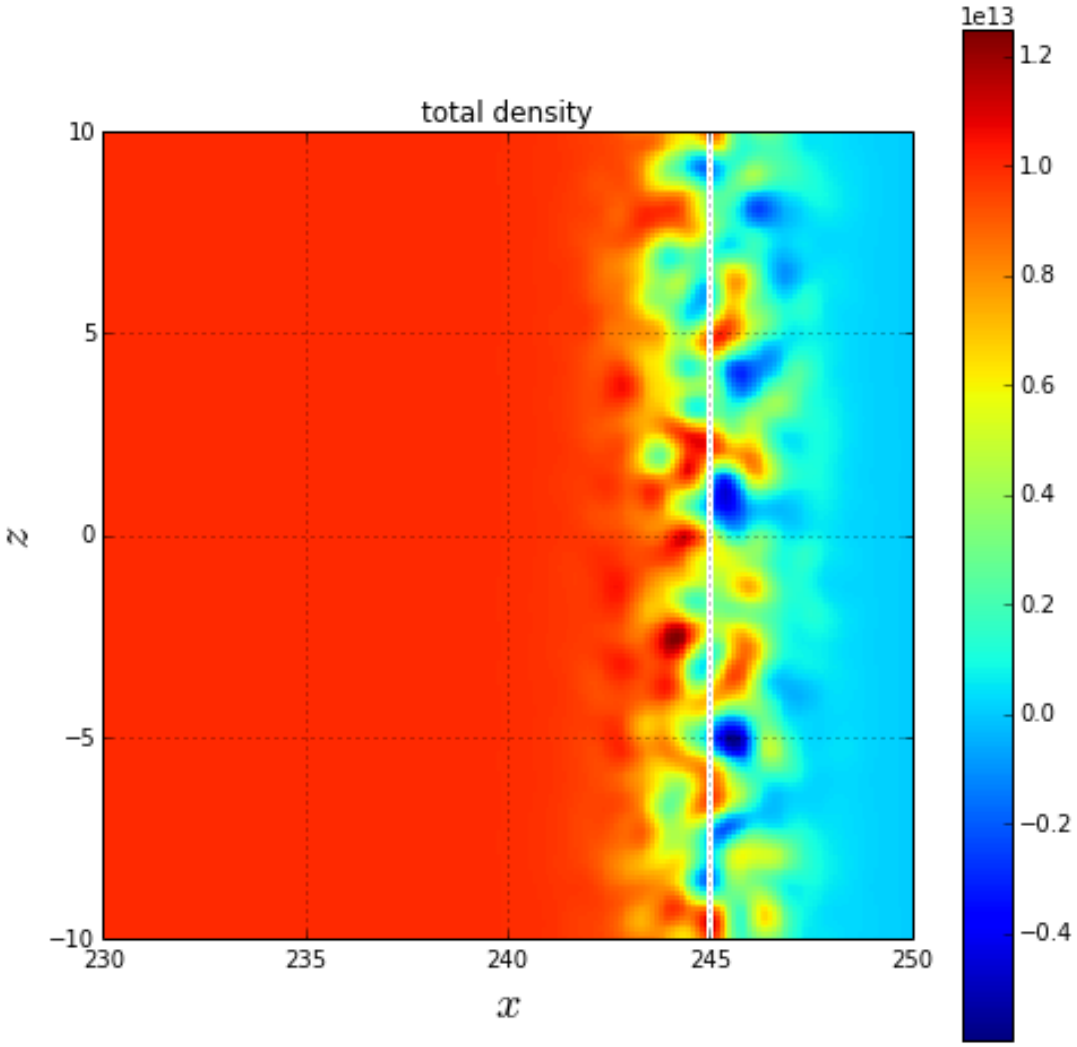
For testing purposes we have considered a slab geometry which have the same spatial dimension of a box including the edge of an ASDEX Upgrade plasma on the outer equatorial plane. The correlation length of the turbulence should be realistic. We also consider a non-trivial background density profile and a Gaussian envelope centered around the pedestal (both function of the horizontal coordinate x only).

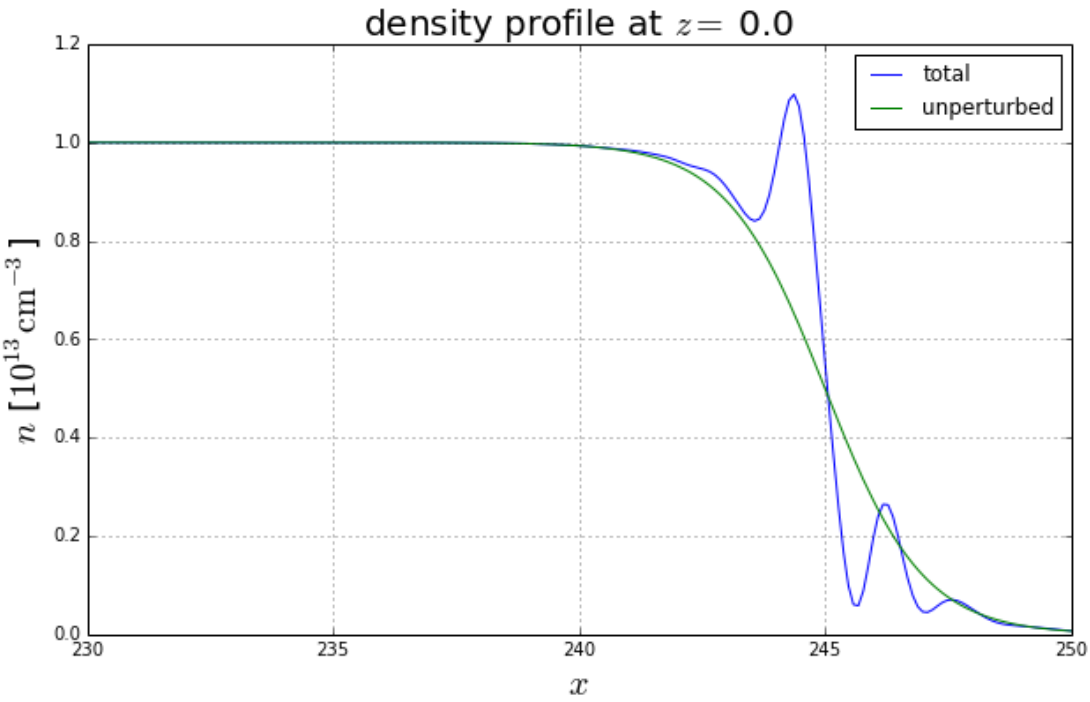
In [7]:
```
# Physics parameters
x1 = 230.          # coordinate of the left boundary of the box in cm
x2 = 250.          # coordinate of the right boundary of the box in cm
z1 = -10.          # lower boundary of the box
z2 = +10.          # upper boundary of the box
Lpedestal = 2.     # scale length (in cm) of the pedestal
Xpedestal = 245.   # coordinate of the center of the pedestal in cm
Npedestal = 1.e13 # density at the pedestal top in cm^-3
dn_n0 = 1.0        # relative level of fluctuations dn/n
Lperp = 0.5        # correlation length in cm
Xfluct = 246.      # position of the turbulence layer
Wfluct = 2.        # width in cm of the turbulence layer

# Background density and fluctuation envelope
n0 = lambda x,z : 0.5 * Npedestal * \
                  (1. - tanh((x-Xpedestal)/Lpedestal))
F = lambda x, z : exp(- (x-Xfluct)**2 / Wfluct**2)

# Construction of the fluctuation object
parameters ={'x1': x1,
             'x2': x2,
             'z1': z1,
             'z2': z2,
             'Lperp': Lperp,
             'n0' : n0,
             'dn_n0': dn_n0,
             'F' : F,
             'tol' : 1.e-4}
deltaN_AUG = Fluctuations_RandomPhases(parameters)

# Generate the grid
nptx = 200
nptz = 201
X, Z = deltaN_AUG.grid(nptx, nptz)
```

In [8]:
```
# Sample the total field
N = deltaN_AUG.full(X, Z)

# Some data
Mx = deltaN_simple.Mx
Mz = deltaN_simple.Mz
Nh = deltaN_simple.Nharm
print( 'Required max. sample frequency Mx in x = {}'.format(Mx) )
print( 'Required max. sample frequency Mz in z = {}'.format(Mz) )
print( 'Number of independent harmonics N = {}'.format(Nh) )

# plot of the full density (profile plus fluctuations)
figure(3, figsize=(8,8))

subplot(111, aspect='equal')
pcolormesh(X, Z, N)
colorbar()
contour(X, Z, X, [Xpedestal], colors='w', linewidths=3)
xlabel('$x$', fontsize=20)
ylabel('$z$', fontsize=20)
title('total density')
grid('on')

# A section at z=constant of the full density
```

```
figure(4, figsize=(10,6))

index = int(nptz / 2)

subplot(111)
plot(X[index,:], N[index,:] * 1.e-13, label='total')
plot(X[index,:], n0(X, Z)[index,:] * 1.e-13, label='unperturbed')
legend()
xlabel('$x$', fontsize=20)
ylabel('$n$ [$10^{13} \mathrm{cm}^{-3}$]', fontsize=20)
title('density profile at $z=$ {}'.format(Z[index,0]), fontsize=20)
grid('on')

# Check positivity of the total density
Nmin = np.min(N)
print( 'Minimum value of the total density = {} cm^-3'.format(Nmin) )
```

```
Required max. sample frequency Mx in x = 16
Required max. sample frequency Mz in z = 8
Number of independent harmonics N = 281
Minimum value of the total density = -5949266626677.477 cm^-3
```





For the ASDEX-like slab, we plot the total density (background plus fluctuations) in cm^{-3}. The position of the pedestal center is indicated by the white contour line.