Solving Systems of Linear Equations

Focus: Practise solving linear systems to build skills that will be used throughout the unit.

PROPERTIES OF LINEAR SYSTEMS

- A linear system of equations can have **zero**, **one** or an **infinite #** of solutions
- A system of equations is **consistent** if it has either **one** solution or an **infinite #** of solutions
- A system of equations is **inconsistent** if it has **no** solutions.
- Two systems are defined as **equivalent** if every solution to one system is also a solution to the second system of equations, and vice versa.

How do we solve a system?
We must create a simplified "equivalent" system using elementary operations.

ELEMENTARY OPERATIONS

1. Equations are allowed to be multiplied by a nonzero constant.
2. Any pair of equations may be interchanged
3. Any equation may be added to another equation in non-zero multiples.

Ex 1.
Use elementary operations to solve the system of equations.

a) ① $3x + 2y = 8$
② $2x - 5y = -12$

①×2 $6x + 4y = 16$
②×(−3) $-6x + 15y = 36$

$19y = 52$

$y = \frac{52}{19}$

Sub y into ①

$3x + 2\left(\frac{52}{19}\right) = 8$

$57x + 104 = 152$

$x = \frac{16}{19}$

$(x, y) = \left(\frac{16}{19}, \frac{52}{19}\right)$

b) $2x - 4y = -3$
$3x - 6y = -5$

$$\left[\begin{array}{cc|c} 2 & -4 & -3 \\ 3 & -6 & -5 \end{array}\right] \begin{array}{l} R_1 \div 2 \\ R_2 \div 3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -\frac{3}{2} \\ 1 & -2 & -5/3 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & -2 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{6} \end{array}\right] \leftarrow 0x + 0y = \frac{-1}{6}$$

- inconsistent system
- i.e. pll lines

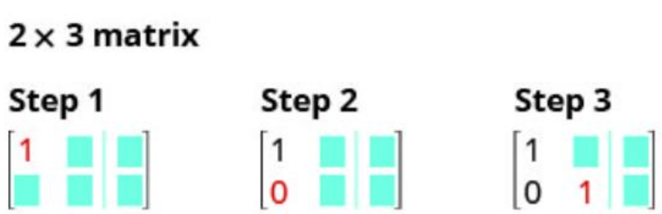
No Sol^n exists

$x - 2y = -\frac{3}{2}$

$\ominus\ x - 2y = -5/3$

$0 \quad 0 = -\frac{5}{3} - \left(-\frac{3}{2}\right)$

$0 = \frac{-1}{6}$



Ex 2.
Use elementary operations to find a solution to the system.

a)
$$\begin{aligned} x - 3y - 2z &= -9 \\ 2x - 5y + z &= 3 \\ -3x + 6y + 2z &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 2 & -5 & 1 & 3 \\ -3 & 6 & 2 & 8 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ 0 & -3 & -4 & -19 \end{array}\right] \xrightarrow{R3+3R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & -9 \\ 0 & 1 & 5 & 21 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

$\to x - 3y - 2z = -9$ (3) solve 3rd
$\to y + 5z = 21$ (2) solve 2nd
$\to 11z = 44$ (1) solve 1st

(1) $11z = 44$, $z = 4$

(2) $y + 5(4) = 21$, $y = 1$

(3) $x - 3(1) - 2(4) = -9$
$x - 3 - 8 = -9$
$x = 2$

A single solution exists
$(x, y, z) = (2, 1, 4)$

b)
$$\begin{aligned} x + y + 2z &= -2 \\ 3x - y + 14z &= 6 \\ x + 2y &= -5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right] \xrightarrow[R3-R1]{R2-3R1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & -4 & 8 & 12 \\ 0 & 1 & -2 & -3 \end{array}\right] -\tfrac{1}{4}R2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 1 & -2 & -3 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to 0x + 0y + 0z = 0$$

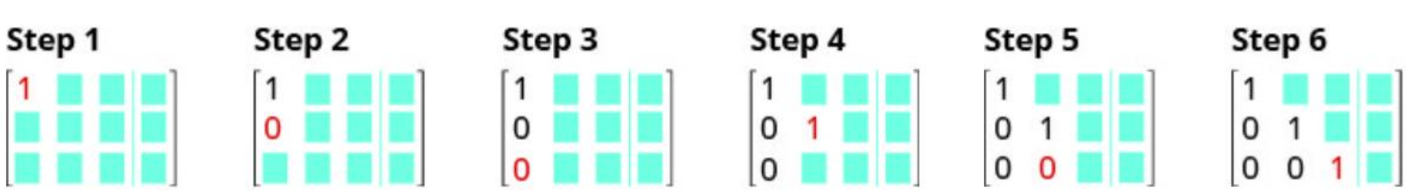
- infinite # of solns

c)
$$\begin{aligned} x - 2y + 3z &= 9 \\ x + y - z &= 4 \\ 2x - 4y + 6z &= 5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & +3 & 9 \\ 1 & +1 & -1 & 4 \\ 2 & -4 & +6 & 5 \end{array}\right] \xrightarrow{R3-2R_1} \left[\begin{array}{ccc|c} -2 & +4 & -6 & 18 \\ 1 & +1 & -1 & 4 \\ 0 & +0 & +0 & -13 \end{array}\right] \leftarrow 0x + 0y + 0z = -13$$

inconsistent
- no solutions

3 × 4 matrix



Ex 3.
Determine the value of k for which this system of equations has:

a) no solutions.
b) one solution.
c) an infinite # of solutions.

① $x + ky = 9$
② $kx + 9y = -27$

eliminate x by k① − ②

$kx + k^2y = 9k$
$-kx - 9y = +27$
$(k^2-9)y = 9k+27$

For no sol^{ns}

$\underbrace{0}_{k^2-9=0}y = \text{some \#}$

$k = \pm 3$

$K = 3$

For ∞ sol^{ns}

$0y = 0$

$k^2-9=0$ or $9k+27=0$

$k=\pm3$ or $k=-3$

$K = -3$

For one sol^{n}

$k \neq \pm 3$

Alternative

① $y = -\frac{1}{k}x + 9$
② $y = -\frac{k}{9}x - 3$

For 0 sol^{ns} + ∞ sol^{ns}

① + ② are pll

$-\frac{1}{k} = \frac{-k}{9}$

$\times(-k) \quad \times(-k)$

$1 = \frac{k^2}{9}$

$9 = k^2$

$\pm 3 = k$

For 1 sol^{n}

$k \neq \pm 3$

Test $k=3$	Test $k=-3$
① $x+3y=9$	① $x-3y=9$
$-3x-9y=-27$	$3x-9y=+27$
② $3x+9y=-27$	② $-3x+9y=-27$
$0+0=-54$	$0+0=0$
-pll	-same line
-no sol^{ns}	-∞ sol^{ns}

$K = 3 \rightarrow$ no sol^{ns}
$K = -3 \rightarrow$ ∞ sol^{ns}
$K \neq \pm 3 \rightarrow$ one sol^{n}

Additional HW:

223. $\begin{cases} x + 2y + 6z = 5 \\ -x + y - 2z = 3 \\ x - 4y - 2z = 1 \end{cases}$

225. $\begin{cases} 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \\ 2x - 2y + 3z = 6 \end{cases}$

227. $\begin{cases} -x - 3y + 2z = 14 \\ -x + 2y - 3z = -4 \\ 3x + y - 2z = 6 \end{cases}$