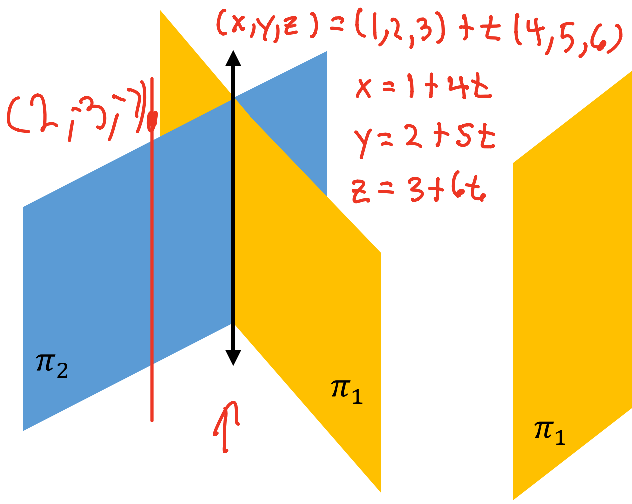
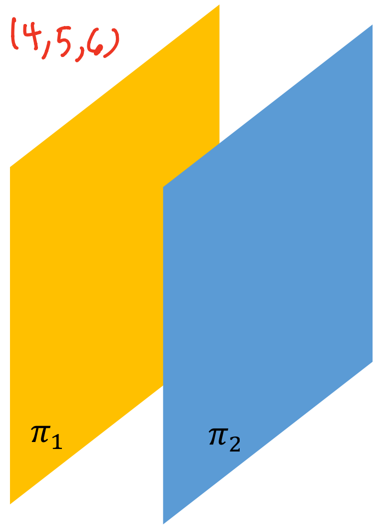
Intersection of Two Planes

Given two planes in R^3, there are three possible geometric models for the intersection of the planes.

CASE 1
Two planes intersecting along a line. There is an infinite number of solutions.



CASE 2
Two parallel planes. Planes are parallel but non-coincident. So there is NO SOLTION!



CASE 3
Two planes are coincident (same plane). There will be an infinite number of solutions.

Ex 1.
Describe how the planes in each pair intersect.

a) π_1: $2x - y + z - 1 = 0$ $\vec{n}_1 = (2,-1,1)$
   π_2: $x + y + z - 6 = 0$ $\vec{n}_2 = (1,1,1)$

not scalar multiples
- must intersect

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{1}{3} & \frac{4}{6} \end{bmatrix}$$

1^{st} - manipulate eq^{ns} to eliminate 1 variable
2^{nd} - set one variable to a parameter
3^{rd} - solve for the line of intersection

$\pi_1 + \pi_2$

$$\begin{aligned} 2x - y + z - 1 &= 0 \\ x + y + z - 6 &= 0 \\ \hline 3x + 2z - 7 &= 0 \end{aligned}$$

t or s

$\vec{r} = (1,2,3) + \underset{\text{Parameter}}{t}(4,5,6)$

Let $z = t$

$3x + 2t - 7 = 0$

$x = \frac{7}{3} - \frac{2}{3}t$

Sub $\underline{x}$ + $\underline{z}$ into π_1

$2\left(\frac{7}{3} - \frac{2}{3}t\right) - y + t = 1$

$\frac{14}{3} - \frac{4t}{3} - 1 + t = y$

$\frac{11}{3} - \frac{t}{3} = y$

$$\left.\begin{aligned} z &= t \\ x &= \frac{7}{3} - \frac{2}{3}t \\ y &= \frac{11}{3} - \frac{1}{3}t \end{aligned}\right\} \text{Parametric}$$

$$\left.\begin{aligned} (x,y,z) &= \left(\frac{7}{3}, \frac{11}{3}, 0\right) + t\left(\frac{-2}{3}, \frac{-1}{3}, 1\right) \\ &= \left(\frac{7}{3}, \frac{11}{3}, 0\right) + t(-2,-1,3) \end{aligned}\right\} \text{vector}$$

- the planes intersect in a line

$-7(1.5) = -10.5$

b) π_3: $2x - 6y + 4z \boxed{-7} = 0$

$\downarrow \times 1.5 \quad \downarrow \times 1.5 \quad \downarrow 1.5 \quad \downarrow \times 1.5$

π_4: $3x - 9y + 6z \boxed{-2} = 0$

$1.5\vec{n}_3 = \vec{n}_4$

$\vec{n}_3 = (2,-6,4)$

$\vec{n}_4 = (3,-9,6)$

$3\vec{n}_3 = 2\vec{n}_4$

- either pll or coincident

is $3D_3 = 2D_4$?

$3(-7)$	$2(-2)$
-21	-4

NO - planes are pll but <u>not</u> coincident

Alternative

$\vec{n}_3 = (2,-6,4) = 2(1,-3,2)$

$\vec{n}_4 = (3,-9,6) = 3(1,-3,2)$

- either pll or coincident
- are π_3 + π_4 multiples of each other?

$\frac{\pi_3}{2}$ $\quad x - 3y + 2z \boxed{-\frac{7}{2}} = 0$

No! Planes are pll

$\frac{\pi_4}{3}$ $\quad x - 3y + 2z \boxed{-\frac{2}{3}} = 0$

$3x + 5y = 1$

$\times 2 \quad \times 2 \quad \times 2$

$6x + 10y = 2$

c) π_5: $1x + 1y + 1z \boxed{= 1}$

$\downarrow \times 2 \quad \downarrow \times 2 \quad \downarrow \times 2 \quad \downarrow \times 2$

π_6: $2x + 2y + 2z \boxed{= 2}$

$\vec{n}_5 = (1,1,1)$

$\vec{n}_6 = (2,2,2)$

$2\vec{n}_5 = \vec{n}_6$

- either pll or coincident

$D_5 = -1$

$D_6 = -2$

same ratio?

yes!

- infinite number of solns
- coincident (same) planes

Alternative

$\vec{n}_5 = (1,1,1)$

$\vec{n}_6 = (2,2,2) = 2(1,1,1)$

- either pll or coincident
- are π_5 + π_6 multiples of each other?

π_5 $\quad x + y + z = \boxed{1}$

yes! planes are coincident

$\frac{\pi_6}{2}$ $\quad x + y + z = \boxed{1}$

$Ax + By + Cz + D = 0$