The Product Rule

$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$\rightarrow$ a single slope

Ex 1.
Calculate the derivative of h(x) = $\underbrace{(x^2 + 1)}_{f(x)}\underbrace{(x^3 – 2)}_{g(x)}$

$h(x) = x^5 + x^3 - 2x^2 - 2$

$h'(x) = 5x^4 + 3x^2 - 4x$

$f(x) = cx^n + x^b$ $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = c(n)x^{n-1} + bx^{b-1}$ $\rightarrow$ a function where $f'(x)$ is all slopes of $f(x)$

$f(x) = \sin(x)\left(\frac{1}{x^2}\right)$

THE PRODUCT RULE

NEWTON:
If p(x) = f(x)g(x), then

$$\mathbf{p'(x) = f'(x)g(x) + f(x)g'(x)}$$

LIEBNIZ:
If u and v are functions of x,

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{uv}) = \frac{\mathbf{du}}{\mathbf{dx}}\mathbf{v} + \frac{\mathbf{dv}}{\mathbf{dx}}\mathbf{u}$$

** assume f(x), g(x), u, and v are differentiable functions*

Ex 1.
Calculate the derivative of h(x) = $\underbrace{(x^2 + 1)}_{f(x)}\underbrace{(x^3 – 2)}_{g(x)}$ using the product rule.

$h'(x) = (2x)(x^3 - 2) + (x^2 + 1)(3x^2)$

$= 2x^4 - 4x + 3x^4 + 3x^2$

$= 5x^4 + 3x^2 - 4x$

PROOF:

$$\frac{[f(x+h) - f(x)]}{h}(g(x+h))$$

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}\right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x+h) + \lim_{h \to 0} f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

Ex 2.

Determine $\frac{d}{dx}$ if y = (2x³ + 5)(3x² – x).

$$\frac{dy}{dx} = (6x^2)(3x^2 - x) + (2x^3+5)(6x-1)$$

x^1 $1x^0$

$$= 18x^4 - 6x^3 + 12x^4 - 2x^3 + 30x - 5$$
$$= 30x^4 - 8x^3 + 30x - 5$$

Ex 3.

Differentiate $f(x) = \sqrt{x}(2 - 3x)$ and simplify.

$$= x^{\frac{1}{2}}(2-3x)$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(2-3x) + x^{\frac{1}{2}}(-3)$$
$$= x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} - 3x^{\frac{1}{2}}$$

$-\frac{9}{2}\frac{\sqrt{x}}{1}$

$$= \frac{1}{x^{\frac{1}{2}}} - \frac{9}{2}x^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{x}} - \frac{9\sqrt{x}}{2}$$
$$= \frac{\sqrt{x}}{x} - \frac{9\sqrt{x}}{2}$$

all forms are acceptable (to me)

$$\frac{1}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}}$$

$$\frac{\sqrt{x}}{x}$$

Ex 4.

Determine the slope of the tangent to the graph of the function $f(x) = (3x^2 + 2)(2x^3 - 1)$ at the point (1,5). (1,5)

$$f'(x) = 6x(2x^3-1) + (3x^2+2)(6x^2)$$
$$= 12x^4 - 6x + 18x^4 + 12x^2$$
$$= 30x^4 + 12x^2 - 6x$$

@(1,5)

$f'(1) = 30 + 12 - 6$

$f'(1) = 36$

m_{tan} at x=1 is 36

EXTENDED PRODUCT RULE
For a Product of Three Functions

If p(x) = f(x)g(x)h(x), then

$$\mathbf{p'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)}$$

The Quotient Rule

THE QUOTIENT RULE

NEWTON:

If $h(x) = \frac{f(x)}{g(x)}$, then $f(x) = \frac{x}{(x^2+5x)'}$

functions of x,

$f'(x)g(x) + f(x)g'(x)$

$$\mathbf{h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

LIEBNIZ:

If u and v are

functions of x,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

* *assume f(x), g(x), y, and u are differentiable functions*

PROOF (using the product rule):

If $h(x) = \frac{f(x)}{g(x)}$, then $h(x)g(x) = f(x)$

Use the product rule to take the derivative of f(x): $h'(x)g(x) + h(x)g'(x) = f'(x)$

Solve for $h'(x)$: $h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)}$

Substitute $h(x) = \frac{f(x)}{g(x)}$: $h'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$

Simplify (multiply by $\frac{g(x)}{g(x)}$): $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Ex 1.

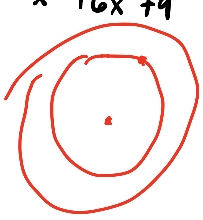
Calculate the derivative. Simplify.

a) $h(x) = \frac{\boxed{2x^2-3x+1}^{\,f(x)}}{\boxed{x^2+3}\; g(x)}$

$$h'(x) = \frac{\overset{f'(x)}{(4x-3)}\overset{g(x)}{(x^2+3)} - \overset{f(x)}{(2x^2-3x+1)}\overset{g'(x)}{(2x)}}{\underset{g(x)}{[x^2+3]^2}}$$

$$= \frac{4x^3 - 3x^2 + 12x - 9 - 4x^3 + 6x^2 - 2x}{x^4 + 6x^2 + 9}$$

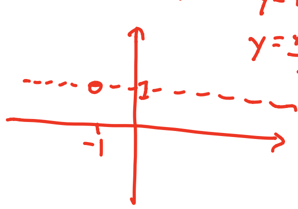
$$= \frac{3x^2 + 10x - 9}{x^4 + 6x^2 + 9}$$



$y = \frac{(x+1)}{(x+1)} = 1$

$y = 1$

$y = \frac{x+1}{x+1}$



b) $g(x) = \dfrac{\overbrace{(2x+1)(x-3)}^{f(x)}}{\underbrace{2-x}_{p(x)}}$

$$\boxed{g'(x) = \frac{f'(x)\,p(x) - f(x)\,p'(x)}{[p(x)]^2}}$$

$$g'(x) = \frac{[(2)(x-3)+(2x+1)(1)](2-x) - (2x+1)(x-3)(-1)}{(2-x)^2}$$

$$= \frac{[4x-5](2-x) + (2x^2-5x-3)}{(2-x)^2}$$

$$= \frac{8x-4x^2-10+5x+2x^2-5x-3}{(2-x)^2}$$

$$= \frac{-2x^2+8x-13}{(2-x)^2}$$

$$g(x) = \frac{\overbrace{2x^2-5x-3}^{f(x)}}{(2-x)}$$

$$g'(x) = \frac{(4x-5)(2-x) - (2x^2-5x-3)(-1)}{(2-x)^2}$$

$$= \frac{8x-4x^2-10+5x+2x^2-5x-3}{(2-x)^2}$$

$$= \frac{-2x^2+8x-13}{(2-x)^2}$$

Ex 2.

Determine the slope of the tangent to h(x) at x = 2

$$h(x) = \frac{5x+2}{x-4}$$

$$h'(x) = \frac{(5)(x-4) - (5x+2)(1)}{(x-4)^2}$$

$$= \frac{5x-20-5x-2}{(x-4)^2}$$

$$= \frac{-22}{(x-4)^2}$$

m_{tan} @ $x=2$

$$h'(2) = \frac{-22}{(2-4)^2}$$

$$= \frac{-22}{4}$$

$$= \frac{-11}{2}$$

Ex 3.

Determine the point(s) on the curve $y = \frac{x}{x-1}$ where the tangent line is parallel to the line x + 4y = 1.

$$\underbrace{y = -\tfrac{1}{4}x + \tfrac{1}{4}}_{\text{has } m = -\frac{1}{4}}$$

$$m_{tan} = y' = \frac{(1)(x-1) - (x)(1)}{(x-1)^2}$$

$$= \frac{x-1-x}{(x-1)^2}$$

$$= \frac{-1}{(x-1)^2}$$

Let $y' = -\frac{1}{4}$

$$-\frac{1}{4} = \frac{-1}{(x-1)^2}$$

$$4 = (x-1)^2$$

$$\pm 2 = x - 1$$

$$1 \pm 2 = x$$

$$x = 3,\ x = -1$$

Points where $m_{tan} = -\frac{1}{4}$

$$y = \frac{x}{x-1}$$

$$\left(3, \frac{3}{2}\right)$$

$$\left(-1, \frac{1}{2}\right)$$