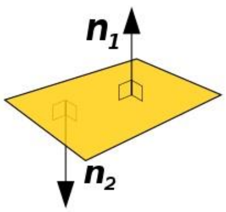
The Cartesian (Scalar) Equation of a Plane



The normal vector is a nonzero vector perpendicular to all vectors in the plane. It is useful when:

- determining the Cartesian equation of a plane
- determining the angle between planes

Cartesian Equation of π

The Cartesian (or scalar) equation of a plane in R^3 is of the form:

$$\mathbf{Ax + By + Cz + D = 0}$$

With a normal vector:

$$\vec{\mathbf{n}} = (\mathbf{A}, \mathbf{B}, \mathbf{C})$$

Line: $Ax + By + C = 0$
$\vec{n}(A, B)$

Ex 1.
Determine the Cartesian equation of a plane with a normal of (2,-3,4) and a point A(1,2,3).

$Ax + By + Cz + D = 0$

$\vec{n} = (2, -3, 4)$

$2x - 3y + 4z + D = 0$

sub in A(1,2,3) to solve for D

$2(1) - 3(2) + 4(3) + D = 0$

$D = -8$

$2x - 3y + 4z - 8 = 0$

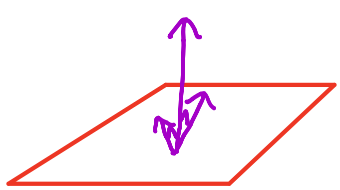
Ex 2.
Given three non-collinear points A(2,3,4), B(6,5,4) and C(–2, 4, –1):

a) Determine two direction vectors.

- many possibilities
- we will choose $\vec{AB}$ + $\vec{AC}$

$\vec{AB} = (4, 2, 0)$

$\vec{AC} = (-4, 1, -5)$



b) Determine the normal to the plane.

$\vec{n} = \vec{AB} \times \vec{AC}$

2 1
0 -5
4 -4
2 1

$\vec{n} = (-10, 20, 12)$

or

$\vec{n} = (-5, 10, 6)$

c) Determine the Cartesian equation of the plane.

$\vec{n} = (-5, 10, 6)$ A(2,3,4)

$Ax + By + Cz + D = 0$

$-5(2) + 10(3) + 6(4) + D = 0$

$D = -44$

$-5x + 10y + 6z - 44 = 0$

REMEMBER:
To determine an equation for a plane you will need two nonzero, non-collinear vectors and a point on the plane. If you are given the vector or parametric equations then you have all the information you need.

Ex 3.

Determine the vector equation of the plane given the Cartesian equation $x - 3y + 5z - 1 = 0$.

① need 2 non-collinear vectors on the plane (if we have 3 points, we can make 2 vectors)

② need a point on the plane (see step ①)

$y = mx + b$

1st - determine 3 points

Let $y=0, z=0$	Let $y=1, z=0$	Let $y=0, z=1$
$x - 3(0) + 5(0) - 1 = 0$	$x - 3(1) + 5(0) - 1 = 0$	$x - 3(0) + 5(1) - 1 = 0$
$x = 1$	$x = 4$	$x = -4$
$(1,0,0)$	$(4,1,0)$	$(-4,0,1)$

$\vec{m}_1 = (3,1,0)$ $\quad$ $\vec{m}_2 = (8,1,-1)$

$\pi \; (x,y,z) = (1,0,0) + s(3,1,0) + t(8,1,-1)$

Check if $\vec{n}$ is the same as cartesian version

$\vec{m}_1 \times \vec{m}_2$

$$\begin{array}{cc} 1 & 1 \\ 0 & -1 \\ 3 & 8 \\ 1 & 1 \end{array}$$

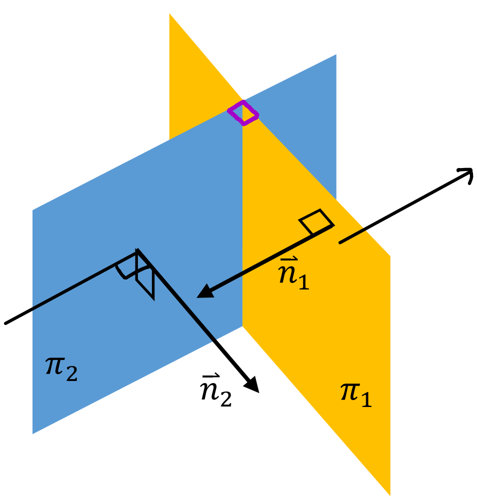
$\vec{n} = (-1, 3, -5)$

$\vec{n} = -1(1,-3,5)$

Yes!

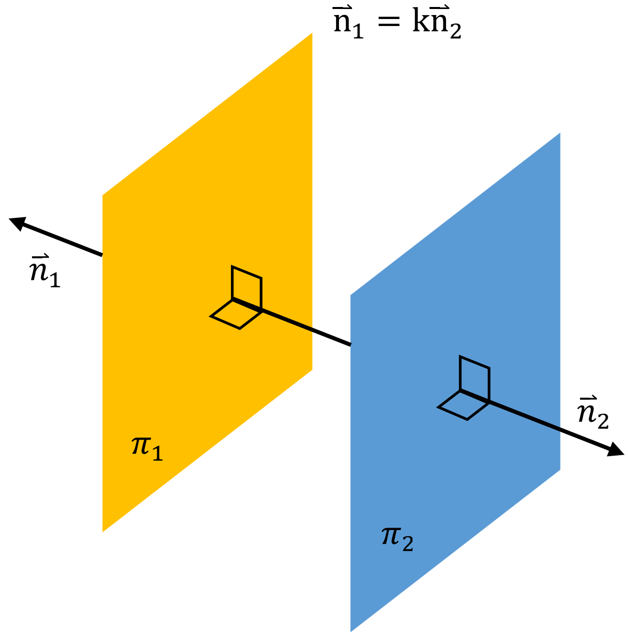
Perpendicular planes have perpendicular normals.

$\vec{n}_1 \cdot \vec{n}_2 = 0$



Parallel planes have Parallel normals.

$\vec{n}_1 = k\vec{n}_2$



Ex 4.

$4x+8y-2z+1=0$

a) What is the value of k that makes $\underbrace{4x + ky - 2z + 1 = 0}_{\vec{n}_1=(4,k,-2)}$ and $\underbrace{2x + 4y - z + 4 = 0}_{\vec{n}_2=(2,4,-1)}$ parallel?

If pll, normals must be scalar multiples of each other.

compare

$\vec{n}_1 = 2\vec{n}_2$

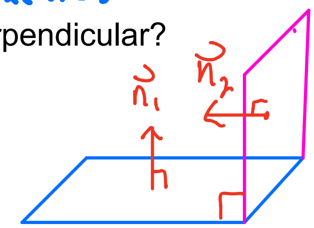
$k = 2(4) = 8$

b) What is the value of k that makes the above planes perpendicular?

$\vec{n}_1 \cdot \vec{n}_2 = 0$

$8 + 4k + 2 = 0$

$k = -\frac{5}{2}$



c) Can these planes ever be coincident (identical)? Explain

Coincident planes must be pll $\underbrace{\text{(have pll normals)}}_{k=8}$ and share all points

- find a point on one plane + check that it satisfies the eq^n of the other plane

π_2 $2x+4y-z+4=0$

Let $x=0, y=0$

$2(0)+4(0)-z+4=0$

$z=4$

$(0,0,4)$

Check $(0,0,4)$ in π_1

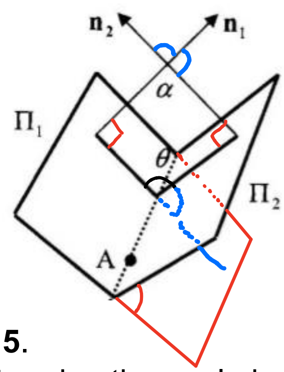
π_1 $4x+8y-2z+1=0$

$4(0)+8(0)-2(4)+1=0$

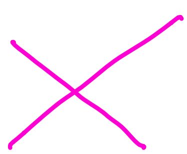
$-7=0$

Can't be coincident

Calculating the Angle between Intersecting Planes



Use the dot product of the normals of two planes to calculate the angle between them.



$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Ex 5.

Determine the angle between two planes x + 2y – 3z –4 = 0 and x + 2y – 1 = 0.

$\vec{n}_1 = (1,2,-3)$ $\vec{n}_2 = (1,2,0)$

$$\cos\alpha = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{5}{\sqrt{14}\sqrt{5}}$$

$\alpha = 53^\circ$

or $\theta = 180 - 53^\circ$

$\theta = 147^\circ$

Aside

$\vec{n}_1 \cdot \vec{n}_2 = 5$

$|\vec{n}_1| = \sqrt{14}$

$|\vec{n}_2| = \sqrt{5}$