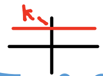
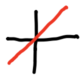
Properties of Limits – Part 1

For any real number "a", f and g are functions with limits that exist at x = a:

1. $\lim_{x\to a} k = $ k, for any constant k
2. $\lim_{x\to a} x = $ a



$\text{L } f(x)=x$



THINK!
What do these functions look like?

3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a} c.f(x) = c\left[\lim_{x\to a} f(x)\right]$, for any constant c
5. $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$
6. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, for any rational number n

$\lim_{x\to 2} x^2\sqrt{x+1} = \left(\lim_{x\to 2} x^2\right)\left(\lim_{x\to 2}\sqrt{x+1}\right)$

$(f(2))(g(2))$

Ex 1.
Use the properties of limits to evaluate.

a) $\lim_{x\to -2}(2x^2 - 7x + 4)$

$= \lim_{x\to -2} 2x^2 - \lim_{x\to -2} 7x + \lim_{x\to -2} 4$

$= 8 + 14 + 4$

$= 26$

b) $\lim_{x\to 1}[-2(x-3)^2 + 4]$

$= -2\left[\lim_{x\to 1}(x-3)\right]^2 + \lim_{x\to 1} 4$

$= -2[-2]^2 + 4$

$= -4$

c) $\lim_{x\to 3}\frac{3x^2-x+4}{2x-3}$

$= \frac{\lim_{x\to 3} 3x^2-x+4}{\lim_{x\to 3} 2x-3} = \frac{28}{3}$

d) $\lim_{x\to 8}\sqrt[3]{5x^2 - 18x - 8}$

$= \sqrt[3]{\lim_{x\to 8} 5x^2-18x-8}$

$= \sqrt[3]{168}$

$= \sqrt[3]{8\times 21}$

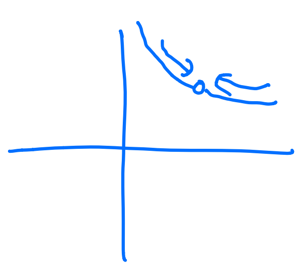
$= 2\sqrt[3]{21}$

Summary:

- In all the above cases the limit can be found by direct substitution… the function is continuous at the limit value so $\lim_{x\to a} f(x) = f(a)$.

$\rightarrow$ hole $\quad f(x) = \frac{\#}{0} \rightarrow$ Asymp.

- When direct substitution of x = a results in $\frac{0}{0}$ this is called an ***indeterminate form***.
- When this happens we look for an equivalent function that has all the same values as f(x) except at x = a



When direct substitution fails try:

1. Factoring

a) $\lim_{x\to 3} \frac{x^2+x-12}{2x^2-5x-3} = \frac{0}{0}$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+4)}{\cancel{(x-3)}(2x+1)}$

$= \frac{7}{7}$

$= 1$

b) $\lim_{x\to 2} \frac{x^3-8}{x-2}$

$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{x-2}}$

$= 12$

$f(x) = \frac{x^3-8}{x-2}$, $f(2) = $ undef.

2. Rationalizing

a) $\lim_{x\to 9} \frac{(\sqrt{x}-3)}{(x-9)} \cdot \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)}$

$= \lim_{x\to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

FORCE FACTOR!

$\lim_{x\to 9} \frac{\sqrt{x}-3}{\underset{(\sqrt{x})^2-9}{x-9}} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)}$

$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

b) $\lim_{x\to 0} \frac{\sqrt{5+x}-\sqrt{5-x}}{x} \cdot \frac{\sqrt{5+x}+\sqrt{5-x}}{\sqrt{5+x}+\sqrt{5-x}}$

$= \lim_{x\to 0} \frac{5+x-(5-x)}{x(\sqrt{5+x}+\sqrt{5-x})}$

$= \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}(\sqrt{5+x}+\sqrt{5-x})}$

$= \frac{2}{2\sqrt{5}}$

$= \frac{1}{\sqrt{5}}$

3. Change of Variable

a) $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{☺\to ☺} \frac{☺^2-1}{(☺-1)}$ Note: this could be rationalized or force factored, but let's learn something new.

$\sqrt{0.9999} = 0.99994999...$

1. Let $\sqrt{x}$ = ☺, so x = ☺^{2} and as x → 1, ☺ → 1
2. Rewrite the question in terms of ☺

$\lim_{☺\to 1} \frac{☺^2-1}{☺-1}$ … now factor and evaluate

$= \lim_{☺\to 1} \frac{(☺-1)(☺+1)}{☺-1}$

$= \lim_{☺\to 1} (☺+1)$

$= 2$

b) $\lim\limits_{x\to2}\frac{(x+25)^{\frac{1}{3}}-3}{x-2}$

$= \lim\limits_{u\to3}\frac{u-3}{u^3-25-2}$

$= \lim\limits_{u\to3}\frac{u-3}{u^3-27}$

$= \lim\limits_{u\to3}\frac{\cancel{u-3}}{(\cancel{u-3})(u^2+3u+9)}$

$= \frac{1}{9+9+9}$

$= \frac{1}{27}$

I chose this

$\downarrow$

$u=(x+25)^{\frac{1}{3}}$

$u^3 = x+25$

$x = u^3-25$

$x=2$

$u=(2+25)^{\frac{1}{3}}$

$u=3$

4. Consider Cases

$\lim\limits_{x\to-3}\frac{|x+3|(x+1)}{x+3}$

$|x+3|\begin{cases}=-(x+3) & \text{if } x\le-3\\ =(x+3) & \text{if } x\ge-3\end{cases}$

<u>LH limit</u>

$= \lim\limits_{x\to-3^-}\frac{-\cancel{(x+3)}(x+1)}{\cancel{(x+3)}}$

$= \lim\limits_{x\to-3^-} -(x+1)$

$= -(-3+1)$

$= 2$

<u>RH Limit</u>

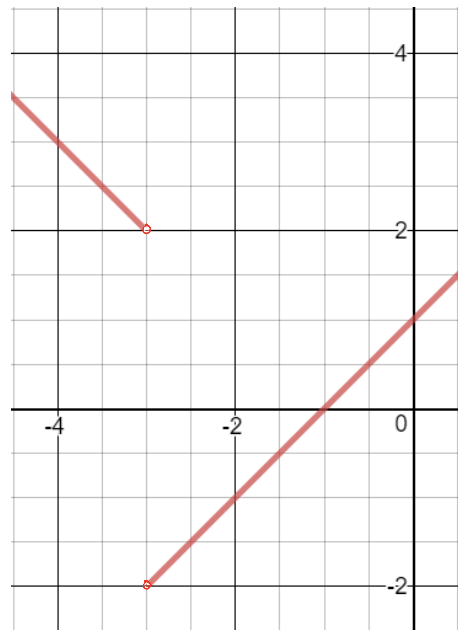
$= \lim\limits_{x\to-3^+}\frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}}$

$= \lim\limits_{x\to-3^+} (x+1)$

$= -2$

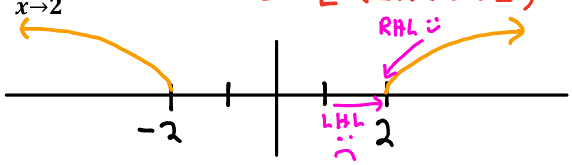
LHL $\neq$ RHL

$\lim\limits_{x\to-3} f(x) = \text{DNE}$



5. Think Graphically/ Reason It Out

$\lim\limits_{x\to2}\sqrt{x^2-4}$ = DNE (LHL DNE)



b) $\lim\limits_{x\to2}\frac{(x+25)^{\frac{1}{3}}-3}{x-2}$

let $u=(x+25)^{\frac{1}{3}}$

$u^3=x+25$

$u^3-25=x$

as $x\to2$

$u=(2+25)^{\frac{1}{3}}$

$=3$

$u\to3$

$=\lim\limits_{u\to3}\frac{u-3}{u^3-25-2}$

$=\lim\limits_{u\to3}\frac{u-3}{u^3-27}$

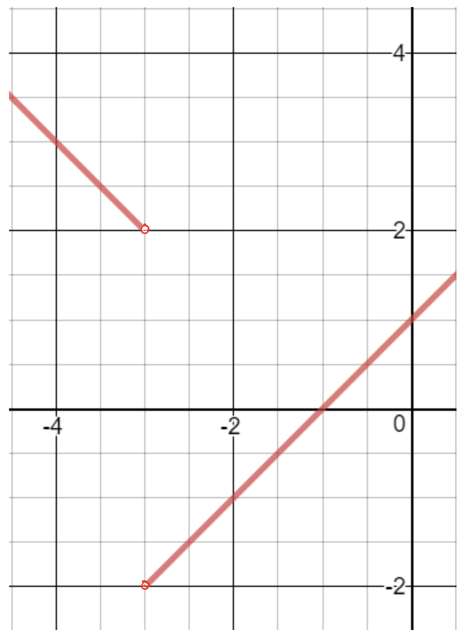
$=\lim\limits_{u\to3}\frac{1\cancel{(u-3)}}{\cancel{(u-3)}(u^2+3u+9)}$

$=\frac{1}{27}$

4. Consider Cases

$\lim\limits_{x\to-3}\frac{|x+3|(x+1)}{x+3}$

$|x+3|\begin{cases}=-(x+3) & \text{if } x<-3\\ =(x+3) & \text{if } x\geq-3\end{cases}$



LH limit

$=\lim\limits_{x\to-3^-}\frac{-\cancel{(x+3)}(x+1)}{\cancel{(x+3)}}$

$=\lim\limits_{x\to-3^-}-(x+1)$

$=-(-3+1)$

$=2$

RH Limit

$=\lim\limits_{x\to-3^+}\frac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}}$

$=\lim\limits_{x\to-3^+}(x+1)$

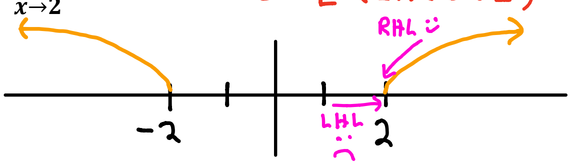
$=-2$

LHL $\neq$ RHL

$\lim\limits_{x\to-3}f(x)=$ DNE

5. Think Graphically/ Reason It Out

$\lim\limits_{x\to2}\sqrt{x^2-4}$ = DNE (LHL DNE)



$$\cancel{g(x)=3x^3}$$

$$\cancel{h(x)=2x}$$

$$k(x)=x$$

$$f(x)=3x^2+2x$$

$$=k(x)(3x+2)$$

$$=k(x)(3k(x)+2)$$

$$=3[k(x)]^2+2k(x)$$