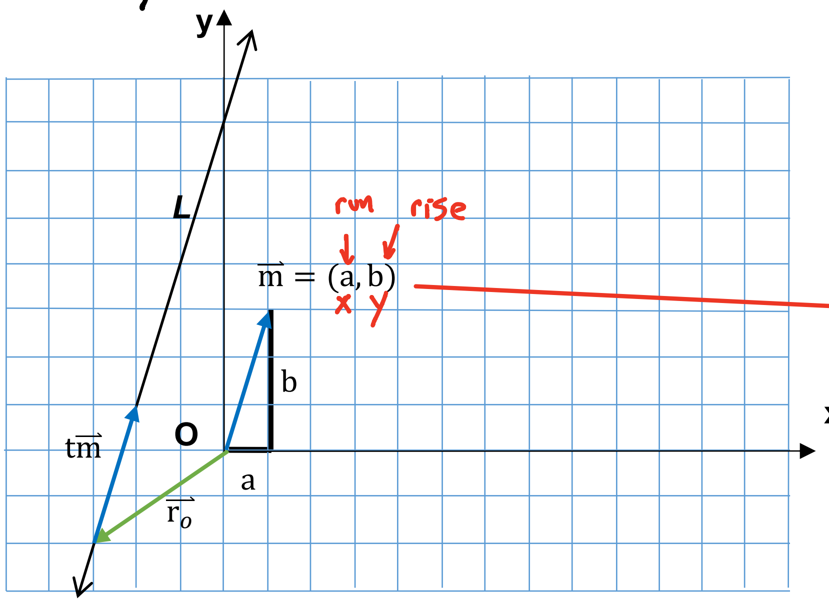
Cartesian Equation of a Line

Focus: Derive vector, parametric, and Cartesian equations of a line.

Recall:

- y = mx + b (called "slope-y-intercept form")
- Ax + By + C = 0 (called "Standard", "Cartesian", or "Scalar Equation of a line")



The direction numbers of the direction vector can be easily used to determine the slope of the line

$$\mathbf{m} = \frac{\mathbf{b}}{\mathbf{a}}$$

Ex 1.
Determine the vector and parametric equations of a line y = –2x + 3.

Need initial point + direction

Choose $x=0$

$y=-2(0)+3$

$y=3$

$(0,3)$

$m = \frac{-2}{1}$

$\vec{m} = (1,-2)$

$(x,y)=(0,3)+t(1,-2),\ t \in \mathbb{R}$

$x=t$

$y=3-2t$, $t \in \mathbb{R}$

Ex 2.
Write the Cartesian form of a line with an equation:

a) $\vec{r} = (2,4) + t\underbrace{(-1,5)}$

$m = \frac{5}{-1}$

Rearrange slope formula

$m = \frac{y-y_1}{x-x_1}$

$-5 = \frac{y-4}{x-2}$

$-5x+10 = y-4$

$0 = 5x+y-14$

OR

$y=mx+b$

$4=-5(2)+b$

$14=b$

$y=-5x+14$

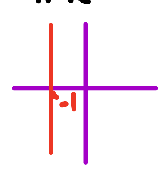
$5x+y-14=0$

b) $\vec{r} = (-1,2) + s(0,-7)$

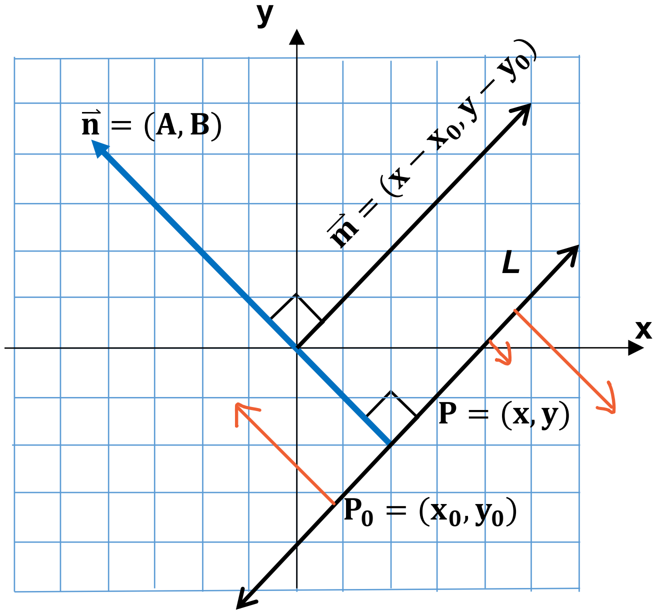
$m = \frac{-7}{0}$ ← Vertical line

$x=-1$

$x+1=0$



Introduction of the Normal Vector ($\vec{n}$)



- A line is drawn perpendicular to the line L from the origin. This perpendicular line is called the NORMAL axis.
- There are an infinite number of normal vectors on the normal axis.
- The normal is perpendicular to any vector on the given line
- The Cartesian equation and the normal are related through the coefficients **A** and **B**. (the proof is in your textbook on page 438)

$$\mathbf{Ax + By + C = 0}, \text{where } \vec{\mathbf{n}} = (\mathbf{A}, \mathbf{B})$$

Ex 3.
Determine the Cartesian equation for a line with a normal vector of (2,3) and passing through the point (–1,4).

$\vec{n} = (2,3)$
A B

$2x + 3y + C = 0$

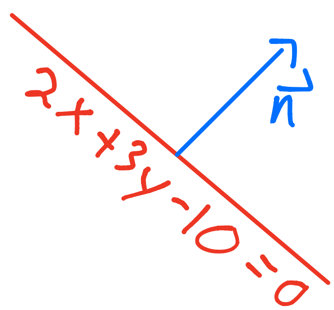
Sub in $(-1,4)$

$2(-1) + 3(4) + C = 0$

$-2 + 12 + C = 0$

$C = -10$

$\boxed{2x + 3y - 10 = 0}$

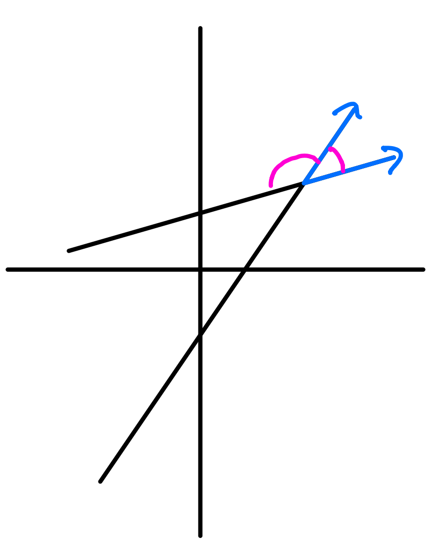


Ex 4.
Determine the acute angle between the two lines **x = 2t, y = –1 –3t** and **(x,y) = (1,1) + s(3,1).**

$\left.\begin{array}{l} x = 0 + 2t \\ y = -1 - 3t \end{array}\right\} (x,y) = (0,-1) + t(2,-3)$

$\vec{m}_2 = (3,1)$

$\vec{m}_1 = (2,-3)$



$\vec{m}_1 \cdot \vec{m}_2 = 2(3) + 1(-3)$

$= 3$

$|\vec{m}_1| = \sqrt{4+9} = \sqrt{13}$

$|\vec{m}_2| = \sqrt{9+1} = \sqrt{10}$

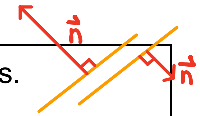
Find angle between direction vectors

$\cos\theta = \dfrac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$

$= \dfrac{3}{\sqrt{13}\sqrt{10}}$

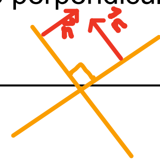
$\theta = 75°$

If L₁ and L₂ are parallel then their normals are scalar multiples.



If L₁ and L₂ are perpendicular then their normals are also perpendicular.

$\overrightarrow{\mathbf{n_1}} \cdot \overrightarrow{\mathbf{n_2}} = \mathbf{0}$



Ex 5.
Determine the Cartesian equation of a line that is perpendicular to $\underbrace{2x - y + 4 = 0}_{\text{given}}$ and passes through the point P(2,3).

$\left.\begin{array}{l}\vec{n}_g = (2,-1) \\ m_{nog} = \dfrac{-1}{2}\end{array}\right\}$ Given line

normal of given

$\vec{n}_g = \vec{m}$ of new line | point (2,3)

$y = mx + b$

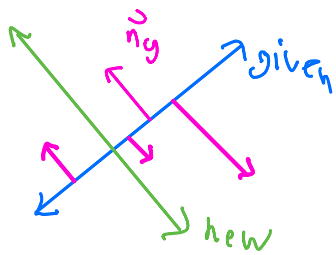
$3 = -\frac{1}{2}(2) + b$

$4 = b$

$y = -\frac{1}{2}x + 4$

$2y = -x + 8$

$x + 2y - 8 = 0$



OR

point (2,3)

point (x,y)

$m = \dfrac{y-3}{x-2}$

$\vec{m} = (x-2, y-3)$

$\vec{n}_{NL} = (1,2)$

new line

$\vec{n}_{NL} \cdot \vec{m} = 0$

$(1,2) \cdot (x-2, y-3) = 0$

$x - 2 + 2y - 6 = 0$

$x + 2y - 8 = 0$

Ex 6.
Are the following lines perpendicular?
L₁ has a normal vector of (1,3) and L₂ has a normal vector of (3, –1)

$(1,3) \cdot (3,-1) = 0$

$3 + (-3) = 0$

$0 = 0$

LS	RS
$1(3) - 1(3)$	0
0	

LS = RS

Ex 5 cont'd

OR

$\left.\begin{array}{l}\vec{n} = (2,-1) \\ m = \dfrac{-1}{2}\end{array}\right\}$ Given line

(of normal)

$m_{\perp} = \dfrac{2}{1}$

(of normal)

$\vec{n}_{\perp} = (\underset{A}{1}, \underset{B}{2})$ | point (2,3)

$1(2) + 2(3) + C = 0$

$C = -8$

$x + 2y - 8 = 0$