The Chain Rule

THE CHAIN RULE

NEWTON:
If $h(x) = f[g(x)]$ then

$$h'(x) = f'[g(x)]g'(x)$$

LIEBNIZ:
If y is a function of u, and u is a function of x, then:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = y'u'$$

** assume f(x), g(x), y, and u are differentiable functions*

$y = (2x+1)^2$
$u = 2x+1$
$y = u^2$

$= 2u2$
$= 4u$
$= 4(2x+1)$

$y = (2x+1)^2$
$y' = 2(2x+1)'(2)$
$= 4(2x+1)$

Ex 1.
Calculate the derivative of $f(x) = (2x^2)^3$ using the following methods.

a) Simplifying and using the power rule
b) Using the chain rule

a) $f(x) = 8x^6$
$f'(x) = 6(8x^5)$
$= 48x^5$

b) $f(x) = (g(x))^3$
$f'(x) = 3(g(x))^2 g'(x)$

$f'(x) = 3(2x^2)^2(4x)$
$= 3(4x^4)(4x)$
$= 48x^5$

$\overbrace{\qquad}^{u}$
$f(u) = u^3$
$f'(u) = 3u^2u'$

Ex 2.
Determine f'(x).

a) $f(x) = (3x^2 - 7x + 4)^3$

$f'(x) = 3(3x^2-7x+4)^2(6x-7)$

b) $f(x) = \sqrt[3]{x^2 - 5}$

$f(u) = (u)^{\frac{1}{3}}$
$f'(u) = \frac{1}{3}u^{-\frac{2}{3}}u'$

$= (x^2-5)^{\frac{1}{3}}$
$f'(x) = \frac{1}{3}(x^2-5)^{-\frac{2}{3}}(2x)$
$= \dfrac{2x}{3(x^2-5)^{\frac{2}{3}}}$

c) $f(x) = \frac{5}{(2x^2-7x+1)^3}$

$= 5(2x^2-7x+1)^{-3}$
$= 5(-3)(2x^2-7x+1)^{-4}(4x-7)$
$= \dfrac{-15(4x-7)}{(2x^2-7x+1)^4}$

d) $f(x) = (3x^2 - 5x - 1)^{\frac{5}{3}}$

$f'(x) = \frac{5}{3}(3x^2-5x-1)^{\frac{2}{3}}(6x-5)$

$= \dfrac{5\sqrt[3]{(3x^2-5x-1)^2}\,(6x-5)}{3}$

$= \dfrac{5\left(\sqrt[3]{(3x^2-5x-1)}\right)^2(6x-5)}{3}$

Ex 3.

Determine y'. Simplify your answer by factoring.

$y = x^2 + x^3$

$= x^2(1+x)$

a) $y = \underbrace{(3x^2+1)^4}_{f(x)}\underbrace{(2-3x)^3}_{g(x)}$ $f'(x)g(x) + f(x)g'(x)$

$$y' = \overbrace{4(3x^2+1)^3(6x)}^{\text{Chain rule}}(2-3x)^3 + (3x^2+1)^4\overbrace{(3)(2-3x)^2(-3)}^{\text{Chain Rule}}$$

$$= 24x(3x^2+1)^3(2-3x)^3 - 9(3x^2+1)^4(2-3x)^2$$

common factoring

$$= 3(3x^2+1)^3(2-3x)^2\left[8x(2-3x) - 3(3x^2+1)\right]$$

$$= 3(3x^2+1)^3(2-3x)^2(16x - 24x^2 - 9x^2 - 3)$$

$$= 3(3x^2+1)^3(2-3x)^2(-33x^2 + 16x - 3)$$

b) $y = \dfrac{(2x^2-5)^6}{(3x-1)^7}$ $\dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

OR

$= (2x^2-5)^6(3x-1)^{-7}$

$g(x) = (3x-7)^7$

$[g(x)]^2 = [(3x-7)^7]^2$

$$y' = \frac{\overbrace{6(2x^2-5)^5(4x)}^{\text{Chain rule}}(3x-1)^7 - (2x^2-5)^6\overbrace{(7)(3x-1)^6(3)}^{\text{Chain rule}}}{(3x-1)^{14}}$$

$$\frac{24x(2x^2-5)^5(3x-1)^7 - (2x^2-5)^6(21)(3x-1)^6}{(3x-1)^{14}}$$

$$= \frac{3(2x^2-5)^5(3x-1)^{\cancel{6}}\left[8x(3x-1) - 7(2x^2-5)\right]}{(3x-1)^{\cancel{14}\,8}}$$

$$= \frac{3(2x^2-5)^5\left[24x^2 - 8x - 14x^2 + 35\right]}{(3x-1)^8}$$

$$= \frac{3(2x^2-5)^5\left[10x^2 - 8x + 35\right]}{(3x-1)^8}$$

$3(x)(2)(x)(1)(x)$

$3(2)(1)(x)(x)(x)$

$6x^3$

Ex 4.

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Use Leibniz notation to determine $\frac{dy}{dx}$ at the give value of x.

$y = 4u^2 - 3u,\ u = \sqrt{x^3+1},\ x = 2$

$y = 4u^2 - 3u$

$\frac{dy}{du} = 8u - 3$

$u = \sqrt{(2)^3+1}$

$u = 3$

$\left.\frac{dy}{du}\right|_{u=3} = 8(3) - 3$

$= 21$

$u = (x^3+1)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}\frac{(x^3+1)^{-\frac{1}{2}}}{1}\frac{(3x^2)}{1}$

$= \frac{3x^2}{2\sqrt{x^3+1}}$

$\left.\frac{du}{dx}\right|_{x=2} = \frac{3(2)^2}{2\sqrt{(2)^3+1}}$

$= 2$

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 21 \cdot 2$

$= 42$

Ex 5.

Determine the equation of the tangent to the curve $y = \frac{1}{\sqrt{20-x^4}}$ at the point $\left(2, \frac{1}{2}\right)$.

$y = (20 - x^4)^{-\frac{1}{2}}$

$y' = -\frac{1}{2}(20 - x^4)^{-\frac{3}{2}}(-4x^3)$

$= \frac{2x^3}{(20-x^4)^{\frac{3}{2}}}$

Let $x = 2$

$y' = \frac{2(8)}{(20-16)^{\frac{3}{2}}}$

$= \frac{16}{8}$

$y' = 2$

$m_{tan} = 2$, point $\left(2, \frac{1}{2}\right)$

$y = mx + b$

$\frac{1}{2} = 2(2) + b$

$-\frac{7}{2} = b$

$y = 2x - \frac{7}{2}$

Alternative (p, q)

$y = m(x-p) + q$

$y = 2(x-2) + \frac{1}{2}$

$= 2x - 4 + \frac{1}{2}$

$y = 2x - \frac{7}{2}$

Intro

$g(x) = x^3 + 7x$ $\quad$ $f(x) = x^5$

$g'(x) = 3x^2 + 7$ $\quad$ $f'(x) = 5(x)^4 \cdot 1$

$= 5x^4$

Composite $F^{\underline{n}}$

(Nested $F^{\underline{n}}$)

$h(x) = (x^3 + 7x)^5$

$= 5(x^3 + 7x)^4 \cdot (3x^2 + 7)$