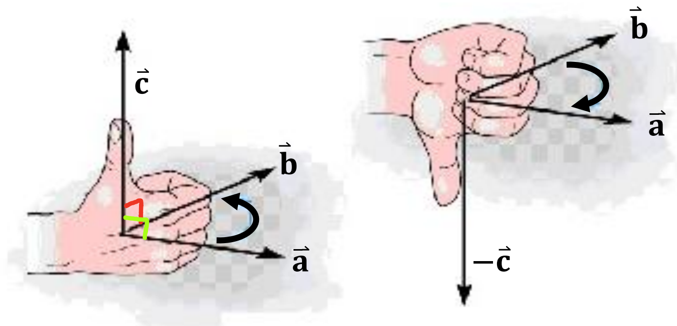
The Cross Product of Algebraic Vectors

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 is the vector that is perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}\times\vec{b}$ forms a right-handed system. The cross product is not commutative, meaning $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$.



$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \vec{\mathbf{c}}$ $\qquad$ $\vec{\mathbf{b}}\times\vec{\mathbf{a}} = -\vec{\mathbf{c}}$

Note:
Solving from first principles is time consuming, so a general formula has been derived to save us time.

Formula for Calculating the Cross Product of Algebraic Vectors

A vector perpendicular to both $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

OR $\vec{a}\times\vec{b} = (a_2b_3 - \mathbf{b_2a_3}, a_3b_1 - \mathbf{b_3a_1}, a_1b_2 - \mathbf{b_1a_2})$

$-(a_1b_3 - a_3b_1)$
$-a_1b_3 + a_3b_1$
$a_3b_1 - a_1b_3$

Cofactor Expansion Method

$$\vec{a}\times\vec{b} = \begin{vmatrix} \overset{+}{\hat{\imath}} & \overset{-}{\hat{\jmath}} & \overset{+}{\hat{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{k}$$

$a_2b_3 - a_3b_2$ $\qquad$ $a_3b_1 - a_1b_3$ $\qquad$ $a_1b_2 - a_2b_1$

Ex 1.
Determine $\vec{a}\times\vec{b}$ given $\vec{a} = (3,2,-1)$ and $\vec{b} = (-1,2,-4)$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ 3 & 2 & -1 \\ -1 & 2 & -4 \end{vmatrix} = +\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ -1 & 2 & -4 \end{vmatrix} - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ -1 & 2 & -4 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ -1 & 2 & -4 \end{vmatrix}$$

$= (-8+2)\hat{i} - (-12-1)\hat{j} + (6+2)\hat{k}$

$= -6\hat{i} + 13\hat{j} + 8\hat{k}$

$= (-6, 13, 8)$

Ex 2.
Determine $\vec{x} \times \vec{y}$ and $\vec{y} \times \vec{x}$ given $\vec{x} = (3,5,-2)$ and $\vec{y} = (-1,2,3)$.

$$\vec{x} \times \vec{y} = \begin{array}{c} + \quad - \quad + \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & -2 \\ -1 & 2 & 3 \end{vmatrix} \end{array}$$

$$= (15+4, -(9-2), 6+5)$$

$$= (19, -7, 11)$$

$$\vec{y} \times \vec{x} = \begin{array}{c} + \quad - \quad + \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ 3 & 5 & -2 \end{vmatrix} \end{array}$$

$$= (-4-15, -(2-9), -5-6)$$

$$= (-19, 7, -11)$$

$$= -(19, -7, 11)$$ * just for comparison to $\vec{x} \times \vec{y}$

$\vec{y} \times \vec{x} = -(\vec{x} \times \vec{y}) \longrightarrow$ Same magnitude, opposite direction
wa wa wee wa!

Ex 3.
Given $\vec{p} = (1,-2,4)$, $\vec{q} = (1,2,7)$, and $\vec{r} = (-1,1,0)$, determine:

a) $\vec{p} \times (\vec{q} + \vec{r})$

$$\vec{q} + \vec{r} = (1,2,7) + (-1,1,0)$$

$$= (0,3,7)$$

$$\vec{p} \times (\vec{q} + \vec{r}) = \begin{array}{c} + \quad - \quad + \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 0 & 3 & 7 \end{vmatrix} \end{array}$$

$$= (-14-12, -(7-0), 3-0)$$

$$= (-26, -7, 3)$$

b) $(\vec{p} \times \vec{q}) + (\vec{p} \times \vec{r})$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 1 & 2 & 7 \end{vmatrix} = (-14-8, -(7-4), 2+2) = (-22, -3, 4)$$

$$\vec{p} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ -1 & 1 & 0 \end{vmatrix} = (0-4, -(0+4), 1-2) = (-4, -4, -1)$$

$$\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (-22,-3,4) + (-4,-4,-1)$$

$$= (-26, -7, 3)$$

distributive property

Properties of the Cross Product
If $\vec{p}$, $\vec{q}$, and $\vec{r}$ are vectors in R^3 and $k \in R$

Scaler Multiplication:	$k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$
Distributive Property:	$\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$
NO COMMUTATIVE PROPERTY:	$\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$

Ex 4.

If $(3,-2,1) \times (1,-1,m) = (5,7,-1)$, determine the value of m.

$$\begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ 3 & -2 & 1 \\ 1 & -1 & m \end{vmatrix} = \left(\overset{-2(m)-(-1)(1)}{-2m+1}, -(3m-1), -3+2\right) = (5,7,-1)$$

$-2m+1 = 5$ OR $-3m+1 = 7$

$-2m = 4$ $\qquad$ $-3m = 6$

$m = -2$ $\qquad$ $m = -2$

The Magnitude of the Cross Product of Geometric Vectors

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\theta$$

$\vec{T} = |\vec{a}||\vec{b}|\sin\theta\,[\quad]$

Ex 5.

Determine the angle between vectors $\vec{a} = (3,2,-1)$ and $\vec{b} = (-1,2,-4)$.

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$$\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \sin\theta$$

$\vec{a} \times \vec{b} = (-6,13,8) \leftarrow$ from ex 1

$|\vec{a} \times \vec{b}| = \sqrt{(-6)^2 + 13^2 + 8^2}$

$= \sqrt{36+169+64}$

$= \sqrt{269}$

$|\vec{a}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(-1)^2+2^2+(-4)^2} = \sqrt{21}$

$$\sqrt{\frac{269}{14 \cdot 21}} = \sin\theta$$

$$\theta = \sin^{-1}\left(\sqrt{\frac{269}{294}}\right) = 73^\circ$$