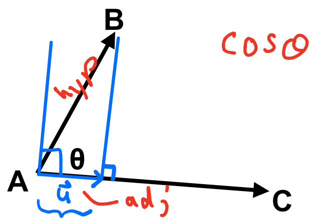
The Dot Product of Geometric Vectors

Focus: Become familiar with the mechanics of a geometric dot product. We will investigate why we use it later ☺

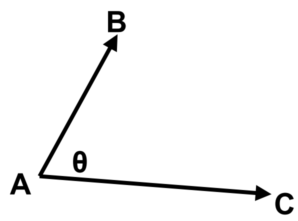
- applies the magnitude of one vector in the direction of another vector
- measures the result as a scalar

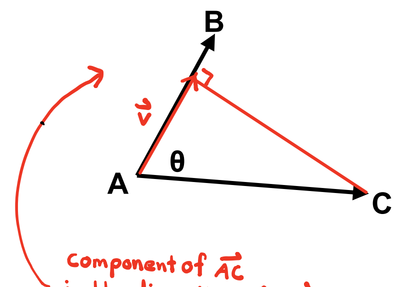
$$\boxed{\overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}| \cos\theta, 0 \le \theta \le 180°}$$



$\cos\theta = \frac{|\vec{u}|}{|\overrightarrow{AB}|}$

Component of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$
$|\vec{u}| = |\overrightarrow{AB}|\cos\theta$





Component of $\overrightarrow{AC}$ in the direction of $\overrightarrow{AB}$
$|\vec{v}| = |\overrightarrow{AC}|\cos\theta$

The Dot Product

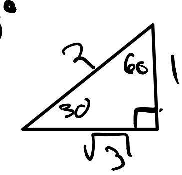
- A.K.A the "scalar product" because the result is a scalar (magnitude only – no associated direction)
- Produces a positive scalar result when $0 \le \theta < 90°$ because $\cos\theta > 0$
- Produces zero when $\theta = 90°$ because $\cos\theta = 0$
- Produces a negative scalar result when $90 < \theta \le 180°$ because $\cos\theta < 0$

Properties of the Dot Product

Commutative Property:	$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
Associative Property:	$(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
Distributive Property:	$\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$
** For any vector $\vec{u}$:	$\vec{u} \cdot \vec{u} = |\vec{u}|^2$ $\quad \vec{u} \cdot \vec{u} = |\vec{u}||\vec{u}|\underbrace{\cos\theta}_{1}$
** For perpendicular vectors $\vec{u}$ and $\vec{v}$:	$\vec{u} \cdot \vec{v} = 0$

Ex 1.
$|\vec{a}| = 3$ and $|\vec{b}| = 6$ with an angle between the vectors of 60°. Calculate:

a) $\vec{a} \cdot \vec{b}$
$= |\vec{a}||\vec{b}|\cos 60°$
$= (3)(6)\left(\frac{1}{2}\right)$
$= \frac{18}{2}$
$= 9$

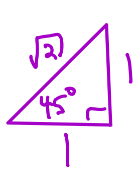


b) $\vec{b} \cdot \vec{a}$
$= (6)(3)\cos 60°$
$= (6)(3)\left(\frac{1}{2}\right)$
$= 9$

c) $\vec{a} \cdot \vec{a}$
$= (3)(3)\cos 0$
$= (3)(3)(1)$
$= 3^2$
$= 9$

commutative!

$\vec{a}$



$\sin 45 = \frac{1}{\sqrt{2}}$

$\cos 45 = \frac{1}{\sqrt{2}}$

$\tan 45 = \frac{1}{1} = 1$

Ex 2.

The vectors $\underbrace{\vec{a}-5\vec{b}}_{\vec{n}_1}$ and $\underbrace{\vec{a}-\vec{b}}_{\vec{n}_2}$ are perpendicular. If $\vec{a}$ and $\vec{b}$ are unit vectors, determine $\vec{a}\cdot\vec{b}$.

$(\hat{a}-5\hat{b})\cdot(\hat{a}-\hat{b})=0$ dot product = 0

$\vec{n}_1$ $\vec{n}_2$ – use distributive prop.

$\hat{a}\cdot\hat{a}-\hat{a}\cdot\hat{b}-5\hat{b}\cdot\hat{a}+5\hat{b}\cdot\hat{b}=0$

$|\hat{a}|^2-6\hat{a}\cdot\hat{b}+5|\hat{b}|^2=0$

$1-6\hat{a}\cdot\hat{b}+5=0$

$-6\hat{a}\cdot\hat{b}=-6$

$\hat{a}\cdot\hat{b}=1$

$\hat{a}\cdot\hat{a}=|\hat{a}||\hat{a}|\cos 0$
$=|\hat{a}||\hat{a}|$

$\hat{a}\cdot\hat{b}=|\hat{a}||\hat{b}|\cos\theta$
$=|1||1|\cos\theta$
$=\cos\theta$

Ex 3.

Given $\vec{x}=\hat{\imath}-2\hat{\jmath}-\hat{k} = (1,-2,-1)$ and $\vec{y}=\hat{\imath}-\hat{\jmath}-\hat{k} = (1,-1,-1)$, determine the value of $2\vec{x}\cdot\vec{y}-5\vec{y}\cdot\vec{x}$.

$2(\vec{x}\cdot\vec{y})$
$\vec{x}\cdot 2\vec{y}$

$2\vec{x}\cdot\vec{y}-5\vec{y}\cdot\vec{x}$
$=-3(\vec{x}\cdot\vec{y})$

$2(\vec{x}\cdot\vec{y})-5(\vec{x}\cdot\vec{y})$

commutative

$=-3(\hat{\imath}-2\hat{\jmath}-\hat{k})\cdot(\hat{\imath}-\hat{\jmath}-\hat{k})$

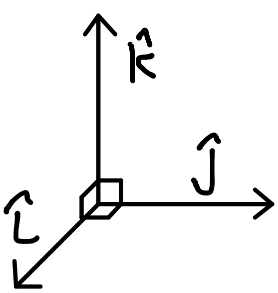
$=-3(\hat{\imath}\cdot\hat{\imath}\underbrace{-\hat{\imath}\cdot\hat{\jmath}}_{\theta=90^\circ}\underbrace{-\hat{\imath}\cdot\hat{k}}_{90^\circ}\underbrace{-2\hat{\jmath}\cdot\hat{\imath}}_{90^\circ}+2\hat{\jmath}\cdot\hat{\jmath}\underbrace{+2\hat{\jmath}\cdot\hat{k}}_{90^\circ}\underbrace{-\hat{k}\cdot\hat{\imath}}_{90^\circ}\underbrace{+\hat{k}\cdot\hat{\jmath}}_{90^\circ}+\hat{k}\cdot\hat{k})$

$=-3(\hat{\imath}\cdot\hat{\imath}+2\hat{\jmath}\cdot\hat{\jmath}+\hat{k}\cdot\hat{k})$

$(a+b+c)(2a+2b+2c)$

$=-3(1+2+1)$

$=-12$



$\hat{\imath}\cdot\hat{\jmath}=0$

$\hat{\imath}\cdot\hat{k}=0$

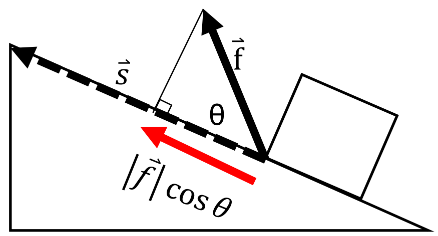
$\hat{\jmath}\cdot\hat{k}=0$

Applications of the Dot Product

Dot Product (Scalar Product): $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Work:

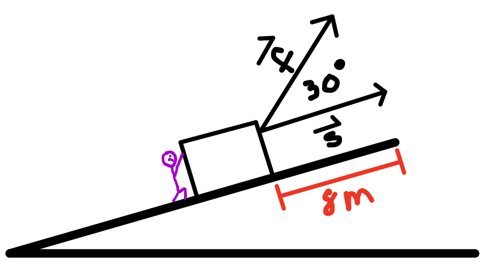
- Work is defined as the product of the displacement, $\vec{s}$, under and applied force, $\vec{f}$, along the line of displacement. The resulting unit is joules (J).
- The component of $\vec{f}$ in the direct of $\vec{s}$ can be calculated using $|\vec{f}|\cos\theta$. This calculation is also referred to as the scalar "projection of $\vec{f}$ onto $\vec{s}$".



$$\mathbf{W = displacement \cdot force}$$

$$\mathbf{W = |\vec{s}||\vec{f}|\cos\theta}$$

Ex 1.
A crate on a ramp is hauled 8 m up the ramp under a constant force of 20 N applied at an angle of 30° to the ramp. Find the work done.



$W = \vec{f} \cdot \vec{s}$

$= 20\cos 30(8) = (20)(8)\cos 30$

$= 138.6$ Joules (J)

There is 138.6 J of work performed.