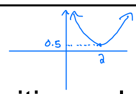
The Limit of a Function

One-Sided Limits



$\lim\limits_{x \to a^-} f(x)$ The limiting value of f(x) as x approaches "a" from the left (a.k.a "left-hand limit").

a   y

$\lim\limits_{x \to a^+} f(x)$ The limiting value of f(x) as x approaches "a" from the right (a.k.a "right-hand limit").

Two-Sided Limit

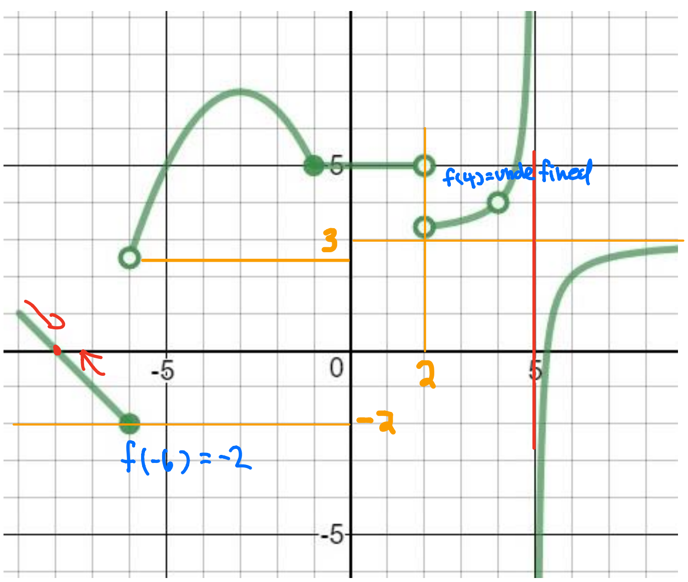
$\lim\limits_{x \to a} f(x)$ The limiting value of f(x) as x approaches "a" from ***the left and right***.

If the left and right limits are different then $\lim\limits_{x \to a} f(x)$ does not exist.

$\pm\infty$ are not considered "limiting values".

LIMITS FROM A GRAPH

Ex 1.
Evaluate the following limits from the graph of y = f(x) shown below:



$\lim\limits_{x \to -6^-} f(x) = -2$ from the left

$\lim\limits_{x \to -6^+} f(x) = 2.5$ from the right

$\lim\limits_{x \to -6} f(x) =$ DNE

$\lim\limits_{x \to -1^-} f(x) = 5$

$\lim\limits_{x \to -1^+} f(x) = 5$

$\lim\limits_{x \to -1} f(x) = 5$

$\lim\limits_{x \to 2^-} f(x) = 5$

$\lim\limits_{x \to 2^+} f(x) = 3.3$

$\lim\limits_{x \to 2} f(x) =$ DNE

$\lim\limits_{x \to 4^-} f(x) = 4$

$\lim\limits_{x \to 4^+} f(x) = 4$

$\lim\limits_{x \to 4} f(x) = 4$

$\lim\limits_{x \to 5^-} f(x) = \infty$

$\lim\limits_{x \to 5^+} f(x) = -\infty$

$\lim\limits_{x \to 5} f(x) =$ DNE

Thought Experiment

Can the limit at "a" exist even if the function does not exist at "a"?

yes! A hole at x = a (ie. x = 4 above)

Is it possible that the function exists at "a", but the limit does not exist at "a"?

Yes. A piecewise function that is discontinuous but defined at a (ie. x = -6 above)

Can the limit at "a" be the same as the function value at "a"?

yes. A function that is continuous at a ...
the point (a, f(a)) is on the graph (ie. x = -1 above)

LIMITS FROM EQUATIONS

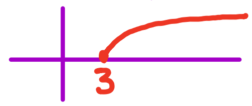
Ex 2.
Evaluate each limit. If the limit does not exist, explain why.

a) $\lim\limits_{x\to 2} 3x^2 - 5x + 1$

$= 3(2)^2 - 5(2) + 1$
$= 3$

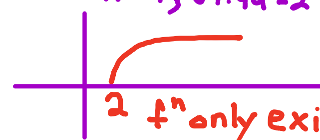
b) $\lim\limits_{x\to 3^+} \sqrt{x-3} = 0$

$x = 3.1, \sqrt{3.1-3} = 0.32$
$x = 3.01, \sqrt{3.01-3} = 0.1$



c) $\lim\limits_{x\to 2^-} \sqrt{x-2} = \text{DNE}$

$x = 1.9, \sqrt{1.9-2} = \mathbb{C}$
$x = 1.99, \sqrt{1.99-2} = \mathbb{C}$



f^n only exists for $x \geq 2$

d) $\lim\limits_{x\to 0} \sqrt{x^2+5}$

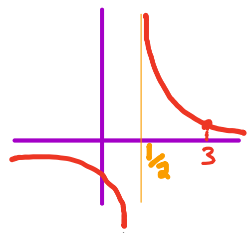
$\lim\limits_{x\to 3} \sqrt{x^2-5}$

$= \sqrt{5}$

$\lim\limits_{x\to 0^+} \sqrt{x^2+5} = \sqrt{5}$

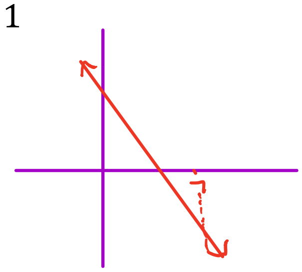
$\lim\limits_{x\to 0^-} \sqrt{x^2+5} = \sqrt{5}$

e) $\lim\limits_{x\to 3} \frac{1}{2x-1} = \frac{1}{5}$



f) $\lim\limits_{x\to 7^-} -3x + 1$

$= -20$



g) $\lim\limits_{x\to 5^+} \frac{1}{x-5}$

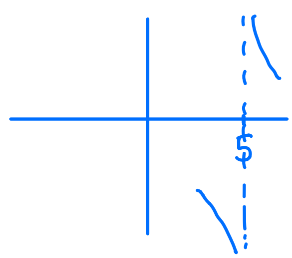
$= +\infty$

$x = 5.1$
$x = 5.01$
$x = 5.00001$

h) $\lim\limits_{x\to 5^-} \frac{1}{x-5}$

$= -\infty$

$x = 4.9 \to y = \frac{1}{-0.1}$
$x = 4.999 = \frac{1}{-0.001}$
$x = 4.999999$



i) $\lim\limits_{x\to 5} \frac{1}{x-5}$

$= \text{DNE}$

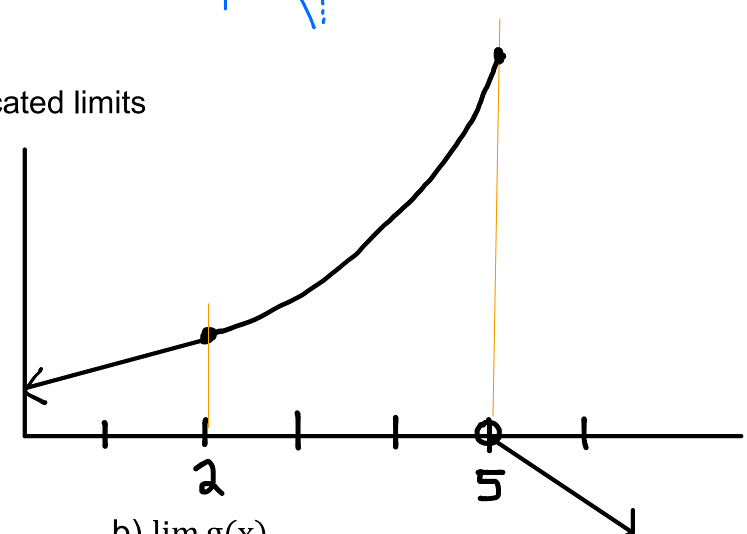
$\lim\limits_{x\to 5^-} \neq \lim\limits_{x\to 5^+}$

PIECEWISE FUNCTIONS

Ex 3.
Given y = g(x), evaluate the indicated limits

$$g(x) = \begin{cases} 2x+1 & x \leq 2 \\ x^2+1 & 2 < x \leq 5 \\ 5 - x & x > 5 \end{cases}$$

$5 =$



a) $\lim\limits_{x\to 2} g(x)$

$\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^+} g(x) = 5$

b) $\lim\limits_{x\to 5} g(x)$

$\lim\limits_{x\to 5^-} g(x) = 26$

$\lim\limits_{x\to 5^+} g(x) = 0$

$\lim\limits_{x\to 5} g(x) = \text{DNE}$