The Dot Product of Algebraic Vectors

Focus: Become familiar with the mechanics of an algebraic dot product.

In R^3, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = |\vec{a}|\,|\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$ where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

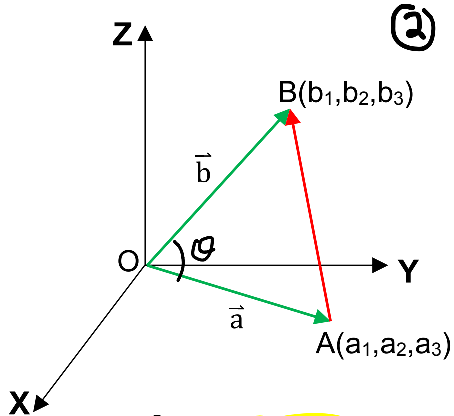
In R^2, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = |\vec{a}|\,|\vec{b}|\cos\theta = a_1b_1 + a_2b_2$ where $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$

PROOF:

① $|\vec{AB}|^2 = |\vec{a}|^2 + |b|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

② $|\vec{AB}|^2 = (b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2$

① = ②



$$|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta = (b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2$$

$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2(\vec{a}\cdot\vec{b}) = b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 + b_3^2 - 2a_3b_3 + a_3^2$$

$$-2(\vec{a}\cdot\vec{b}) = -2a_1b_1 - 2a_2b_2 - 2a_3b_3$$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Ex 1.

Determine $\vec{b}\cdot\vec{a}$ given $\vec{a} = (4,5,6)$ and $\vec{b} = (7,3,-1)$.

$$\vec{b}\cdot\vec{a} = (7,3,-1)\cdot(4,5,6)$$
$$= 7(4) + 3(5) - 1(6)$$
$$= 28 + 15 - 6$$
$$= 37$$

Q: What is the utility of the dot product?

A: The dot product is "handy dandy" when investigating or manipulating the angle between two vectors.

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

Ex 2.

Determine the angle between $\vec{u} = (-3,1,2)$ and $\vec{v} = (5,-4,-1)$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$$= \frac{-3(5)+1(-4)+2(-1)}{\sqrt{14}\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{-21}{\sqrt{588}}\right)$$

$$\theta = 150^\circ$$

aside

$$|\vec{u}| = \sqrt{(-3)^2+(1)^2+(2)^2}$$
$$= \sqrt{14}$$

$$|\vec{v}| = \sqrt{5^2+(-4)^2+(-1)^2}$$
$$= \sqrt{42}$$

Ex 3.

For what values of p are the vectors $\vec{a} = (p,-3,14)$ and $\vec{b} = (p,p,-2)$ perpendicular?

$\vec{a}\cdot\vec{b} = 0$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= a_1b_1 + a_2b_2 + a_3b_3$

$\vec{a}\cdot\vec{b} = 0$

$p(p) - 3(p) + 14(-2) = 0$

$p^2 - 3p - 28 = 0$

$(p-7)(p+4) = 0$

$p = 7$ + $p = -4$

$\vec{a} = (7,-3,14)$, $\vec{b} = (7,7,-2)$ | $\vec{a} = (-4,-3,14)$

Ex 4.

Determine a unit vector that is parallel to the xy-plane and perpendicular to the vector $4\hat{\imath} - 3\hat{\jmath} + \hat{k}$

$(4,-3,1)$

z coordinate must be zero

Let our vector be $\vec{r} = (a,b,0)$

$\because (a,b,0) \perp (4,-3,1)$

$(a,b,0)\cdot(4,-3,1) = 0$

$4a - 3b + 0 = 0$

$4a = 3b$

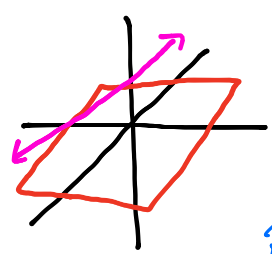
$a = \frac{3}{4}b$ ∞ number of solutions

ex

b	a
1	$\frac{3}{4}$
4	3

$\vec{r} = \left(\frac{3}{4}, 1, 0\right)$

$\vec{r} = (3,4,0)$



$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

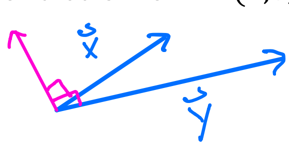
Unit vector

$|\vec{r}| = |(3,4,0)| = \sqrt{9+16+0}$
$= 5$

$$\hat{r} = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

Ex 5.

Find 3 vectors perpendicular to $\vec{x} = (1,3,-4)$ and $\vec{y} = (-1,-2,3)$.



Let $\vec{v} = (a, b, c)$

$(a,b,c)\cdot(-1,2,3) = 0$

① $-a - 2b + 3c = 0$

$(a,b,c)\cdot(1,3,-4) = 0$

② $a + 3b - 4c = 0$

Solve the system

$$\begin{array}{r} -a - 2b + 3c = 0 \\ a + 3b - 4c = 0 \quad \oplus \\ \hline b - c = 0 \end{array}$$

$b = c$

Sub $b = c$ into ②

$a + 3c - 4c = 0$

$a - c = 0$

$a = c$

$b = c = a$

$\vec{v} = (a, b, c)$

$= (a, a, a)$

$= (1, 1, 1)$

or

$= (-5, -5, -5)$

or

$= (3, 3, 3)$