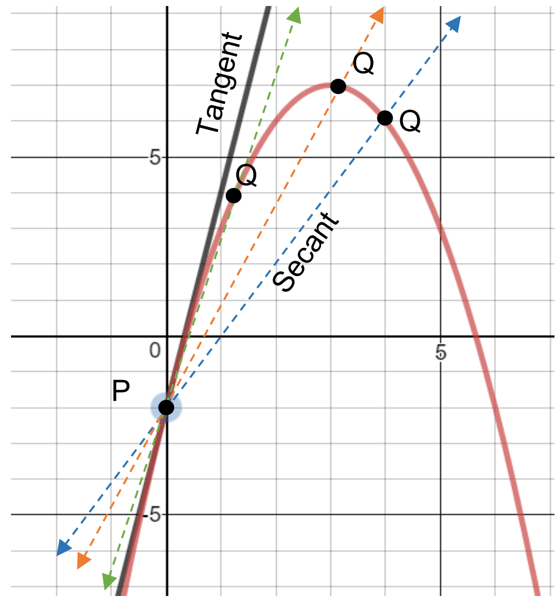
The Slope of the Tangent and Rate of Change

How do we determine the tangent of a curve at point P?

- The tangent is a straight line that is constructed at point P on the curve in such a way, that it has the exact slope of the curve at that particular point.

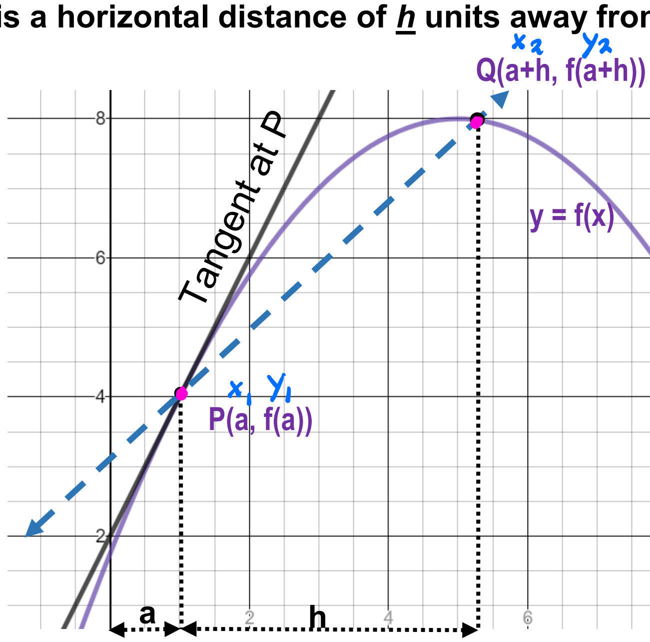


How do we get the tangent at just ONE point if the slope requires TWO points?

- Recall: The slope of a line $\mathbf{m} = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{0}{0}$ indet.
- Since two points are needed, we use secants with point Q sliding closer to point P.
- The slope of a tangent to a curve at point P is simply the limit of the slopes of the secants PQ as Q moves closer to P

$$\mathbf{m}_{\text{tangent}} = \lim_{Q \to P} \mathbf{m}_{\text{secant PQ}}$$

Let Q be a point that is a horizontal distance of *h* units away from point P (see Desmos Simulator)



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{a+h-a}$$

Slope of the secant

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(a+h) - f(a)}{(a+h) - a}$$ a+h-a
$$= \frac{f(a+h) - f(a)}{h}$$

Slope of the tangent

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the limit exists.

h=0

$\frac{f(a)-f(a)}{0}$

$\frac{0}{0}$

THOUGHT EXPERIMENT

If the above graph is a distance vs. time graph,

- What would the slope of the secant represent?
  Average Speed
- What would the slope of the tangent represent?
  Instantaneous Speed

Ex 1.

a) Find the slope of the tangent to the curve $y = 2x^3$ at the point $x = 1$.
b) Determine the equation of the tangent line.

$(1,2)$ $\quad a$

$(1+h)^3$
$(1+h)(1+h)(1+h)$
$(1+2h+h^2)(1+h)$

a) $y = 2x^3$ @ $x = 1$
$f(x) = 2x^3$ $\quad \hookrightarrow a = 1$

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$(1+h)^3$
- Binomial Th$^{\underline{m}}$

$$= \binom{3}{0}1^3h^0 + \binom{3}{1}1^2h^1 + \binom{3}{2}1^1h^2 + \binom{3}{3}1^0h^3$$

$$= (1)(1)(1) + (3)(1)(h) + 3(1)(h^2) + (1)(1)(h^3)$$

$$= 1 + 3h + 3h^2 + h^3$$

$$= \lim_{h\to 0} \frac{2(1+h)^3 - 2(1)^3}{h}$$

$$= \lim_{h\to 0} \frac{2(1+3h+3h^2+h^3)-2}{h}$$

$$= \lim_{h\to 0} \frac{6h+6h^2+2h^3}{h}$$

$$\frac{6\cancel{h}}{\cancel{h}} + \frac{6h^{\cancel{2}}}{\cancel{h}} + \frac{2h^{\cancel{3}2}}{\cancel{h}}$$

$$= \lim_{h\to 0} 6+6h+2h^2$$

$$= 6+6(0)+2(0)^2$$

$m = 6$

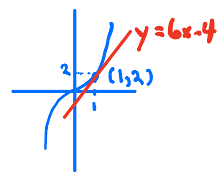
b) $y = mx + b$

Point	m
$y = 2(1)^3$	$m = 6$
$y = 2$	
$(1,2)$	

$y = mx + b$
$2 = 6(1) + b$
$-4 = b$

$y = 6x - 4$

eq^n of tangent of $y = 2x^3$ @ $(1,2)$



Ex 2.

Find the slope of the tangent to the curve $y = \sqrt{x-3}$ at the point (12, 3).

$y = \sqrt{x-3}$, $a = 12$

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$\sqrt{x-3}$
$\sqrt{(a+h)-3}$
$\sqrt{(12+h)-3}$
$\sqrt{12+h-3}$

$$= \lim_{h\to 0} \frac{\sqrt{12+h-3} - \sqrt{12-3}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{9+h}-3)}{(h)} \cdot \frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0} \frac{1\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)}$$

$$= \frac{1}{(\sqrt{9}+3)}$$

$m = \frac{1}{6}$

$\sqrt{x-3}$
$\sqrt{9-3}$
$\sqrt{12-3}$

Ex 3.
Recall:

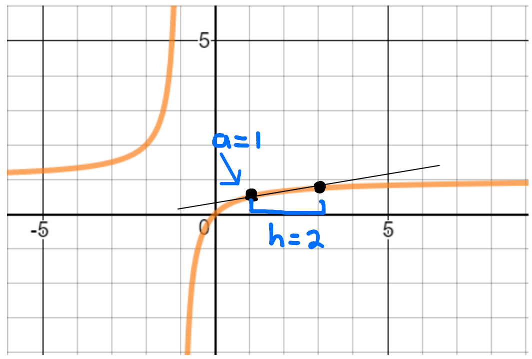
Slope of the secant

$$m = \frac{f(a+h) - f(a)}{h}$$

Slope of the tangent

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$, if the limit exists.

Given the function $s(t) = \frac{t}{t+1}$



a) Calculate the average rate of change for the interval $1 \le t \le 3$.
b) Calculate the instantaneous rate of change at t = 4.

h = 2
a = 1

$$m = \frac{S(3) - S(1)}{3 - 1}$$

$$a)\ m_{avg} = \frac{f(a+h) - f(a)}{h}$$

$$= \frac{f(1+2) - f(1)}{2}$$

$$= \frac{\frac{3}{4} - \frac{1}{2}}{2}$$

$$= \frac{\frac{1}{4}}{2}$$

$$m_{avg} = \frac{1}{8} \leftarrow$$ Secant Slope

Aside

$$f(t) = \frac{t}{t+1}$$

$$f(1+2) = f(3) = \frac{3}{3+1} = \frac{3}{4}$$

$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$b)\ m_{inst} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4+h}{5+h} - \frac{4}{5}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{20+5h-20-4h}{25+5h}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{25+5h} \times \frac{1}{\cancel{h}}$$

$$m_{inst} = \frac{1}{25}$$

Aside

a = 4

$$f(4+h) = \frac{4+h}{4+h+1} = \frac{4+h}{5+h}$$

$$f(4) = \frac{4}{4+1} = \frac{4}{5}$$

$$\underbrace{\frac{4+h}{5+h} \cdot \frac{5}{5}}_{(5+h)5} - \underbrace{\frac{4}{5} \cdot \frac{5+h}{5+h}}_{5(5+h)}$$

common denominator

Determine $\lim_{h \to 0} \dfrac{(3+h)^2 - 3^2}{h}$

$$= \lim_{h \to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6+h$$

$$= 6 + 0$$

$$= 6$$

$$\frac{(3+h)^2 - 3^2}{h} = \text{undef.}$$

@ $h=0$