Random Variables

Definition of Random Variable

- A random variable is a function that associates a. real number with each element in the sample space.

Example : Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values *y of the random variable Y, where Y is the number of red balls, are*

Sample Space	Y
RR	2
RB	1
BR	1
BB	0

- If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a discrete sample space.
- If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous sample space.

A random variable: is called a discrete random variable if its set. of possible outcomes is countable.

Example: Slide 2

The set of ordered pairs $(x, f(x))$ is a **probability function, probability mass function, or probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

Continuous Random Variable

A Random Variable 'X' which takes all possible values in a given interval is called continuous random variable.

1. $f(x) \geqq 0$

2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$P(a < X < b) = \int_a^b f(x)\, dx$$

A function $f(x)$ which satisfies the above requirements is called a *probability function* or *probability distribution* for a continuous random variable, it is more often called a *probability density function*

DISTRIBUTION FUNCTIONS FOR CONTINUOUS RANDOM VARIABLES

$$F(x) = P(X \leqq x) = P(-\infty < X \leqq x) = \int_{-\infty}^{x} f(u)\,du$$

Properties of c.d.f of a random variable X

i) $0 \le F(x) \le 1 \qquad -\infty < x < \infty$

ii) $\lim_{x\to-\infty} F(x) = 0$

iii) $\lim_{x\to\infty} F(x) = 1$

iv) $P(a \le x \le b) = F(b) - F(a)$

v) $F'(x) = \frac{dF(x)}{dx} = f(x) \ge 0 \qquad$ [f(x) is a p.d.f]

(or)

$dF(x) = f(x)dx$

dF(x) is called the probability differential of the random variable X.

By differentiating F(x),we get p.d.f f(x)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can be any of the numbers 0, 1, and 2. Now,

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{5}{2}}{\text{®}} = \frac{10}{28},$$

$$f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = \frac{15}{28},$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}.$$

Thus the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

2. If the random variable X takes the values 1,2,3 and 4 such that

$$\mathbf{2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)}$$

find the probability distribution and cumulative distribution function of X

$$2P(X = 1) = k \Rightarrow \quad P(X = 1) = \frac{k}{2}$$

$$3P(X = 2) = k \Rightarrow \quad P(X = 2) = \frac{k}{3}$$

$$P(X = 3) = k$$

$$5P(X = 4) = k \Rightarrow P(X = 4) = \frac{k}{5}$$

$$\sum_{i=1}^{n} p(x_i) = 1$$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{3} = 1 \Rightarrow k\left[\frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{5}\right] = 1$$

$$\frac{61}{30}k = 1 \quad \therefore k = \frac{30}{61}$$

x_i	$p(x_i)$	$F(X)$
1	$p(1) = \frac{k}{2} = \frac{15}{61}$	$F(1) = p(1) = \frac{15}{61}$
2	$p(2) = \frac{k}{3} = \frac{10}{61}$	$F(2) = F(1) + p(2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$p(3) = k = \frac{30}{61}$	$F(3) = F(2) + p(3) = \frac{25}{61} + \frac{30}{61} = \frac{55}{61}$
4	$p(4) = k$/[illegible] $= 6$/[illegible]	$F(4) = F(3) + p(4) = 55$/[illegible] $+ 6$/[illegible] $= 61$/[illegible] $= 1$

A Random variable X has the following probability distribution

x	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	3k

i)Find the value of K

ii)Evaluate $P(X<2)$ and $P(-2<X<2)$

iii)Cumulative distribution of X

iv)Mean of X

1. $\sum_{i=1}^{n} p(x_i) = 1$

2. $F(x) = P[X \le x]$

i.e., e.g., $P[X \le 4] = F[4]$

$P[X \le 5] = F[5]$

$F[1] = P[0] + P[1]$

$F[2] = P[0] + P[1] + P[2] = F[1] + P[2]$

$F[3] = P[0] + P[1] + P[2] + P[3] = F[2] + P[3]$

... ...

3. $P[1] = F[1] - F[0]$

$P[2] = F[2] - F[1]$

$P[3] = F[3] - F[2]$

4. Mean $= E[X] = \Sigma x_i p(x_i) =$ Expected value

5. $E[X^2] = \Sigma x_i^2 p(x_i)$

6. Variance $= \mathrm{Var}[X] = E[X^2] - [E[X]]^2$

7. $E[aX + b] = a\,E[X] + b$

8. $\mathrm{Var}\,[aX \pm b] = a^2\,\mathrm{Var}\,X$

$E\,(ax + b) = a\,E\,(X) + b.$

$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}\,(X)$

Practice Problems

1.A r.v X has the following probability function

X	0	1	2	3	4	5	6	7
p(x)	**0**	**K**	**2K**	**2K**	**3K**	**K^2**	**$2K^2$**	**$7K^2 + K$**

i. Find K
ii. Evaluate P(X<6)
iii. If $P(X \leq C) > \frac{1}{2}$ find the minimum value of C.
iv. Evaluate $P(1.5 < X < 4.5 / X > 2)$
v. Find $P(X < 2)$, $P(X > 3)$, $P(1 < X < 5)$

1.A continuous random variable 'X' has a p.d.f $f(x) = 3x^2, 0 \le x \le 1$.

Find 'a' and 'b' such that

i)$P(x \le a) = P(x > a)$ and ii)$P(x > b) = 0.05$.

2.Show that the function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & otherwise \end{cases} \text{ is a p.d.f}$$

3.The amount of time ,in hours that a computer functions before breaking down is a continuous random variable with p.d.f is given by

$$f(x) = \begin{cases} \lambda e^{\frac{-x}{100}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

what is the probability that

a)a computer will function between 50 and 150 hrs before breaking down.

b)it will function less than 500 hrs.

4.If a random variable 'X' has the p.d.f

$$f(x) = \begin{cases} \frac{1}{2}(x+1), & if -1 < x < 1 \\ 0, & otherwise \end{cases}$$

- Find mean and variance of x

Summary: Mean & Variance

	Definition	Discrete R.V.s	Continuous R.V.s
Mean: μ	$E(X)$	$\sum_i p(x_i)x_i$	$\int_{-\infty}^{\infty} \mathbf{f}(x)x\,dx$
Variance: σ^2	$E((X-\mu)^2)$	$\sum_i p(x_i)(x_i-\mu)^2$	$\int_{-\infty}^{\infty} \mathbf{f}(x)(x-\mu)^2\,dx$

Practice Problems

1..If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & otherwise \end{cases}$$

i. Find the value of a.

ii. The cumulative distribution functions of X.

1.Verify whether the following is a distribution function

$$F(x)=\begin{cases} 0, & x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & -a < x < a \\ 1 & x > 1 \end{cases}$$

2. Given the p.d.f of a continuous random variable 'X' as

$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find cdf for 'X'

3.A random variable 'X' has the density function

$$f(x) = \begin{cases} K\dfrac{1}{1+x^2} & -\infty < x < \infty \\ 0, & otherwise \end{cases}$$

a)Find 'K' and the distribution function F(x)

b)Find P(X>0)

1. A random sample ‘X’ has the probability function

X	0	1	2	3	4	5	6	7	8
p(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

Determine the value of a

If X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & otherwise \end{cases}$$ what is the value of c?

Given that the p.d.f of a R.V X is $f(x) = Kx, 0 < x < 1$ find K and P(X>0.5)

4.

In a continuous random variable X having the p.d.f

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & otherwise \end{cases}$$ Find $P(0<X\leq 1)$

For the following c.d.f $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$

find (i) $P(X>0.2)$ (ii) $P(0.2<X\leq 0.5)$

MATHEMATICAL EXPECTATION

Let 'X' be a random variable with p.d.f (or p.m.f).

Then the mathematical expection of 'X' is denoted by E(X)

and is given by $E(X) = \int_{-\infty}^{\infty} xf(x)dx$ (for a continuous random variable)

$$E(X) = \sum_{x} xP(x) \qquad \text{(for a discrete random variable)}$$

r^{th} moment (about origin)

Consider a continuous r.v. 'X' with p.d.f (or p.m.f) f(x) then the r^{th} moment (about origin) of the probability distribution is defined as

$$E(X) = \int_{-\infty}^{\infty} x^r f(x)dx \,.$$

It is denoted by $\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx$

$$\mu_1' = E(X)$$

$$\mu_2' = E(X^2)$$

$$Mean = \overline{X} = \mu_1' = E(X)$$

$$Variance = \mu_2 = \mu_2' - (\mu_1')^2$$

$$Variance = E(X^2) - [E(X)]^2$$

rth moment about mean is denoted by μ_r

$$\mu_r = \int_{-\infty}^{\infty} (x - \overline{X})^r f(x) dx$$

Variance in terms of expectation

$$Variance = \mu_2 = E[\{X - E(X)\}^2]$$

$$Mean = \mu_1^{'} = \sum xP(x)$$

$$Variance = \mu_2 = \sum_x (x - \overline{X})^2 P(x)$$