Probability distributions- Discrete

Bernoulli Trial

A Bernoulli trial is a random experiment in which there are exactly two possible outcomes - success or failure

Example

Head or Tail ; Defective or Non-defective; Survived or Died; True or False

Mean and Variance of a Bernoulli r.v.

Let X be a Bernoulli random variable. $X = \{1, 0\}$. Let

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p = q$$

$$E(X) = \sum xP(x) = 1 * p + 0 * q = p$$

$$Var(X) = \sum x^2 * P(x) - (\sum x * P(x))^2 = p - p^2 = p(1 - p) = pq$$

Binomial Distribution

Binomial distribution results when it satisfies the following:

- It has n repeated Bernoulli trials (i.e. outcomes are either success or failure in each trial)
- The trials are independent
- The probability of success in each trial is constant

Notations

n	number of trials
x	number of successes in n trials
p	Probability of success in each trial
q	Probability of failure in each trial
$P(X = x)$	Probability of getting exactly x successes among n trials
$P(X <= x)$	Probability of getting at most x successes
$P(X >= x)$	Probability of getting at least x successes

Pmf of a Binomail distribution

Let X denote the discrete random variable that counts the number of successes in n trials.
$X = \{0, 1, 2, \ldots, n\}$
The probability mass function of a binomial random variable X with parameters n and p is

$$P(X = x) = nC_x p^x q^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

Note that among n trials, getting x successes can be obtained in nC_k ways.

Mean and Variance

Let $X_1, X_2, X_3, \ldots, X_n$ be the bernoulli random variables.
Then $X = X_1 + X_2 + X_3 + \ldots + X_n$ is a binomial random variable.

$$E[X] = E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

$$= p + p + \ldots + p = np$$

Similarly,

$$Var[X] = Var[X_1 + X_2 + \ldots + X_n]$$

Since $X_1, X_2, \ldots, X_n$ are independent variables,

$$Var[X] = Var[X_1] + Var[X_2] + \ldots + Var[X_n] = pq + pq + \ldots + pq$$

$$Var[X] = npq$$

Recall

$$P(X = x) = nC_x p^x q^{n-x}, \qquad x = 0, 1, \ldots, n$$

MGF

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= nC_0 (pe^t)^0 q^n + nC_1 (pe^t)^1 q^{n-1} + \ldots + nC_n (pe^t)^n q^0$$

$$M_X(t) = (pe^t + q)^n$$

Characteristic function

$$\phi_X(t) = (pe^{it} + q)^n$$

Problems

1. A biased coin is tossed 6 times. The probability of heads on any toss is 0.3. Let X denote the number of heads that come up. Calculate (i) $P(X = 2)$, (ii) $P(X \leq 3)$, (iii) $P(1 < X \leq 5)$.

Solution

$n = 6, p = 0.3, q = 0.7$

$$(i) \quad P(X = 2) = 6C_2(0.3)^2(0.7)^{6-2} = 0.3241$$

$$(ii) \quad P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 6C_0(0.3)^0(0.7)^6 + 6C_1(0.3)(0.7)^5 + 6C_2(0.3)^2(0.7)^4 + 6C_3(0.3)^3(0.7)^3$$

$$0.9295$$

$$(iii) \quad P(1 < X \leq 5) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

$$= 0.5791$$

Problems

2. The incidence of a certain disease is such that 20% of workers suffer from it. If 10 workers are selected at random, find the probability that (i) exactly 2 workers suffer from the disease, (ii) not more that 2 workers suffer from the disease. ((i) 0.302, (ii) 0.678)

3. Comment on the following: Mean of a binomial distribution is 3 and Variance is 4. (Not possible)

4. Bottled sweet milk stored in a godown is reported to have gone sour. A test check has revealed that milk in 25 percent of the bottles is bad and thus unfit for consumption. The salesman at a retail outlet offers 5 bottles for sale on demand. Find the probability that milk will be unfit for consumption (a) exactly in 2 bottles, (b) at least in 2 bottles, (c) at most in 2 bottles, and (d) between 2 and 4 bottles. ((a) 0.2636, (b) 0.3672, (c) 0.8964 (d) 0.3662)

Try these

5. For a binomial distribution, the mean is 6 and standard deviation is $\sqrt{2}$. Find $P(X = 0)$. (0.00005)

6. A specific type of screws are manufactured on a machine. It has been found through experience that 80 percent of the screws are strictly according to the given specification. Before placing order for bulk buying, a merchant draws a sample of 15 screws and is interested in knowing the probability of getting (a) at least 10 screws strictly according to specifications, and (b) at most 5 screws meeting specifications. ((a) 0.9389 ,(b) 0.0001)

7. The probability of a man hitting a target is $\frac{1}{4}$. How many times must he fire so that the probability of hitting the target at least once is more than $\frac{2}{3}$? ($n = 4$ approx)

Poisson Distribution

If n is large, p is small and np is a constant, then the Binomial distribution can be approximated as Poisson distribution.

- Count the number of occurrences of an event in a given interval of time, distance, area, or volume.
- Events are occuring independently
- The probability that an event occurs in a given length of time does not change through time.

Example

- The number of incoming flights at an international airport during a particular time interval
- The number of road accidents on a particular day
- The number of typos in 10 pages

Pmf

Let X be a random variable representing the number of occurrences in a given interval for which the average rate of occurrence is λ, then the probability of x occurrences in that interval is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0, 1, 2, \dots$$

Mean

$$
\begin{aligned}
E[x] &= \sum xP(x) \\
&= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}
$$

Variance

$$E[x^2] = \sum x^2 P(x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda}\lambda \sum_{x=1}^{\infty} [(x-1)+1] \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda \sum_{x=1}^{\infty} (x-1) \frac{\lambda^{x-1}}{(x-1)!} + e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda}\lambda^2 e^{\lambda} + e^{-\lambda}\lambda e^{\lambda}$$

$$= \lambda^2 + \lambda$$

$$Var(x) = E(x^2) - (E(x))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$M_X(t) = e^{\lambda(e^t - 1)}, \phi_X(t) = e^{\lambda(e^{it} - 1)}$$

Problems

1. Consider a Computer system with Poisson job arrival stream at an average of 2 per minute. Determine the probability that in any one minute interval there will be (i) Zero jobs, (ii) exactly 3 jobs, (iii) at most 3 jobs, (iv) less than 3 job arrivals. ((i) 0.135, (ii) 0.18, (iii) 0.8571, (iv) 0.6767)

2. The number of calls coming per minute into a hotel reservation center is a Poisson random variable with mean 3. (i) Find the probability that no calls come in a given one minute period. (ii) Find the probability that at least two calls will arrive in a given two minute period. ((i) 0.04979, (ii) 0.8009)

Problems

3. In a certain Poisson distribution, the frequency corresponding to 2 successes is half the frequency corresponding to 3 successes. Find the mean and standard deviation. (Mean=6, SD=2.45)

4. It is 1 in 1000 that a birth is a case of twins. If there are 100 births in a town one day, what is the chance that two or more pairs of twins are born? Compare the results obtained by (a) Binomial distribution, (b) Poisson distribution. ((a) 0.0046, (b) 0.0047)

5. Calculate $P(X = 3)$ given that the variance of X is 0.5 (0.0126)

Problems

6. Given that an inspection of a random sample of 100 pages has revealed 20 printing errors, find the probabilities that a page contains (a) exactly 3 errors, (b) at most 3 errors, (c) at least 3 errors, and (d) less than 3 errors. ((a) 0.001, (b) 0.9999, (c) 0.0011, (d) 0.9989)

7. A certain type of cloth was inspected for the number of defects per square metre. It was found that on an average a square metre of cloth had 0.06 defects. Find the probabilities that a randomly selected one square metre of cloth has (a) no defect, (b) two or less defects, and (c) one or more defects. ((a) 0.5488, (b) 0.9769, (c) 0.4512)

8. Suppose a fast food restaurant can expect two customers every 3 minutes on an average. What is the probability that four or fewer customers will ente the restaurant in a 9 min period ? (0.2851)

More Problems - Binomial and Poisson

1. Ten percent of computer parts produced by a certain supplier are defective. What is the probability that a sample of 10 parts contains more than 3 defective ones?
   (0.0128)

2. Suppose that the number of inquiries arriving at a certain interactive system follows a Poisson distribution with arrival rate of 12 inquiries per minute. Find the probability of 10 inquiries arriving (a) in a 1-minute interval, (b) in a 3-minute interval, (c) What is the expectation and variance of the number of arrivals during each of these intervals?
   ((a) 0.1048, (b) 0.0000002337, (c) for 1-min interval, E(X)=V(X)=12, for 3-min interval, E(X)=V(X)=36)

Problems

1. On an average, 1 computer in 800 crashes during a severe thunderstorm. A certain company had 4000 working computers when the area was hit by a severe thunderstorm.
(a) Compute the expected value and variance of the number of crashed computers
(b) Compute the probability that less than 10 computers crashed.
(c) Compute the probability that exactly 10 computers crashed.
((a) $E(X)=5$, $V(X)=4.994$, (b) 0.968, (c) 0.018)

Problems

1. Suppose the number of babies born during an 8-hour shift at a hospital's maternity wing follows a Poisson distribution with a mean of 6 an hour. Find the probability that five babies are born during a particular 1-hour period in this maternity wing.
(0.1606)
2. A small life insurance company has determined that on the average it receives 6 death claims per day. Find the probability that the company receives at least seven death claims on a randomly selected day.
(0.3937)