

Exercice 1 (2-2-2-2-2 points)

Déterminez les primitives suivantes :

1) $\int (\sin(x))^3 dx$

2) $\int \frac{1}{4x^2+1} dx$

3) $\int \frac{x^3-x+1}{x^2-1} dx$

4) $\int x \arctan(x) dx$

5) $\int \frac{1-\sqrt{x}}{\sqrt{x}} dx$

Exercice 2 (4 points)

Calculer l'intégrale suivante :

$$\int_0^4 \sqrt{16-x^2} dx$$

- 1) En faisant un changement de variable
- 2) En donnant une interprétation géométrique explicite

Exercice 3 (2 - 2 points)

- 1) Représenter le domaine :

$$D = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \text{ et } x^2 \leq y \leq 4 - x^3\}$$

- 2) Calculer son aire.

Exercice 4 (2 - 2 points)

Soit le domaine

$$D = \{(x, y) \in \mathbb{R}^2; x \geq 0, y \geq 0 \text{ et } x^2 + y^2 \leq 4\}$$

- 1) Représenter D.
- 2) Intégrer $f(x, y) = \frac{xy}{x^2+y^2}$ sur D.

① $\int \sin^3 x \, dx$

$\sinh x = \frac{e^{ix} + e^{-ix}}{2i}$

1
1 1
1 2 1
1 3 3 1

$\sin^3 x = \left(\frac{1}{2i}\right)^3 [e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix}]$

$= \frac{1}{8(-i)} [(e^{3ix} - e^{-3ix}) - 3(e^{ix}e^{-ix} - e^{-ix}e^{ix})]$

$= \frac{1}{-8i} [2i(\sin(3x)) - 3(2i)(\sin x)]$

$= -\frac{1}{4} \sin(3x) + \frac{3}{4} \sin x$

Can integrate: $\int \sin^3 x = \int (-\frac{1}{4} \sin 3x + \frac{3}{4} \sin x) dx$

$= -\frac{1}{4} \left[\frac{\cos 3x}{-3} \right] + \frac{3}{4} [-\cos x]$

$= \frac{\cos 3x}{12} - \frac{3}{4} \cos x$

(2)

or more:

$\int \sin^3 x \, dx = \int \sin^2 \sin x \, dx$

$= \int (1 - \cos^2) \sin x \, dx$

$= \int \sin x - \cos^2 \sin x \, dx$

$= [-\cos x] + \left[\frac{\cos^3 x}{3} \right]$

② $\int \frac{1}{4x^2+1} dx$

Sub $X = 2x \rightarrow dx = 2dx$

(2)

$\hookrightarrow \int \frac{1}{2} dX \frac{1}{X^2+1} = \frac{1}{2} \int \frac{dX}{X^2+1} = \frac{1}{2} \arctan(X) = \frac{1}{2} \arctan(2x)$

(3) Example des cours!

$\int \frac{x^3 - x + 1}{x^2 - 1} dx$

(2)

$= \int \left(x + \frac{1}{x^2-1} \right) dx$

$x^3 - x + 1 \mid x^2 - 1$
1 $\mid x$

$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$\frac{A(x-1) + B(x+1)}{(x+1)(x-1)} = \frac{(A+B)x - A + B}{(x+1)(x-1)}$

$\rightarrow \begin{cases} A+B=0 \\ -A+B=1 \end{cases} \Rightarrow \begin{cases} A=-B \\ A=-1/2 \end{cases} \Rightarrow B=1/2$

$= \int \left(x + \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1} \right) dx$

$= \frac{x^2}{2} + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1|$

④ $\int x \arctan x$

IPP $u' = x \quad v = \arctan x$

$u = \frac{x^2}{2} \quad v' = \frac{1}{1+x^2}$

$= [uv] - \int uv' =$

$= \frac{x^2}{2} \arctan x - \int \frac{1}{2} \frac{x^2}{1+x^2} dx$

$\rightarrow \frac{x^2}{1+x^2} = 1 - \frac{1}{x^2+1}$

anyway, $\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}$

$$\rightarrow \int x \operatorname{arctan} x = \frac{x^2}{2} \operatorname{arctan} x - \frac{1}{2} \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= -\frac{1}{2} \left[x - \operatorname{arctan} x \right]$$

$$\rightarrow \int = \frac{x^2}{2} \operatorname{arctan} x - \frac{1}{2} x + \frac{\operatorname{arctan} x}{2}$$

$$= \frac{1}{2} \left[(x^2 + 1) \operatorname{arctan} x - x \right].$$

(2)

(5) $\int \frac{1 - \sqrt{x}}{\sqrt{x}} dx$

or pose $u = \sqrt{x}$

$$u^2 = x$$

$$\frac{dx}{du} = 2u \quad dx = 2u du.$$

$$\rightarrow \int \frac{1 - u}{u} (2u du).$$

$$= \int \frac{2u - 2u^2}{u^{\frac{1}{2}}} du = \int \frac{2u - 2u^2}{u^{\frac{1}{2}}} du$$

(2)

$$= 2 \left(u - \frac{u^2}{2} \right) = 2u - u^2$$

$$= 2\sqrt{x} - x$$

Exo 2

$$\int_0^4 \sqrt{16-x^2} dx = \int_0^4 \sqrt{16\left(1-\left(\frac{x}{4}\right)^2\right)} dx = 4 \int_0^4 \sqrt{1-\left(\frac{x}{4}\right)^2} dx$$

1) On pose $\frac{x}{4} = \sin t$ $x = 4 \sin t$ $\frac{dx}{dt} = 4 \cos t$

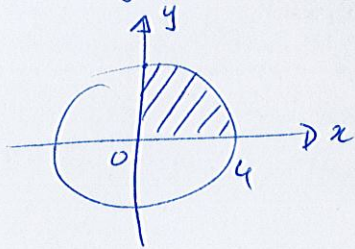
$$1 - \left(\frac{x}{4}\right)^2 = \cos^2 t$$

pour $x=0 \rightarrow \sin t = 0 \rightarrow t=0$
 $x=4 \rightarrow \sin t = 1 \rightarrow t = \frac{\pi}{2}$

(2)

$$\begin{aligned} I &= 4 \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t} \cdot 4 \cos t dt = 16 \int_0^{\frac{\pi}{2}} \cos^2 t dt = 16 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt \\ &= \frac{16}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{\frac{\pi}{2}} = \frac{16}{2} \frac{\pi}{2} = \frac{16}{4} (4\pi) \end{aligned}$$

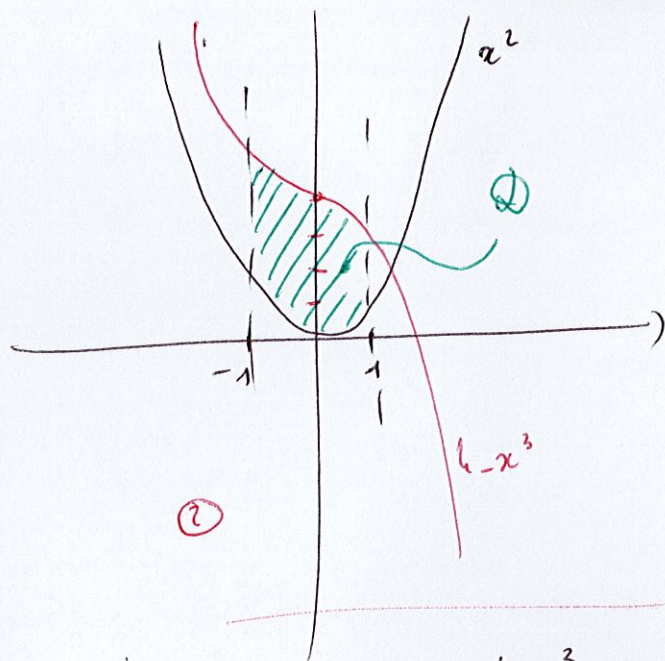
2) Soit $y^2 = 16 - x^2 \Leftrightarrow x^2 + y^2 = 16 \rightarrow$ cercle centré en $(0,0)$ de rayon 4. Pour $x \in [0,4] \rightarrow 1/4$ de disque.



$$I = \frac{1}{4} (\pi 4^2) = 4\pi$$

(2)

①



$$\left. \begin{array}{l} \text{en } -1 : x^2 = 1 \\ 4 - x^3 = 5 \\ \text{en } 1 : x^2 = 1 \\ 4 - x^3 = 3 \end{array} \right\}$$

entier -1 et 1 on a bjs
 $x^2 < 4 - x^3$

②

$$x^2 = 4 - x^3$$

$$x^2 + x^3 = 4$$

$$\begin{aligned} \rightarrow A &= \int_{-1}^1 \int_{x^2}^{4-x^3} dy dx = \int_{-1}^1 [y]_{x^2}^{4-x^3} dx \\ &= \int_{-1}^1 (4 - x^3 - x^2) dx \\ &= \left[4x - \frac{x^4}{4} - \frac{x^3}{3} \right]_{-1}^1 = \left(4 - \frac{1}{4} - \frac{1}{3} - \left(-4 - \frac{1}{4} + \frac{1}{3} \right) \right) \\ &= \left(8 - \frac{2}{3} \right) = \frac{22}{3} \end{aligned}$$

②

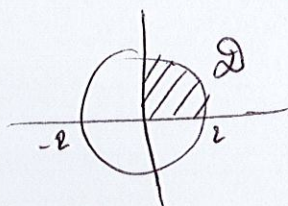
② Soit $f(x, y) = \frac{xy}{x^2 + y^2}$

- Reprendre D
 - Intégrer

sur D $\left\{ \begin{array}{l} x > 0 \\ y > 0 \end{array} \right. \quad x^2 + y^2 \leq 4$

$x^2 + y^2 = 4 \rightarrow$ cercle de rayon 2

$D = \frac{1}{4}$ de disque



②

↳ Polar !

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dx dy = r dr d\theta$$

$\rightarrow \iint_D f(x, y) dx dy$

$$= \int_0^2 \int_0^{\frac{\pi}{2}} \frac{x \cos \theta \cdot y \sin \theta}{r^2} r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \cos \theta \sin \theta dr d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin(2\theta) d\theta \int_0^2 r dr$$

$$= \left[\frac{1}{2} \frac{\cos(2\theta)}{(-2)} \right]_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} \right]_0^2 = -\frac{1}{4} (-1 - 1) \left(\frac{4}{2} \right) = 1$$

②