UNIVERSITÉ DE MONTPELLIER **FACULTÉ DES SCIENCES**

L2 - Techniques mathématiques EEA - HAE304X

Feuille de TD n° 2

Primitives, intégrales, sommes de Riemann

Exercice 1

Déterminer les primitives suivantes

1. $\displaystyle\int xe^{x^2}dx$ 2. $\displaystyle\int \frac{\ln|x|}{x}dx$ 3. $\displaystyle\int \frac{dx}{x\ln|x|}$ 4. $\displaystyle\int \frac{\sin x}{1+\cos^2 x}dx$

5. $\displaystyle\int \ln|x|dx$ 6. $\displaystyle\int x\ln|x|dx$; 7. $\displaystyle\int xe^{3x}dx$ 8. $\displaystyle\int \frac{dx}{(2x+3)^2}$ 9. $\displaystyle\int \frac{dx}{x^2+4}$

10. $\displaystyle\int \frac{dx}{2x^2+8x+10}$ 11. $\displaystyle\int \frac{2x+4}{2x^2+8x+10}dx$ 12. $\displaystyle\int \frac{2x+5}{2x^2+8x+10}dx$ 13. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

Exercice 2

Calculer les intégrales suivantes :

1. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$ 2. $\displaystyle\int_0^1 x^2 \arctan x\, dx$ 3. $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\, dx$

Exercice 3

Calculer les intégrales suivantes :

1. $\displaystyle\int_0^{\pi/2} \sin^2 x\cos^2 x\, dx$ 2. $\displaystyle\int_0^{\pi/4} \sin^4 x\, dx$ 3.(*) $\displaystyle\int_0^{\pi/3} \sin^3 x\cos^2 x\, dx$

Exercice 4

(*)Calculer de deux manières différentes les intégrales

$I = \displaystyle\int_0^{+\infty} e^{-t}\cos t\ dt$ et $J = \displaystyle\int_0^{+\infty} e^{-t}\sin t\ dt.$

Exercice 5

Déterminer les primitives des fractions rationnelles suivantes :

1. $\displaystyle\frac{1}{x(x+1)}$ 2. $\displaystyle\frac{1}{x^2(x^2+1)}$ 3. $\displaystyle\frac{x}{x^2-4}$ 4. $\displaystyle\frac{x^3}{x^2-4}$ 5. $\displaystyle\frac{1}{(x-1)^2(x+2)}$ 6. $\displaystyle\frac{x^4}{x^3-x^2+x-1}$

Exercice 6 (Sommes de Riemann)

Calculer les limites des sommes de Riemann suivantes :

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}}\ ,\quad (*)T_n = \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2-k^2}}\ ,\quad (*)U_n = \sum_{k=1}^{n} \frac{k}{n^2}\ln(1+\frac{k}{n})\,.$$

Exercice 7

Une application

Calculer la valeur efficace sur l'intervalle $[0,1]$ du signal $s(t) = \dfrac{1}{2t+3}$.

On rappelle que la valeur efficace d'un signal $s(t)$ sur l'intervalle $[0,T]$ est Veff $= \sqrt{\frac{1}{T}\int_0^T s^2(t)\, dt}$.

Dérivées partielles

Exercice 8
Soit la fonction $f : \mathbb{R}^2 \to \mathbb{R}$ définie par $f(x,y) = x \sin y$. Calculer $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}$ et $\dfrac{\partial^2 f}{\partial y \partial x}$.

Intégrales doubles

Exercice 9
Calculer $I = \displaystyle\iint_{[0,1]\times[0,1]} \frac{1}{1+x+y}\, dx\, dy$.

Exercice 10
Soit Δ, le domaine du plan délimitée par les paraboles d'équations $y = x^2$ et $x = y^2$.
a) Calculer $I = \displaystyle\iint_{\Delta} xy\, dx\, dy$.
b) Calculer l'aire de Δ.

Exercice 11
On consière le disque centré en O et de rayon R : $D(O,R) = \{(x,y) \in \mathbb{R}^2, \quad x^2 + y^2 \leq R^2\}$.
Retrouver le fait que son aire vaut πR^2 :
a) en réalisant un découpage par tranches verticales de ce disque.
b) en utilisant les coordonnées polaires.

Exercice 12
Calculer $I = \displaystyle\iint_{D(O,1)} \frac{1}{1+x^2+y^2}\, dx\, dy \quad (D(O,1)$ est le disque unité).

Ex 1

1) $\int x e^{x^2} dx$

$u = e^{x^2}$

$du = 2x e^{x^2} dx$

$dx = \frac{du}{2x e^{x^2}}$

$\rightarrow \int \frac{\cancel{x} u}{2 \cancel{x} u} du$

$\frac{1}{2} \int 1 \, du$

$\frac{1}{2} u$ $\qquad \frac{1}{2} e^{x^2} + C$

2) $\int \frac{\ln(x)}{x} dx$

$u = \ln(x)$

$du = \frac{1}{x} dx$

$dx = du \times x$

$\int \frac{u}{\cancel{x}} du \times \cancel{x}$

$\int u \, du$

$\frac{u^2}{2} = \frac{\ln(x)^2}{2} + C$

3) $\int \frac{dx}{x \ln(x)} = \int \frac{1}{x} \frac{1}{\ln(x)} dx$

$u = \ln(x)$

$du = \frac{1}{x} dx$

$dx = du \, x$

$= \int \frac{1}{\cancel{x}} \frac{1}{u} du \cancel{x}$

$= \int \frac{1}{u} du$

$= \ln(u)$

$= \ln(\ln(x)) + C$

4) $\int \frac{\sin(x)}{1+\cos^2(x)}$

$t = \cos(x)$

$dt = -\sin(x)\,dx$

$dx = \frac{dt}{-\sin(x)}$

$-\int \frac{1}{1+t^2} = -\arctan(t)$

$= -\arctan(\cos(x)) + C$

5) $\int 1 \times \ln(x)\,dx =$

	D	I
+	$\ln(x)$	1
−	$\frac{1}{x}$	x

$\ln(x) \times x - \int \frac{1}{\cancel{x}} \cancel{x}\,dx$

$\ln x \times x - x + C$

$x(\ln(x) - 1) + C$

$I = \int x^2 \sin(3x)\ dx$

	D	I
+	x^2	$\sin(3x)$
	$2x$	$\cos(3x)$

$- \quad 9x \qquad -\cos(3x)$

$x^2 \times -\frac{\cos(3x)}{3} + 2x\frac{\sin(3x)}{9} + \frac{2\cos(3x)}{27} + C$

$\frac{}{3}$

$+ \quad 2 \qquad \frac{-\sin(3x)}{9}$

$- \quad 0 \qquad \frac{\cos(3x)}{27}$

$I = \int x^4 \ln(x)\, dx$

	D	I
+	$\ln(x)$	x^4
−	$\frac{1}{x}$	$\frac{x^5}{5}$

$\ln(x) \times \frac{x^5}{5} - \int \frac{1}{x} \frac{x^5}{5}$

$\ln(x)\frac{x^5}{5} - \frac{x^5}{25} + C$

$I = \int e^x \sin x \; dx$

	D	I
+	e^x	$\sin x$
−	e^x	$-\cos x$
+	e^x	$-\sin x$
−		

$\frac{1}{2}e^x \times \cos x \frac{-1}{2} e^x \sin(x)$

6) $\int x \ln(x)\; dx$

	D	I
+	$\ln(x)$	x

$\ln(x) \times \frac{x^2}{} \quad \int \frac{1}{} \times \frac{x^2}{}$

$\ln(x) \times \frac{x^2}{2} - \int \frac{x}{2}$ ... $\frac{x}{2}$

$- \quad \frac{1}{x} \qquad \frac{x^2}{2}$

$\ln(x) \times \frac{x^2}{2} \frac{1}{2} - \int x \, dx$

$\ln(x) \times \frac{x^2}{2} - \frac{x^4}{4}$

7) $\int x e^{3x}$

	D	I
+	x	e^{3x}
−	1	$\frac{e^{3x}}{3}$
+	0	$\frac{e^{3x}}{9}$

$\frac{e^{3x}}{3} x - \frac{e^{3x}}{9}$

8) $\int \frac{dx}{(2x+3)^2}$

C.V $\quad u = 2x+3$

$du = 2$

$dx = \frac{du}{2}$

$\frac{1}{2} \int \frac{1}{u^2} \, du$

$\frac{1}{2} \times \left(-\frac{1}{u}\right) = \frac{1}{2} \times \frac{1}{2x+3} = \frac{-1}{4x+6}$

9) $\int \frac{dx}{x^2+4}$

$\int \frac{1}{\left(\frac{x^2}{4}+1\right)\times 4} \Rightarrow \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}$

$u = \frac{x}{2}$

$du = \frac{1}{2}\, dx$

$dx = du \times 2$

$\frac{1}{2}\int \frac{1}{u^2+1}\, du$

$\frac{1}{2} \arctan(u)$

$\frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$

2° méthode ?

10) $\int \frac{dx}{2x^2+8x+10}$

$\frac{1}{2}\int \frac{1}{\ldots} = \frac{1}{2}\int \frac{1}{\ldots}$

$$= \frac{1}{2} \int \frac{}{x^2 + 4x + 5} = \frac{}{2} \int \frac{}{(x^2 + 4x + 4 + 1)}$$

$$= \frac{1}{2} \int \frac{1}{((x+2)^2 + 1)}$$

$$= \frac{1}{2} \int \frac{1}{t^2 + 1} dt \qquad t = x+2 \quad dt = dx$$

$$= \frac{1}{2} \arctan(x+2) + C$$

11) $\int \frac{2x+4}{2x^2+8x+10} = \frac{1}{2} \int \frac{2x+4}{x^2+4x+5}$

$$= \frac{1}{2} \int \frac{u'}{u} \qquad u = x^2 + 4x + 5$$

$$= \frac{1}{2} \ln(u) = \frac{1}{2} \ln(x^2 + 4x + 5)$$

12) $\int \frac{2x+5}{2x^2+8x+10}$

$$\frac{1}{2} \int \frac{2x+5}{x^2+4x+5} \Rightarrow \int \frac{2x+4}{x^2+4x+5} + \int \frac{1}{x^2+4x+5}$$

$$\boxed{a^2 + 2ab + b^2 = (a+b)^2}$$

$$\frac{1}{2}\ln(x^2+4x+5 + \int \frac{1}{x^2+4x+5}$$

$$+ \int \frac{1}{x^2+4x+4+1}$$

$$u = x+2$$
$$du = dx$$

$$+ \int \frac{1}{(x+2)^2+1}$$

$$+ \int \frac{1}{u^2+1}$$

$$\frac{1}{2}\ln(x^2+4x+5) + \arctan(x+2) + C$$

13) $$\int \frac{dx}{\sqrt{4-x^2}} = \frac{dx}{\sqrt{4} \times \sqrt{1-\frac{x^2}{4}}} = \frac{dx}{2 \times \sqrt{1-u^2}}$$

$$u = \frac{x}{2}$$
$$du = \frac{1}{2}\,dx$$
$$dx = du \times 2$$

$$\frac{1}{2} \times 2 \int \frac{du}{\sqrt{1-u^2}} = \arcsin\left(\frac{x}{2}\right) + C$$

Ex 2

1) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \Big[\arctan x\Big]_{-\infty}^{+\infty}$

$= \frac{\pi}{2} - -\frac{\pi}{2} = \pi$

2) $\displaystyle\int_0^1 x^2 \arctan(x)\, dx$

D	I
$\arctan(x)$	x^2
$\frac{1}{1+x^2}$	$\frac{x^3}{3}$

$\arctan(x) \times \frac{x^3}{3} - \displaystyle\int \frac{1}{1+x^2} \times \frac{x^3}{3}$

Indefinite integrals:

STEP 1

Take the integral:

$$\int x^2 \tan^{-1}(x)\, dx$$

STEP 2 ■ Multiple intermediate steps

For the integrand $x^2 \tan^{-1}(x)$, integrate by parts, $\int f\, dg = f\, g - \int g\, df$, where

$f = \tan^{-1}(x), \quad dg = x^2\, dx,$

$\boxed{df = \frac{1}{x^2+1}\, dx}$, $\boxed{g = \frac{x^3}{3}}$:

$$= \frac{1}{3} x^3 \tan^{-1}(x) - \int \frac{x^3}{3(x^2+1)}\, dx$$

STEP 3

Factor out constants:

$$= \frac{1}{3} x^3 \tan^{-1}(x) - \frac{1}{3} \int \frac{x^3}{x^2+1}\, dx$$

STEP 4

For the integrand $\frac{x^3}{x^2+1}$, substitute $u = x^2$ and $du = 2\, x\, dx$:

$$= \frac{1}{3} x^3 \tan^{-1}(x) - \frac{1}{6} \int \frac{u}{u+1}\, du$$

STEP 5

Show intermediate steps

For the integrand $\frac{u}{u+1}$, do long division:

$$= \frac{1}{3} x^3 \tan^{-1}(x) - \frac{1}{6} \int \boxed{1 - \frac{1}{u+1}} du$$

STEP 6

Integrate the sum term by term and factor out constants:

$$= \frac{1}{3} x^3 \tan^{-1}(x) + \frac{1}{6} \int \frac{1}{u+1} du - \frac{1}{6} \times \int 1 \, du$$

STEP 7

For the integrand $\frac{1}{u+1}$, substitute $s = u + 1$ and $ds =$ a:

$$= \frac{1}{3} x^3 \tan^{-1}(x) + \frac{1}{6} \int \frac{1}{s} ds - \frac{1}{6} \times \int 1 \, du$$

STEP 8

The integral of $\frac{1}{s}$ is $\log(s)$:

$$= \frac{1}{3} x^3 \tan^{-1}(x) + \frac{\log(s)}{6} - \frac{1}{6} \times \int 1 \, du$$

STEP 9

The integral of 1 is u:

$$= \frac{\log(s)}{6} - \frac{u}{6} + \frac{1}{3} x^3 \tan^{-1}(x) + \text{constant}$$

STEP 10

Substitute back for $s = u + 1$:

$$= -\frac{u}{6} + \frac{1}{6} \log(u + 1) + \frac{1}{3} x^3 \tan^{-1}(x) + \text{constant}$$

STEP 11

Substitute back for $u = x^2$:

$$= \frac{1}{3} x^3 \tan^{-1}(x) - \frac{x^2}{6} + \frac{1}{6} \log(x^2 + 1) + \text{constant}$$

STEP 12

Which is equal to:

Answer:

$$= \frac{1}{6} \left(2 x^3 \tan^{-1}(x) - x^2 + \log(x^2 + 1)\right) + \text{constant}$$

3) $\int^{1} \sqrt{\quad}$

$\int_{-1} \sqrt{1-x^2}\, dx$

STEP 1

Take the integral:

$$\int \sqrt{1-x^2}\; dx$$

STEP 2

For the integrand $\sqrt{1-x^2}$, substitute $x = \sin(u)$ and $dx = \cos(u)\,du$. Then $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \cos(u)$ and $u = \sin^{-1}(x)$:

$$= \int \cos^2(u)\, du$$

STEP 3

Write $\cos^2(u)$ as $\frac{1}{2}\cos(2u) + \frac{1}{2}$:

$$= \int \left(\frac{1}{2}\cos(2u) + \frac{1}{2}\right) du$$

STEP 4

Integrate the sum term by term and factor out constants:

$$= \frac{1}{2}\int \cos(2u)\, du + \frac{1}{2} \times \int 1\, du$$

STEP 5

For the integrand $\cos(2u)$, substitute $s = 2u$ and $ds = 2\,du$:

$$= \frac{1}{4}\int \cos(s)\, ds + \frac{1}{2} \times \int 1\, du$$

STEP 6

The integral of $\cos(s)$ is $\sin(s)$:

$$= \frac{\sin(s)}{4} + \frac{1}{2} \times \int 1\, du$$

STEP 7

The integral of 1 is u:

$$= \frac{\sin(s)}{4} + \frac{u}{2} + \text{constant}$$

STEP 8

Substitute back for $s = 2u$:

$$= \frac{u}{2} + \frac{1}{4}\sin(2u) + \text{constant}$$

STEP 9

Apply the double angle formula $\sin(2u) = 2\sin(u)\cos(u)$:

$$= \frac{u}{2} + \frac{1}{2}\sin(u)\cos(u) + \text{constant}$$

STEP 10

Express $\cos(u)$ in terms of $\sin(u)$ using $\cos^2(u) = 1 - \sin^2(u)$:

$$= \frac{u}{2} + \frac{1}{2}\sin(u)\sqrt{1-\sin^2(u)} + \text{constant}$$

STEP 11

Substitute back for $u = \sin^{-1}(x)$:

$$= \frac{1}{2}\sqrt{1-x^2}\,x + \frac{1}{2}\sin^{-1}(x) + \text{constant}$$

STEP 12

Which is equal to:

Answer:

$$= \frac{1}{2}\left(\sqrt{1-x^2}\,x + \sin^{-1}(x)\right) + \text{constant}$$

Ex 3

$$\int_0^{\pi/2} \sin^2 x \cos^2 x \, dx \quad \rightarrow \text{euler}$$

$$\frac{1}{4}\left(e^{2ix} - 2 + e^{-2ix}\right) \times \frac{1}{4}\left(e^{2ix} + 2 + e^{-2ix}\right)$$

$$\frac{1}{4}\left(2\cos(2x) - 2\right) \times \frac{1}{4}\left(2\cos(2x) + 2\right)$$

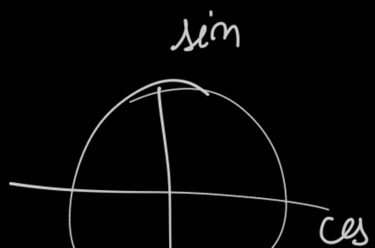
$$\left(\frac{1}{2}\cos(2x) - \frac{1}{2}\right) \times \left(\frac{1}{2}\cos(2x) + \frac{1}{2}\right)$$

$$\cos(2x) + \cancel{\frac{1}{4}\cos(2x)} - \cancel{\frac{1}{4}\cos(2x)} - \frac{1}{4}$$

$$\int \cos(2x) - \frac{1}{4}$$

$$\int \cos(2x) - \int \frac{1}{4}$$

$$\left[\frac{1}{2}\sin(2x) - \frac{1}{4}x\right]_0^{\pi/2}$$



$$\left[\frac{1}{2}\sin\left(2\times\frac{\pi}{8}\right)-\frac{1}{4}\times\frac{\pi}{2}\right]-(0)$$

$$\frac{1}{2}\times 0-\frac{\pi}{8}-0$$

$$=-\frac{\pi}{8}$$

Pas le bon resultat avant PK

$$\int_0^{\pi/2}\sin^2 x\times\cos^2 x\,dx$$

?

$$\int^{\pi/4}\sin^4(x)$$

!

$$\int_0^{\pi/3}\sin^3(x)\cos^2(x)$$

$u=\cos(x)$

$du=-\sin(x)\,dx$

$dx=\dfrac{du}{-\sin x}$

$$\int\sin(x)\times(1-\cos^2(x))\times\cos^2(x)$$

$$\int\sin x\times(1-u^2)\times u^2\times\frac{du}{-\sin}$$

$$-\int u^2\times(1-u^2)$$

$$\int u^2\ u^4\ du=-\int u^2+\int u^4=-\frac{u^3}{3}+\frac{u^5}{5}$$

$-\int u^{2}-u$

$$\left[-\frac{\cos(x)^3}{3}+\frac{\cos(x)^5}{5}\right]_0^{\pi/3} = \frac{-17}{480} - \frac{-2}{15}$$

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4)

$$\int e^{-t}\cos t \, dt$$

	D	I
+	$\cos t$	e^{-t}
−	$-\sin t$	$-e^{-t}$
+	$-\cos t$	e^{-t}

$-\cos t \times e^{-t} + \sin t \times e^{-t} - \int \cos \; e^{-t}$

$$2\int e^{-t}\cos t \, dt = e^{-t}\sin(t) - \cos t \times e^{-t} + C$$

$$\int e^{-t}\cos t \, dt = \frac{1}{2}\left(e^{-t}\sin(t) - \cos \times e^{-t} + C\right)$$

$$\int e^{-t}\sin(t) = \frac{-\sin(t)+\cos(t)}{2e^{t}}$$

Ex 5

$1) \; \frac{1}{\quad} = \frac{A}{\quad} + \frac{B}{\quad} = \frac{Ax + A + Bx}{\quad}$

$$\frac{1}{x(x+1)} = \frac{}{x} + \frac{}{(x+1)} = \frac{}{x(x+1)}$$

$$\ln(x) - \ln(x+1)$$

2) $$\frac{1}{x^2(x^2+1)} =$$

3) $$\frac{x}{x^2-4} = \frac{1}{2}\ln(x^2-4)$$

$$\begin{array}{r|l} x^3 & x^2-4 \\ \hline -\ x^3+4x & x \\ 0\ \ 4x & \end{array}$$

4)

$$\frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4} = 2\ \frac{2x}{x^2-4}$$

$$2\ln(x^2-4) + \frac{x^2}{2} + C$$

5) $$\frac{1}{(x-1)^2(x+2)} =$$

6) $\dfrac{x^4}{x^3 - x^2 + x - 1} =$?

Ex 7)

Calculer la valeur sur l'intervalle [0, 1] du signal

$$s(t) = \frac{1}{2t+3}.$$

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T s^2(t)\,dt}$$

$$V_{eff} = \sqrt{\frac{1}{1}\int_0^1 s^2(t)\,dt}$$

$$\sqrt{\int_0^1 s^2(t)\,dt}$$

$$\sqrt{\left[\frac{s^3}{3}\right]_0^1}$$

$$\sqrt{\frac{1^3}{3} - 0}$$

$$\sqrt{\frac{1}{3}} \simeq 0,57 = V_{eff}$$

Ex 8

$$f(x,y) = x \sin y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}(1 \times \sin y)$$

$$= 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}(x \times \cos y)$$

$$= -x \sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial y}\left(\sin y \right)$$

$$= \cos y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial x}\left(x \times \cos y \right)$$

$$= \left(\cos y \right)$$

Ex 9

$$I = \int_0^1 \int_0^1 \frac{1}{1+x+y}\, dx\, dy$$

$$I = \int_0^1 \left| \ln(1+x+y) \right|_0^1 dy$$

$$I = \int_0^1 \ln(2+y) - \ln(1+y)\, dy$$

$$\int_0^1 \ln(2+y) - \int \ln(1+y)$$

$$\int \ln(2+y) \times (y+2) - \int \frac{y+2}{2+y}$$

$$\ln(2+y)(y+2) - y \Big|_0^1 - \ln(1+y) \times (y+1) + y \Big|$$

D	I
$\ln(2+y)$	1
$\frac{1}{2+y}$	$y+2$

$\ln(3)(3) - 1 - \ln(2)(2)$

$\ln \frac{(3)^3}{(2)^2} - 1$

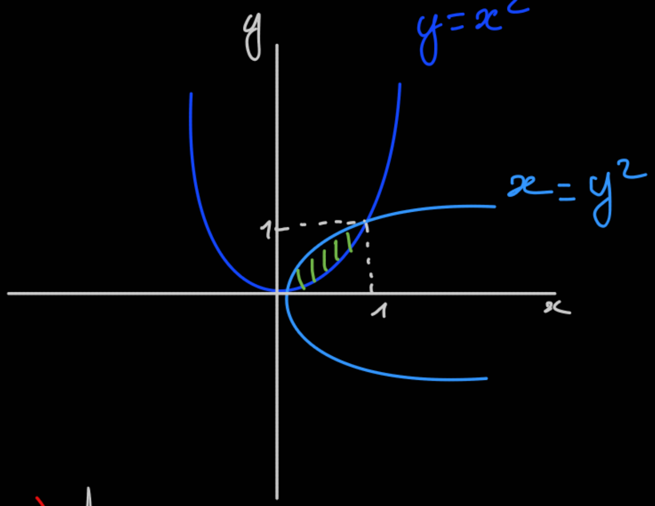
$\ln \frac{(27)}{4} \cancel{-1} \quad - \quad \ln(2) \times 2 \cancel{+1} - \ln(1) \times (1)$

$- \quad \ln(4) + 1 - \ln(1)$

$\ln\left(\frac{27}{4} \times \frac{1}{4} \times \frac{1}{1}\right)$

$\ln\left(\frac{27}{16}\right)$

<u>Exc 10</u>



$$I = \iint_\Delta x\, y\, dx\, dy = \int_0^1 x \left(\int_{x^2}^{\sqrt{x}} y\, dy \right) dx$$